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Research article

Some new estimates of well known inequalities for (h_1, h_2)-Godunova-Levin functions by means of center-radius order relation

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Abstract: In this manuscript, we aim to establish a connection between the concept of inequalities and the novel Center-Radius order relation. The idea of a Center-Radius (CR)-order interval-valued Godunova-Levin (GL) function is introduced by referring to a total order relation between two intervals. Consequently, convexity and nonconvexity contribute to different kinds of inequalities. In spite of this, convex theory turns to Godunova-Levin functions because they are more efficient at determining inequality terms than other convexity classes. Our application of these new definitions has led to many classical and novel special cases that illustrate the key findings of the paper. Using total order relations between two intervals, this study introduces CR-(h_1, h_2)-Goduova-Levin functions. It is clear from their properties and widespread usage that the Center-Radius order relation is an ideal tool for studying inequalities. This paper discusses various inequalities based on the Center-Radius order relation. With the CR-order relation, we can first derive Hermite-Hadamard $(\mathcal{H}.\mathcal{H})$ inequalities and then develop Jensen-type inequality for interval-valued functions $(\mathcal{IVFS})$ of type (h_1, h_2)-Godunova-Levin function. Furthermore, the study includes examples to support its conclusions.

Keywords: Jensen inequality; Hermite-Hadamard inequality; Godunova-Levin function; CR-order relation; interval-valued function
Mathematics Subject Classification: 26A48, 26A51, 33B10, 39A12, 39B62

1. Introduction

The field of interval analysis is a subfield of set-valued analysis, which focuses on sets in mathematics and topology. Historically, Archimede's method included calculating the circumference of a circle, which is an example of interval enclosure. By focusing on interval variables instead of point variables, and expressing computation results as intervals, this method eliminates errors that cause misleading conclusions. An initial objective of the interval-valued analysis was to estimate error estimates for numerical solutions to finite state machines. In 1966, Moore [2], published the first book on interval analysis, which is credited with being the first to use intervals in computer mathematics in order to improve calculation results. There are many situations where the interval analysis can be used to solve uncertain problems because it can be expressed in terms of uncertain variables. In spite of this, interval analysis remains one of the best approaches to solving interval uncertain structural systems and has been used for over fifty years in mathematical modeling such as computer graphics [3], decision-making analysis [4], multi-objective optimization, [5], error analysis [40]. In summary, interval analysis research has yielded numerous excellent results, and readers can consult Refs. [7–9], for additional information.

Convexity has been recognized for many years as a significant factor in such fields as probability theory, economics, optimal control theory, and fuzzy analysis. On the other hand, generalized convexity of mappings is a powerful tool for solving numerous nonlinear analysis and applied analysis problems, including a wide range of mathematical physics problems. A number of rigorous generalizations of convex functions have recently been investigated, see Refs. [10–13]. An interesting topic in mathematical analysis is integral inequalities. Convexity plays a significant role in inequality theory. During the last few decades, generalized convexity has played a prominent role in many disciplines and applications of $\mathcal{IVFS}$, see Refs. [14–19]. Several recent applications have addressed these inequalities, see Refs. [20–22]. First, Breckner describes the idea of continuity for $\mathcal{IVFS}$, see Ref. [23]. Using the generalized Hukuhara derivative, Chalco-Cano et al. [24], and Costa et al. [25], derived some Ostrowski and Opial type inequality for $\mathcal{IVFS}$, respectively. Bai et al. [26], formulated an interval-based Jensen inequality. First, Zhao [27], and co-authors established $(\mathcal{H}.\mathcal{H})$ and Jensen inequality using h-convexity for $\mathcal{IVFS}$. In general, the traditional $(\mathcal{H}.\mathcal{H})$ inequality has the following definition:

$$\frac{\Theta(g)+\Theta(h)}{2} \geq \frac{1}{h-g}\int_g^h \Theta(\gamma)d\gamma \geq \Theta\left(\frac{g+h}{2}\right). \tag{1.1}$$

Because of the nature of its definition, it is the first geometrical interpretation of convex mappings in elementary mathematics, and has attracted a large amount of attention. Several generalizations of this inequality are presented here, see Refs. [28–31]. Initially, Awan et al. explored (h_1, h_2)-convex functions and proved the following inequality [32]. Several authors have developed $\mathcal{H}.\mathcal{H}$ and Jensen-type inequalities utilizing (h_1, h_2)-convexity. Ruonan Liu [33] developed $\mathcal{H}.\mathcal{H}$ inequalities for harmonically (h_1, h_2)-convex functions. Wengui Yang [34] developed $\mathcal{H}.\mathcal{H}$ inequalities on the coordinates for (p_1, h_1)-(p_2, h_2)-convex functions. Shi et al. [35] developed $\mathcal{H}.\mathcal{H}$ inequalities for (m, h_1, h_2)-convex functions via Riemann Liouville fractional integrals. Sahoo et al. [36] established $\mathcal{H}.\mathcal{H}$ and Jensen-type inequalities for harmonically (h_1, h_2)-Godunova-Levin functions. Afzal et al. [37] developed these inequalities for a generalized class of Godunova-Levin functions using inclusion relation. An et al. [38] developed $\mathcal{H}.\mathcal{H}$ type inequalities for interval-valued (h_1, h_2)-convex functions. Results are now influenced by less accurate inclusion relation and interval LU-order relation. For some recent developments using the inclusion relation for the generalized class of Godunova-Levin functions, see Refs. [39, 40, 44]. It is clear from comparing the examples presented in this literature that the inequalities obtained using these old partial order relations are not as precise as those obtained by using CR-order relation. As a result, it is critically important that we are able to study inequalities and convexity by using a total order relation. Therefore, we use Bhunia's [41], CR-order, which is total interval order relation. The notions of CR-convexity and CR-order relation were used by several authors in 2022, in an attempt to prove a number of recent developments in these inequalities, see Refs. [42, 43]. Afzal et al. using the notion of the h-GL function, proves the following result [45].

Theorem 1.1. (See [45]) Consider $\Theta : [g, h] \to \mathcal{R}_{\mathcal{I}}^{+}$. Define $h : (0,1) \to \mathcal{R}^{+}$ and $h\left(\frac{1}{2}\right) \neq 0$. If $\Theta \in SX(\text{CR-h}, [g, h], \mathcal{R}_{\mathcal{I}}^{+})$ and $\Theta \in \mathcal{IR}_{[g,h]}$, then

$$\frac{h\left(\frac{1}{2}\right)}{2}\Theta\left(\frac{g+h}{2}\right) \preceq_{CR} \frac{1}{h-g}\int_{g}^{h}\Theta(\gamma)d\gamma \preceq_{CR} [\Theta(g)+\Theta(h)]\int_{0}^{1}\frac{dx}{h(x)}. \tag{1.2}$$

Also, by using the notion of the h-GL function Jensen-type inequality was also developed.

