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Research article

The l_∞-induced norm of multivariable discrete-time linear systems: Upper and lower bounds with convergence rate analysis

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Abstract: This paper develops a method for computing the l_∞-induced norm of a multivariable discrete-time linear system, for which an infinite-dimensional matrix should be intrinsically concerned with. To make such a computation feasible, we treat the infinite-dimensional matrix in a truncated fashion, and an upper bound and a lower bound on the l_∞-induced norm of the original multivariable discrete-time linear system are derived. More precisely, the matrix ∞-norm of the (infinite-dimensional) tail part can be approximately computed by deriving its upper and lower bounds, while that of the (finite-dimensional) truncated part can be exactly obtained. With these values, an upper bound and a lower bound on the original l_∞-induced norm can be computed. Furthermore, these bounds are shown to converge to each other within an exponential order of N, where N is the corresponding truncation parameter. Finally, some numerical examples are provided to demonstrate the theoretical validity and practical effectiveness of the developed computation method.

Keywords: multivariable discrete-time linear systems; truncated fashion; l_∞-induced norm; generalized H_2 norm; convergence rate analysis
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1. Introduction

The problem that evaluates the effects of external disturbance on the regulated output in control systems has been regarded as one of the most important issues in the control system society. For the purpose of quantitatively evaluating some desired performances of the control systems, various system norms have been taken depending on the nature of the disturbances and the control objectives.

For instance, in [1–6], the H_∞ norm is concerned with evaluating the possible maximum ratio between the energies of the disturbance and the regulated output, i.e., the L_2-induced norm (or the l_2-induced norm) for the continuous-time case (or the discrete-time case). To consider the input/output behavior of control systems in a mixed fashion, the generalized H_2 norm is taken in [6–11] for assessing the maximum magnitude of the regulated output for the worst disturbance with a unit energy, i.e., the induced norm from L_2 to L_∞ (or from l_2 to l_∞) for the continuous-time case (or the discrete-time case).

Even though these two induced norms are sometimes practically meaningful, they do not take into account the effects of bounded persistent disturbances on the regulated output. To address this issue, the L_∞-induced norm or the l_∞-induced norm could be employed for evaluating the effects of the above disturbances on the regulated output in continuous-time systems [12–14] or discrete-time systems [14–17], respectively. However, in contrast to the cases of the H_∞ norm and the generalized H_2 norm, whose analytic computations are possible for both the pure continuous-time and discrete-time systems, it is a non-trivial task to explicitly compute the L_∞-induced norm and the l_∞-induced norm, even for the pure continuous-time and discrete-time systems. This is because the integrals and sums of absolute values for impulse responses should be computed in the analysis problems of the L_∞-induced and l_∞-induced norms, respectively. In a similar line, the L_∞-induced norms in previous studies [12–14] are confined to the case of single-input/single-output (SISO) systems, and no possible extension for the case of general multivariable systems is discussed in those studies. To solve this difficulty in the case of continuous-time multivariable systems, an upper bound and a lower bound on the L_∞-induced norm are derived in [18, 19] with the associated convergence analysis. These results are further considered in an extensive fashion to more involved systems such as sampled data systems [20–23], nonlinear systems [24] and so on.

Similarly to the problems of dealing with the discrete-time l_∞-induced norm [14–17], on the other hand, an infinite number of variables or constraints should be concerned with. This intrinsic characteristic also leads to an infinite number of computations for the treatment of the l_∞-induced norm of discrete-time linear systems, even for the analysis problem. As a pioneering study to alleviate this difficulty, the relationship between the l_∞-induced norm and the root mean square (RMS) gain for discrete-time linear systems is discussed in [25]. Based on the results of that study, a sophisticated method for computing the l_∞-induced norm is subsequently introduced in [26]. However, the methods in [25, 26] are limited to the case of SISO discrete-time linear systems, and no argument on their extensions to the general multivariable case is given. Furthermore, no clear convergence order is derived in [25, 26], although a sort of approximate computation is used in those studies.

With this in mind, we develop a new approximate method for computing the l_∞-induced norm of discrete-time linear systems, compatible with the general multivariable case. Motivated by the fact that the l_∞-induced norm coincides with the ∞-norm of an infinite-dimensional matrix, a truncation scheme of the infinite-dimensional matrix is provided. Here, the ∞-norm of the (infinite-dimensional) tail part can be approximately computed within any degree of accuracy, while that of the finite-dimensional truncated part can be exactly obtained. More precisely, we derive an upper bound and a lower bound on the ∞-norm of the tail part, in terms of the sum of a geometric sequence and the generalized H_2 norm of discrete-time linear systems, respectively. They are further shown to converge to each other within an exponential order of N, where N is the corresponding truncation parameter. To summarize, these arguments can lead to computable upper and lower bounds on the l_∞-induced norm with the convergence rate no smaller than ρ^N for a constant $0 < \rho < 1$.

