



Research article

A mathematical approach for studying the fractal-fractional hybrid Mittag-Leffler model of malaria under some control factors

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Abstract: Malaria disease, which is of parasitic origin, has always been one of the challenges for human societies in areas with poor sanitation. The lack of proper distribution of drugs and lack of awareness of people in such environments cause us to see many deaths every year, especially in children under the age of five. Due to the importance of this issue, in this paper, a new five-compartmental (c_1, c_2) -fractal-fractional $SIR-SI$ -model of malaria disease for humans and mosquitoes is presented. We use the generalized Mittag-Leffler fractal-fractional derivatives to design such a mathematical model. In different ways, we study all theoretical aspects of solutions such as the existence, uniqueness and stability. A Newton polynomial that works in fractal-fractional settings is shown, which allows us to get some numerical trajectories. From the trajectories, we saw that an increase in antimalarial treatment in consideration to memory effects reduces the peak of sick individuals, and mosquito insecticide spraying minimizes the disease burden in all compartments.

Keywords: fractal-fractional hybrid operator; malaria; Newton polynomial; fixed point; stability

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1. Introduction

The well-known disease malaria is one of the illnesses threatening human health that appears under the influence of a parasitic infectious agent in female *Anopheles* mosquitoes. In fact, *Plasmodium* parasites are the main cause of this infection, which causes disease by implanting in the red blood cells of an infected person, and among the different types of these parasites, two of them are among the most common pathogenic parasites: *Plasmodium vivax* and *Plasmodium falciparum* [1]. The malaria parasite is transmitted to the human body and its bloodstream by biting infected female *Anopheles* mosquitoes. Also, the process of transmitting the parasite can occur through blood transfusions or even infection of the fetus through its pregnant mother who carries the infection [2].

Although this disease will not be dangerous if it is detected in time and treated properly, it has negative effects on human health and social life. The disease is most commonly reported in the tropical regions of Africa and Asia and imposes heavy financial burdens on families and governments. According to extensive studies on the spread and control of the disease by various medical institutes and associations, it still threatens public health and causes deaths in children under five in disadvantaged and less developed countries [3]. Irregular vaccination or improper distribution of antimalarial drugs can spread the disease. Even due to the nature of the parasites, sometimes, with widespread changes in the climate and the surrounding environment, the drug resistance of the parasites is impaired and causes the drugs to lose their effectiveness. Currently, the simplest advice for people in high-risk areas is to use bedside nets and window sills, which, to a large extent, prevent the number of bites during sleep or indoors. With widespread social and environmental changes and the effects of climate change in recent years, there has been a need to study the biology of host parasites carefully. The study of behaviors and dynamics of these parasites in the context of different methods of disease control and treatment has been one of the important points in recent research studies that have attracted the attention of researchers.

In this regard, various mathematical models came to the aid of researchers to simulate the exact dynamic behaviors of the transmission and spread of different types of viruses and infectious parasites. Also, by providing and designing different treatment and control methods, one can evaluate and predict the amount of prevalence during a specific time period. In 2001, Yang [4] considered a mathematical structure to model the transmission of malaria based on two factors, i.e., global warming and socio-economic conditions; further, Yang completed the study by providing the sensitivity analysis. In 2008, Chiyaka et al. [5] turned to control strategies on a deterministic model of malaria during two latent periods and analyzed some qualitative criteria to calculate the vaccination rate. One year later, Rafikov et al. [6] implemented some strategies on malaria to compute the optimal control index with the help of some genetic modifications on vector mosquitoes. In 2011, Mandal et al. [7] published a review paper on the different models of malaria disease. After that, in 2012, Agosto et al. [8] extended control strategies on malaria by adding three indexes, including bed nets, treatment and the use of the spray. In 2013, Abdullahi et al. [9] designed their mathematical model of malaria by investigating the effectiveness of drugs. Recently, in 2017, Senthamarai et al. [10] gave a multi-compartmental model of malaria and predicted the behavior of solutions based on numerical algorithms obtained by the homotopy method.

However, for the sake of the existing limitations in the above methods and models, and due to the locality of the integer-order operators, in recent years, the tools and operators in fractional calculus

have begun to provide considerable simulation and prediction power for mathematicians and physicians to be able to evaluate a variety of biological mathematical models in the form of finite-dimensional systems of fractional differential equations. The extent and increasing efficiency of such mathematical fractional modeling can be seen in the study of various types of diseases, and even in various fields of engineering, such as the models presented for COVID-19 [11–14], HIV [15], anthrax [16], hepatitis C [17], circuits of memristor type [18], hyperchaotic modeling [19], persistence of infections in the environment [20], the Langevin systems [21], genetic networks [22], well-known viruses such as mumps [23] and Zika [24, 25], mosaic disease [26], some models of viruses in computers [27], thermostat control [28, 29], pantograph system [30, 31], canine virus [32], Q-fever [33], p-Laplacian systems [34], the co-dynamics of diabetes and COVID-19 [35], the modeling of glucose [36], Navier equations [37], etc.

Numerical methods and simulation techniques and the application of fractional calculus are not limited to the above-mentioned cases. This fractionalization has also been used to design various research models of malaria. For example, Badshah and Akbar [38] chose a region in Pakistan called Khyber Pakhtunkhwa; they designed a four-compartmental fractional model of malaria based on the given data in that city and investigated the qualitative properties of the model via the Routh-Hurwitz condition. In the same year, Pawar et al. [39] gave another fractional model of malaria for two classes of non-immune and semi-immune peoples; they analyzed it analytically and solved it numerically via the generalized Euler criterion. In [40], ul Rehman et al. considered two factors of the temporary immunity and relapse of disease. They then defined a new model of malaria in the context of the fractional Caputo operator. Recently, in 2022, Cui et al. [41] presented an advanced fractional model for malaria, entitled the delayed Ross-Macdonald model, and reviewed all properties of the solutions. The role of insecticides and treatment was investigated in a new fractional Atangana-Baleanu model of malaria designed by Sinan et al. [42] in 2022.

By studying the published manuscripts of the last few years, we found an article of Atangana [43] in which a newly structured operator of generalized derivatives has been defined, i.e., fractal-fractional derivatives, which is considered a junction between fractal calculus and fractional calculus. He also discussed fractal-fractional integrals. These derivatives have the same form as the convolution of the generalized Mittag-Leffler law, exponential law and power-law with fractal derivatives. Such operators include two components, i.e., fractional order and fractal dimension. Taking into account the good accuracy of operators in the simulation of fractal-fractional systems, most researchers have focused on such fractal-fractional models. Gomez-Aguilar et al. [44] turned to a model of malaria using these fractal-fractional derivatives. The behavior of the coronavirus was discussed in Pakistan by Shah and his colleagues, who used fractal-fractional operators [45]. Ali et al. examined a fractal-fractional model of COVID-19 based on some statistical data in Wuhan [46], as well as other manuscripts like [47–51].

In this paper, we aim to present a new model of malaria disease in the framework of the Atangana-Baleanu fractal-fractional derivatives for two groups of populations, i.e., humans and mosquitoes. Our model is a generalization of standard integer-order and fractional-order models to a (c_1, c_2) -fractal-fractional-order structure for the first time. In our new model, the fractal dimension c_2 and fractional order c_1 play considerable roles in the simulations of solutions. In addition, we use a new numerical method to solve these generalized fractal-fractional operators. We use Newton polynomials to give numerical solutions with two fractal and fractional parameters.

This forms the novelty of our model and method in consideration of the other models presented for

malaria until now. In relation to our main contribution to the present study, it is necessary to emphasize that we divide our target population into five different groups of humans and mosquitoes and then provide parameters and rates for which we can measure the effects of vaccination, antimalarial drugs and spraying on the control and reduction of this disease. Our model will be discussed from several perspectives. Because our fractal-fractional model is newly structured, we investigate existence theory via the Leray-Schauder alternative fixed-point theorem. Furthermore, the Banach contraction is utilized to obtain a unique solution. Further, other types of stable solutions are studied here for the suggested model. To simulate it, we use the new method of Newton polynomials in the fractal-fractional version. The findings and effects of fractal-fractional orders on the dynamics of solutions are analyzed, and the graphs are plotted using MATLAB.

2. Preliminaries

In the present section, we state some definitions of the generalized fractal-fractional operators. We refer the readers to [43] for more information.

Let a continuous map $z : (t_0, b) \rightarrow [0, \infty)$ be fractal-differentiable of dimension c_2 . The (c_1, c_2) -fractal-fractional derivative of the function z of the generalized Mittag-Leffler-type kernel in the Riemann-Liouville (RL) sense is given by

$${}^{\text{FFML}}\mathcal{D}_{t_0,t}^{(c_1,c_2)}z(t) = \frac{\mathbf{AB}(c_1)}{1-c_1} \frac{d}{dt^{c_2}} \int_{t_0}^t E_{c_1} \left[-\frac{c_1}{1-c_1}(t-q)^{c_1} \right] z(q) dq, \quad 0 < c_1, c_2 \leq 1, \quad (2.1)$$

with

$$\frac{dz(q)}{dq^{c_2}} = \lim_{t \rightarrow q} \frac{z(t) - z(q)}{t^{c_2} - q^{c_2}},$$

which is the fractal derivative; also,

$$\mathbf{AB}(c_1) = 1 - c_1 + \frac{c_1}{\Gamma(c_1)}$$

and $\mathbf{AB}(0) = \mathbf{AB}(1) = 1$ [43].

Simply, we see that the fractal-fractional derivative ${}^{\text{FFML}}\mathcal{D}_{t_0,t}^{(c_1,c_2)}$ is transformed into the standard c_1^{th} -RL derivative ${}^{\text{RL}}\mathcal{D}_{t_0,t}^{c_1}$ by assuming $c_2 = 1$.

By considering such a function z with the above properties, the (c_1, c_2) -fractal-fractional integral with the Mittag-Leffler-type kernel is defined by

$${}^{\text{FFML}}\mathcal{I}_{t_0,t}^{(c_1,c_2)}z(t) = \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_{t_0}^t q^{c_2-1} (t-q)^{c_1-1} z(q) dq + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} z(t) \quad (2.2)$$

if the integral is finite-valued, where $c_1, c_2 > 0$ [43].

3. Description of (c_1, c_2) -fractal-fractional $SIR-SI$ -model

In 2019, Kumar et al. [52] designed a new model of malaria with the operators involving the exponential law. In fact, they were motivated by the standard system of differential equations given by

$$\left\{ \begin{array}{l} \frac{d\mathbb{H}^S(t)}{dt} = \Theta_{\mathbb{H}} + \delta\mathbb{H}^R(t) - (\eta a_1 \mathbb{H}^I(t) + \lambda a_2 \mathbb{M}^I(t))\mathbb{H}^S(t) - (p + f_{\mathbb{H}})\mathbb{H}^S(t), \\ \frac{d\mathbb{H}^I(t)}{dt} = \varpi\mathbb{H}^I(t) + (\eta a_1 \mathbb{H}^I(t) + \lambda a_2 \mathbb{M}^I(t))\mathbb{H}^S(t) - (f_{\mathbb{H}} + \omega + \gamma\beta)\mathbb{H}^I(t), \\ \frac{d\mathbb{H}^R(t)}{dt} = \gamma\beta\mathbb{H}^I(t) - (f_{\mathbb{H}} + \delta)\mathbb{H}^R(t) + p\mathbb{H}^S(t), \\ \frac{d\mathbb{M}^S(t)}{dt} = \Theta_{\mathbb{M}} - (\kappa a_3 \mathbb{H}^I(t) + f_{\mathbb{M}} + \alpha)\mathbb{M}^S(t), \\ \frac{d\mathbb{M}^I(t)}{dt} = \kappa a_3 \mathbb{H}^I(t)\mathbb{M}^S(t) - (f_{\mathbb{M}} + \alpha)\mathbb{M}^I(t), \end{array} \right. \quad (3.1)$$

where the total population of the humans and the total population of mosquitoes are divided into three categories and two categories, respectively. Kumar et al. [52] generalized the standard model (3.1) to a form of the fractional SIRS-SI model with the Caputo-Fabrizio derivative, which involves memory effects, as

$$\left\{ \begin{array}{l} {}^{\text{CF}}\mathcal{D}_{0,t}^c \mathbb{H}^S(t) = \Theta_{\mathbb{H}} + \delta\mathbb{H}^R(t) - (\eta a_1 \mathbb{H}^I(t) + \lambda a_2 \mathbb{M}^I(t))\mathbb{H}^S(t) - (p + f_{\mathbb{H}})\mathbb{H}^S(t), \\ {}^{\text{CF}}\mathcal{D}_{0,t}^c \mathbb{H}^I(t) = \varpi\mathbb{H}^I(t) + (\eta a_1 \mathbb{H}^I(t) + \lambda a_2 \mathbb{M}^I(t))\mathbb{H}^S(t) - (f_{\mathbb{H}} + \omega + \gamma\beta)\mathbb{H}^I(t), \\ {}^{\text{CF}}\mathcal{D}_{0,t}^c \mathbb{H}^R(t) = \gamma\beta\mathbb{H}^I(t) - (f_{\mathbb{H}} + \delta)\mathbb{H}^R(t) + p\mathbb{H}^S(t), \\ {}^{\text{CF}}\mathcal{D}_{0,t}^c \mathbb{M}^S(t) = \Theta_{\mathbb{M}} - (\kappa a_3 \mathbb{H}^I(t) + f_{\mathbb{M}} + \alpha)\mathbb{M}^S(t), \\ {}^{\text{CF}}\mathcal{D}_{0,t}^c \mathbb{M}^I(t) = \kappa a_3 \mathbb{H}^I(t)\mathbb{M}^S(t) - (f_{\mathbb{M}} + \alpha)\mathbb{M}^I(t). \end{array} \right. \quad (3.2)$$

In the above model, the authors introduced the categories $\mathbb{H}^S(t)$, $\mathbb{H}^I(t)$ and $\mathbb{H}^R(t)$ as the number of susceptible, infected and recovered persons, respectively, and also introduced two categories $\mathbb{M}^S(t)$ and $\mathbb{M}^I(t)$ as the number of susceptible and infected mosquitoes, respectively, at the time $t \in I := [0, \tau]$, $(\tau > 0)$.

To achieve more accurate numerical results in an efficient manner, due to the important role of two parameters of fractional order and dimension order in fractal-fractional operators for exact simulations, and motivated by both the standard and fractional models (3.1) and (3.2), we design and give a mathematical five-compartmental (c_1, c_2) -fractal-fractional $SIR-SI$ -model of malaria disease under antimalarial treatments with the generalized Mittag-Leffler-type kernel between two populations of humans (\mathbb{H}) and mosquitoes (\mathbb{M}) (shortly, (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $SIR-SI$ -model of

malaria), which takes a form

$$\left\{ \begin{array}{l} \text{FFML } \mathcal{D}_{0,t}^{(c_1,c_2)} \mathbb{H}^S(t) = \Theta_{\mathbb{H}} + \delta \mathbb{H}^R(t) - (\eta a_1 \mathbb{H}^I(t) + \lambda a_2 \mathbb{M}^I(t)) \mathbb{H}^S(t) - (p + f_{\mathbb{H}}) \mathbb{H}^S(t), \\ \text{FFML } \mathcal{D}_{0,t}^{(c_1,c_2)} \mathbb{H}^I(t) = \varpi \mathbb{H}^I(t) + (\eta a_1 \mathbb{H}^I(t) + \lambda a_2 \mathbb{M}^I(t)) \mathbb{H}^S(t) - (f_{\mathbb{H}} + \omega + \gamma \beta) \mathbb{H}^I(t), \\ \text{FFML } \mathcal{D}_{0,t}^{(c_1,c_2)} \mathbb{H}^R(t) = \gamma \beta \mathbb{H}^I(t) - (f_{\mathbb{H}} + \delta) \mathbb{H}^R(t) + p \mathbb{H}^S(t), \\ \text{FFML } \mathcal{D}_{0,t}^{(c_1,c_2)} \mathbb{M}^S(t) = \Theta_{\mathbb{M}} - (\kappa a_3 \mathbb{H}^I(t) + f_{\mathbb{M}} + \alpha) \mathbb{M}^S(t), \\ \text{FFML } \mathcal{D}_{0,t}^{(c_1,c_2)} \mathbb{M}^I(t) = \kappa a_3 \mathbb{H}^I(t) \mathbb{M}^S(t) - (f_{\mathbb{M}} + \alpha) \mathbb{M}^I(t), \end{array} \right. \quad (3.3)$$

subject to

$$\mathbb{H}^S(0) = \varkappa_1, \quad \mathbb{H}^I(0) = \varkappa_2, \quad \mathbb{H}^R(0) = \varkappa_3, \quad \mathbb{M}^S(0) = \varkappa_4, \quad \mathbb{M}^I(0) = \varkappa_5,$$

where all the state functions $\mathbb{H}^S(t)$, $\mathbb{H}^I(t)$, $\mathbb{H}^R(t)$, $\mathbb{M}^S(t)$ and $\mathbb{M}^I(t)$ are similar to those introduced in the Caputo-Fabrizio model (3.2). Also, $\text{FFML } \mathcal{D}_{0,t}^{(c_1,c_2)}$ is the (c_1, c_2) -fractal-fractional derivation operator equipped with the fractional order c_1 and fractal order c_2 so that $c_1, c_2 \in (0, 1]$; and, the kernel of the operator is of the generalized Mittag-Leffler type.

In the mentioned (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $SIR-SI$ -model (3.3) of malaria, all parameters are positive and the initial values $\varkappa_1, \dots, \varkappa_5$ are nonnegative. Also, $p \in [0, 1]$, $\beta \in [0.01, 1]$ and $\alpha \in [0, 1]$. The structure of the (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $SIR-SI$ -model (3.3) of malaria can be described as follows.

The symbol η shows the average number of blood transfusions among the infected and susceptible categories over a fixed period of time. The symbol κ states the average number of susceptible mosquito bites to the infected individuals over a fixed period of time. Accordingly, λ shows the average number of infected mosquito bites on a susceptible individual over a fixed period of time. Moreover, the newly born individuals have belonged to the susceptible category \mathbb{H}^S via a constant rate $\Theta_{\mathbb{H}}$ per unit of time. Similarly, newly born mosquitoes have belonged to the susceptible category \mathbb{M}^S via a constant rate $\Theta_{\mathbb{M}}$ per unit of time.

The symbol a_1 interprets the probability of transmission of the disease to susceptible individuals through infected individuals, a_2 is the probability of transmission of the disease from infected mosquitoes to susceptible peoples and a_3 stands for the probability of transmission of the disease to susceptible mosquitoes through the infected population. The rate of personal healing is denoted by γ , and the symbol β denotes the power of antimalarial medicines.

In this model, we see that the people belonging to the susceptible category \mathbb{H}^S are transferred to the infected categories \mathbb{H}^I and \mathbb{M}^I by way of the blood exchanging via the rate ηa_1 , or due to an infected mosquito bite via the rate λa_2 for each unit of time, respectively. Also, due to vaccination at the rate p for each unit of time, the people belonging to the susceptible category \mathbb{H}^S transfer into the recovered category \mathbb{H}^R . The people belonging to the susceptible category \mathbb{H}^S die at the rate $f_{\mathbb{H}}$. Moreover, the individuals belonging to the infected category \mathbb{H}^I die at the rate $f_{\mathbb{H}}$ and expire due to the malaria disease at the rate ω for each unit of time.