Theorem 1.2. (See [45]) Let $u_i \in \mathcal{R}^{+}$, $j_i \in [g, h]$. If h is non-negative super multiplicative function and $\Theta \in SX(\text{CR-h}, [g, h], \mathcal{R}_{\mathcal{I}}^{+})$, then this holds :

$$\Theta\left(\frac{1}{U_k}\sum_{i=1}^{k}u_i j_i\right) \preceq_{CR} \sum_{i=1}^{k}\frac{\Theta(j_i)}{h\left(\frac{u_i}{U_k}\right)}. \tag{1.3}$$

In addition, it introduces a new concept of interval-valued GL-functions pertaining to a total order relation, the Center-Radius order, which is unique as far as the literature goes. With the example presented in this article, we are able to show how CR-$\mathcal{IVFS}$ can be used to analyze various integral inequalities. In contrast to classical interval-valued analysis, CR-order interval-valued analysis differs from it. Using the concept of Centre and Radius, we calculate intervals as follows: $\mathcal{M}_C = \frac{\underline{\mathcal{M}}+\overline{\mathcal{M}}}{2}$ and $\mathcal{M}_R = \frac{\overline{\mathcal{M}}-\underline{\mathcal{M}}}{2}$, respectively, where $\mathcal{M} = [\underline{\mathcal{M}}, \bar{\mathcal{M}}]$. Inspired by the concepts of interval valued analysis and the strong literature that has been discussed above with particular articles, see e.g., Zhang et al. [39], Bhunia and Samanta [41], Shi et al. [42], Liu et al. [43] and Afzal et al. [44, 45], we introduced the idea of CR-(h_1, h_2)-GL function. By using this new concept we developed $\mathcal{H}.\mathcal{H}$ and Jensen-type inequalities. The study also includes useful examples to back up its findings.

Finally, the article is designed as follows: In Section 2, preliminary is provided. The main problems and applications are provided in Section 3 and 4. Finally, Section 5 provides the conclusion.

2. Preliminaries

As for the notions used in this paper but not defined, see Refs. [42, 43, 45]. It is a good idea to familiarize yourself with some basic arithmetic related to interval analysis in this section since it will prove very helpful throughout the paper.

$$[\mathcal{M}] = [\underline{\mathcal{M}}, \overline{\mathcal{M}}] \qquad (x \in \mathcal{R},\ \underline{\mathcal{M}} \leqq x \leqq \overline{\mathcal{M}};\ x \in \mathcal{R})$$

$$[\mathcal{N}] = [\underline{\mathcal{N}}, \overline{\mathcal{N}}] \qquad (x \in \mathcal{R},\ \underline{\mathcal{N}} \leqq x \leqq \overline{\mathcal{N}};\ x \in \mathcal{R})$$

$$[\mathcal{M}] + [\mathcal{N}] = [\underline{\mathcal{M}}, \overline{\mathcal{M}}] + [\underline{\mathcal{N}}, \overline{\mathcal{N}}] = [\underline{\mathcal{M}} + \underline{\mathcal{N}}, \overline{\mathcal{M}} + \overline{\mathcal{N}}]$$

$$\eta\mathcal{M} = \eta[\underline{\mathcal{M}}, \overline{\mathcal{M}}] = \begin{cases} \left[\eta\underline{\mathcal{M}}, \eta\overline{\mathcal{M}}\right] & (\eta > 0) \\ \{0\} & (\eta = 0) \\ \left[\eta\overline{\mathcal{M}}, \eta\underline{\mathcal{M}}\right] & (\eta < 0), \end{cases}$$

where $\eta \in \mathcal{R}$.

Let $\mathcal{R}_I$ and $\mathcal{R}_I^+$ be the set of all closed and all positive compact intervals of $\mathcal{R}$, respectively. Several algebraic properties of interval arithmetic will now be discussed.

Consider $\mathcal{M} = [\underline{\mathcal{M}}, \bar{\mathcal{M}}] \in \mathcal{R}_I$, then $\mathcal{M}_c = \frac{\overline{\mathcal{M}}+\underline{\mathcal{M}}}{2}$ and $\mathcal{M}_r = \frac{\overline{\mathcal{M}}-\underline{\mathcal{M}}}{2}$ are the center and radius of interval $\mathcal{M}$ respectively. The CR form of interval $\mathcal{M}$ can be defined as:

$$\mathcal{M} = \left\langle \mathcal{M}_c, \mathcal{M}_r \right\rangle = \left\langle \frac{\overline{\mathcal{M}} + \underline{\mathcal{M}}}{2}, \frac{\overline{\mathcal{M}} - \underline{\mathcal{M}}}{2} \right\rangle.$$

Following are the order relations for the center and radius of intervals:

Definition 2.1. The CR-order relation for $\mathcal{M} = [\underline{\mathcal{M}}, \overline{\mathcal{M}}] = \langle \mathcal{M}_c, \mathcal{M}_r \rangle$, $\mathcal{N} = [\underline{\mathcal{N}}, \overline{\mathcal{N}}] = \langle \mathcal{N}_c, \mathcal{N}_r \rangle \in \mathcal{R}_I$ represented as:

$$\mathcal{M} \preceq_{cr} \mathcal{N} \Longleftrightarrow \begin{cases} \mathcal{M}_c < \mathcal{N}_c, & if \quad \mathcal{M}_c \neq \mathcal{N}_c; \\ \mathcal{M}_r \leq \mathcal{N}_r, & if \quad \mathcal{M}_c = \mathcal{N}_c. \end{cases}$$

Note: For arbitrary two intervals $\mathcal{M}, \mathcal{N} \in \mathcal{R}_I$, we have either $\mathcal{M} \preceq_{cr} \mathcal{N}$ or $\mathcal{N} \preceq_{cr} \mathcal{M}$.
Riemann integral operators for $\mathcal{IVFS}$ are presented here.

Definition 2.2. (See [45]) Let $\mathcal{D} : [g, h]$ be an $\mathcal{IVF}$ such that $\mathcal{D} = [\underline{\mathcal{D}}, \overline{\mathcal{D}}]$. Then $\mathcal{D}$ is Riemann integrable ($\mathcal{IR}$) on $[g, h]$ if $\underline{\mathcal{D}}$ *and* $\overline{\mathcal{D}}$ are $\mathcal{IR}$ on $[g, h]$, that is,

$$(\mathcal{IR}) \int_g^h \mathcal{D}(\mathrm{s})ds = \left[(\mathcal{R}) \int_g^h \underline{\mathcal{D}}(\mathrm{s})ds, (\mathcal{R}) \int_g^h \overline{\mathcal{D}}(\mathrm{s})ds\right].$$

The collection of all ($\mathcal{IR}$) $\mathcal{IVFS}$ on $[g, h]$ is represented by $\mathcal{IR}_{([g,h])}$.

Shi et al. [42] proved that the based on CR-order relations, the integral preserves order.

Theorem 2.1. *Let $\mathcal{D}, \mathcal{F} : [g, h]$ be $\mathcal{IVFS}$ given by $\mathcal{D} = [\underline{\mathcal{D}}, \overline{\mathcal{D}}]$ and $\mathcal{F} = [\underline{\mathcal{F}}, \overline{\mathcal{F}}]$. If $\mathcal{D}(\mathrm{s}) \preceq_{CR} \mathcal{F}(\mathrm{s})$, $\forall\ i \in [g, h]$, then*

$$\int_g^h \mathcal{D}(\mathrm{s})ds \preceq_{C\mathcal{R}} \int_g^h \mathcal{F}(\mathrm{s})ds.$$

We'll now provide an illustration to support the aforementioned Theorem.