The people belonging to the recovered category \mathbb{H}^R also die at the rate $f_{\mathbb{H}}$ for each unit of time. The susceptible and infected mosquitoes die due to the exercise of spraying at the rate α for each unit of time. The infected mosquitoes belonging to \mathbb{M}^I die at the rate $f_{\mathbb{M}}$ for each unit of time.

The newly born children are infected by malaria disease through the mother at the rate ϖ for each unit of time. The individuals in the infected category \mathbb{H}^I can join the recovered category as a result of using antimalarial drugs at the rate $\gamma\beta$ for each unit of time. The mosquitoes belonging to the susceptible category \mathbb{M}^S transfer to the infected category \mathbb{M}^I by being bitten by an infected mosquito at the rate κa_3 for each unit of time, or they die at the rate $f_{\mathbb{M}}$ for each unit of time. The symbol δ is used for the mean value per capita rate of loss of immunity for each unit of time.

4. Existence results

In this section, we get help from the well-known theorems of fixed-point theory to investigate the existence property. Take the Banach space $\mathcal{X} = \mathbb{F}^5$, and then assume $\mathbb{F} = C(I, \mathbb{R})$ with the supremum norm

$$\|\mathbf{K}\|_{\mathcal{X}} = \|(\mathbb{H}^S, \mathbb{H}^I, \mathbb{H}^R, \mathbb{M}^S, \mathbb{M}^I)\|_{\mathcal{X}} = \sup \{|P(t)| : t \in I\}$$

for

$$|P| := |\mathbb{H}^S| + |\mathbb{H}^I| + |\mathbb{H}^R| + |\mathbb{M}^S| + |\mathbb{M}^I|.$$

Further, for simplicity, the right-hand side of the (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $SIR-SI$ -model (3.3) of malaria can be rewritten as

$$\begin{cases} \mathbf{X}_1(t, \mathbb{H}^S(t)) = \Theta_{\mathbb{H}} + \delta \mathbb{H}^R(t) - (\eta a_1 \mathbb{H}^I(t) + \lambda a_2 \mathbb{M}^I(t)) \mathbb{H}^S(t) - (p + f_{\mathbb{H}}) \mathbb{H}^S(t), \\ \mathbf{X}_2(t, \mathbb{H}^I(t)) = \varpi \mathbb{H}^I(t) + (\eta a_1 \mathbb{H}^I(t) + \lambda a_2 \mathbb{M}^I(t)) \mathbb{H}^S(t) - (f_{\mathbb{H}} + \omega + \gamma\beta) \mathbb{H}^I(t), \\ \mathbf{X}_3(t, \mathbb{H}^R(t)) = \gamma\beta \mathbb{H}^I(t) - (f_{\mathbb{H}} + \delta) \mathbb{H}^R(t) + p \mathbb{H}^S(t), \\ \mathbf{X}_4(t, \mathbb{M}^S(t)) = \Theta_{\mathbb{M}} - (\kappa a_3 \mathbb{H}^I(t) + f_{\mathbb{M}} + \alpha) \mathbb{M}^S(t), \\ \mathbf{X}_5(t, \mathbb{M}^I(t)) = \kappa a_3 \mathbb{H}^I(t) \mathbb{M}^S(t) - (f_{\mathbb{M}} + \alpha) \mathbb{M}^I(t). \end{cases} \quad (4.1)$$

Accordingly, the (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $SIR-SI$ -model (3.3) of malaria is reformulated as

$$\begin{cases} {}^{\text{ABR}}\mathcal{D}_{0,t}^{c_1} \mathbb{H}^S(t) = c_2 t^{c_2-1} \mathbf{X}_1(t, \mathbb{H}^S(t)), \\ {}^{\text{ABR}}\mathcal{D}_{0,t}^{c_1} \mathbb{H}^I(t) = c_2 t^{c_2-1} \mathbf{X}_2(t, \mathbb{H}^I(t)), \\ {}^{\text{ABR}}\mathcal{D}_{0,t}^{c_1} \mathbb{H}^R(t) = c_2 t^{c_2-1} \mathbf{X}_3(t, \mathbb{H}^R(t)), \\ {}^{\text{ABR}}\mathcal{D}_{0,t}^{c_1} \mathbb{M}^S(t) = c_2 t^{c_2-1} \mathbf{X}_4(t, \mathbb{M}^S(t)), \\ {}^{\text{ABR}}\mathcal{D}_{0,t}^{c_1} \mathbb{M}^I(t) = c_2 t^{c_2-1} \mathbf{X}_5(t, \mathbb{M}^I(t)). \end{cases} \quad (4.2)$$

Here, take into account the system (4.2), and reconstruct it as a compact initial value problem, like

$$\begin{cases} {}^{\text{ABR}}\mathcal{D}_{0,s}^{c_1} \mathbf{K}(t) = c_2 t^{c_2-1} \mathbf{X}(t, \mathbf{K}(t)), \\ \mathbf{K}(0) = \mathbf{K}_0, \end{cases} \quad (4.3)$$

by assuming

$$\mathbf{K}(t) = (\mathbb{H}^S(t), \mathbb{H}^I(t), \mathbb{H}^R(t), \mathbb{M}^S(t), \mathbb{M}^I(t))^T,$$

$$\mathbf{K}_0 = (\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4, \varkappa_5)^T, \quad c_1, c_2 \in (0, 1], \quad (4.4)$$

and

$$\mathbf{X}(t, \mathbf{K}(t)) = \begin{cases} \mathbf{X}_1(t, \mathbb{H}^S(t)), \\ \mathbf{X}_2(t, \mathbb{H}^I(t)), \\ \mathbf{X}_3(t, \mathbb{H}^R(t)), \\ \mathbf{X}_4(t, \mathbb{M}^S(t)), \\ \mathbf{X}_5(t, \mathbb{M}^I(t)), \quad t \in I. \end{cases} \quad (4.5)$$

The non-singular Atangana-Baleanu-Reimann-Liouville fractional derivative changes (4.3) to

$$\frac{\mathbf{AB}(c_1)}{1-c_1} \frac{d}{dt} \int_0^t E_{c_1} \left[-\frac{c_1}{1-c_1} (t-q)^{c_1} \right] \mathbf{K}(q) dq = c_2 t^{c_2-1} \mathbf{X}(t, \mathbf{K}(t)). \quad (4.6)$$

The Atangana-Baleanu fractal-fractional integral on (4.6) gives

$$\mathbf{K}(t) = \mathbf{K}(0) + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}(t, \mathbf{K}(t)) + \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathbf{X}(q, \mathbf{K}(q)) dq. \quad (4.7)$$

The following extensions of the base system of fractal-fractional integral equations are given as

$$\left\{ \begin{array}{l} \mathbb{H}^S(t) = \varkappa_1 + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_1(t, \mathbb{H}^S(t)) \\ \quad + \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathbf{X}_1(q, \mathbb{H}^S(q)) dq, \\ \mathbb{H}^I(t) = \varkappa_2 + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_2(t, \mathbb{H}^I(t)) \\ \quad + \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathbf{X}_2(q, \mathbb{H}^I(q)) dq, \\ \mathbb{H}^R(t) = \varkappa_3 + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_3(t, \mathbb{H}^R(t)) \\ \quad + \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathbf{X}_3(q, \mathbb{H}^R(q)) dq, \\ \mathbb{M}^S(t) = \varkappa_4 + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_4(t, \mathbb{M}^S(t)) \\ \quad + \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathbf{X}_4(q, \mathbb{M}^S(q)) dq, \\ \mathbb{M}^I(t) = \varkappa_5 + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_5(t, \mathbb{M}^I(t)) \\ \quad + \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathbf{X}_5(q, \mathbb{M}^I(q)) dq. \end{array} \right.$$

A new map, to make a fixed-point problem, is defined by $T : \mathcal{X} \rightarrow \mathcal{X}$, which has been formulated as

$$T(\mathbf{K}(t)) = \mathbf{K}(0) + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}(t, \mathbf{K}(t)) + \frac{c_1 c_2}{\Gamma(c_1)\mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathbf{X}(q, \mathbf{K}(q)) dq. \quad (4.8)$$

To study the existence of solutions of the (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $SIR-SI$ -model (3.3) of malaria, we use the following theorem:

Theorem 4.1 ([53]). (*Leray-Schauder theorem*) Let \mathcal{X} be a Banach space, $\mathcal{A} \subset \mathcal{X}$ a convex closed bounded set, $\mathbb{G} \subset \mathcal{A}$ an open set and $0 \in \mathbb{G}$. Then, for the continuous and compact function $T : \mathbb{G} \rightarrow \mathcal{A}$, we have one of the following:

- (hy1) There is $x \in \mathbb{G}$ such that $x = T(x)$, or
 (hy2) There is $x \in \partial\mathbb{G}$ and $0 < \mu < 1$ such that $x = \mu T(x)$.

Theorem 4.2. Let $\mathbf{X} \in C(I \times \mathcal{X}, \mathcal{X})$, and we have the following:

- (g1) There are $\mathcal{F} \in L^1(I, \mathbb{R}^+)$ and a nondecreasing function $\Psi \in C([0, \infty), (0, \infty))$ such that, for each $t \in I$ and $\mathbf{K} \in \mathcal{X}$,

$$|\mathbf{X}(t, \mathbf{K}(t))| \leq \mathcal{F}(t)\Psi(|\mathbf{K}(t)|);$$

- (g2) There is $\mathfrak{R} > 0$ such that

$$\frac{\mathfrak{R}}{\left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2)\mathbf{AB}(c_1)} \right] \mathcal{F}_0^* \Psi(\mathfrak{R}) + \mathbf{K}_0} > 1, \quad (4.9)$$

with $\mathcal{F}_0^* = \sup_{t \in I} |\mathcal{F}(t)|$.

Then, for the FF-system (4.3) and the (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $SIR-SI$ -model (3.3) of malaria, at least one solution exists on I .

Proof. Consider $T : \mathcal{X} \rightarrow \mathcal{X}$ defined by (4.8) and

$$N_R = \{\mathbf{K} \in \mathcal{X} : \|\mathbf{K}\|_{\mathcal{X}} \leq R\}, \quad R > 0.$$

The continuity of \mathbf{X} implies the same property for T . The existing inequality in (g1) gives

$$\begin{aligned} |T(\mathbf{K}(t))| &\leq |\mathbf{K}(0)| + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} |\mathbf{X}(t, \mathbf{K}(t))| \\ &\quad + \frac{c_1 c_2}{\Gamma(c_1)\mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} |\mathbf{X}(q, \mathbf{K}(q))| dq \\ &\leq \mathbf{K}_0 + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathcal{F}(t)\Psi(|\mathbf{K}(t)|) \\ &\quad + \frac{c_1 c_2}{\Gamma(c_1)\mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathcal{F}(q)\Psi(|\mathbf{K}(q)|) dq \end{aligned}$$

$$\begin{aligned} &\leq \mathbf{K}_0 + \frac{(1-c_1)c_2\tau^{c_2-1}}{\mathbf{AB}(c_1)}\mathcal{F}_0^*\Psi(R) + \frac{c_1c_2\tau^{c_1+c_2-1}B(c_1, c_2)}{\Gamma(c_1)\mathbf{AB}(c_1)}\mathcal{F}_0^*\Psi(R) \\ &= \mathbf{K}_0 + \frac{(1-c_1)c_2\tau^{c_2-1}}{\mathbf{AB}(c_1)}\mathcal{F}_0^*\Psi(R) + \frac{c_1c_2\tau^{c_1+c_2-1}\Gamma(c_2)}{\mathbf{AB}(c_1)\Gamma(c_1+c_2)}\mathcal{F}_0^*\Psi(R) \end{aligned}$$

for $\mathbf{K} \in N_R$. We get

$$\|T\mathbf{K}\|_{\mathbb{X}} \leq \mathbf{K}_0 + \left[\frac{(1-c_1)c_2\tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1c_2\tau^{c_1+c_2-1}\Gamma(c_2)}{\mathbf{AB}(c_1)\Gamma(c_1+c_2)} \right] \mathcal{F}_0^*\Psi(R) < \infty. \quad (4.10)$$

So, $\|T\mathbf{K}\|_{\mathbb{X}} < \infty$ and T is uniformly bounded on \mathcal{X} . Let $t, z \in [0, \tau]$ with $t < z$ and $\mathbf{K} \in N_R$. Let

$$\sup_{(t, \mathbf{K}) \in I \times N_R} |\mathbf{X}(t, \mathbf{K}(t))| = \mathbf{X}^* < \infty.$$

Then,

$$\begin{aligned} |T(\mathbf{K}(z)) - T(\mathbf{K}(t))| &\leq \left| \frac{(1-c_1)c_2z^{c_2-1}}{\mathbf{AB}(c_1)}\mathbf{X}(z, \mathbf{K}(z)) - \frac{(1-c_1)c_2t^{c_2-1}}{\mathbf{AB}(c_1)}\mathbf{X}(t, \mathbf{K}(t)) \right. \\ &\quad + \frac{c_1c_2}{\Gamma(c_1)\mathbf{AB}(c_1)} \int_0^z q^{c_2-1}(z-q)^{c_1-1}\mathbf{X}(q, \mathbf{K}(q)) \, dq \\ &\quad \left. - \frac{c_1c_2}{\Gamma(c_1)\mathbf{AB}(c_1)} \int_0^t q^{c_2-1}(t-q)^{c_1-1}\mathbf{X}(q, \mathbf{K}(q)) \, dq \right| \\ &\leq \frac{(1-c_1)c_2\mathbf{X}^*}{\mathbf{AB}(c_1)}(z^{c_2-1} - t^{c_2-1}) \\ &\quad + \frac{c_1c_2\mathbf{X}^*}{\Gamma(c_1)\mathbf{AB}(c_1)} \left| \int_0^z q^{c_2-1}(z-q)^{c_1-1} \, dq - \int_0^t q^{c_2-1}(t-q)^{c_1-1} \, dq \right| \\ &\leq \frac{(1-c_1)c_2\mathbf{X}^*}{\mathbf{AB}(c_1)}(z^{c_2-1} - t^{c_2-1}) + \frac{c_1c_2\mathbf{X}^*B(c_1, c_2)}{\Gamma(c_1)\mathbf{AB}(c_1)}[z^{c_1+c_2-1} - t^{c_1+c_2-1}] \\ &= \frac{(1-c_1)c_2\mathbf{X}^*}{\mathbf{AB}(c_1)}(z^{c_2-1} - t^{c_2-1}) + \frac{c_1c_2\mathbf{X}^*\Gamma(c_2)}{\Gamma(c_1+c_2)\mathbf{AB}(c_1)}[z^{c_1+c_2-1} - t^{c_1+c_2-1}], \end{aligned} \quad (4.11)$$

where (independent of \mathbf{K}) (4.11) converges to 0 if $z \rightarrow t$. So,

$$\lim_{z \rightarrow t} \|T(\mathbf{K}(z)) - T(\mathbf{K}(t))\|_{\mathcal{X}} = 0,$$

and T is equicontinuous and compact on N_R by the Arzelà–Ascoli theorem. By Theorem 4.1, either (hy1) or (hy2) is to be held. From (g2), take

$$\Phi := \{ \mathbf{K} \in \mathcal{X} : \|\mathbf{K}\|_{\mathcal{X}} < \mathfrak{R} \}$$

for some $\mathfrak{R} > 0$, with $\mathbf{K}_0 + \left[\frac{(1-c_1)c_2\tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1c_2\tau^{c_1+c_2-1}\Gamma(c_2)}{\Gamma(c_1+c_2)\mathbf{AB}(c_1)} \right] \mathcal{F}_0^* \Psi(\mathfrak{R}) < \mathfrak{R}$. With the help of (g1) and by (4.10), we estimate

$$\|T\mathbf{K}\|_{\mathcal{X}} \leq \mathbf{K}_0 + \left[\frac{(1-c_1)c_2\tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1c_2\tau^{c_1+c_2-1}\Gamma(c_2)}{\Gamma(c_1+c_2)\mathbf{AB}(c_1)} \right] \mathcal{F}_0^* \Psi(\|\mathbf{K}\|_{\mathcal{X}}). \quad (4.12)$$

By taking into account the existence of $\mathbf{K} \in \partial\Phi$ and $0 < \mu < 1$ with $\mathbf{K} = \mu T(\mathbf{K})$, for these choices of \mathbf{K} and μ , and by (4.12), we have

$$\begin{aligned} \mathfrak{R} = \|\mathbf{K}\|_{\mathcal{X}} = \mu\|T\mathbf{K}\|_{\mathcal{X}} &< \mathbf{K}_0 + \left[\frac{(1-c_1)c_2\tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1c_2\tau^{c_1+c_2-1}\Gamma(c_2)}{\Gamma(c_1+c_2)\mathbf{AB}(c_1)} \right] \mathcal{F}_0^* \Psi(\|\mathbf{K}\|_{\mathcal{X}}) \\ &< \mathbf{K}_0 + \left[\frac{(1-c_1)c_2\tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1c_2\tau^{c_1+c_2-1}\Gamma(c_2)}{\Gamma(c_1+c_2)\mathbf{AB}(c_1)} \right] \mathcal{F}_0^* \Psi(\mathfrak{R}) < \mathfrak{R}, \end{aligned}$$

which is impossible. Thus, the item (hy2) is not fulfilled and T has a fixed point in $\bar{\Phi}$ (from Theorem 4.1), which is the solution of the (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional SIR - SI -model (3.3) of malaria. \square

5. Uniqueness result

To establish the uniqueness of the solution of the given (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional SIR - SI -model (3.3) of malaria, we investigate the Lipschitz property for the functions \mathbf{X}_j , ($j = 1, \dots, 5$) defined in (4.1).

Lemma 5.1. Let $\mathbb{H}^S, \mathbb{H}^I, \mathbb{H}^R, \mathbb{M}^S, \mathbb{M}^I, \mathbb{H}_*^S, \mathbb{H}_*^I, \mathbb{H}_*^R, \mathbb{M}_*^S, \mathbb{M}_*^I \in \mathbb{F} := C(I, \mathbb{R})$ and

(G3) $\|\mathbb{H}^S\| \leq v_1$, $\|\mathbb{H}^I\| \leq v_2$, $\|\mathbb{H}^R\| \leq v_3$, $\|\mathbb{M}^S\| \leq v_4$ and $\|\mathbb{M}^I\| \leq v_5$ for some real constants $v_1, v_2, v_3, v_4, v_5 > 0$.

Then, the functions $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4$ and \mathbf{X}_5 defined by (4.1) are Lipschitz if $L_1, L_2, L_3, L_4, L_5 > 0$, with

$$\begin{aligned} L_1 &= \eta a_1 v_2 + \lambda a_2 v_5 + p + f_{\mathbb{H}}, & L_2 &= \varpi + \eta a_1 v_1 + (f_{\mathbb{H}} + \omega + \gamma\beta), \\ L_3 &= f_{\mathbb{H}} + \delta, & L_4 &= \kappa a_3 v_4 + f_{\mathbb{M}} + \alpha, \\ L_5 &= f_{\mathbb{M}} + \alpha. \end{aligned} \quad (5.1)$$

Proof. For the function \mathbf{X}_1 , we choose $\mathbb{H}^S, \mathbb{H}_*^S \in \mathbb{F} := C(I, \mathbb{R})$ arbitrarily. Then,

$$\begin{aligned} &\|\mathbf{X}_1(t, \mathbb{H}^S(t)) - \mathbf{X}_1(t, \mathbb{H}_*^S(t))\| \\ &= \left\| (\Theta_{\mathbb{H}} + \delta\mathbb{H}^R(t) - (\eta a_1 \mathbb{H}^I(t) + \lambda a_2 \mathbb{M}^I(t))\mathbb{H}^S(t) - (p + f_{\mathbb{H}})\mathbb{H}^S(t) \right. \\ &\quad \left. - (\Theta_{\mathbb{H}} + \delta\mathbb{H}^R(t) - (\eta a_1 \mathbb{H}^I(t) + \lambda a_2 \mathbb{M}^I(t))\mathbb{H}_*^S(t) - (p + f_{\mathbb{H}})\mathbb{H}_*^S(t)) \right\| \\ &\leq [\eta a_1 \|\mathbb{H}^I(t)\| + \lambda a_2 \|\mathbb{M}^I(t)\| + p + f_{\mathbb{H}}] \|\mathbb{H}^S(t) - \mathbb{H}_*^S(t)\| \end{aligned}$$

$$\begin{aligned} &\leq [\eta a_1 v_2 + \lambda a_2 v_5 + p + f_{\mathbb{H}}] \|\mathbb{H}^{\mathbb{S}}(t) - \mathbb{H}_*^{\mathbb{S}}(t)\| \\ &= L_1 \|\mathbb{H}^{\mathbb{S}}(t) - \mathbb{H}_*^{\mathbb{S}}(t)\|. \end{aligned}$$

This states that \mathbf{X}_1 is Lipschitz with respect to $\mathbb{H}^{\mathbb{S}}$ with the Lipschitz constant $L_1 > 0$. Regarding \mathbf{X}_2 , for each $\mathbb{H}^{\mathbb{I}}, \mathbb{H}_*^{\mathbb{I}} \in \mathbb{F} := C(I, \mathbb{R})$, we have

$$\begin{aligned} &\|\mathbf{X}_2(t, \mathbb{H}^{\mathbb{I}}(t)) - \mathbf{X}_2(t, \mathbb{H}_*^{\mathbb{I}}(t))\| \\ &= \left\| (\varpi \mathbb{H}^{\mathbb{I}}(t) + (\eta a_1 \mathbb{H}^{\mathbb{I}}(t) + \lambda a_2 \mathbb{M}^{\mathbb{I}}(t)) \mathbb{H}^{\mathbb{S}}(t) - (f_{\mathbb{H}} + \omega + \gamma \beta) \mathbb{H}^{\mathbb{I}}(t)) \right. \\ &\quad \left. - (\varpi \mathbb{H}_*^{\mathbb{I}}(t) + (\eta a_1 \mathbb{H}_*^{\mathbb{I}}(t) + \lambda a_2 \mathbb{M}^{\mathbb{I}}(t)) \mathbb{H}^{\mathbb{S}}(t) - (f_{\mathbb{H}} + \omega + \gamma \beta) \mathbb{H}_*^{\mathbb{I}}(t)) \right\| \\ &\leq [\varpi + \eta a_1 \|\mathbb{H}^{\mathbb{S}}(t)\| + (f_{\mathbb{H}} + \omega + \gamma \beta)] \|\mathbb{H}^{\mathbb{I}}(t) - \mathbb{H}_*^{\mathbb{I}}(t)\| \\ &\leq [\varpi + \eta a_1 v_1 + (f_{\mathbb{H}} + \omega + \gamma \beta)] \|\mathbb{H}^{\mathbb{I}}(t) - \mathbb{H}_*^{\mathbb{I}}(t)\| \\ &= L_2 \|\mathbb{H}^{\mathbb{I}}(t) - \mathbb{H}_*^{\mathbb{I}}(t)\|. \end{aligned}$$

This means that \mathbf{X}_2 is Lipschitz with respect to $\mathbb{H}^{\mathbb{I}}$ with the Lipschitz constant $L_2 > 0$. Now, for each $\mathbb{H}^{\mathbb{R}}, \mathbb{H}_*^{\mathbb{R}} \in \mathbb{F} := C(I, \mathbb{R})$, we estimate

$$\begin{aligned} &\|\mathbf{X}_3(t, \mathbb{H}^{\mathbb{R}}(t)) - \mathbf{X}_3(t, \mathbb{H}_*^{\mathbb{R}}(t))\| \\ &= \left\| (\gamma \beta \mathbb{H}^{\mathbb{I}}(t) - (f_{\mathbb{H}} + \delta) \mathbb{H}^{\mathbb{R}}(t) + p \mathbb{H}^{\mathbb{S}}(t)) \right. \\ &\quad \left. - (\gamma \beta \mathbb{H}^{\mathbb{I}}(t) - (f_{\mathbb{H}} + \delta) \mathbb{H}_*^{\mathbb{R}}(t) + p \mathbb{H}^{\mathbb{S}}(t)) \right\| \\ &\leq [f_{\mathbb{H}} + \delta] \|\mathbb{H}^{\mathbb{R}}(t) - \mathbb{H}_*^{\mathbb{R}}(t)\| \\ &= L_3 \|\mathbb{H}^{\mathbb{R}}(t) - \mathbb{H}_*^{\mathbb{R}}(t)\|. \end{aligned}$$

Thus, \mathbf{X}_3 is Lipschitz with respect to $\mathbb{H}^{\mathbb{R}}$ with the Lipschitz constant $L_3 > 0$. For each $\mathbb{M}^{\mathbb{S}}, \mathbb{M}_*^{\mathbb{S}} \in \mathbb{F} := C(I, \mathbb{R})$, we have

$$\begin{aligned} &\|\mathbf{X}_4(t, \mathbb{M}^{\mathbb{S}}(t)) - \mathbf{X}_4(t, \mathbb{M}_*^{\mathbb{S}}(t))\| \\ &= \left\| (\Theta_{\mathbb{M}} - (\kappa a_3 \mathbb{H}^{\mathbb{I}}(t) + f_{\mathbb{M}} + \alpha) \mathbb{M}^{\mathbb{S}}(t)) \right. \\ &\quad \left. - (\Theta_{\mathbb{M}} - (\kappa a_3 \mathbb{H}^{\mathbb{I}}(t) + f_{\mathbb{M}} + \alpha) \mathbb{M}_*^{\mathbb{S}}(t)) \right\| \\ &\leq [\kappa a_3 \|\mathbb{H}^{\mathbb{I}}\| + f_{\mathbb{M}} + \alpha] \|\mathbb{M}^{\mathbb{S}}(t) - \mathbb{M}_*^{\mathbb{S}}(t)\| \\ &\leq [\kappa a_3 v_4 + f_{\mathbb{M}} + \alpha] \|\mathbb{M}^{\mathbb{S}}(t) - \mathbb{M}_*^{\mathbb{S}}(t)\| \\ &= L_4 \|\mathbb{M}^{\mathbb{S}}(t) - \mathbb{M}_*^{\mathbb{S}}(t)\|, \end{aligned}$$

which implies that \mathbf{X}_4 is Lipschitz with respect to \mathbb{M}^S with the Lipschitz constant $L_4 > 0$. Lastly, for each $\mathbb{M}^I, \mathbb{M}_*^I \in \mathbb{F} := C(I, \mathbb{R})$, we have

$$\begin{aligned} & \| \mathbf{X}_5(t, \mathbb{M}^I(t)) - \mathbf{X}_5(t, \mathbb{M}_*^I(t)) \| \\ &= \| (\kappa a_3 \mathbb{H}^I(t) \mathbb{M}^S(t) - (f_{\mathbb{M}} + \alpha) \mathbb{M}^I(t) \\ &\quad - (\kappa a_3 \mathbb{H}^I(t) \mathbb{M}^S(t) - (f_{\mathbb{M}} + \alpha) \mathbb{M}_*^I(t)) \| \\ &\leq [f_{\mathbb{M}} + \alpha] \| \mathbb{M}^I(t) - \mathbb{M}_*^I(t) \| \\ &= L_5 \| \mathbb{M}^I(t) - \mathbb{M}_*^I(t) \|. \end{aligned}$$

Therefore, \mathbf{X}_5 is Lipschitz with respect to \mathbb{M}^I with the Lipschitz constant $L_5 > 0$; the proof is completed. \square

Theorem 5.2. *Let (G3) be held. Then, the given (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional SIR - SI -model (3.3) of malaria has a unique solution if*

$$\left[\frac{(1 - c_1)c_2\tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1c_2\tau^{c_1+c_2-1}\Gamma(c_2)}{\Gamma(c_1 + c_2)\mathbf{AB}(c_1)} \right] L_j < 1, \quad (j \in \{1, 2, 3, 4, 5\}), \quad (5.2)$$

where L_j is introduced in (5.1).

Proof. We consider this fact that the theorem is not true. Hence, there exists another solution for the given (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional SIR - SI -model (3.3) of malaria. Assume that $(\mathbb{H}_*^S(t), \mathbb{H}_*^I(t), \mathbb{H}_*^R(t), \mathbb{M}_*^S(t), \mathbb{M}_*^I(t))$ is another solution with the conditions

$$(\mathbb{H}_*^S(0), \mathbb{H}_*^I(0), \mathbb{H}_*^R(0), \mathbb{M}_*^S(0), \mathbb{M}_*^I(0)) = (\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4, \varkappa_5).$$

Then, by (4.8), we have

$$\begin{aligned} \mathbb{H}_*^S(t) &= \varkappa_1 + \frac{(1 - c_1)c_2t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_1(t, \mathbb{H}_*^S(t)) \\ &\quad + \frac{c_1c_2}{\Gamma(c_1)\mathbf{AB}(c_1)} \int_0^t q^{c_2-1}(t - q)^{c_1-1} \mathbf{X}_1(q, \mathbb{H}_*^S(q)) \, dq, \end{aligned}$$

and

$$\begin{aligned} \mathbb{H}_*^I(t) &= \varkappa_2 + \frac{(1 - c_1)c_2t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_2(t, \mathbb{H}_*^I(t)) \\ &\quad + \frac{c_1c_2}{\Gamma(c_1)\mathbf{AB}(c_1)} \int_0^t q^{c_2-1}(t - q)^{c_1-1} \mathbf{X}_2(q, \mathbb{H}_*^I(q)) \, dq, \end{aligned}$$

and

$$\begin{aligned}\mathbb{H}_*^{\mathcal{R}}(t) &= \varkappa_3 + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_3(t, \mathbb{H}_*^{\mathcal{R}}(t)) \\ &\quad + \frac{c_1 c_2}{\Gamma(c_1)\mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathbf{X}_3(q, \mathbb{H}_*^{\mathcal{R}}(q)) \, dq,\end{aligned}$$

and

$$\begin{aligned}\mathbb{M}_*^{\mathcal{S}}(t) &= \varkappa_4 + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_4(t, \mathbb{M}_*^{\mathcal{S}}(t)) \\ &\quad + \frac{c_1 c_2}{\Gamma(c_1)\mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathbf{X}_4(q, \mathbb{M}_*^{\mathcal{S}}(q)) \, dq,\end{aligned}$$

and

$$\begin{aligned}\mathbb{M}_*^{\mathcal{I}}(t) &= \varkappa_5 + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_5(t, \mathbb{M}_*^{\mathcal{I}}(t)) \\ &\quad + \frac{c_1 c_2}{\Gamma(c_1)\mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathbf{X}_5(q, \mathbb{M}_*^{\mathcal{I}}(q)) \, dq.\end{aligned}$$

Now, we can estimate

$$\begin{aligned}\|\mathbb{H}^{\mathcal{S}}(t) - \mathbb{H}_*^{\mathcal{S}}(t)\| &\leq \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \left| \mathbf{X}_1(t, \mathbb{H}^{\mathcal{S}}(t)) - \mathbf{X}_1(t, \mathbb{H}_*^{\mathcal{S}}(t)) \right| \\ &\quad + \frac{c_1 c_2}{\Gamma(c_1)\mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \left| \mathbf{X}_1(w, \mathbb{H}^{\mathcal{S}}(w)) - \mathbf{X}_1(w, \mathbb{H}_*^{\mathcal{S}}(w)) \right| \, dw \\ &\leq \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} L_1 \|\mathbb{H}^{\mathcal{S}} - \mathbb{H}_*^{\mathcal{S}}\| + \frac{c_1 c_2}{\Gamma(c_1)\mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} L_1 \|\mathbb{H}^{\mathcal{S}} - \mathbb{H}_*^{\mathcal{S}}\| \, dw \\ &\leq \left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2)\mathbf{AB}(c_1)} \right] L_1 \|\mathbb{H}^{\mathcal{S}} - \mathbb{H}_*^{\mathcal{S}}\|,\end{aligned}$$

and so

$$\left(1 - \left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2)\mathbf{AB}(c_1)} \right] L_1 \right) \|\mathbb{H}^{\mathcal{S}} - \mathbb{H}_*^{\mathcal{S}}\| \leq 0.$$

The above inequality holds when $\|\mathbb{H}^{\mathcal{S}} - \mathbb{H}_*^{\mathcal{S}}\| = 0$ or $\mathbb{H}^{\mathcal{S}} = \mathbb{H}_*^{\mathcal{S}}$. From the inequality

$$\|\mathbb{H}^{\mathcal{I}} - \mathbb{H}_*^{\mathcal{I}}\| \leq \left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2)\mathbf{AB}(c_1)} \right] L_2 \|\mathbb{H}^{\mathcal{I}} - \mathbb{H}_*^{\mathcal{I}}\|,$$

we reach

$$\left(1 - \left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2)\mathbf{AB}(c_1)} \right] L_2 \right) \|\mathbb{H}^{\mathcal{I}} - \mathbb{H}_*^{\mathcal{I}}\| \leq 0.$$

This is true when $\|\mathbb{H}^I - \mathbb{H}_*^I\| = 0$ or $\mathbb{H}^I = \mathbb{H}_*^I$. Moreover, the inequality

$$\|\mathbb{H}^R - \mathbb{H}_*^R\| \leq \left[\frac{(1-c_1)c_2\tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1c_2\tau^{c_1+c_2-1}\Gamma(c_2)}{\Gamma(c_1+c_2)\mathbf{AB}(c_1)} \right] L_3 \|\mathbb{H}^R - \mathbb{H}_*^R\|$$

yields

$$\left(1 - \left[\frac{(1-c_1)c_2\tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1c_2\tau^{c_1+c_2-1}\Gamma(c_2)}{\Gamma(c_1+c_2)\mathbf{AB}(c_1)} \right] L_3 \right) \|\mathbb{H}^R - \mathbb{H}_*^R\| \leq 0.$$

Hence, $\mathbb{H}^I = \mathbb{H}_*^I$. Accordingly, we get $\mathbb{M}^S = \mathbb{M}_*^S$ and $\mathbb{M}^I = \mathbb{M}_*^I$. In consequence, we get

$$(\mathbb{H}^S(t), \mathbb{H}^I(t), \mathbb{H}^R(t), \mathbb{M}^S(t), \mathbb{M}^I(t)) = (\mathbb{H}_*^S(t), \mathbb{H}_*^I(t), \mathbb{H}_*^R(t), \mathbb{M}_*^S(t), \mathbb{M}_*^I(t)).$$

This means that the (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $\mathcal{SIR-SI}$ -model (3.3) of malaria has a unique solution if (5.2) is satisfied. \square

6. Ulam-Hyers-Rassias stability

In the sequel, we investigate the stability notion of four types including the Ulam-Hyers, Ulam-Hyers-Rassias and their generalized versions in relation to the system of the (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $\mathcal{SIR-SI}$ -model (3.3) of malaria.

Definition 6.1. The (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $\mathcal{SIR-SI}$ -model (3.3) of malaria is Ulam-Hyers-stable if there are the constants $0 < Q_{X_j} \in \mathbb{R}$, $j \in \{1, \dots, 5\}$ such that, for each $R_j > 0$, and for each $(\mathbb{H}_*^S, \mathbb{H}_*^I, \mathbb{H}_*^R, \mathbb{M}_*^S, \mathbb{M}_*^I) \in \mathcal{X}$ satisfying

$$\begin{cases} \left| \mathbf{FFML} \mathcal{D}_{0,t}^{(c_1, c_2)} \mathbb{H}_*^S(t) - \mathbf{X}_1(t, \mathbb{H}_*^S(t)) \right| < R_1, \\ \left| \mathbf{FFML} \mathcal{D}_{0,t}^{(c_1, c_2)} \mathbb{H}_*^I(t) - \mathbf{X}_2(t, \mathbb{H}_*^I(t)) \right| < R_2, \\ \left| \mathbf{FFML} \mathcal{D}_{0,t}^{(c_1, c_2)} \mathbb{H}_*^R(t) - \mathbf{X}_3(t, \mathbb{H}_*^R(t)) \right| < R_3, \\ \left| \mathbf{FFML} \mathcal{D}_{0,t}^{(c_1, c_2)} \mathbb{M}_*^S(t) - \mathbf{X}_4(t, \mathbb{M}_*^S(t)) \right| < R_4, \\ \left| \mathbf{FFML} \mathcal{D}_{0,t}^{(c_1, c_2)} \mathbb{M}_*^I(t) - \mathbf{X}_5(t, \mathbb{M}_*^I(t)) \right| < R_5, \end{cases} \quad (6.1)$$

there is $(\mathbb{H}^S, \mathbb{H}^I, \mathbb{H}^R, \mathbb{M}^S, \mathbb{M}^I) \in \mathcal{X}$ satisfying the (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $\mathcal{SIR-SI}$ -model (3.3) of malaria with

$$\begin{cases} \left| \mathbb{H}_*^S(t) - \mathbb{H}^S(t) \right| \leq Q_{X_1} R_1, \\ \left| \mathbb{H}_*^I(t) - \mathbb{H}^I(t) \right| \leq Q_{X_2} R_2, \\ \left| \mathbb{H}_*^R(t) - \mathbb{H}^R(t) \right| \leq Q_{X_3} R_3, \\ \left| \mathbb{M}_*^S(t) - \mathbb{M}^S(t) \right| \leq Q_{X_4} R_4, \\ \left| \mathbb{M}_*^I(t) - \mathbb{M}^I(t) \right| \leq Q_{X_5} R_5. \end{cases}$$

Definition 6.2. The given (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $SIR-SI$ -model (3.3) of malaria is generalized Ulam-Hyers-stable if there are the functions $Q_{X_j} \in C(\mathbb{R}^+, \mathbb{R}^+)$, ($j \in \{1, \dots, 5\}$), with $Q_{X_j}(0) = 0$, so that, for each $R_j > 0$ and each $(\mathbb{H}_*^S, \mathbb{H}_*^I, \mathbb{H}_*^R, \mathbb{M}_*^S, \mathbb{M}_*^I) \in \mathcal{X}$ satisfying (6.1), there is $(\mathbb{H}^S, \mathbb{H}^I, \mathbb{H}^R, \mathbb{M}^S, \mathbb{M}^I) \in \mathcal{X}$ as a solution of the given (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $SIR-SI$ -model (3.3) of malaria with

$$\begin{cases} |\mathbb{H}_*^S(t) - \mathbb{H}^S(t)| \leq Q_{X_1}(R_1), \\ |\mathbb{H}_*^I(t) - \mathbb{H}^I(t)| \leq Q_{X_2}(R_2), \\ |\mathbb{H}_*^R(t) - \mathbb{H}^R(t)| \leq Q_{X_3}(R_3), \\ |\mathbb{M}_*^S(t) - \mathbb{M}^S(t)| \leq Q_{X_4}(R_4), \\ |\mathbb{M}_*^I(t) - \mathbb{M}^I(t)| \leq Q_{X_5}(R_5). \end{cases}$$

Notice that Definition 6.2 is obtained from Definition 6.1.