Example 2.1. Let $\mathcal{D} = [\mathrm{s}, 2\mathrm{s}]$ and $\mathcal{F} = [\mathrm{s}^2, \mathrm{s}^2 + 2]$, then for $\mathrm{s} \in [0, 1]$.

$$\mathcal{D}_C = \frac{3s}{2}, \mathcal{D}_{\mathcal{R}} = \frac{\mathrm{s}}{2}, \mathcal{F}_C = \mathrm{s}^2 + 1 \text{ and } \mathcal{F}_{\mathcal{R}} = 1.$$

From Definition 2.1, we have $\mathcal{D}(\mathrm{s}) \preceq_{CR} \mathcal{F}(\mathrm{s})$, $\mathrm{s} \in [0, 1]$.
Since,

$$\int_0^1 [\mathrm{s}, 2\mathrm{s}]ds = \left[\frac{1}{2}, 1\right]$$

and

$$\int_0^1 [\mathrm{s}^2, \mathrm{s}^2 + 2]ds = \left[\frac{1}{3}, \frac{7}{3}\right].$$

Also, from above Theorem 2.1, we have

$$\int_0^1 \mathcal{D}(\mathrm{s})ds \preceq_{CR} \int_0^1 \mathcal{F}(\mathrm{s})ds.$$

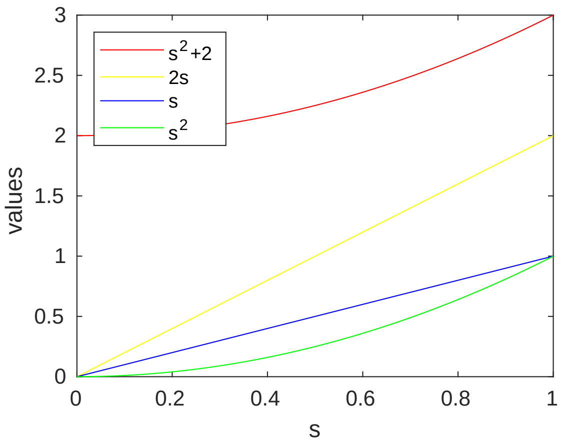

Figure 1. A clear indication of the validity of the CR-order relationship can be seen in the graph.

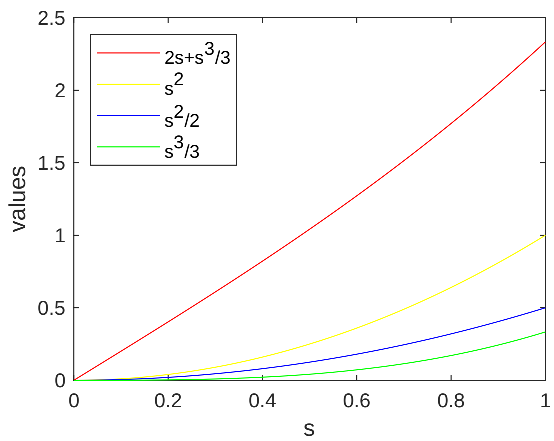

Figure 2. As can be seen from the graph, the Theorem 2.1 is valid.

Definition 2.3. (See [42]) Define $h_1, h_2 : [0,1] \to \mathcal{R}^+$. We say that $\Theta : [g,h] \to \mathcal{R}^+$ is called (h_1, h_2)-convex function, or that $\Theta \in SX((h_1,h_2),[g,h],\mathcal{R}^+)$, if $\forall\ g_1, h_1 \in [g,h]$ and $\gamma \in [0,1]$, we have

$$\Theta(\gamma g_1 + (1-\gamma)h_1) \leq\ h_1(\gamma)h_2(1-\gamma)\Theta(g_1) + h_1(1-\gamma)h_2(\gamma)\Theta(h_1). \tag{2.1}$$

If in (2.1) "$\leq$" replaced with "$\geq$" it is called (h_1,h_2)-concave function or $\Theta \in SV((h_1,h_2),[g,h],\mathcal{R}^+)$.

Definition 2.4. (See [42]) Define $h_1, h_2 : (0,1) \to \mathcal{R}^+$. We say that $\Theta : [g,h] \to \mathcal{R}^+$ is called (h_1,h_2)-GL convex function, or that $\Theta \in SGX((h_1,h_2),[g,h],\mathcal{R}^+)$, if $\forall\ g_1, h_1 \in [g,h]$ and $\gamma \in [0,1]$, we have

$$\Theta(\gamma g_1 + (1-\gamma)h_1) \leq \frac{\Theta(g_1)}{h_1(\gamma)h_2(1-\gamma)} + \frac{\Theta(h_1)}{h_1(1-\gamma)h_2(\gamma)}. \tag{2.2}$$

If in (2.2) "$\leq$" replaced with "$\geq$" it is called (h_1,h_2)-GL concave function or $\Theta \in SGV((h_1,h_2),[g,h],\mathcal{R}^+)$.

Now let's introduce the concept for CR-order form of convexity.

Definition 2.5. (See [42]) Define $h_1, h_2 : [0,1] \to \mathcal{R}^+$. We say that $\Theta : [g,h] \to \mathcal{R}^+$ is called $CR-(h_1,h_2)$-convex function, or that $\Theta \in SX(CR\text{-}(h_1,h_2),[g,h],\mathcal{R}^+)$, if $\forall\ g_1, h_1 \in [g,h]$ and $\gamma \in [0,1]$, we have

$$\Theta(\gamma g_1 + (1-\gamma)h_1) \preceq_{CR}\ h_1(\gamma)h_2(1-\gamma)\Theta(g_1) + h_1(1-\gamma)h_2(\gamma)\Theta(h_1). \tag{2.3}$$

If in (2.3) "$\preceq_{CR}$" replaced with "$\succeq_{CR}$" it is called CR-(h_1,h_2)-concave function or $\Theta \in SV(CR\text{-}(h_1,h_2),[g,h],\mathcal{R}^+)$.

Definition 2.6. (See [42]) Define $h_1, h_2 : (0,1) \to \mathcal{R}^+$. We say that $\Theta : [g,h] \to \mathcal{R}^+$ is called CR-(h_1,h_2)-GL convex function, or that $\Theta \in SGX(CR\text{-}(h_1,h_2),[g,h],\mathcal{R}^+)$, if $\forall\ g_1, h_1 \in [g,h]$ and $\gamma \in [0,1]$, we have

$$\Theta(\gamma g_1 + (1-\gamma)h_1) \preceq_{CR} \frac{\Theta(g_1)}{h_1(\gamma)h_2(1-\gamma)} + \frac{\Theta(h_1)}{h_1(1-\gamma)h_2(\gamma)}. \tag{2.4}$$

If in (2.4) "$\preceq_{CR}$" replaced with "$\succeq_{CR}$" it is called CR-(h_1,h_2)-GL concave function or $\Theta \in SGV(CR\text{-}(h_1,h_2),[g,h],\mathcal{R}^+)$.