Remark 1. $(\mathbb{H}_*^S, \mathbb{H}_*^I, \mathbb{H}_*^R, \mathbb{M}_*^S, \mathbb{M}_*^I) \in \mathcal{X}$ is a solution of (6.1) if and only if there are $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_5 \in C([0, \tau], \mathbb{R})$ (depending on $\mathbb{H}_*^S, \mathbb{H}_*^I, \mathbb{H}_*^R, \mathbb{M}_*^S, \mathbb{M}_*^I$, respectively) such that, for each $t \in I$,

- (i) $|\mathcal{G}_j(t)| < R_j$,
- (ii) we have

$$\begin{cases} \text{FFML } \mathcal{D}_{0,t}^{(c_1, c_2)} \mathbb{H}_*^S(t) = \mathbf{X}_1(t, \mathbb{H}_*^S(t)) + \mathcal{G}_1(t), \\ \text{FFML } \mathcal{D}_{0,t}^{(c_1, c_2)} \mathbb{H}_*^I(t) = \mathbf{X}_2(t, \mathbb{H}_*^I(t)) + \mathcal{G}_2(t), \\ \text{FFML } \mathcal{D}_{0,t}^{(c_1, c_2)} \mathbb{H}_*^R(t) = \mathbf{X}_3(t, \mathbb{H}_*^R(t)) + \mathcal{G}_3(t), \\ \text{FFML } \mathcal{D}_{0,t}^{(c_1, c_2)} \mathbb{M}_*^S(t) = \mathbf{X}_4(t, \mathbb{M}_*^S(t)) + \mathcal{G}_4(t), \\ \text{FFML } \mathcal{D}_{0,t}^{(c_1, c_2)} \mathbb{M}_*^I(t) = \mathbf{X}_5(t, \mathbb{M}_*^I(t)) + \mathcal{G}_5(t). \end{cases}$$

Definition 6.3. The (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $SIR-SI$ -model (3.3) of malaria is Ulam-Hyers-Rassias-stable with respect to the functions χ_j , ($j \in \{1, \dots, 5\}$) if there are the constants $0 < Q_{(X_j, \chi_j)} \in \mathbb{R}$ such that, for each $R_j > 0$ and each $(\mathbb{H}_*^S, \mathbb{H}_*^I, \mathbb{H}_*^R, \mathbb{M}_*^S, \mathbb{M}_*^I) \in \mathcal{X}$ satisfying

$$\begin{cases} \left| \text{FFML } \mathcal{D}_{0,t}^{(c_1, c_2)} \mathbb{H}_*^S(t) - \mathbf{X}_1(t, \mathbb{H}_*^S(t)) \right| < R_1 \chi_1(t), \\ \left| \text{FFML } \mathcal{D}_{0,t}^{(c_1, c_2)} \mathbb{H}_*^I(t) - \mathbf{X}_2(t, \mathbb{H}_*^I(t)) \right| < R_2 \chi_2(t), \\ \left| \text{FFML } \mathcal{D}_{0,t}^{(c_1, c_2)} \mathbb{H}_*^R(t) - \mathbf{X}_3(t, \mathbb{H}_*^R(t)) \right| < R_3 \chi_3(t), \\ \left| \text{FFML } \mathcal{D}_{0,t}^{(c_1, c_2)} \mathbb{M}_*^S(t) - \mathbf{X}_4(t, \mathbb{M}_*^S(t)) \right| < R_4 \chi_4(t), \\ \left| \text{FFML } \mathcal{D}_{0,t}^{(c_1, c_2)} \mathbb{M}_*^I(t) - \mathbf{X}_5(t, \mathbb{M}_*^I(t)) \right| < R_5 \chi_5(t), \end{cases} \quad (6.2)$$

there is $(\mathbb{H}^S, \mathbb{H}^I, \mathbb{H}^R, \mathbb{M}^S, \mathbb{M}^I) \in \mathcal{X}$ as a solution of the (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $SIR-SI$ -model (3.3) of malaria with

$$\begin{cases} |\mathbb{H}_*^S(t) - \mathbb{H}^S(t)| \leq R_1 Q_{(\mathbf{x}_1, \chi_1)} \chi_1(t), & \forall t \in I, \\ |\mathbb{H}_*^I(t) - \mathbb{H}^I(t)| \leq R_2 Q_{(\mathbf{x}_2, \chi_2)} \chi_2(t), & \forall t \in I, \\ |\mathbb{H}_*^R(t) - \mathbb{H}^R(t)| \leq R_3 Q_{(\mathbf{x}_3, \chi_3)} \chi_3(t), & \forall t \in I, \\ |\mathbb{M}_*^S(t) - \mathbb{M}^S(t)| \leq R_4 Q_{(\mathbf{x}_4, \chi_4)} \chi_4(t), & \forall t \in I, \\ |\mathbb{M}_*^I(t) - \mathbb{M}^I(t)| \leq R_5 Q_{(\mathbf{x}_5, \chi_5)} \chi_5(t), & \forall t \in I, \end{cases}$$

where $\chi_1, \chi_2, \chi_3, \chi_4, \chi_5 \in C([0, \tau], \mathbb{R}^+)$.

Definition 6.4. The (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $SIR-SI$ -model (3.3) of malaria is generalized Ulam-Hyers-Rassias-stable with respect to the function χ_j if there is $0 < Q_{(\mathbf{x}_j, \chi_j)} \in \mathbb{R}$ provided that, for each $(\mathbb{H}_*^S, \mathbb{H}_*^I, \mathbb{H}_*^R, \mathbb{M}_*^S, \mathbb{M}_*^I) \in \mathcal{X}$ satisfying

$$\begin{cases} \left| \text{FFML} \mathcal{D}_{0,t}^{(c_1, c_2)} \mathbb{H}_*^S(t) - \mathbf{X}_1(t, \mathbb{H}_*^S(t)) \right| < \chi_1(t), \\ \left| \text{FFML} \mathcal{D}_{0,t}^{(c_1, c_2)} \mathbb{H}_*^I(t) - \mathbf{X}_2(t, \mathbb{H}_*^I(t)) \right| < \chi_2(t), \\ \left| \text{FFML} \mathcal{D}_{0,t}^{(c_1, c_2)} \mathbb{H}_*^R(t) - \mathbf{X}_3(t, \mathbb{H}_*^R(t)) \right| < \chi_3(t), \\ \left| \text{FFML} \mathcal{D}_{0,t}^{(c_1, c_2)} \mathbb{M}_*^S(t) - \mathbf{X}_4(t, \mathbb{M}_*^S(t)) \right| < \chi_4(t), \\ \left| \text{FFML} \mathcal{D}_{0,t}^{(c_1, c_2)} \mathbb{M}_*^I(t) - \mathbf{X}_5(t, \mathbb{M}_*^I(t)) \right| < \chi_5(t), \end{cases}$$

there is $(\mathbb{H}^S, \mathbb{H}^I, \mathbb{H}^R, \mathbb{M}^S, \mathbb{M}^I) \in \mathcal{X}$ as a solution of the (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $SIR-SI$ -model (3.3) of malaria with

$$\begin{cases} |\mathbb{H}_*^S(t) - \mathbb{H}^S(t)| \leq Q_{(\mathbf{x}_1, \chi_1)} \chi_1(t), \\ |\mathbb{H}_*^I(t) - \mathbb{H}^I(t)| \leq Q_{(\mathbf{x}_2, \chi_2)} \chi_2(t), \\ |\mathbb{H}_*^R(t) - \mathbb{H}^R(t)| \leq Q_{(\mathbf{x}_3, \chi_3)} \chi_3(t), \\ |\mathbb{M}_*^S(t) - \mathbb{M}^S(t)| \leq Q_{(\mathbf{x}_4, \chi_4)} \chi_4(t), \\ |\mathbb{M}_*^I(t) - \mathbb{M}^I(t)| \leq Q_{(\mathbf{x}_5, \chi_5)} \chi_5(t). \end{cases}$$

Remark 2. $(\mathbb{H}_*^S, \mathbb{H}_*^I, \mathbb{H}_*^R, \mathbb{M}_*^S, \mathbb{M}_*^I) \in \mathcal{X}$ is a solution of (6.2) if and only if there are $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_5 \in C([0, \tau], \mathbb{R})$ (depending on $\mathbb{H}_*^S, \mathbb{H}_*^I, \mathbb{H}_*^R, \mathbb{M}_*^S, \mathbb{M}_*^I$) such that, for each $t \in I$,

$$(i) |\mathcal{G}_j(t)| < R_j \chi_j(t),$$

(ii) we have

$$\begin{cases} \text{FFML } \mathcal{D}_{0,t}^{(c_1,c_2)} \mathbb{H}_*^S(t) = \mathbf{X}_1(t, \mathbb{H}_*^S(t)) + \mathcal{G}_1(t), \\ \text{FFML } \mathcal{D}_{0,t}^{(c_1,c_2)} \mathbb{H}_*^I(t) = \mathbf{X}_2(t, \mathbb{H}_*^I(t)) + \mathcal{G}_2(t), \\ \text{FFML } \mathcal{D}_{0,t}^{(c_1,c_2)} \mathbb{H}_*^R(t) = \mathbf{X}_3(t, \mathbb{H}_*^R(t)) + \mathcal{G}_3(t), \\ \text{FFML } \mathcal{D}_{0,t}^{(c_1,c_2)} \mathbb{M}_*^S(t) = \mathbf{X}_4(t, \mathbb{M}_*^S(t)) + \mathcal{G}_4(t), \\ \text{FFML } \mathcal{D}_{0,t}^{(c_1,c_2)} \mathbb{M}_*^I(t) = \mathbf{X}_5(t, \mathbb{M}_*^I(t)) + \mathcal{G}_5(t). \end{cases}$$

The following lemmas are useful for our main theorems.

Lemma 6.5. For every $R_1, R_2, R_3, R_4, R_5 > 0$, suppose that $(\mathbb{H}_*^S, \mathbb{H}_*^I, \mathbb{H}_*^R, \mathbb{M}_*^S, \mathbb{M}_*^I) \in \mathcal{X}$ is a solution of (6.1). Then, $\mathbb{H}_*^S, \mathbb{H}_*^I, \mathbb{H}_*^R, \mathbb{M}_*^S, \mathbb{M}_*^I \in \mathbb{F}$ satisfy the inequalities

$$\begin{aligned} & \left| \mathbb{H}_*^S(t) - \left(\varkappa_1 + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_1(t, \mathbb{H}_*^S(t)) \right. \right. \\ & \quad \left. \left. + \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathbf{X}_1(q, \mathbb{H}_*^S(q)) \, dq \right) \right| \\ & \leq \left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2) \mathbf{AB}(c_1)} \right] R_1, \end{aligned} \quad (6.3)$$

and

$$\begin{aligned} & \left| \mathbb{H}_*^I(t) - \left(\varkappa_2 + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_2(t, \mathbb{H}_*^I(t)) \right. \right. \\ & \quad \left. \left. + \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathbf{X}_2(q, \mathbb{H}_*^I(q)) \, dq \right) \right| \\ & \leq \left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2) \mathbf{AB}(c_1)} \right] R_2, \end{aligned} \quad (6.4)$$

and

$$\begin{aligned} & \left| \mathbb{H}_*^R(t) - \left(\varkappa_3 + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_3(t, \mathbb{H}_*^R(t)) \right. \right. \\ & \quad \left. \left. + \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathbf{X}_3(q, \mathbb{H}_*^R(q)) \, dq \right) \right| \\ & \leq \left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2) \mathbf{AB}(c_1)} \right] R_3, \end{aligned} \quad (6.5)$$

and

$$\begin{aligned} & \left| \mathbb{M}_*^S(t) - \left(\varkappa_4 + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_4(t, \mathbb{M}_*^S(t)) \right. \right. \\ & \quad \left. \left. + \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathbf{X}_4(q, \mathbb{M}_*^S(q)) \, dq \right) \right| \\ & \leq \left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2) \mathbf{AB}(c_1)} \right] R_4, \end{aligned} \quad (6.6)$$

and

$$\begin{aligned} & \left| \mathbb{M}_*^I(t) - \left(\varkappa_5 + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_5(t, \mathbb{M}_*^I(t)) \right. \right. \\ & \quad \left. \left. + \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathbf{X}_5(q, \mathbb{M}_*^I(q)) \, dq \right) \right| \\ & \leq \left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2) \mathbf{AB}(c_1)} \right] R_5. \end{aligned} \quad (6.7)$$

Proof. Let $R_1 > 0$ be arbitrary. Since $\mathbb{H}_*^S \in \mathbb{F}$ satisfies

$$\left| \mathbf{FFML} \mathcal{D}_{0,t}^{(c_1,c_2)} \mathbb{H}_*^S(t) - \mathbf{X}_1(t, \mathbb{H}_*^S(t)) \right| < R_1,$$

via Remark 1, we are allowed to select $\mathcal{G}_1(t)$ such that

$$\mathbf{FFML} \mathcal{D}_{0,t}^{(c_1,c_2)} \mathbb{H}_*^S(t) = \mathbf{X}_1(t, \mathbb{H}_*^S(t)) + \mathcal{G}_1(t)$$

and $|\mathcal{G}_1(t)| \leq R_1$. It follows that

$$\begin{aligned} \mathbb{H}_*^S(t) &= \varkappa_1 + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \left[\mathbf{X}_1(t, \mathbb{H}_*^S(t)) + \mathcal{G}_1(t) \right] \\ & \quad + \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \left[\mathbf{X}_1(q, \mathbb{H}_*^S(q)) + \mathcal{G}_1(q) \right] \, dq. \end{aligned}$$

Then, we estimate

$$\begin{aligned} & \left| \mathbb{H}_*^S(t) - \left(\varkappa_1 + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_1(t, \mathbb{H}_*^S(t)) \right. \right. \\ & \quad \left. \left. + \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathbf{X}_1(q, \mathbb{H}_*^S(q)) \, dq \right) \right| \\ & \leq \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} |\mathcal{G}_1(t)| + \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} |\mathcal{G}_1(q)| \, dq \\ & \leq \frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} R_1 + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2) \mathbf{AB}(c_1)} R_1 \end{aligned}$$

$$= \left[\frac{(1-c_1)c_2\tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1c_2\tau^{c_1+c_2-1}\Gamma(c_2)}{\Gamma(c_1+c_2)\mathbf{AB}(c_1)} \right] R_1.$$

It is found that the inequality (6.3) is obtained. We get the inequalities (6.4) and (6.5) similarly. \square

To prove the next result, we regard the following:

(G4) There are increasing functions $\chi_J \in C([0, \tau], \mathbb{R}^+)$, ($J \in \{1, \dots, 5\}$) and there is $\Delta_{\chi_J} > 0$ such that

$$\mathbf{FFML} \mathcal{I}_{0,t}^{(c_1,c_2)} \chi_J(t) < \Delta_{\chi_J} \chi_J(t), \quad (J \in \{1, \dots, 5\}), \forall t \in I. \quad (6.8)$$

Lemma 6.6. *Let (G4) be held. For each $R_1, R_2, R_3, R_4, R_5 > 0$, suppose that*

$$(\mathbb{H}_*^S, \mathbb{H}_*^I, \mathbb{H}_*^R, \mathbb{M}_*^S, \mathbb{M}_*^I) \in \mathcal{X}$$

satisfies (6.2). Then, the functions $\mathbb{H}_^S, \mathbb{H}_*^I, \mathbb{H}_*^R, \mathbb{M}_*^S, \mathbb{M}_*^I \in \mathbb{F}$ fulfill the inequalities*

$$\begin{aligned} \left| \mathbb{H}_*^S(t) - \left(\varkappa_1 + \mathbf{FFML} \mathcal{I}_{0,t}^{(c_1,c_2)} \mathbf{X}_1(t, \mathbb{H}_*^S(t)) \right) \right| &\leq R_1 \Delta_{\chi_1} \chi_1(t), \\ \left| \mathbb{H}_*^I(t) - \left(\varkappa_2 + \mathbf{FFML} \mathcal{I}_{0,t}^{(c_1,c_2)} \mathbf{X}_2(t, \mathbb{H}_*^I(t)) \right) \right| &\leq R_2 \Delta_{\chi_2} \chi_2(t), \\ \left| \mathbb{H}_*^R(t) - \left(\varkappa_3 + \mathbf{FFML} \mathcal{I}_{0,t}^{(c_1,c_2)} \mathbf{X}_3(t, \mathbb{H}_*^R(t)) \right) \right| &\leq R_3 \Delta_{\chi_3} \chi_3(t), \\ \left| \mathbb{M}_*^S(t) - \left(\varkappa_4 + \mathbf{FFML} \mathcal{I}_{0,t}^{(c_1,c_2)} \mathbf{X}_4(t, \mathbb{M}_*^S(t)) \right) \right| &\leq R_4 \Delta_{\chi_4} \chi_4(t), \\ \left| \mathbb{M}_*^I(t) - \left(\varkappa_5 + \mathbf{FFML} \mathcal{I}_{0,t}^{(c_1,c_2)} \mathbf{X}_5(t, \mathbb{M}_*^I(t)) \right) \right| &\leq R_5 \Delta_{\chi_5} \chi_5(t). \end{aligned}$$

Proof. Let $R_1 > 0$ be arbitrary. Since $\mathbb{H}_*^S \in \mathbb{F}$ satisfies

$$\left| \mathbf{FFML} \mathcal{D}_{0,t}^{(c_1,c_2)} \mathbb{H}_*^S(t) - \mathbf{X}_1(t, \mathbb{H}_*^S(t)) \right| < R_1 \chi_1(t),$$

by Remark 2, we are allowed to select a function $\mathcal{G}_1(t)$ so that

$$\mathbf{FFML} \mathcal{D}_{0,t}^{(c_1,c_2)} \mathbb{H}_*^S(t) = \mathbf{X}_1(t, \mathbb{H}_*^S(t)) + \mathcal{G}_1(t)$$

and $|\mathcal{G}_1(t)| \leq R_1 \chi_1(t)$. It follows that

$$\begin{aligned} \mathbb{H}_*^S(t) &= \varkappa_1 + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \left[\mathbf{X}_1(t, \mathbb{H}_*^S(t)) + \mathcal{G}_1(t) \right] \\ &\quad + \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \left[\mathbf{X}_1(q, \mathbb{H}_*^S(q)) + \mathcal{G}_1(q) \right] dq. \end{aligned}$$

Then, we estimate

$$\begin{aligned}
& \left| \mathbb{H}_*^S(t) - \left(\varkappa_1 + {}^{\text{FFML}}\mathcal{I}_{0,t}^{(c_1,c_2)} \mathbf{X}_1(t, \mathbb{H}_*^S(t)) \right) \right| \\
&= \left| \mathbb{H}_*^S(t) - \left(\varkappa_1 + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_1(t, \mathbb{H}_*^S(t)) \right. \right. \\
&\quad \left. \left. + \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathbf{X}_1(q, \mathbb{H}_*^S(q)) \, dq \right) \right| \\
&\leq \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} |\mathcal{G}_1(t)| + \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} |\mathcal{G}_1(q)| \, dq \\
&= {}^{\text{FFML}}\mathcal{I}_{0,t}^{(c_1,c_2)} |\mathcal{G}_1(t)| \\
&\leq {}^{\text{FFML}}\mathcal{I}_{0,t}^{(c_1,c_2)} R_1 \chi_1(t) \\
&\leq R_1 \Delta_{\chi_1} \chi_1(t).
\end{aligned}$$

Similarly, we can obtain the remaining inequalities. \square

The Ulam-Hyers stability is checked for the (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $SIR-SI$ -model (3.3) of malaria.

Theorem 6.7. *Let (G3) be fulfilled. Then, the (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $SIR-SI$ -model (3.3) of malaria is Ulam-Hyers-stable on $I := [0, \tau]$ and is also generalized Ulam-Hyers-stable such that*

$$\left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2) \mathbf{AB}(c_1)} \right] L_J < 1, \quad J \in \{1, \dots, 5\}.$$

Proof. Let $R_1 > 0$ and $\mathbb{H}_*^S \in \mathbb{F}$ be an arbitrary solution of (6.1). By Theorem 5.2, we take $\mathbb{H}^S \in \mathbb{F}$ as a unique solution of the (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $SIR-SI$ -model (3.3) of malaria. Then, $\mathbb{H}^S(t)$ is defined as

$$\begin{aligned}
\mathbb{H}^S(t) &= \varkappa_1 + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_1(t, \mathbb{H}^S(t)) \\
&\quad + \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathbf{X}_1(q, \mathbb{H}^S(q)) \, dq.
\end{aligned}$$

Via the triangle inequality, and by Lemma 6.5, estimate

$$\begin{aligned}
|\mathbb{H}_*^S(t) - \mathbb{H}^S(t)| &\leq \left| \mathbb{H}_*^S(t) - \varkappa_1 - \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_1(t, \mathbb{H}_*^S(t)) \right. \\
&\quad \left. - \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathbf{X}_1(q, \mathbb{H}_*^S(q)) \, dq \right| \\
&\leq \left| \mathbb{H}_*^S(t) - \left(\varkappa_1 + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_1(t, \mathbb{H}_*^S(t)) \right) \right.
\end{aligned}$$

$$\begin{aligned}
& \left. + \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathbf{X}_1(q, \mathbb{H}_*^S(q)) \, dq \right| \\
& + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} |\mathbf{X}_1(t, \mathbb{H}_*^S(t)) - \mathbf{X}_1(t, \mathbb{H}^S(t))| \\
& + \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} |\mathbf{X}_1(q, \mathbb{H}_*^S(q)) - \mathbf{X}_1(q, \mathbb{H}^S(q))| \, dq \\
\leq & \left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2) \mathbf{AB}(c_1)} \right] R_1 + \frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} L_1 \|\mathbb{H}_*^S - \mathbb{H}^S\| \\
& + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2) \mathbf{AB}(c_1)} L_1 \|\mathbb{H}_*^S - \mathbb{H}^S\| \\
\leq & \left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2) \mathbf{AB}(c_1)} \right] R_1 \\
& + \left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2) \mathbf{AB}(c_1)} \right] L_1 \|\mathbb{H}_*^S - \mathbb{H}^S\|.
\end{aligned}$$

Hence, we get

$$\|\mathbb{H}_*^S - \mathbb{H}^S\| \leq \frac{\left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2) \mathbf{AB}(c_1)} \right] R_1}{1 - \left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2) \mathbf{AB}(c_1)} \right] L_1}.$$

If we let $Q_{X_1} = \frac{\left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2) \mathbf{AB}(c_1)} \right]}{1 - \left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2) \mathbf{AB}(c_1)} \right] L_1}$, then $\|\mathbb{H}_*^S - \mathbb{H}^S\| \leq Q_{X_1} R_1$. Similarly, we

have

$$\|\mathbb{H}_*^I - \mathbb{H}^I\| \leq Q_{X_2} R_2,$$

$$\|\mathbb{H}_*^R - \mathbb{H}^R\| \leq Q_{X_3} R_3,$$

$$\|\mathbb{M}_*^S - \mathbb{M}^S\| \leq Q_{X_4} R_4,$$

$$\|\mathbb{M}_*^I - \mathbb{M}^I\| \leq Q_{X_5} R_5,$$

where

$$Q_{X_j} = \frac{\left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2) \mathbf{AB}(c_1)} \right]}{1 - \left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2) \mathbf{AB}(c_1)} \right] L_j}, \quad (j \in \{2, 3, 4, 5\}).$$

Hence, the Ulam-Hyers stability of the (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $SIR-SI$ -model (3.3) of malaria is fulfilled. Next, by assuming

$$Q_{X_j}(R_j) = \frac{\left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2) \mathbf{AB}(c_1)} \right] R_j}{1 - \left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2) \mathbf{AB}(c_1)} \right] L_j}, \quad (j \in \{1, \dots, 5\}),$$

with $Q_{\mathbf{X}}(0) = 0$, clearly, the generalized Ulam-Hyers stability is confirmed. \square

The Ulam-Hyers-Rassias stability is checked for the (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $SIR-SI$ -model (3.3) of malaria in the next theorem.

Theorem 6.8. *The hypotheses (G3) and (G4) are considered to be held. Then, the given (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $SIR-SI$ -model (3.3) of malaria is Ulam-Hyers-Rassias- and generalized Ulam-Hyers-Rassias-stable.*

Proof. Let $R_1 > 0$ and $\mathbb{H}_*^S \in \mathbb{F}$ satisfy (6.2). By Theorem 5.2, let $\mathbb{H}^S \in \mathbb{F}$ be the unique solution of the given (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $SIR-SI$ -model (3.3) of malaria. Then, $\mathbb{H}^S(t)$ becomes

$$\begin{aligned} \mathbb{H}^S(t) &= \varkappa_1 + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_1(t, \mathbb{H}^S(t)) \\ &\quad + \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathbf{X}_1(q, \mathbb{H}^S(q)) dq. \end{aligned}$$

Via the triangle inequality, and by Lemma 6.6, estimate

$$\begin{aligned} |\mathbb{H}_*^S(t) - \mathbb{H}^S(t)| &\leq \left| \mathbb{H}_*^S(t) - \varkappa_1 - \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_1(t, \mathbb{H}_*^S(t)) \right. \\ &\quad \left. - \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathbf{X}_1(q, \mathbb{H}_*^S(q)) dq \right| \\ &\leq \left| \mathbb{H}_*^S(t) - \left(\varkappa_1 + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_1(t, \mathbb{H}_*^S(t)) \right. \right. \\ &\quad \left. \left. + \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathbf{X}_1(q, \mathbb{H}_*^S(q)) dq \right) \right| \\ &\quad + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} |\mathbf{X}_1(t, \mathbb{H}_*^S(t)) - \mathbf{X}_1(t, \mathbb{H}^S(t))| \\ &\quad + \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} |\mathbf{X}_1(q, \mathbb{H}_*^S(q)) - \mathbf{X}_1(q, \mathbb{H}^S(q))| dw \\ &\leq \left| \mathbb{H}_*^S(t) - \left(\varkappa_1 + {}^{\mathbf{FFML}} \mathcal{I}_{0,t}^{(c_1, c_2)} \mathbf{X}_1(t, \mathbb{H}_*^S(t)) \right) \right| \\ &\quad + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} |\mathbf{X}_1(t, \mathbb{H}_*^S(t)) - \mathbf{X}_1(t, \mathbb{H}^S(t))| \\ &\quad + \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} |\mathbf{X}_1(q, \mathbb{H}_*^S(q)) - \mathbf{X}_1(q, \mathbb{H}^S(q))| dw \\ &\leq R_1 \Delta_{\mathcal{X}_1, \mathcal{X}_1}(t) + \frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} L_1 \|\mathbb{H}_*^S - \mathbb{H}^S\| + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2) \mathbf{AB}(c_1)} L_1 \|\mathbb{H}_*^S - \mathbb{H}^S\| \\ &\leq R_1 \Delta_{\mathcal{X}_1, \mathcal{X}_1}(t) + \left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2) \mathbf{AB}(c_1)} \right] L_1 \|\mathbb{H}_*^S - \mathbb{H}^S\|. \end{aligned}$$

Accordingly, it gives

$$\|\mathbb{H}_*^S - \mathbb{H}^S\| \leq \frac{R_1 \Delta_{\chi_1} \chi_1(t)}{1 - \left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2) \mathbf{AB}(c_1)} \right] L_1}.$$

If we let

$$Q_{(\mathbf{x}_1, \chi_1)} = \frac{\Delta_{\chi_1}}{1 - \left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2) \mathbf{AB}(c_1)} \right] L_1},$$

then $\|\mathbb{H}_*^S - \mathbb{H}^S\| \leq R_1 Q_{(\mathbf{x}_1, \chi_1)} \chi_1(t)$. Similarly, we have

$$\|\mathbb{H}_*^I - \mathbb{H}^I\| \leq R_2 Q_{(\mathbf{x}_2, \chi_2)} \chi_2(t),$$

$$\|\mathbb{H}_*^R - \mathbb{H}^R\| \leq R_3 Q_{(\mathbf{x}_3, \chi_3)} \chi_3(t),$$

$$\|\mathbb{M}_*^S - \mathbb{M}^S\| \leq R_4 Q_{(\mathbf{x}_4, \chi_4)} \chi_4(t),$$

$$\|\mathbb{M}_*^I - \mathbb{M}^I\| \leq R_5 Q_{(\mathbf{x}_5, \chi_5)} \chi_5(t),$$

where

$$Q_{(\mathbf{x}_J, \chi_J)} = \frac{\Delta_{\chi_J}}{1 - \left[\frac{(1-c_1)c_2 \tau^{c_2-1}}{\mathbf{AB}(c_1)} + \frac{c_1 c_2 \tau^{c_1+c_2-1} \Gamma(c_2)}{\Gamma(c_1+c_2) \mathbf{AB}(c_1)} \right] L_J}, \quad (J \in \{2, 3, 4, 5\}).$$

Hence, the given (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $SIR-SI$ -model (3.3) of malaria is stable (Ulam-Hyers-Rassias). By setting $R_j = 1$, ($J \in \{1, \dots, 5\}$), the mentioned (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $SIR-SI$ -model (3.3) of malaria is generalized Ulam-Hyers-Rassias-stable. \square

7. Numerical scheme via the Newton polynomial method

In this section, we give a numerical scheme for solutions of our (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $SIR-SI$ -model (3.3) of malaria which was presented by Atangana and Araz in their book [54] in 2021. For this purpose, we need the compact form of the initial value problem (4.3) again under the conditions (4.4). Thus,

$$\mathbf{K}(t) - \mathbf{K}(0) = \frac{c_1 c_2}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t q^{c_2-1} (t-q)^{c_1-1} \mathbf{X}(q, \mathbf{K}(q)) dq + \frac{(1-c_1)c_2 t^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}(t, \mathbf{K}(t)).$$

Take $\mathbf{X}^*(t, \mathbf{K}(t)) = c_2 t^{c_2-1} \mathbf{X}(t, \mathbf{K}(t))$. Then,

$$\mathbf{K}(t) - \mathbf{K}(0) = \frac{c_1}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^t (t-q)^{c_1-1} \mathbf{X}^*(q, \mathbf{K}(q)) dq + \frac{(1-c_1)}{\mathbf{AB}(c_1)} \mathbf{X}^*(t, \mathbf{K}(t)).$$

By discretizing the above equation at $t = t_{k+1} = (k+1)\mathbf{h}$, we get

$$\mathbf{K}(t_{k+1}) - \mathbf{K}(0) = \frac{c_1}{\Gamma(c_1) \mathbf{AB}(c_1)} \int_0^{t_{k+1}} (t_{k+1}-q)^{c_1-1} \mathbf{X}^*(q, \mathbf{K}(q)) dq + \frac{(1-c_1)}{\mathbf{AB}(c_1)} \mathbf{X}^*(t_k, \mathbf{K}(t_k)).$$

If we approximate the above integral, then it becomes

$$\begin{aligned} \mathbf{K}(t_{k+1}) &= \mathbf{K}_0 + \frac{(1-c_1)}{\mathbf{AB}(c_1)} \mathbf{X}^*(t_k, \mathbf{K}(t_k)) \\ &+ \frac{c_1}{\Gamma(c_1)\mathbf{AB}(c_1)} \sum_{\ell=2}^k \int_{t_\ell}^{t_{\ell+1}} (t_{k+1}-q)^{c_1-1} \mathbf{X}^*(q, \mathbf{K}(q)) \, dq. \end{aligned} \quad (7.1)$$

In this step, the function $\mathbf{X}^*(t, \mathbf{K}(t))$ is approximated by the Newton polynomial as

$$\begin{aligned} P_k^*(q) &= \mathbf{X}^*(t_{k-2}, \mathbf{K}(t_{k-2})) + \frac{\mathbf{X}^*(t_{k-1}, \mathbf{K}(t_{k-1})) - \mathbf{X}^*(t_{k-2}, \mathbf{K}(t_{k-2}))}{\mathbf{h}} (q - t_{k-2}) \\ &+ \frac{\mathbf{X}^*(t_k, \mathbf{K}(t_k)) - 2\mathbf{X}^*(t_{k-1}, \mathbf{K}(t_{k-1})) + \mathbf{X}^*(t_{k-2}, \mathbf{K}(t_{k-2}))}{2\mathbf{h}^2} (q - t_{k-2})(q - t_{k-1}). \end{aligned} \quad (7.2)$$

Substitute (7.2) into (7.1):

$$\begin{aligned} \mathbf{K}_{k+1} &= \mathbf{K}_0 + \frac{(1-c_1)}{\mathbf{AB}(c_1)} \mathbf{X}^*(t_k, \mathbf{K}(t_k)) + \frac{c_1}{\Gamma(c_1)\mathbf{AB}(c_1)} \sum_{\ell=2}^k \int_{t_\ell}^{t_{\ell+1}} \left[\mathbf{X}^*(t_{\ell-2}, \mathbf{K}_{\ell-2}) \right. \\ &+ \frac{\mathbf{X}^*(t_{\ell-1}, \mathbf{K}_{\ell-1}) - \mathbf{X}^*(t_{\ell-2}, \mathbf{K}_{\ell-2})}{\mathbf{h}} (q - t_{\ell-2}) \\ &+ \left. \frac{\mathbf{X}^*(t_\ell, \mathbf{K}_\ell) - 2\mathbf{X}^*(t_{\ell-1}, \mathbf{K}_{\ell-1}) + \mathbf{X}^*(t_{\ell-2}, \mathbf{K}_{\ell-2})}{2\mathbf{h}^2} (q - t_{\ell-2})(q - t_{\ell-1}) \right] \\ &\times (t_{k+1} - q)^{c_1-1} \, dq. \end{aligned}$$

We simplify the above relations, and we get

$$\begin{aligned} \mathbf{K}_{k+1} &= \mathbf{K}_0 + \frac{(1-c_1)}{\mathbf{AB}(c_1)} \mathbf{X}^*(t_k, \mathbf{K}(t_k)) \\ &+ \frac{c_1}{\Gamma(c_1)\mathbf{AB}(c_1)} \sum_{\ell=2}^k \left[\int_{t_\ell}^{t_{\ell+1}} \mathbf{X}^*(t_{\ell-2}, \mathbf{K}_{\ell-2}) (t_{k+1} - q)^{c_1-1} \, dq \right. \\ &+ \int_{t_\ell}^{t_{\ell+1}} \frac{\mathbf{X}^*(t_{\ell-1}, \mathbf{K}_{\ell-1}) - \mathbf{X}^*(t_{\ell-2}, \mathbf{K}_{\ell-2})}{\mathbf{h}} (q - t_{\ell-2}) (t_{k+1} - q)^{c_1-1} \, dq \\ &+ \int_{t_\ell}^{t_{\ell+1}} \frac{\mathbf{X}^*(t_\ell, \mathbf{K}_\ell) - 2\mathbf{X}^*(t_{\ell-1}, \mathbf{K}_{\ell-1}) + \mathbf{X}^*(t_{\ell-2}, \mathbf{K}_{\ell-2})}{2\mathbf{h}^2} (q - t_{\ell-2}) (q - t_{\ell-1}) \\ &\left. \times (t_{k+1} - q)^{c_1-1} \, dq \right]. \end{aligned}$$

In consequence,

$$\begin{aligned}
\mathbf{K}_{k+1} &= \mathbf{K}_0 + \frac{(1-c_1)}{\mathbf{AB}(c_1)} \mathbf{X}^*(t_k, \mathbf{K}(t_k)) \\
&+ \frac{c_1}{\Gamma(c_1)\mathbf{AB}(c_1)} \sum_{\ell=2}^k \mathbf{X}^*(t_{\ell-2}, \mathbf{K}_{\ell-2}) \int_{t_\ell}^{t_{\ell+1}} (t_{k+1} - q)^{c_1-1} dq \\
&+ \frac{c_1}{\Gamma(c_1)\mathbf{AB}(c_1)} \sum_{\ell=2}^k \frac{\mathbf{X}^*(t_{\ell-1}, \mathbf{K}_{\ell-1}) - \mathbf{X}^*(t_{\ell-2}, \mathbf{K}_{\ell-2})}{\mathbf{h}} \int_{t_\ell}^{t_{\ell+1}} (q - t_{\ell-2})(t_{k+1} - q)^{c_1-1} dq \\
&+ \frac{c_1}{\Gamma(c_1)\mathbf{AB}(c_1)} \sum_{\ell=2}^k \frac{\mathbf{X}^*(t_\ell, \mathbf{K}_\ell) - 2\mathbf{X}^*(t_{\ell-1}, \mathbf{K}_{\ell-1}) + \mathbf{X}^*(t_{\ell-2}, \mathbf{K}_{\ell-2})}{2\mathbf{h}^2} \\
&\times \int_{t_\ell}^{t_{\ell+1}} (q - t_{\ell-2})(q - t_{\ell-1}) \times (t_{k+1} - q)^{c_1-1} dq. \tag{7.3}
\end{aligned}$$

On the other hand, we compute the above three integrals separately, and we get

$$\int_{t_\ell}^{t_{\ell+1}} (t_{k+1} - q)^{c_1-1} dq = \frac{\mathbf{h}^{c_1}}{c_1} [(k - \ell + 1)^{c_1} - (k - \ell)^{c_1}], \tag{7.4}$$

and

$$\begin{aligned}
\int_{t_\ell}^{t_{\ell+1}} (q - t_{\ell-2})(t_{k+1} - q)^{c_1-1} dq &= \frac{\mathbf{h}^{c_1+1}}{c_1(c_1+1)} [(k - \ell + 1)^{c_1}(k - \ell + 3 + 2c_1) \\
&- (k - \ell + 1)^{c_1}(k - \ell + 3 + 3c_1)], \tag{7.5}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t_\ell}^{t_{\ell+1}} (q - t_{\ell-2})(q - t_{\ell-1})(t_{k+1} - q)^{c_1-1} dq \\
&= \frac{\mathbf{h}^{c_1+2}}{c_1(c_1+1)(c_1+2)} \left((k - \ell + 1)^{c_1} [2(k - \ell)^2 \right. \\
&\quad + (3c_1 + 10)(k - \ell) + 2c_1^2 + 9c_1 + 12] - (k - \ell)^{c_1} [2(k - \ell)^2 \\
&\quad \left. + (5c_1 + 10)(k - \ell) + 6c_1^2 + 18c_1 + 12] \right). \tag{7.6}
\end{aligned}$$