Remark 2.1.
- *If $h_1 = h_2 = 1$, Definition 2.6 becomes a CR-P-function [45].*
- *If $h_1(\gamma) = \frac{1}{h_1(\gamma)}$, $h_2 = 1$ Definition 2.6 becomes a CR-h-convex function [45].*
- *If $h_1(\gamma) = h_1(\gamma)$, $h_2 = 1$ Definition 2.6 becomes a CR-h-GL function [45].*
- *If $h_1(\gamma) = \frac{1}{\gamma^s}$, $h_2 = 1$ Definition 2.6 becomes a CR-s-convex function [45].*
- *If $h(\gamma) = \gamma^s$, Definition 2.6 becomes a CR-s-GL function [45].*

3. Main results

Proposition 3.1. *Consider $\Theta : [g,h] \to \mathcal{R}_\mathcal{I}$ given by $[\underline{\Theta}, \overline{\Theta}] = (\Theta_C, \Theta_R)$. If Θ_C and Θ_R are (h_1,h_2)-GL over $[g,h]$, then Θ is called CR-(h_1,h_2)-GL function over $[g,h]$.*

Proof. Since Θ_C and Θ_R are (h_1,h_2)-GL over $[g,h]$, then for each $\gamma \in (0,1)$ and for all $g_1, h_1 \in [g,h]$, we have

$$\Theta_C(\gamma g_1 + (1-\gamma)h_1) \preceq_{CR} \frac{\Theta_C(g_1)}{h_1(\gamma)h_2(1-\gamma)} + \frac{\Theta_C(h_1)}{h_1(1-\gamma)h_2(\gamma)},$$

and

$$\Theta_R(\gamma g_1 + (1-\gamma)h_1) \preceq_{CR} \frac{\Theta_R(g_1)}{h_1(\gamma)h_2(1-\gamma)} + \frac{\Theta_R(h_1)}{h_1(1-\gamma)h_2(\gamma)}.$$

Now, if

$$\Theta_C(\gamma g_1 + (1-\gamma)h_1) \neq \frac{\Theta_C(g_1)}{h_1(\gamma)h_2(1-\gamma)} + \frac{\Theta_C(h_1)}{h_1(1-\gamma)h_2(\gamma)},$$

then for each $\gamma \in (0,1)$ and for all $g_1, h_1 \in [g,h]$,

$$\Theta_C(\gamma g_1 + (1-\gamma)h_1) < \frac{\Theta_C(g_1)}{h_1(\gamma)h_2(1-\gamma)} + \frac{\Theta_C(h_1)}{h_1(1-\gamma)h_2(\gamma)}.$$

Accordingly,

$$\Theta_C(\gamma g_1 + (1-\gamma)h_1) \preceq_{CR} \frac{\Theta_C(g_1)}{h_1(\gamma)h_2(1-\gamma)} + \frac{\Theta_C(h_1)}{h_1(1-\gamma)h_2(\gamma)}.$$

Otherwise, for each $\gamma \in (0,1)$ and for all $g_1, h_1 \in [g,h]$,

$$\Theta_R(\nu g_1 + (1-\gamma)h_1) \leq \frac{\Theta_R(g_1)}{h_1(\gamma)h_2(1-\gamma)} + \frac{\Theta_R(h_1)}{h_1(1-\gamma)h_2(\gamma)}$$

$$\Rightarrow \Theta(\gamma g_1 + (1-\gamma)h_1) \preceq_{CR} \frac{\Theta(g_1)}{h_1(\gamma)h_2(1-\gamma)} + \frac{\Theta(h_1)}{h_1(1-\gamma)h_2(\gamma)}.$$

Taking all of the above into account, and Definition 2.6 this can be written as

$$\Theta(\gamma g_1 + (1-\gamma)h_1) \preceq_{CR} \frac{\Theta(g_1)}{h_1(\gamma)h_2(1-\gamma)} + \frac{\Theta(h_1)}{h_1(1-\gamma)h_2(\gamma)}$$

for each $\gamma \in (0,1)$ and for all $g_1, h_1 \in [g,h]$.
This completes the proof. □

The next step is to establish the $\mathcal{H}.\mathcal{H}$ inequality for the CR-(h_1, h_2)-GL function.

Theorem 3.1. *Define $h_1, h_2 : (0,1) \to \mathcal{R}^+$ and $h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \neq 0$. Let $\Theta : [g,h] \to \mathcal{R}_\mathcal{I}{}^+$, if $\Theta \in SGX(CR\text{-}(h_1,h_2), [t,u], R_I{}^+)$ and $\Theta \in IR_{[t,u]}$, we have*

$$\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]}{2}\Theta\left(\frac{g+h}{2}\right) \preceq_{CR} \frac{1}{h-g}\int_g^h \Theta(\gamma)d\gamma \preceq_{CR} [\Theta(g)+\Theta(h)]\int_0^1 \frac{dx}{H(x,1-x)}.$$

Proof. Since $\Theta \in SGX(CR\text{-}(h_1.h_2), [g,h], \mathcal{R}_\mathcal{I}{}^+)$, we have

$$\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]\Theta\left(\frac{g+h}{2}\right) \preceq_{CR} \Theta(xg+(1-x)h) + \Theta((1-x)g+xh).$$

Integration over (0,1), we have

$$\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]\Theta\left(\frac{g+h}{2}\right) \preceq_{CR} \left[\int_0^1 \Theta(xg+(1-x)h)dx + \int_0^1 \Theta((1-x)g+xh)dx\right]$$

$$= \Big[\int_0^1 \underline{\Theta}(xg + (1-x)h)dx + \int_0^1 \underline{\Theta}((1-x)g + xh)dx,$$

$$\int_0^1 \overline{\Theta}(xg + (1-x)h)dx + \int_0^1 \overline{\Theta}((1-x)g + xh)dx\Big]$$

$$= \left[\frac{2}{h-g}\int_g^h \underline{\Theta}(\gamma)d\gamma, \frac{2}{h-g}\int_g^h \overline{\Theta}(\gamma)d\gamma\right]$$

$$= \frac{2}{h-g}\int_g^h \Theta(\gamma)d\gamma. \tag{3.1}$$

By Definition 2.6, we have

$$\Theta(xg + (1-x)h) \preceq_{CR} \frac{\Theta(g)}{h_1(x)h_2(1-x)} + \frac{\Theta(h)}{h_1(1-x)h_2(x)}.$$

Integration over (0,1), we have

$$\int_0^1 \Theta(xg + (1-x)h)dx \preceq_{CR} \Theta(g)\int_0^1 \frac{dx}{h_1(x)h_2(1-x)} + \Theta(h)\int_0^1 \frac{dx}{h_1(1-x)h_2(x)}.$$

Accordingly,

$$\frac{1}{h-g}\int_g^h \Theta(\gamma)d\gamma \preceq_{CR} [\Theta(g) + \Theta(h)]\int_0^1 \frac{dx}{H(x, 1-x)}. \tag{3.2}$$

Now combining (3.1) and (3.2), we get required result

$$\frac{\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]}{2}\Theta\left(\frac{t+u}{2}\right) \preceq_{CR} \frac{1}{h-g}\int_g^h \Theta(\gamma)d\gamma \preceq_{CR} [\Theta(g) + \Theta(h)]\int_0^1 \frac{dx}{H(x, 1-x)}.$$

□

Remark 3.1.

- *If $h_1(x) = h_2(x) = 1$, Theorem 3.1 becomes result for CR- P-function:*

$$\frac{1}{2}\Theta\left(\frac{g+h}{2}\right) \preceq_{CR} \frac{1}{h-g}\int_g^h \Theta(\gamma)d\gamma \preceq_{CR} [\Theta(g) + \Theta(h)].$$

- *If $h_1(x) = h(x)$, $h_2(x) = 1$ Theorem 3.1 becomes result for CR-h-GL-function:*

$$\frac{h\left(\frac{1}{2}\right)}{2}\Theta\left(\frac{g+h}{2}\right) \preceq_{CR} \frac{1}{h-g}\int_g^h \Theta(\gamma)d\gamma \preceq_{CR} \int_0^1 \frac{dx}{h(x)}.$$

- *If $h_1(x) = \frac{1}{h(x)}$, $h_2(x) = 1$ Theorem 3.1 becomes result for CR-h-convex function:*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\Theta\left(\frac{g+h}{2}\right) \preceq_{CR} \frac{1}{h-g}\int_g^h \Theta(\gamma)d\gamma \preceq_{CR} \int_0^1 h(x)dx.$$

- *If $h_1(x) = \frac{1}{h_1(x)}$, $h_2(x) = \frac{1}{h_2(x)}$ Theorem 3.1 becomes result for CR-(h_1, h_2)-convex function:*

$$\frac{1}{2\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]}\Theta\left(\frac{g+h}{2}\right) \preceq_{CR} \frac{1}{h-g}\int_g^h \Theta(\gamma)d\gamma \preceq_{CR} \int_0^1 \frac{dx}{H(x,1-x)}.$$

Example 3.1. Consider $[t,u] = [0,1]$, $h_1(x) = \frac{1}{x}$, $h_2(x) = 1$, $\forall\ x \in\ (0,1)$. $\Theta : [g,h] \to \mathcal{R}_{\mathcal{I}}{}^{+}$ is defined as

$$\Theta(\gamma) = [-\gamma^2, 2\gamma^2 + 1].$$

where

$$\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]}{2}\Theta\left(\frac{g+h}{2}\right) = \Theta\left(\frac{1}{2}\right) = \left[\frac{-1}{4},\frac{3}{2}\right],$$

$$\frac{1}{h-g}\int_g^h \Theta(\gamma)d\gamma = \left[\int_0^1(-\gamma^2)d\gamma, \int_0^1(2\gamma^2+1)d\gamma\right] = \left[\frac{-1}{3},\frac{5}{3}\right],$$

$$[\Theta(g)+\Theta(h)]\int_0^1 \frac{dx}{H(x,1-x)} = \left[\frac{-1}{2},2\right].$$

As a result,

$$\left[\frac{-1}{4},\frac{3}{2}\right] \preceq_{CR} \left[\frac{-1}{3},\frac{5}{3}\right] \preceq_{CR} \left[\frac{-1}{2},2\right].$$

This proves the above theorem.

Theorem 3.2. *Define $h_1, h_2 : (0,1) \to \mathcal{R}^{+}$ and $h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \neq 0$. Let $\Theta : [g,h] \to \mathcal{R}_{\mathcal{I}}{}^{+}$, if $\Theta \in SGX(CR\text{-}(h_1,h_2), [t,u], R_I{}^{+})$ and $\Theta \in\ IR_{[g,h]}$, we have*

$$\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{4}\Theta\left(\frac{g+h}{2}\right) \preceq_{CR} \triangle_1 \preceq_{CR} \frac{1}{h-g}\int_g^h \Theta(\gamma)d\gamma \preceq_{CR} \triangle_2$$

$$\preceq_{CR} \left\{[\Theta(g)+\Theta(h)]\left[\frac{1}{2} + \frac{1}{H\left(\frac{1}{2},\frac{1}{2}\right)}\right]\right\}\int_0^1 \frac{dx}{H(x,1-x)},$$

where

$$\triangle_1 = \frac{H\left(\frac{1}{2},\frac{1}{2}\right)}{4}\left[\Theta\left(\frac{3g+h}{4}\right) + \Theta\left(\frac{3h+g}{4}\right)\right],$$

$$\triangle_2 = \left[\Theta\left(\frac{g+h}{2}\right) + \frac{\Theta(g)+\Theta(h)}{2}\right]\int_0^1 \frac{dx}{H(x,1-x)}.$$

Proof. Take $[g, \frac{g+h}{2}]$, we have

$$\Theta\left(\frac{g+\frac{g+h}{2}}{2}\right) = \Theta\left(\frac{3g+h}{2}\right) \preceq_{CR} \frac{\Theta\left(xg+(1-x)\frac{g+h}{2}\right)}{H\left(\frac{1}{2},\frac{1}{2}\right)} + \frac{\Theta\left((1-x)g+x\frac{g+h}{2}\right)}{H\left(\frac{1}{2},\frac{1}{2}\right)}.$$

Integration over (0,1), we have

$$\Theta\left(\frac{3g+h}{2}\right) \preceq_{CR} \frac{1}{H\left(\frac{1}{2},\frac{1}{2}\right)}\left[\int_0^1 \Theta\left(xg+(1-x)\frac{g+h}{2}\right)dx + \int_0^1 \Theta\left(x\frac{g+h}{2}+(1-x)h\right)dx\right]$$

$$= \frac{1}{H\left(\frac{1}{2},\frac{1}{2}\right)}\left[\frac{2}{h-g}\int_g^{\frac{g+h}{2}} \eta(\gamma)d\gamma + \frac{2}{h-g}\int_g^{\frac{g+h}{2}} \Theta(\gamma)d\gamma\right]$$

$$= \frac{4}{H\left(\frac{1}{2},\frac{1}{2}\right)}\left[\frac{1}{h-g}\int_g^{\frac{g+h}{2}} \Theta(\gamma)d\gamma\right].$$

Accordingly,

$$\frac{H\left(\frac{1}{2},\frac{1}{2}\right)}{4}\Theta\left(\frac{3g+h}{2}\right) \preceq_{CR} \frac{1}{h-g}\int_g^{\frac{g+h}{2}} \Theta(\gamma)d\gamma. \tag{3.3}$$

Similarly for interval $[\frac{g+h}{2}, h]$, we have

$$\frac{H\left(\frac{1}{2},\frac{1}{2}\right)}{4}\Theta\left(\frac{3h+g}{2}\right) \preceq_{CR} \frac{1}{h-g}\int_g^{\frac{g+h}{2}} \Theta(\gamma)d\gamma. \tag{3.4}$$

Adding inequalities (3.3) and (3.4), we get

$$\triangle_1 = \frac{H\left(\frac{1}{2},\frac{1}{2}\right)}{4}\left[\Theta\left(\frac{3g+h}{4}\right)+\Theta\left(\frac{3h+g}{4}\right)\right] \preceq_{CR} \left[\frac{1}{h-g}\int_g^h \Theta(\gamma)d\gamma\right].$$