By putting (7.4)–(7.6) in (7.3), we obtain

$$\begin{aligned}
\mathbf{K}_{k+1} &= \mathbf{K}_0 + \frac{(1-c_1)}{\mathbf{AB}(c_1)} \mathbf{X}^*(t_k, \mathbf{K}(t_k)) \\
&+ \frac{c_1 \mathbf{h}^{c_1}}{\Gamma(c_1+1)\mathbf{AB}(c_1)} \sum_{\ell=2}^k \mathbf{X}^*(t_{\ell-2}, \mathbf{K}_{\ell-2}) [(k - \ell + 1)^{c_1} - (k - \ell)^{c_1}]
\end{aligned}$$

$$\begin{aligned}
& + \frac{c_1 \mathbf{h}^{c_1}}{\Gamma(c_1 + 2) \mathbf{AB}(c_1)} \sum_{\ell=2}^k [\mathbf{X}^*(t_{\ell-1}, \mathbf{K}_{\ell-1}) - \mathbf{X}^*(t_{\ell-2}, \mathbf{K}_{\ell-2})] \\
& \times [(k - \ell + 1)^{c_1} (k - \ell + 3 + 2c_1) - (k - \ell + 1)^{c_1} (k - \ell + 3 + 3c_1)] \\
& + \frac{c_1 \mathbf{h}^{c_1}}{2\Gamma(c_1 + 3) \mathbf{AB}(c_1)} \sum_{\ell=2}^k [\mathbf{X}^*(t_{\ell}, \mathbf{K}_{\ell}) - 2\mathbf{X}^*(t_{\ell-1}, \mathbf{K}_{\ell-1}) + \mathbf{X}^*(t_{\ell-2}, \mathbf{K}_{\ell-2})] \\
& \times [(k - \ell + 1)^{c_1} [2(k - \ell)^2 + (3c_1 + 10)(k - \ell) + 2c_1^2 + 9c_1 + 12] - (k - \ell)^{c_1} [2(k - \ell)^2 \\
& + (5c_1 + 10)(k - \ell) + 6c_1^2 + 18c_1 + 12]]. \tag{7.7}
\end{aligned}$$

Finally, we replace $\mathbf{X}^*(t, \mathbf{K}(t)) = c_2 t^{c_2-1} \mathbf{X}(t, \mathbf{K}(t))$ in (7.7), and we get

$$\begin{aligned}
\mathbf{K}_{k+1} & = \mathbf{K}_0 + \frac{(1 - c_1) c_2 t_k^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}(t_k, \mathbf{K}(t_k)) \\
& + \frac{c_1 c_2 \mathbf{h}^{c_1}}{\Gamma(c_1 + 1) \mathbf{AB}(c_1)} \sum_{\ell=2}^k t_{\ell-2}^{c_2-1} \mathbf{X}(t_{\ell-2}, \mathbf{K}_{\ell-2}) \hat{A}_1(k, \ell, c_1) \\
& + \frac{c_1 c_2 \mathbf{h}^{c_1}}{\Gamma(c_1 + 2) \mathbf{AB}(c_1)} \sum_{\ell=2}^k [t_{\ell-1}^{c_2-1} \mathbf{X}(t_{\ell-1}, \mathbf{K}_{\ell-1}) - t_{\ell-2}^{c_2-1} \mathbf{X}(t_{\ell-2}, \mathbf{K}_{\ell-2})] \hat{A}_2(k, \ell, c_1) \\
& + \frac{c_1 c_2 \mathbf{h}^{c_1}}{2\Gamma(c_1 + 3) \mathbf{AB}(c_1)} \sum_{\ell=2}^k [t_{\ell}^{c_2-1} \mathbf{X}(t_{\ell}, \mathbf{K}_{\ell}) - 2t_{\ell-1}^{c_2-1} \mathbf{X}(t_{\ell-1}, \mathbf{K}_{\ell-1}) + t_{\ell-2}^{c_2-1} \mathbf{X}(t_{\ell-2}, \mathbf{K}_{\ell-2})] \hat{A}_3(k, \ell, c_1),
\end{aligned} \tag{7.8}$$

where

$$\begin{aligned}
\hat{A}_1(k, \ell, c_1) & = (k - \ell + 1)^{c_1} - (k - \ell)^{c_1}, \\
\hat{A}_2(k, \ell, c_1) & = (k - \ell + 1)^{c_1} (k - \ell + 3 + 2c_1) - (k - \ell + 1)^{c_1} (k - \ell + 3 + 3c_1), \\
\hat{A}_3(k, \ell, c_1) & = (k - \ell + 1)^{c_1} [2(k - \ell)^2 + (3c_1 + 10)(k - \ell) + 2c_1^2 + 9c_1 + 12] \\
& \quad - (k - \ell)^{c_1} [2(k - \ell)^2 + (5c_1 + 10)(k - \ell) + 6c_1^2 + 18c_1 + 12]. \tag{7.9}
\end{aligned}$$

Based on the numerical scheme obtained in (7.8), and based on (4.1), by assuming

$$\mathbf{X}(t, \mathbf{K}(t)) := \begin{cases} \mathbf{X}_1(t, \mathbb{H}^S(t)) := \mathbf{X}_1(t, \mathbb{H}^S(t), \mathbb{H}^I(t), \mathbb{H}^R(t), \mathbb{M}^S(t), \mathbb{M}^I(t)), \\ \mathbf{X}_2(t, \mathbb{H}^I(t)) := \mathbf{X}_2(t, \mathbb{H}^S(t), \mathbb{H}^I(t), \mathbb{H}^R(t), \mathbb{M}^S(t), \mathbb{M}^I(t)), \\ \mathbf{X}_3(t, \mathbb{H}^R(t)) := \mathbf{X}_3(t, \mathbb{H}^S(t), \mathbb{H}^I(t), \mathbb{H}^R(t), \mathbb{M}^S(t), \mathbb{M}^I(t)), \\ \mathbf{X}_4(t, \mathbb{M}^S(t)) := \mathbf{X}_4(t, \mathbb{H}^S(t), \mathbb{H}^I(t), \mathbb{H}^R(t), \mathbb{M}^S(t), \mathbb{M}^I(t)), \\ \mathbf{X}_5(t, \mathbb{M}^I(t)) := \mathbf{X}_5(t, \mathbb{H}^S(t), \mathbb{H}^I(t), \mathbb{H}^R(t), \mathbb{M}^S(t), \mathbb{M}^I(t)), \end{cases}$$

numerical solutions of the (\mathbb{H}, \mathbb{M}) - (c_1, c_2) -fractal-fractional $SIR-SI$ -model (3.3) of malaria are given by

$$\begin{aligned}
\mathbb{H}_{k+1}^S &= \varkappa_1 + \frac{(1-c_1)c_2 t_k^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_1(t_k, \mathbb{H}^S(t_k), \mathbb{H}^I(t_k), \mathbb{H}^R(t_k), \mathbb{M}^S(t_k), \mathbb{M}^I(t_k)) \\
&+ \frac{c_1 c_2 \mathbf{h}^{c_1}}{\Gamma(c_1+1)\mathbf{AB}(c_1)} \sum_{\ell=2}^k t_{\ell-2}^{c_2-1} \mathbf{X}_1(t_{\ell-2}, \mathbb{H}_{\ell-2}^S, \mathbb{H}_{\ell-2}^I, \mathbb{H}_{\ell-2}^R, \mathbb{M}_{\ell-2}^S, \mathbb{M}_{\ell-2}^I) \hat{A}_1(k, \ell, c_1) \\
&+ \frac{c_1 c_2 \mathbf{h}^{c_1}}{\Gamma(c_1+2)\mathbf{AB}(c_1)} \sum_{\ell=2}^k \left[t_{\ell-1}^{c_2-1} \mathbf{X}_1(t_{\ell-1}, \mathbb{H}_{\ell-1}^S, \mathbb{H}_{\ell-1}^I, \mathbb{H}_{\ell-1}^R, \mathbb{M}_{\ell-1}^S, \mathbb{M}_{\ell-1}^I) \right. \\
&- \left. t_{\ell-2}^{c_2-1} \mathbf{X}_1(t_{\ell-2}, \mathbb{H}_{\ell-2}^S, \mathbb{H}_{\ell-2}^I, \mathbb{H}_{\ell-2}^R, \mathbb{M}_{\ell-2}^S, \mathbb{M}_{\ell-2}^I) \right] \hat{A}_2(k, \ell, c_1) \\
&+ \frac{c_1 c_2 \mathbf{h}^{c_1}}{2\Gamma(c_1+3)\mathbf{AB}(c_1)} \sum_{\ell=2}^k \left[t_{\ell}^{c_2-1} \mathbf{X}_1(t_{\ell}, \mathbb{H}_{\ell}^S, \mathbb{H}_{\ell}^I, \mathbb{H}_{\ell}^R, \mathbb{M}_{\ell}^S, \mathbb{M}_{\ell}^I) \right. \\
&- \left. 2t_{\ell-1}^{c_2-1} \mathbf{X}_1(t_{\ell-1}, \mathbb{H}_{\ell-1}^S, \mathbb{H}_{\ell-1}^I, \mathbb{H}_{\ell-1}^R, \mathbb{M}_{\ell-1}^S, \mathbb{M}_{\ell-1}^I) \right. \\
&+ \left. t_{\ell-2}^{c_2-1} \mathbf{X}_1(t_{\ell-2}, \mathbb{H}_{\ell-2}^S, \mathbb{H}_{\ell-2}^I, \mathbb{H}_{\ell-2}^R, \mathbb{M}_{\ell-2}^S, \mathbb{M}_{\ell-2}^I) \right] \hat{A}_3(k, \ell, c_1), \tag{7.10}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{H}_{k+1}^I &= \varkappa_2 + \frac{(1-c_1)c_2 t_k^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_2(t_k, \mathbb{H}^S(t_k), \mathbb{H}^I(t_k), \mathbb{H}^R(t_k), \mathbb{M}^S(t_k), \mathbb{M}^I(t_k)) \\
&+ \frac{c_1 c_2 \mathbf{h}^{c_1}}{\Gamma(c_1+1)\mathbf{AB}(c_1)} \sum_{\ell=2}^k t_{\ell-2}^{c_2-1} \mathbf{X}_2(t_{\ell-2}, \mathbb{H}_{\ell-2}^S, \mathbb{H}_{\ell-2}^I, \mathbb{H}_{\ell-2}^R, \mathbb{M}_{\ell-2}^S, \mathbb{M}_{\ell-2}^I) \hat{A}_1(k, \ell, c_1) \\
&+ \frac{c_1 c_2 \mathbf{h}^{c_1}}{\Gamma(c_1+2)\mathbf{AB}(c_1)} \sum_{\ell=2}^k \left[t_{\ell-1}^{c_2-1} \mathbf{X}_2(t_{\ell-1}, \mathbb{H}_{\ell-1}^S, \mathbb{H}_{\ell-1}^I, \mathbb{H}_{\ell-1}^R, \mathbb{M}_{\ell-1}^S, \mathbb{M}_{\ell-1}^I) \right. \\
&- \left. t_{\ell-2}^{c_2-1} \mathbf{X}_2(t_{\ell-2}, \mathbb{H}_{\ell-2}^S, \mathbb{H}_{\ell-2}^I, \mathbb{H}_{\ell-2}^R, \mathbb{M}_{\ell-2}^S, \mathbb{M}_{\ell-2}^I) \right] \hat{A}_2(k, \ell, c_1) \\
&+ \frac{c_1 c_2 \mathbf{h}^{c_1}}{2\Gamma(c_1+3)\mathbf{AB}(c_1)} \sum_{\ell=2}^k \left[t_{\ell}^{c_2-1} \mathbf{X}_2(t_{\ell}, \mathbb{H}_{\ell}^S, \mathbb{H}_{\ell}^I, \mathbb{H}_{\ell}^R, \mathbb{M}_{\ell}^S, \mathbb{M}_{\ell}^I) \right. \\
&- \left. 2t_{\ell-1}^{c_2-1} \mathbf{X}_2(t_{\ell-1}, \mathbb{H}_{\ell-1}^S, \mathbb{H}_{\ell-1}^I, \mathbb{H}_{\ell-1}^R, \mathbb{M}_{\ell-1}^S, \mathbb{M}_{\ell-1}^I) \right. \\
&+ \left. t_{\ell-2}^{c_2-1} \mathbf{X}_2(t_{\ell-2}, \mathbb{H}_{\ell-2}^S, \mathbb{H}_{\ell-2}^I, \mathbb{H}_{\ell-2}^R, \mathbb{M}_{\ell-2}^S, \mathbb{M}_{\ell-2}^I) \right] \hat{A}_3(k, \ell, c_1), \tag{7.11}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{H}_{k+1}^{\mathcal{R}} &= \varkappa_3 + \frac{(1-c_1)c_2 t_k^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_3(t_k, \mathbb{H}^{\mathcal{S}}(t_k), \mathbb{H}^{\mathcal{I}}(t_k), \mathbb{H}^{\mathcal{R}}(t_k), \mathbb{M}^{\mathcal{S}}(t_k), \mathbb{M}^{\mathcal{I}}(t_k)) \\
&+ \frac{c_1 c_2 \mathbf{h}^{c_1}}{\Gamma(c_1+1)\mathbf{AB}(c_1)} \sum_{\ell=2}^k t_{\ell-2}^{c_2-1} \mathbf{X}_3(t_{\ell-2}, \mathbb{H}_{\ell-2}^{\mathcal{S}}, \mathbb{H}_{\ell-2}^{\mathcal{I}}, \mathbb{H}_{\ell-2}^{\mathcal{R}}, \mathbb{M}_{\ell-2}^{\mathcal{S}}, \mathbb{M}_{\ell-2}^{\mathcal{I}}) \hat{A}_1(k, \ell, c_1) \\
&+ \frac{c_1 c_2 \mathbf{h}^{c_1}}{\Gamma(c_1+2)\mathbf{AB}(c_1)} \sum_{\ell=2}^k \left[t_{\ell-1}^{c_2-1} \mathbf{X}_3(t_{\ell-1}, \mathbb{H}_{\ell-1}^{\mathcal{S}}, \mathbb{H}_{\ell-1}^{\mathcal{I}}, \mathbb{H}_{\ell-1}^{\mathcal{R}}, \mathbb{M}_{\ell-1}^{\mathcal{S}}, \mathbb{M}_{\ell-1}^{\mathcal{I}}) \right. \\
&- \left. t_{\ell-2}^{c_2-1} \mathbf{X}_3(t_{\ell-2}, \mathbb{H}_{\ell-2}^{\mathcal{S}}, \mathbb{H}_{\ell-2}^{\mathcal{I}}, \mathbb{H}_{\ell-2}^{\mathcal{R}}, \mathbb{M}_{\ell-2}^{\mathcal{S}}, \mathbb{M}_{\ell-2}^{\mathcal{I}}) \right] \hat{A}_2(k, \ell, c_1) \\
&+ \frac{c_1 c_2 \mathbf{h}^{c_1}}{2\Gamma(c_1+3)\mathbf{AB}(c_1)} \sum_{\ell=2}^k \left[t_{\ell}^{c_2-1} \mathbf{X}_3(t_{\ell}, \mathbb{H}_{\ell}^{\mathcal{S}}, \mathbb{H}_{\ell}^{\mathcal{I}}, \mathbb{H}_{\ell}^{\mathcal{R}}, \mathbb{M}_{\ell}^{\mathcal{S}}, \mathbb{M}_{\ell}^{\mathcal{I}}) \right. \\
&- \left. 2t_{\ell-1}^{c_2-1} \mathbf{X}_3(t_{\ell-1}, \mathbb{H}_{\ell-1}^{\mathcal{S}}, \mathbb{H}_{\ell-1}^{\mathcal{I}}, \mathbb{H}_{\ell-1}^{\mathcal{R}}, \mathbb{M}_{\ell-1}^{\mathcal{S}}, \mathbb{M}_{\ell-1}^{\mathcal{I}}) \right. \\
&+ \left. t_{\ell-2}^{c_2-1} \mathbf{X}_3(t_{\ell-2}, \mathbb{H}_{\ell-2}^{\mathcal{S}}, \mathbb{H}_{\ell-2}^{\mathcal{I}}, \mathbb{H}_{\ell-2}^{\mathcal{R}}, \mathbb{M}_{\ell-2}^{\mathcal{S}}, \mathbb{M}_{\ell-2}^{\mathcal{I}}) \right] \hat{A}_3(k, \ell, c_1), \tag{7.12}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{M}_{k+1}^{\mathcal{S}} &= \varkappa_4 + \frac{(1-c_1)c_2 t_k^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_4(t_k, \mathbb{H}^{\mathcal{S}}(t_k), \mathbb{H}^{\mathcal{I}}(t_k), \mathbb{H}^{\mathcal{R}}(t_k), \mathbb{M}^{\mathcal{S}}(t_k), \mathbb{M}^{\mathcal{I}}(t_k)) \\
&+ \frac{c_1 c_2 \mathbf{h}^{c_1}}{\Gamma(c_1+1)\mathbf{AB}(c_1)} \sum_{\ell=2}^k t_{\ell-2}^{c_2-1} \mathbf{X}_4(t_{\ell-2}, \mathbb{H}_{\ell-2}^{\mathcal{S}}, \mathbb{H}_{\ell-2}^{\mathcal{I}}, \mathbb{H}_{\ell-2}^{\mathcal{R}}, \mathbb{M}_{\ell-2}^{\mathcal{S}}, \mathbb{M}_{\ell-2}^{\mathcal{I}}) \hat{A}_1(k, \ell, c_1) \\
&+ \frac{c_1 c_2 \mathbf{h}^{c_1}}{\Gamma(c_1+2)\mathbf{AB}(c_1)} \sum_{\ell=2}^k \left[t_{\ell-1}^{c_2-1} \mathbf{X}_4(t_{\ell-1}, \mathbb{H}_{\ell-1}^{\mathcal{S}}, \mathbb{H}_{\ell-1}^{\mathcal{I}}, \mathbb{H}_{\ell-1}^{\mathcal{R}}, \mathbb{M}_{\ell-1}^{\mathcal{S}}, \mathbb{M}_{\ell-1}^{\mathcal{I}}) \right. \\
&- \left. t_{\ell-2}^{c_2-1} \mathbf{X}_4(t_{\ell-2}, \mathbb{H}_{\ell-2}^{\mathcal{S}}, \mathbb{H}_{\ell-2}^{\mathcal{I}}, \mathbb{H}_{\ell-2}^{\mathcal{R}}, \mathbb{M}_{\ell-2}^{\mathcal{S}}, \mathbb{M}_{\ell-2}^{\mathcal{I}}) \right] \hat{A}_2(k, \ell, c_1) \\
&+ \frac{c_1 c_2 \mathbf{h}^{c_1}}{2\Gamma(c_1+3)\mathbf{AB}(c_1)} \sum_{\ell=2}^k \left[t_{\ell}^{c_2-1} \mathbf{X}_4(t_{\ell}, \mathbb{H}_{\ell}^{\mathcal{S}}, \mathbb{H}_{\ell}^{\mathcal{I}}, \mathbb{H}_{\ell}^{\mathcal{R}}, \mathbb{M}_{\ell}^{\mathcal{S}}, \mathbb{M}_{\ell}^{\mathcal{I}}) \right. \\
&- \left. 2t_{\ell-1}^{c_2-1} \mathbf{X}_4(t_{\ell-1}, \mathbb{H}_{\ell-1}^{\mathcal{S}}, \mathbb{H}_{\ell-1}^{\mathcal{I}}, \mathbb{H}_{\ell-1}^{\mathcal{R}}, \mathbb{M}_{\ell-1}^{\mathcal{S}}, \mathbb{M}_{\ell-1}^{\mathcal{I}}) \right. \\
&+ \left. t_{\ell-2}^{c_2-1} \mathbf{X}_4(t_{\ell-2}, \mathbb{H}_{\ell-2}^{\mathcal{S}}, \mathbb{H}_{\ell-2}^{\mathcal{I}}, \mathbb{H}_{\ell-2}^{\mathcal{R}}, \mathbb{M}_{\ell-2}^{\mathcal{S}}, \mathbb{M}_{\ell-2}^{\mathcal{I}}) \right] \hat{A}_3(k, \ell, c_1), \tag{7.13}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{M}_{k+1}^I &= \varkappa_5 + \frac{(1-c_1)c_2 t_k^{c_2-1}}{\mathbf{AB}(c_1)} \mathbf{X}_5(t_k, \mathbb{H}^S(t_k), \mathbb{H}^I(t_k), \mathbb{H}^R(t_k), \mathbb{M}^S(t_k), \mathbb{M}^I(t_k)) \\
&+ \frac{c_1 c_2 \mathbf{h}^{c_1}}{\Gamma(c_1+1)\mathbf{AB}(c_1)} \sum_{\ell=2}^k t_{\ell-2}^{c_2-1} \mathbf{X}_5(t_{\ell-2}, \mathbb{H}_{\ell-2}^S, \mathbb{H}_{\ell-2}^I, \mathbb{H}_{\ell-2}^R, \mathbb{M}_{\ell-2}^S, \mathbb{M}_{\ell-2}^I) \hat{A}_1(k, \ell, c_1) \\
&+ \frac{c_1 c_2 \mathbf{h}^{c_1}}{\Gamma(c_1+2)\mathbf{AB}(c_1)} \sum_{\ell=2}^k \left[t_{\ell-1}^{c_2-1} \mathbf{X}_5(t_{\ell-1}, \mathbb{H}_{\ell-1}^S, \mathbb{H}_{\ell-1}^I, \mathbb{H}_{\ell-1}^R, \mathbb{M}_{\ell-1}^S, \mathbb{M}_{\ell-1}^I) \right. \\
&- \left. t_{\ell-2}^{c_2-1} \mathbf{X}_5(t_{\ell-2}, \mathbb{H}_{\ell-2}^S, \mathbb{H}_{\ell-2}^I, \mathbb{H}_{\ell-2}^R, \mathbb{M}_{\ell-2}^S, \mathbb{M}_{\ell-2}^I) \right] \hat{A}_2(k, \ell, c_1) \\
&+ \frac{c_1 c_2 \mathbf{h}^{c_1}}{2\Gamma(c_1+3)\mathbf{AB}(c_1)} \sum_{\ell=2}^k \left[t_{\ell}^{c_2-1} \mathbf{X}_5(t_{\ell}, \mathbb{H}_{\ell}^S, \mathbb{H}_{\ell}^I, \mathbb{H}_{\ell}^R, \mathbb{M}_{\ell}^S, \mathbb{M}_{\ell}^I) \right. \\
&- \left. 2t_{\ell-1}^{c_2-1} \mathbf{X}_5(t_{\ell-1}, \mathbb{H}_{\ell-1}^S, \mathbb{H}_{\ell-1}^I, \mathbb{H}_{\ell-1}^R, \mathbb{M}_{\ell-1}^S, \mathbb{M}_{\ell-1}^I) \right. \\
&+ \left. t_{\ell-2}^{c_2-1} \mathbf{X}_5(t_{\ell-2}, \mathbb{H}_{\ell-2}^S, \mathbb{H}_{\ell-2}^I, \mathbb{H}_{\ell-2}^R, \mathbb{M}_{\ell-2}^S, \mathbb{M}_{\ell-2}^I) \right] \hat{A}_3(k, \ell, c_1), \tag{7.14}
\end{aligned}$$