Now

$$\begin{aligned}
&\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{4}\Theta\left(\frac{g+h}{2}\right)\\
&= \frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{4}\Theta\left(\frac{1}{2}\left(\frac{3g+h}{4}\right)+\frac{1}{2}\left(\frac{3h+g}{4}\right)\right)\\
&\preceq_{CR} \frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{4}\left[\frac{\Theta\left(\frac{3g+h}{4}\right)}{h(\frac{1}{2})}+\frac{\Theta\left(\frac{3h+g}{4}\right)}{h(\frac{1}{2})}\right]\\
&= \frac{H\left(\frac{1}{2},\frac{1}{2}\right)}{4}\left[\Theta\left(\frac{3g+h}{4}\right)+\Theta\left(\frac{3h+g}{4}\right)\right]\\
&= \triangle_1\\
&\preceq_{CR} \frac{H\left(\frac{1}{2},\frac{1}{2}\right)}{4}\left\{\frac{1}{H\left(\frac{1}{2},\frac{1}{2}\right)}\left[\Theta(g)+\Theta\left(\frac{g+h}{2}\right)\right]+\frac{1}{H\left(\frac{1}{2},\frac{1}{2}\right)}\left[\Theta(h)+\Theta\left(\frac{g+h}{2}\right)\right]\right\}\\
&= \frac{1}{2}\left[\frac{\Theta(g)+\Theta(h)}{2}+\Theta\left(\frac{g+h}{2}\right)\right]
\end{aligned}$$

$$\preceq_{CR} \left[\frac{\Theta(g)+\Theta(h)}{2} + \Theta\left(\frac{g+h}{2}\right)\right]\int_0^1 \frac{dx}{H(x,1-x)}$$
$$= \triangle_2$$
$$\preceq_{CR} \left[\frac{\Theta(g)+\Theta(h)}{2} + \frac{\Theta(g)}{H\left(\frac{1}{2},\frac{1}{2}\right)} + \frac{\Theta(h)}{H\left(\frac{1}{2},\frac{1}{2}\right)}\right]\int_0^1 \frac{dx}{H(x,1-x)}$$
$$\preceq_{CR} \left[\frac{\Theta(g)+\Theta(h)}{2} + \frac{1}{H\left(\frac{1}{2},\frac{1}{2}\right)}[\Theta(g)+\Theta(h)]\right]\int_0^1 \frac{dx}{H(x,1-x)}$$
$$\preceq_{CR} \left\{[\Theta(g)+\Theta(h)]\left[\frac{1}{2} + \frac{1}{H\left(\frac{1}{2},\frac{1}{2}\right)}\right]\right\}\int_0^1 \frac{dx}{H(x,1-x)}.$$

□

Example 3.2. Thanks to Example 3.1, we have

$$\frac{\left[H(\frac{1}{2},\frac{1}{2})\right]^2}{4}\Theta\left(\frac{g+h}{2}\right) = \Theta\left(\frac{1}{2}\right) = \left[\frac{-1}{4},\frac{3}{2}\right],$$
$$\triangle_1 = \frac{1}{2}\left[\Theta\left(\frac{1}{4}\right)+\Theta\left(\frac{3}{4}\right)\right] = \left[\frac{-5}{16},\frac{13}{8}\right],$$
$$\triangle_2 = \left[\frac{\Theta(0)+\Theta(1)}{2}+\Theta\left(\frac{1}{2}\right)\right]\int_0^1\frac{dx}{H(x,1-x)},$$
$$\triangle_2 = \frac{1}{2}\left(\left[\frac{-1}{4},\frac{3}{2}\right]+\left[\frac{-1}{2},2\right]\right),$$
$$\triangle_2 = \left[\frac{-3}{8},\frac{7}{4}\right],$$
$$\left\{[\Theta(g)+\Theta(h)]\left[\frac{1}{2}+\frac{1}{H(\frac{1}{2},\frac{1}{2})}\right]\right\}\int_0^1\frac{dx}{H(x,1-x)} = \left[\frac{-1}{2},2\right].$$

Thus we obtain

$$\left[\frac{-1}{4},\frac{3}{2}\right] \preceq_{CR} \left[\frac{-5}{16},\frac{13}{8}\right] \preceq_{CR} \left[\frac{-1}{3},\frac{5}{3}\right] \preceq_{CR} \left[\frac{-3}{8},\frac{7}{4}\right] \preceq_{cr} \left[\frac{-1}{2},2\right].$$

This proves the above theorem.

Theorem 3.3. *Let $\Theta,\theta : [g,h] \to \mathcal{R}_\mathcal{I}{}^+, h_1, h_2 : (0,1) \to \mathcal{R}^+$ such that $h_1, h_2 \neq 0$. If $\Theta \in SGX(CR\text{-}h_1,[g,h],R_I{}^+)$, $\theta \in SGX(CR\text{-}h_2,[g,h],\mathcal{R}_\mathcal{I}{}^+)$ and $\Theta,\theta \in IR_{[g,h]}$ then, we have*

$$\frac{1}{h-g}\int_g^h \Theta(\gamma)\theta(\gamma)d\gamma \preceq_{CR} M(g,h)\int_0^1\frac{dx}{H^2(x,1-x)} + N(g,h)\int_0^1\frac{dx}{H(x,x)H(1-x,1-x)},$$

where

$$M(g,h) = \Theta(g)\theta(g)+\Theta(h)\theta(h), N(g,h) = \Theta(g)\theta(h)+\Theta(h)\theta(g).$$

Proof. Conider $\Theta \in SGX(CR\text{-}h_1,[g,h],\mathcal{R}_I{}^+)$, $\theta \in SGX(CR\text{-}h_2,[g,h],\mathcal{R}_I{}^+)$ then, we have

$$\Theta\Big(gx+(1-x)h\Big) \preceq_{CR} \frac{\Theta(g)}{h_1(x)h_2(1-x)} + \frac{\Theta(h)}{h_1(1-x)h_2(x)},$$

$$\theta\Big(gx+(1-x)h\Big) \preceq_{CR} \frac{\theta(g)}{h_1(x)h_2(1-x)} + \frac{\theta(h)}{h_1(1-x)h_2(x)}.$$

Then,

$$\Theta\Big(gx+(1-x)h\Big)\theta\Big(tx+(1-x)u\Big)$$
$$\preceq_{CR} \frac{\Theta(g)\theta(g)}{H^2(x,1-x)} + \frac{\Theta(g)\theta(h)+\Theta(g)\theta(g)}{H^2(1-x,x)} + \frac{\Theta(h)\theta(h)}{H(x,x)H(1-x,1-x)}.$$

Integration over (0,1), we have

$$\int_0^1 \Theta\Big(gx+(1-x)h\Big)\theta\Big(gx+(1-x)h\Big)dx$$
$$= \left[\int_0^1 \underline{\Theta}\Big(gx+(1-x)h\Big)\underline{\theta}\Big(gx+(1-x)h\Big)dx, \int_0^1 \overline{\Theta}\Big(gx+(1-x)h\Big)\overline{\theta}\Big(gx+(1-x)h\Big)dx\right]$$
$$= \left[\frac{1}{h-g}\int_g^h \underline{\Theta}(\gamma)\underline{\theta}(\gamma)d\gamma, \frac{1}{h-g}\int_g^h \overline{\Theta}(\gamma)\overline{\theta}(\gamma d\gamma\right] = \frac{1}{h-g}\int_g^h \Theta(\gamma)\theta(\gamma)d\gamma$$
$$\preceq_{CR} \int_0^1 \frac{[\Theta(g)\theta(g)+\Theta(h)\theta(h)]}{H^2(x,1-x)}dx + \int_0^1 \frac{[\Theta(g)\theta(h)+\Theta(h)\theta(g)]}{H(x,x)H(1-x,1-x)}dx.$$

It follows that

$$\frac{1}{h-g}\int_g^h \Theta(\gamma)\theta(\gamma)d\gamma \preceq_{CR} M(g,h)\int_0^1 \frac{dx}{H^2(x,1-x)} + N(g,h)\int_0^1 \frac{dx}{H(x,x)H(1-x,1-x)}.$$