where the constants $\hat{A}_j(k, \ell, c_1)$ are introduced in (7.9) for $j = 1, 2, 3$.

8. Simulations

We discuss the behavior of the model on different plotted simulations by assuming the numerical data for the parameters computed in [52]. According to this source, we take $\Theta_{\mathbb{H}} = 0.027$, $\delta = 1/730$, $\eta = 0.038$, $a_1 = 0.02$, $\lambda = 0.13$, $a_2 = 0.01$, $p = 0.05$, $f_{\mathbb{H}} = 0.0004$, $\varpi = 0.005$, $\omega = 0.05$, $\gamma = 0.611$, $\beta = 0.25$, $\Theta_{\mathbb{M}} = 0.13$, $\kappa = 0.022$, $a_3 = 0.072$, $f_{\mathbb{M}} = 0.04$ and $\alpha = 0.05$. The initial values are $\mathbb{H}^S(0) = \varkappa_1 = 100$, $\mathbb{H}^I(0) = \varkappa_2 = 2$, $\mathbb{H}^R(0) = \varkappa_3 = 0$, $\mathbb{M}^S(0) = \varkappa_4 = 800$ and $\mathbb{M}^I(0) = \varkappa_5 = 10$. In some figures, we shall illustrate the behaviors of five state functions \mathbb{H}^S , \mathbb{H}^I , \mathbb{H}^R , \mathbb{M}^S and \mathbb{M}^I by considering some values for the fractal dimensions and fractional orders $c_1 = c_2 = 1.00, 0.98, 0.96, 0.94, 0.92, 0.90$ based on numerical algorithms given by (7.10)–(7.14). The compartmental trajectories of the human and vector populations are depicted in Figures 1–7 with the time step $h = 0.01$ and an initial time of $t = 0.000001$ in weeks. Figure 1a shows that, when the fractal-fractional order is 1, the number of susceptible humans decreases rapidly, but not when the fractal-fractional order is 0.5 or less. It also suggests that, when the fractal-fractional order is 0.95, a change in dynamical behavior in the infected and recovered individuals reduces the number of recoveries after week 5, resulting in an increase in the total number of infected humans after week 25. Similarly, we notice in Figure 1b that the susceptibility of mosquitoes decreases rapidly when the fractal and fractional order is 1, as opposed to when the fractal and fractional order is 0.95. In contrast, the fractal-fractional dynamical behavior of the infected mosquitoes increases above the integer order after week 17.

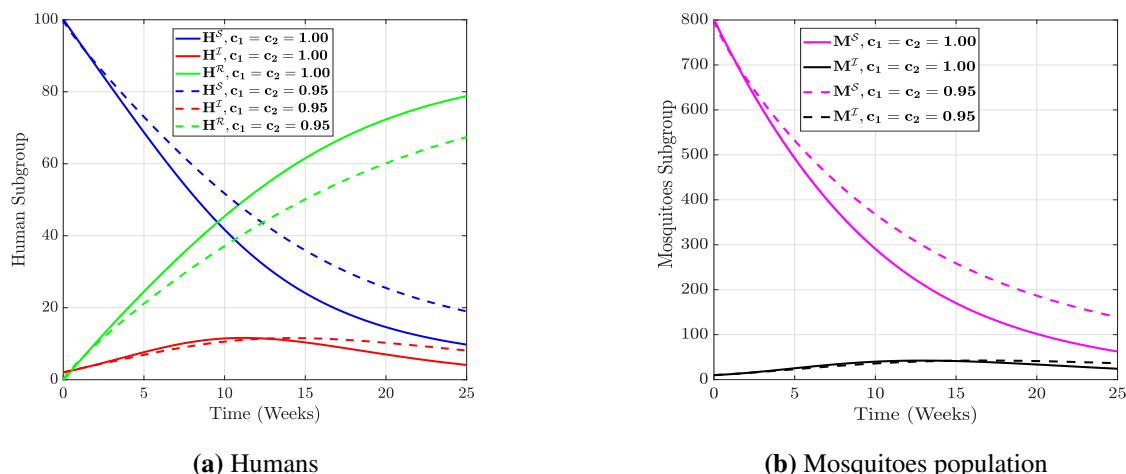
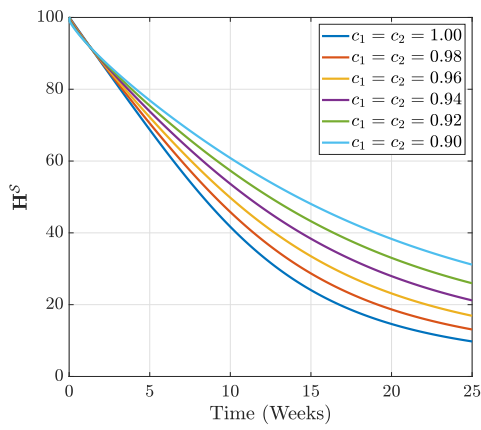


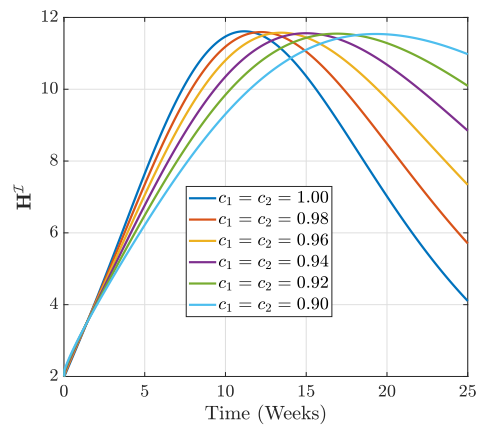
Figure 1. Effects of the fractal and fractional orders $c_1 = c_2 = 1.00, 0.95$ for all five state functions $\mathbb{H}^S(t), \mathbb{H}^I(t), \mathbb{H}^R(t), \mathbb{M}^S(t), \mathbb{M}^I(t)$ in one graph.

Figure 2a through 2e depict the fractal-fractional behaviors of susceptible humans, infected humans, recovered humans, susceptible mosquitoes and infected mosquitoes when the fractal-fractional derivative is 1 and the fractal-fractional order is 0.98, 0.96, 0.94, 0.92 or 0.90, respectively. In Figure 2a, 2d, you see that the number of susceptible humans and susceptible mosquitoes increases as the fractal-fractional order reduces. In Figure 2b, 2e, you see that the number of infected humans decreases as the fractal-fractional order decreases for 13 weeks; it then surpasses the integer order from week 14 onwards. In Figure 2c, you see that the number of recovered individuals reduces as the fractal-fractional order reduces. Figure 3a through 3e show the effect of antimalarial drugs on the human population, and that on the mosquito population. Figure 3b, 3e show that antimalarial drugs have a greater effect on the number of infected humans and mosquitos than the other compartmental classes. Figure 3a shows that an increase in the efficacy of antimalarial drug increases the number of susceptible humans after week 8 when the fractal-fractional order is 0.99. Figure 3c shows that an increase in the efficacy of antimalarial drugs increases the number of recovered humans after week 5 when the fractal-fractional order is 0.99. Figure 3d shows that an increase in the efficacy of antimalarial drugs increases the number of susceptible mosquitoes after week 9 when the fractal-fractional order is 0.99.

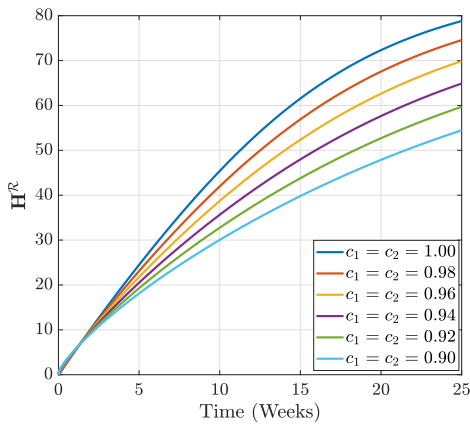
Figure 4a through 4e show the effects of vaccination on all five state functions. Figure 4b, 4e show that vaccination has a weaker effect on the numbers of infected humans and mosquitos than the other compartmental classes as compared to antimalarial drugs. Figure 4a shows that an increase in vaccination increases the number of susceptible humans for 20 weeks more than antimalarial drugs when the fractal-fractional order is 0.99. Still, after 20 weeks, the antimalarial drugs increase the number of susceptible humans more than vaccination. Figure 4c shows that an increase in vaccination increases the number of recovered humans after week 2 when the fractal-fractional order is 0.99. Figure 4d shows that an increase in vaccination does not produce any significant change in the number of susceptible mosquitoes when the fractal-fractional order is 0.99. Figure 5a through 5e show the effects of spraying on all five state functions.



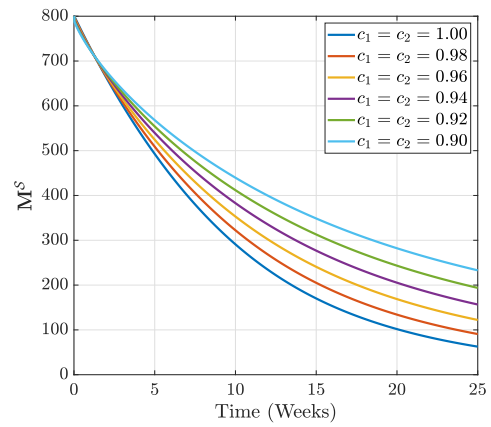
(a) Susceptible humans



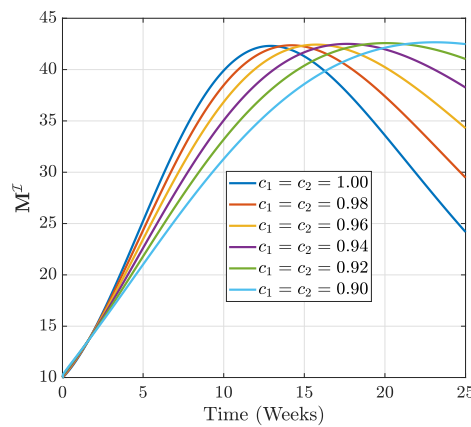
(b) Infected humans



(c) Recovered humans



(d) Susceptible mosquitoes



(e) Infected mosquitoes

Figure 2. Effects of the fractal and fractional orders $c_1 = c_2 = 1.00, 0.98, 0.96, 0.94, 0.92, 0.90$ for all five state functions $H^S(t), H^I(t), H^R(t), M^S(t), M^I(t)$, based on Figure 1.

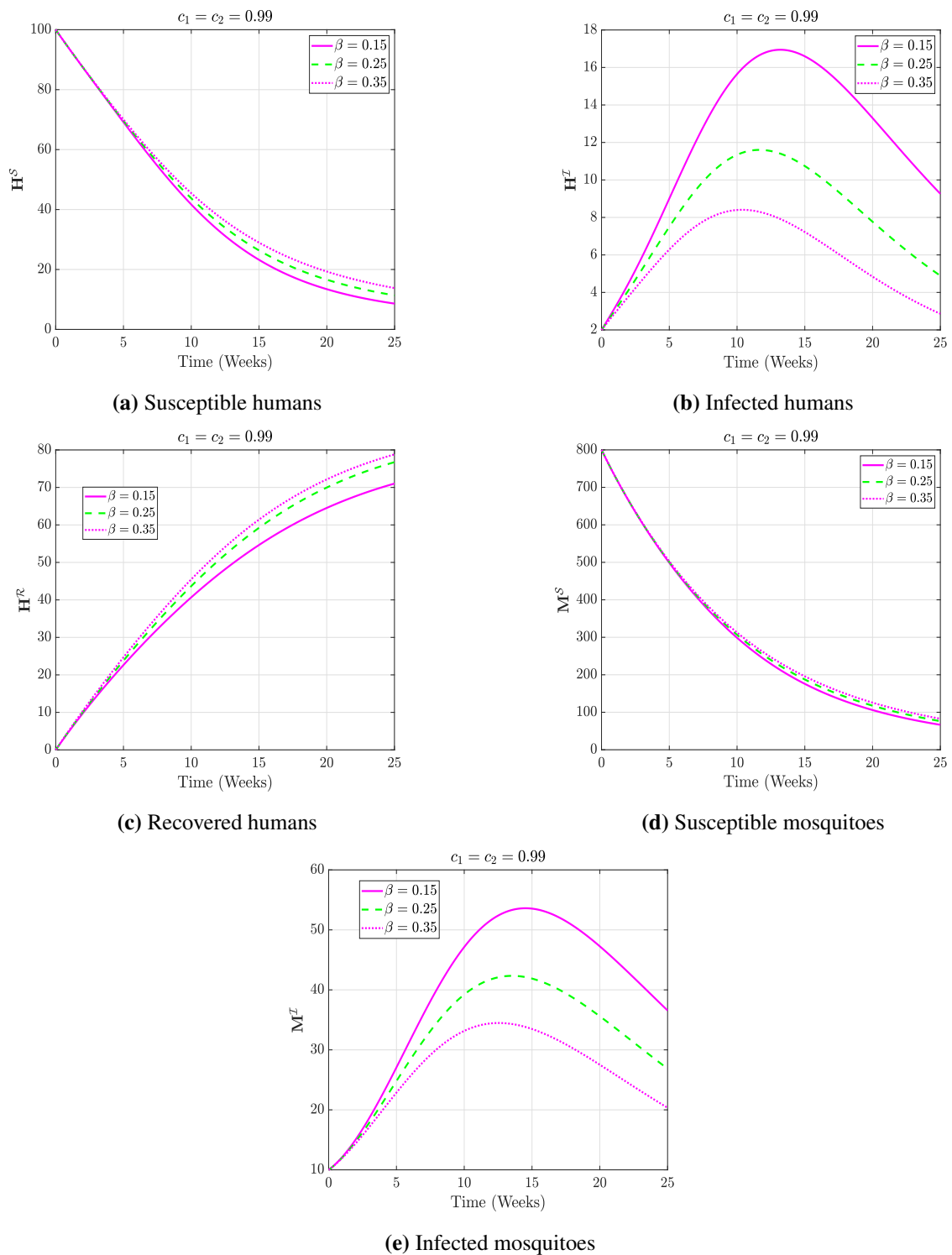


Figure 3. Effects of antimalarial drugs on all five state functions $H^S(t), H^I(t), H^R(t), M^S(t), M^I(t)$ in five distinct graphs when $c_1 = c_2 = 0.99$, $\beta = 0.15, 0.25, 0.35$ and $p = \alpha = 0.05$.

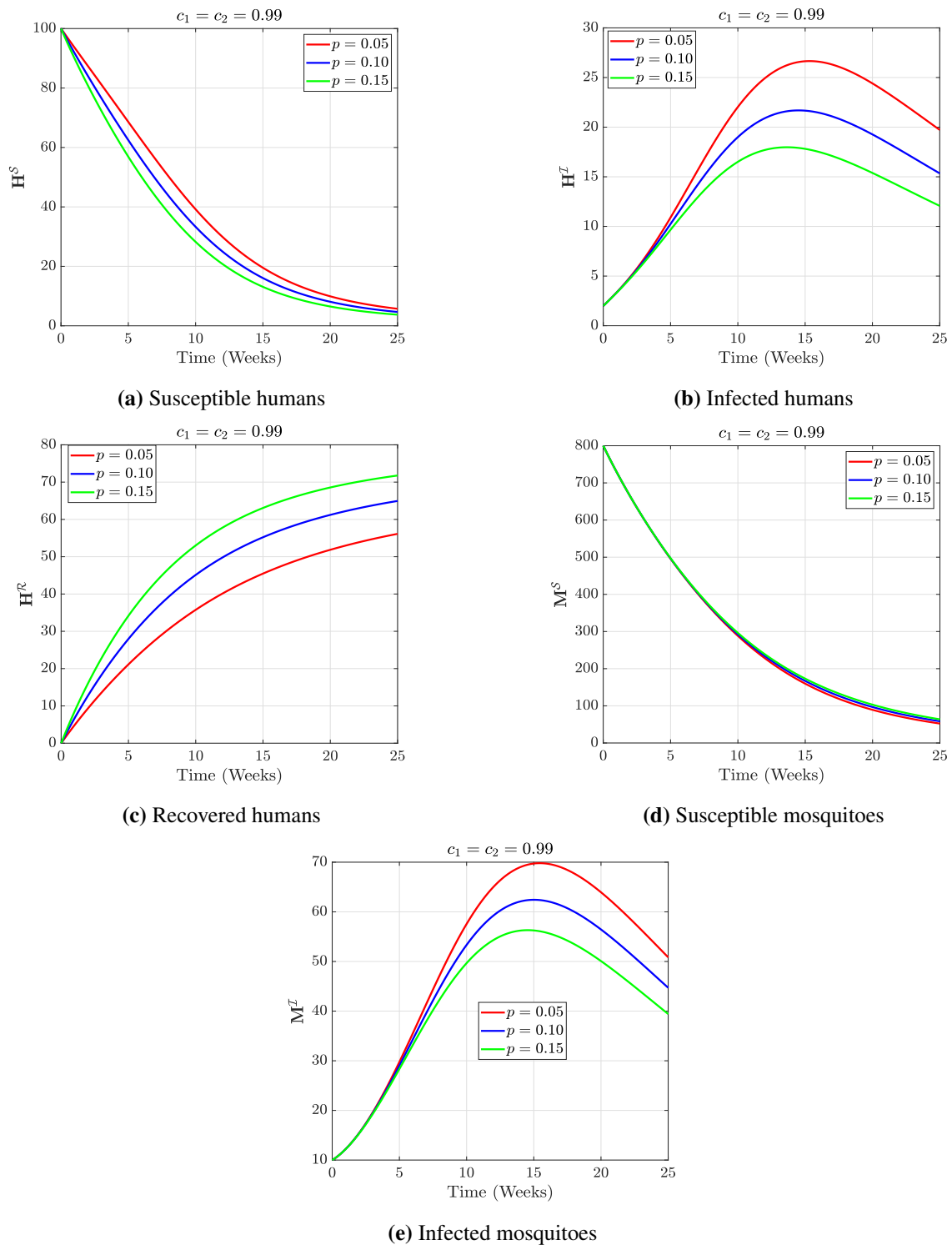


Figure 4. Effects of vaccination on all five state functions $H^S(t)$, $H^I(t)$, $H^R(t)$, $M^S(t)$, $M^I(t)$ in five distinct graphs when $c_1 = c_2 = 0.99$, $p = 0.05, 0.07, 0.09$ and $\beta = \alpha = 0.05$.

Figure 5b, 5e show that spraying has a more significant effect on the number of infected mosquitoes than that of infected humans; hence, it suggests that an increase in spraying potentially reduces the

number of malarial incidence more significantly than antimalarial drugs. Figure 5a shows that an increase in spraying increases the number of susceptible humans more significantly than antimalarial drugs and vaccination when the fractal-fractional order is 0.99. Figure 5c shows that an increase in spraying produces fewer recovered humans as compared to antimalarial drugs and vaccination when the fractal-fractional order is 0.99. Figure 5d shows that an increase in spraying reduces the number of susceptible mosquitoes when the fractal-fractional order is 0.99. The relative importance of fractional order only and fractal dimension only on the epidemic model is shown in Figures 6 and 7, with the corresponding numerical values shown in Table 1.

Table 1. Numerical comparison of fractional order only and fractal dimension only at the end of the simulation time with $c_1 = 1.00, 0.98, 0.96, 0.94, 0.92, 0.90$ and $c_2 = 1.00, 0.98, 0.96, 0.94, 0.92, 0.90$, based on Figures 6 and 7.

Compartment	Order	Only fractional order: c_1	Only fractal order: c_2
$\sum_{0,000001}^{25} \mathbb{H}^S(t)$	1.00	100876	100876
$\sum_{0,000001}^{25} \mathbb{H}^S(t)$	0.98	105146	104622
$\sum_{0,000001}^{25} \mathbb{H}^S(t)$	0.96	109424	108513
$\sum_{0,000001}^{25} \mathbb{H}^S(t)$	0.94	113707	112537
$\sum_{0,000001}^{25} \mathbb{H}^S(t)$	0.92	117995	116681
$\sum_{0,000001}^{25} \mathbb{H}^S(t)$	0.90	122287	120924
$\sum_{0,000001}^{25} \mathbb{H}^I(t)$	1.00	20084	20084
$\sum_{0,000001}^{25} \mathbb{H}^I(t)$	0.98	20692	20742
$\sum_{0,000001}^{25} \mathbb{H}^I(t)$	0.96	21238	21333
$\sum_{0,000001}^{25} \mathbb{H}^I(t)$	0.94	21718	21840
$\sum_{0,000001}^{25} \mathbb{H}^I(t)$	0.92	22125	22252
$\sum_{0,000001}^{25} \mathbb{H}^I(t)$	0.90	22456	22559
$\sum_{0,000001}^{25} \mathbb{H}^R(t)$	1.00	122498	122498
$\sum_{0,000001}^{25} \mathbb{H}^R(t)$	0.98	118128	118719
$\sum_{0,000001}^{25} \mathbb{H}^R(t)$	0.96	113809	114868
$\sum_{0,000001}^{25} \mathbb{H}^R(t)$	0.94	109546	110964
$\sum_{0,000001}^{25} \mathbb{H}^R(t)$	0.92	105345	107029
$\sum_{0,000001}^{25} \mathbb{H}^R(t)$	0.90	101210	103088
$\sum_{0,000001}^{25} \mathbb{M}^S(t)$	1.00	729906	729906
$\sum_{0,000001}^{25} \mathbb{M}^S(t)$	0.98	765866	757474
$\sum_{0,000001}^{25} \mathbb{M}^S(t)$	0.96	801813	785906
$\sum_{0,000001}^{25} \mathbb{M}^S(t)$	0.94	837703	815126
$\sum_{0,000001}^{25} \mathbb{M}^S(t)$	0.92	873500	845048
$\sum_{0,000001}^{25} \mathbb{M}^S(t)$	0.90	909164	875565
$\sum_{0,000001}^{25} \mathbb{M}^I(t)$	1.00	79304	79304
$\sum_{0,000001}^{25} \mathbb{M}^I(t)$	0.98	80724	80688
$\sum_{0,000001}^{25} \mathbb{M}^I(t)$	0.96	81923	81751
$\sum_{0,000001}^{25} \mathbb{M}^I(t)$	0.94	82890	82469
$\sum_{0,000001}^{25} \mathbb{M}^I(t)$	0.92	83612	82828
$\sum_{0,000001}^{25} \mathbb{M}^I(t)$	0.90	84084	82825

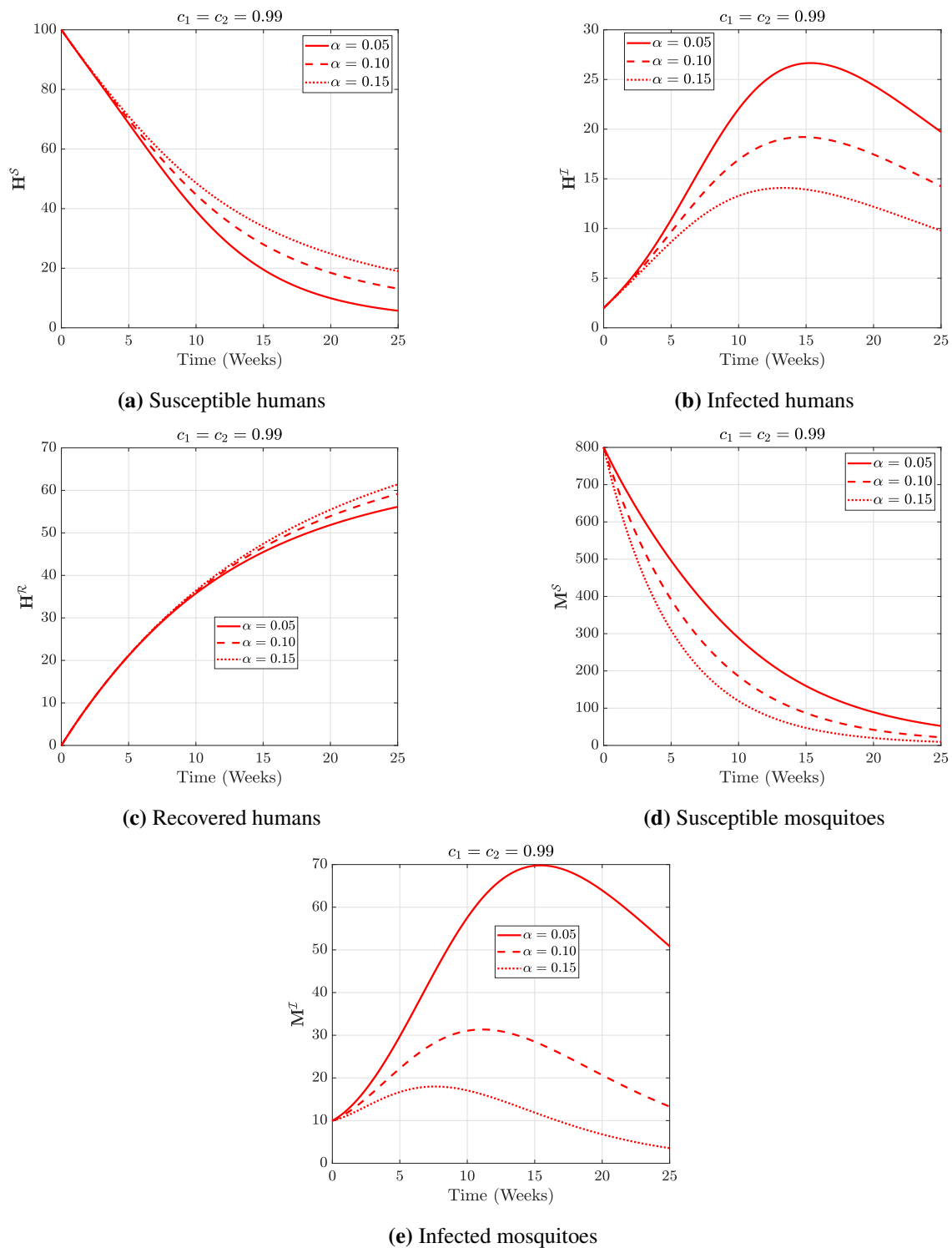


Figure 5. Effects of spraying on all five state functions $H^S(t)$, $H^I(t)$, $H^R(t)$, $M^S(t)$, $M^I(t)$ in five distinct graphs when $c_1 = c_2 = 0.99$, $\alpha = 0.05, 0.10, 0.15$ and $p = \beta = 0.05$.

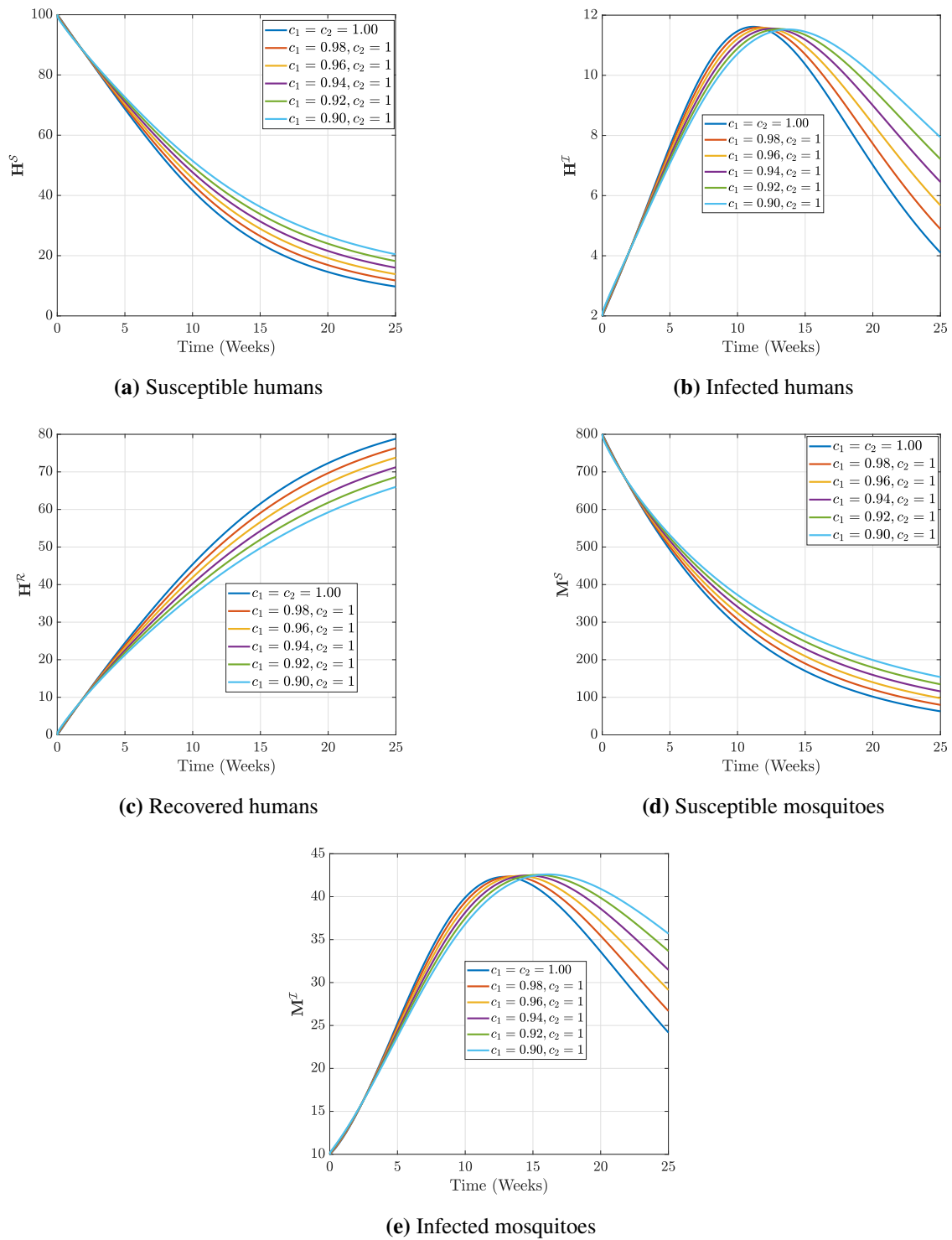
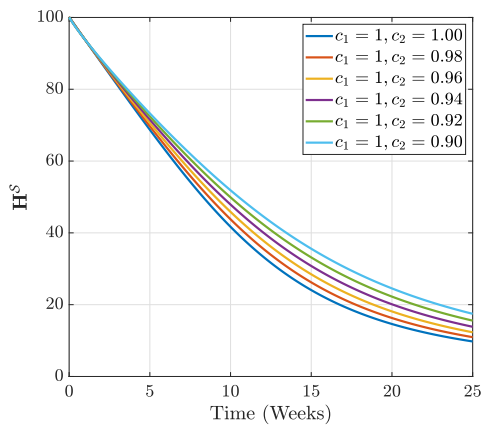
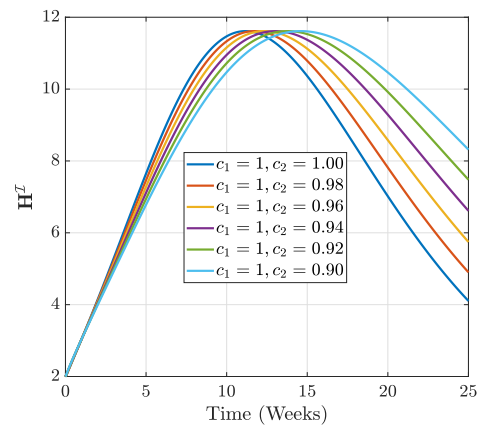


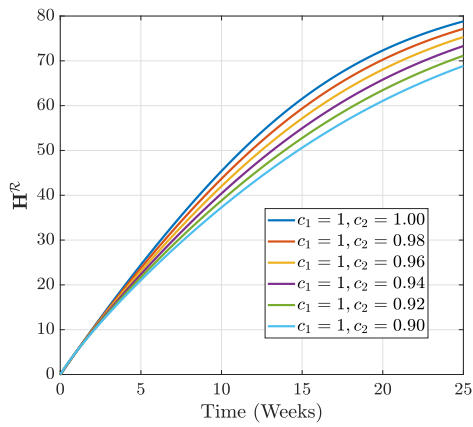
Figure 6. Effects of the fractional orders $c_1 = 1.00, 0.98, 0.96, 0.94, 0.92, 0.90$ for all five state functions $H^S(t), H^I(t), H^R(t), M^S(t), M^I(t)$ when the fractal order is kept constant at $c_2 = 1$.



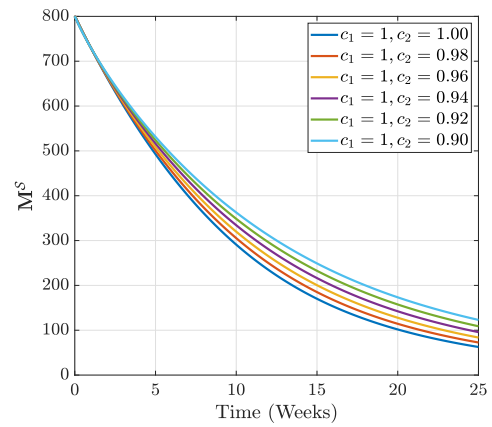
(a) Susceptible humans



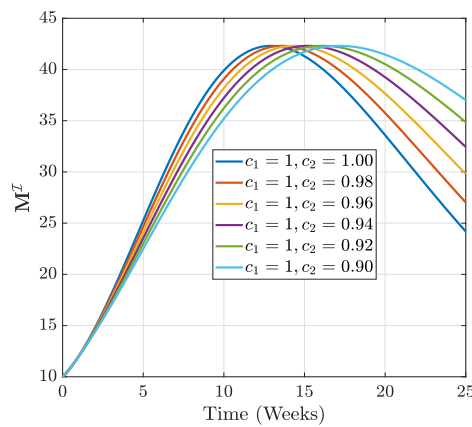
(b) Infected humans



(c) Recovered humans



(d) Susceptible mosquitoes



(e) Infected mosquitoes

Figure 7. Effects of the fractional orders $c_1 = 1.00, 0.98, 0.96, 0.94, 0.92, 0.90$ for all five state functions $H^S(t), H^I(t), H^R(t), M^S(t), M^I(t)$ when the fractional order is kept constant at $c_1 = 1$.

9. Conclusions

In this paper, we analyzed an $STR-SI$ -model of malaria disease analytically and numerically in the context of a five-dimensional system of the Atangana-Baleanu (c_1, c_2) -fractal-fractional differential equations. We introduced all parameters of the model and then derived an equivalent compact fractal-fractional IVP. Then, in order, we examined some properties of the solutions of this system in detail, including the existence, Lipschitz property and uniqueness criterion. Also, stable solutions were defined and proved in the sense of Hyers-Ulam and Hyers-Ulam-Rassias. The Newton polynomials were applied for the first time to derive numerical solutions to the given system in the context of the fractal-fractional version of the malaria disease. Using the simulations, we have studied the role and impact of the fractal dimension c_2 and the fractional order c_1 on the behavior of the system. The effects of some parameters and fractal-fractional orders on the vaccination rates, antimalaria drugs and spraying were analyzed in all graphs. Therefore, if we can consider these processes, then the rate of disease outbreaks will be largely controlled. In subsequent studies, we can implement other simulations with the help of different newly defined numerical methods and compare the results together.

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Conflict of interest

The authors declare no conflicts of interest.

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