Theorem is proved. □

Example 3.3. Consider $[g,h]=[1,2]$, $h_1(x)=\frac{1}{x}$, $h_2(x)=1\ \forall\ x\in\ (0,1)$. $\Theta,\theta:[g,h]\to\mathcal{R}_I{}^+$ be defined as

$$\Theta(\gamma)=[-\gamma^2,2\gamma^2+1], \theta(\gamma)=[-\gamma,\gamma].$$

Then,

$$\frac{1}{h-g}\int_g^h \Theta(\gamma)\theta(\gamma)d\gamma = \left[\frac{15}{4},9\right],$$
$$M(g,h)\int_0^1 \frac{1}{H^2(x,1-x)}dx = M(1,2)\int_0^1 x^2dx = [-7,7],$$
$$N(g,h)\int_0^1 \frac{1}{H(x,x)H(1-x,1-x)}dx = N(1,2)\int_0^1 x(1-x)dx = \left[\frac{-15}{6},\frac{15}{6}\right].$$

It follows that

$$\left[\frac{15}{4},9\right] \preceq_{CR} [-7,7] + \left[\frac{-15}{6},\frac{15}{6}\right] = \left[\frac{-19}{2},\frac{19}{2}\right].$$

It follows that the theorem above is true.

Theorem 3.4. *Let $\Theta, \theta : [g, h] \to \mathcal{R}_{\mathcal{I}}{}^{+}, h_1, h_2 : (0, 1) \to \mathcal{R}^{+}$ such that $h_1, h_2 \neq 0$. If $\Theta \in SGX(CR\text{-}h_1, [g, h], R_I{}^{+})$, $\theta \in SGX(CR\text{-}h_2, [g, h], \mathcal{R}_{\mathcal{I}}{}^{+})$ and $\Theta, \theta \in IR_{[g,h]}$ then, we have*

$$\frac{\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]^2}{2}\Theta\left(\frac{g+h}{2}\right)\theta\left(\frac{g+h}{2}\right)$$
$$\preceq_{CR} \frac{1}{h-g}\int_g^h \Theta(\gamma)\theta(\gamma)d\gamma + M(g,h)\int_0^1 \frac{dx}{H(x,x)H(1-x,1-x)} + N(g,h)\int_0^1 \frac{dx}{H^2(x,1-x)}.$$

Proof. Since $\Theta \in SGX(CR\text{-}h_1, [g, h], \mathcal{R}_{\mathcal{I}}{}^{+})$, $\theta \in SGX(CR\text{-}h_2, [g, h], R_I{}^{+})$, we have

$$\Theta\left(\frac{g+h}{2}\right) \preceq_{CR} \frac{\Theta\left(gx+(1-x)h\right)}{H\left(\frac{1}{2}, \frac{1}{2}\right)} + \frac{\Theta\left(g(1-x)+xh\right)}{H\left(\frac{1}{2}, \frac{1}{2}\right)},$$

$$\theta\left(\frac{g+h}{2}\right) \preceq_{CR} \frac{\theta\left(gx+(1-x)h\right)}{H\left(\frac{1}{2}, \frac{1}{2}\right)} + \frac{\theta\left(g(1-x)+xh\right)}{H\left(\frac{1}{2}, \frac{1}{2}\right)}.$$

Then,

$$\Theta\left(\frac{g+h}{2}\right)\theta\left(\frac{g+h}{2}\right)$$
$$\preceq_{CR} \frac{1}{\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]^2}\left[\Theta\left(gx+(1-x)h\right)\theta\left(gx+(1-x)h\right) + \Theta\left(g(1-x)+xh\right)\theta(g(1-x)+xh)\right]$$
$$+ \frac{1}{\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]^2}\left[\Theta\left(gx+(1-x)h\right)\theta(g(1-x)+xh) + \Theta(g(1-x)+xh)\theta(gx+(1-x)h)\right]$$
$$+ \preceq_{CR} \frac{1}{\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]^2}\left[\Theta\left(gx+(1-x)h\right)\theta(gx+(1-x)h) + \Theta\left(g(1-x)+(xh\right)\theta(g(1-x)+xh)\right]$$
$$+ \frac{1}{\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]^2}\left[\left(\frac{\Theta(g)}{H(x,1-x)} + \frac{\theta(h)}{H(1-x,x)}\right)\left(\frac{\theta(h)}{H(1-x,x)} + \frac{\theta(h)}{H(x,1-x)}\right)\right]$$
$$+ \left[\left(\frac{\Theta(g)}{H(1-x,x)} + \frac{\Theta(h)}{H(x,1-x)}\right)\left(\frac{\theta(g)}{H(x,1-x)} + \frac{\theta(h)}{H(1-x,x)}\right)\right]$$
$$\preceq_{CR} \frac{1}{\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]^2}\left[\Theta\left(gx+(1-x)h\right)\theta(gx+(1-x)h) + \Theta\left(g(1-x)+xh\right)\theta(g(1-x)+xh)\right]$$
$$+ \frac{1}{\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]^2}\left[\left(\frac{2}{H(x,x)H(1-x,1-x)}\right)M(g,h) + \left(\frac{1}{H^2(x,1-x)} + \frac{1}{H^2(1-x,x)}\right)N(g,h)\right].$$

Integration over $(0, 1)$, we have

$$\int_0^1 \Theta\left(\frac{g+h}{2}\right)\theta\left(\frac{g+h}{2}\right)dx = \left[\int_0^1 \underline{\Theta}\left(\frac{g+h}{2}\right)\underline{\theta}\left(\frac{g+h}{2}\right)dx, \int_0^1 \overline{\Theta}\left(\frac{g+h}{2}\right)\overline{\theta}\left(\frac{g+h}{2}\right)dx\right]$$
$$= \Theta\left(\frac{g+h}{2}\right)\theta\left(\frac{g+h}{2}\right)dx$$

$$\preceq_{CR} \frac{2}{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\left[\frac{1}{h-g}\int_g^h \Theta(\gamma)\theta(\gamma)d\gamma\right]$$
$$+\frac{2}{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\left[M(g,h)\int_0^1 \frac{1}{H(x,x)H(1-x,1-x)}dx\right.$$
$$\left.+N(g,h)\int_0^1 \frac{1}{H^2(x,1-x)}dx\right].$$

Multiply both sides by $\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{2}$ above equation, we get the required result

$$\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{2}\Theta\left(\frac{g+h}{2}\right)\theta\left(\frac{g+h}{2}\right)$$
$$\preceq_{CR} \frac{1}{h-g}\int_g^h \Theta(\gamma)\theta(\gamma)d\gamma + M(g,h)\int_0^1 \frac{dx}{H(x,x)H(1-x,1-x)} + N(g,h)\int_0^1 \frac{dx}{H^2(x,1-x)}.$$

As a result, the proof is complete.

□

Example 3.4. Consider $[g,h] = [1,2]$, $h_1(x) = \frac{1}{x}$, $h_2(x) = \frac{1}{4}$, $\forall\ x \in\ (0,1)$. $\Theta, \theta : [g,h] \to \mathcal{R}_{\mathcal{I}}{}^{+}$ be defined as

$$\Theta(\gamma) = [-\gamma^2, 2\gamma^2+1], \theta(\gamma) = [-\gamma, \gamma].$$

Then,

$$\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{2}\Theta\left(\frac{g+h}{2}\right)\theta\left(\frac{g+h}{2}\right) = \frac{1}{8}\Theta\left(\frac{3}{2}\right)\theta\left(\frac{3}{2}\right) = \left[\frac{-33}{32}, \frac{33}{32}\right],$$

$$\frac{1}{h-g}\int_g^h \Theta(\gamma)\theta(\gamma)d\gamma = \left[\frac{15}{4}, 9\right],$$

$$M(g,h)\int_0^1 \frac{dx}{H(x,x)H(1-x,1-x)} = (16)M(1,2)\int_0^1 x(1-x)dx = [-56,56],$$

$$N(g,h)\int_0^1 \frac{dx}{H^2(x,1-x)} = (16)N(1,2)\int_0^1 x^2dx = [-80,80].$$

It follows that

$$\left[\frac{-33}{32}, \frac{33}{32}\right] \preceq_{CR} \left[\frac{15}{4}, 9\right] + [-56,56] + [-80,80] = \left[\frac{-529}{4}, 145\right].$$

This proves the above theorem. Next, we will develop the Jensen-type inequality for CR-(h_1,h_2)-GL functions.

4. Jensen type inequality

Theorem 4.1. *Let $u_i \in \mathcal{R}^+$, $j_i \in [g,h]$. If h_1, h_2 is super multiplicative non-negative functions and if $\Theta \in SGX(CR\text{-}(h_1,h_2),[g,h],\mathcal{R}_{\mathcal{I}}{}^+)$. Then the inequality become as :*

$$\Theta\left(\frac{1}{U_k}\sum_{i=1}^{k} u_i j_i\right) \preceq_{CR} \sum_{i=1}^{k}\left[\frac{\Theta(j_i)}{H\left(\frac{u_i}{U_k},\frac{U_{k-1}}{E_k}\right)}\right], \tag{4.1}$$

where $U_k = \sum_{i=1}^{k} u_i$

Proof. When $k = 2$, then (4.1) holds. Suppose that (4.1) is also valid for $k-1$, then

$$\Theta\left(\frac{1}{U_k}\sum_{i=1}^{k} u_i j_i\right) = \Theta\left(\frac{u_k}{U_k}v_k + \sum_{i=1}^{k-1}\frac{u_i}{U_k}j_i\right)$$

$$\preceq_{CR} \frac{\Theta(j_k)}{h_1\left(\frac{u_k}{U_k}\right)h_2\left(\frac{U_{k-1}}{U_k}\right)} + \frac{\Theta\left(\sum_{i=1}^{k-1}\frac{u_i}{U_k}j_i\right)}{h_1\left(\frac{U_k-1}{U_k}\right)h_2\left(\frac{u_k}{U_k}\right)}$$

$$\preceq_{CR} \frac{\Theta(j_k)}{h_1\left(\frac{u_k}{U_k}\right)h_2\left(\frac{U_{k-1}}{U_k}\right)} + \sum_{i=1}^{k-1}\left[\frac{\Theta(j_i)}{H\left(\frac{u_i}{U_k},\frac{U_k-2}{U_{k-1}}\right)}\right]\frac{1}{h_1\left(\frac{U_{k-1}}{U_k}\right)h_2\left(\frac{u_k}{U_k}\right)}$$

$$\preceq_{CR} \frac{\Theta(j_k)}{h_1\left(\frac{u_k}{U_k}\right)h_2\left(\frac{U_{k-1}}{U_k}\right)} + \sum_{i=1}^{k-1}\left[\frac{\Theta(j_i)}{H\left(\frac{u_i}{U_k},\frac{U_k-2}{U_{k-1}}\right)}\right]$$

$$\preceq_{CR} \sum_{i=1}^{k}\left[\frac{\Theta(j_i)}{H\left(\frac{u_i}{U_k},\frac{U_{k-1}}{U_k}\right)}\right].$$

It follows from mathematical induction that the conclusion is correct. □

Remark 4.1.

- *If $h_1(x) = h_2(x) = 1$, Theorem 4.1 becomes result for CR- P-function:*

$$\Theta\left(\frac{1}{U_k}\sum_{i=1}^{k} u_i j_i\right) \preceq_{CR} \sum_{i=1}^{k}\Theta(j_i).$$

- *If $h_1(x) = \frac{1}{h_1(x)}$, $h_2(x) = \frac{1}{h_2(x)}$ Theorem 4.1 becomes result for CR-(h_1,h_2)-convex function:*

$$\Theta\left(\frac{1}{U_k}\sum_{i=1}^{k} u_i j_i\right) \preceq_{CR} \sum_{i=1}^{k} H\left(\frac{u_i}{U_k},\frac{U_{k-1}}{U_k}\right)\Theta(j_i).$$

- *If $h_1(x) = \frac{1}{x}$, $h_2(x) = 1$ Theorem 4.1 becomes result for CR-convex function:*

$$\Theta\left(\frac{1}{U_k}\sum_{i=1}^{k} u_i j_i\right) \preceq_{CR} \sum_{i=1}^{k}\frac{u_i}{U_k}\Theta(j_i).$$

- *If $h_1(x) = \frac{1}{h(x)}$, $h_2(x) = 1$ Theorem 4.1 becomes result for CR-h-convex function:*

$$\Theta\left(\frac{1}{U_k}\sum_{i=1}^{k} u_i j_i\right) \preceq_{CR} \sum_{i=1}^{k} h\left(\frac{u_i}{U_k}\right)\Theta(j_i).$$

- *If $h_1(x) = h(x)$, $h_2(x) = 1$ Theorem 4.1 becomes result for CR-h-GL-function:*

$$\Theta\left(\frac{1}{U_k}\sum_{i=1}^{k} u_i j_i\right) \preceq_{CR} \sum_{i=1}^{k}\left[\frac{\Theta(j_i)}{h\left(\frac{u_i}{U_k}\right)}\right].$$

- *If $h_1(x) = \frac{1}{(x)^s}$, $h_2(x) = 1$ Theorem 4.1 becomes result for CR-s-convex function:*

$$\eta\left(\frac{1}{U_k}\sum_{i=1}^{k} u_i j_i\right) \preceq_{CR} \sum_{i=1}^{k}\left(\frac{u_i}{U_k}\right)^s \Theta(j_i).$$

5. Conclusions

A useful alternative for incorporating uncertainty into prediction processes is $\mathcal{IVFS}$. The present study introduces the (h_1, h_2)-GL concept for $\mathcal{IVFS}$ using the CR-order relation. As a result of utilizing this new concept, we observe that the inequality terms derived from this class of convexity and pertaining to Cr-order relations give much more precise results than other partial order relations. These findings are generalized from the very recent results described in [37, 42, 43, 45]. There are many new findings in this study that extend those already known. In addition, we provide some numerical examples to demonstrate the validity of our main conclusions. Future research could include determining equivalent inequalities for different types of convexity utilizing various fractional integral operators, including Katugampola, Riemann-Liouville and generalized K-fractional operators. The fact that these are the most active areas of study for integral inequalities will encourage many mathematicians to examine how different types of interval-valued analysis can be applied. We anticipate that other researchers working in a number of scientific fields will find this idea useful.

Conflict of interest

The authors declare that there is no conflict of interest in publishing this paper.

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