This paper is organized as follows. The l_∞-induced norm of multivariable discrete-time linear systems and the issues to be tackled are introduced in Section 2. The main results of this paper, i.e., a method for computing an upper bound and a lower bound on the l_∞-induced norm and the relevant convergence analysis, are provided in Section 3. Based on the arguments in Section 3, a bisection-based computation method is also introduced in Section 4. Some numerical examples are given in Section 5 to demonstrate the overall arguments derived in this paper. Some concluding remarks are provided in Section 6. On the other hand, the notations used in this paper are summarized in Table 1.

Table 1. The notations used in this paper.

Notation	Meaning
$\mathbb{N}_0$	The set of nonnegative integers
$\mathbb{R}^\nu$	The set of ν-dimensional real-valued vectors
$\lambda_{\max}(\cdot)$	The maximum eigenvalue of $(\cdot)$
ceil$(\cdot)$	The ceiling function of a scalar, i.e., ceil(v) := min{n $\in \mathbb{N}_0$: n $\geq$ v}
$\lvert\cdot\rvert_1$	The 1-norm of a real-valued matrix, i.e., $\lvert A\rvert_1 := \max_j \sum_i \lvert A_{ij}\rvert$
$\lvert\cdot\rvert_2$	The 2-norm of a real-valued matrix, i.e., $\lvert A\rvert_2 := \lambda_{\max}(A^T A)^{\frac{1}{2}}$
$\lvert\cdot\rvert_\infty$	The ∞-norm of a real-valued matrix, i.e., $\lvert A\rvert_\infty := \max_i \sum_j \lvert A_{ij}\rvert$
$\lVert\cdot\rVert_2$	The l_2-norm of a real-valued sequence, i.e., $\lVert f\rVert_2^2 := \sum_{k=0}^{\infty} f_k^T f_k$
$\lVert\cdot\rVert_\infty$	The l_∞-norm of a real-valued sequence, or the l_∞-induced norm of an operator, i.e., $\lVert f\rVert_\infty := \sup_{k\in\mathbb{N}\cup 0} \lvert f_k\rvert_\infty$, $\lVert\mathbf{T}\rVert_\infty := \sup_{\lVert w\rVert_\infty\leq 1} \lVert\mathbf{T}w\rVert_\infty$
$\lVert\cdot\rVert_{GH_2}$	The generalized H_2 norm, i.e., $\lVert\mathbf{T}\rVert_{GH_2} := \sup_{\lVert w\rVert_2\leq 1} \lVert\mathbf{T}w\rVert_\infty$

2. Truncated fashion to the l_∞-induced norm of discrete-time linear systems

Let us consider the multivariable discrete-time linear system G given by

$$G : \begin{cases} x_{k+1} = Ax_k + Bu_k, \\ y_k = Cx_k + Du_k, \end{cases} \tag{2.1}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^{n_u}$ is the exogenous disturbance and $y_k \in \mathbb{R}^{n_y}$ is the regulated output, with which the sizes of the matrices are determined by $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_u}$, $C \in \mathbb{R}^{n_y\times n}$ and $D \in \mathbb{R}^{n_y\times n_u}$. To ensure that the l_∞-induced norm of G is bounded and well-defined, we assume that A is Schur stable (i.e., all the eigenvalues of A are located inside the unit circle).

For this system, we first note that

$$y_k = \sum_{i=0}^{k-1} CA^{k-i-1}Bu_i + Du_k + A^k x_0. \tag{2.2}$$

Based on the fact that the Schur stability assumption ensures the convergence of A^k to 0 as k becomes larger, let us denote the corresponding input/output operator by $\mathbf{G}$, while ignoring the third term in the right-hand-side (RHS) of (2.2), equivalently to the corresponding existing studies [16, 27], i.e.,

$$(\mathbf{G}u)_k := y_k = \sum_{i=0}^{k-1} CA^{k-i-1}Bu_i + Du_k. \tag{2.3}$$

This further admits the Toeplitz matrix-based representation described by

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} D & 0 & \cdots & \cdots & \cdots \\ CB & D & \cdots & \cdots & \cdots \\ CAB & CB & D & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \end{bmatrix}. \tag{2.4}$$

On the other hand, we next note that the l_∞-induced norm is defined as the supremum of the ratio between the exogenous disturbance and regulated output in terms of their l_∞ norms, i.e.,

$$||\mathbf{G}||_\infty := \sup_{||u||_\infty \neq 0} \frac{||y||_\infty}{||u||_\infty} = \sup_{||u||_\infty \leq 1} \frac{||y||_\infty}{||u||_\infty} = \sup_{||u||_\infty = 1} ||y||_\infty. \tag{2.5}$$

From the point of view of (2.5) in (2.4), it immediately follows that the l_∞-induced norm coincides with the matrix ∞-norm of the last block row matrix of the Toeplitz matrix (while conversely reordering the elements) defined as

$$P := \begin{bmatrix} D & CB & CAB & CA^2B & \cdots & \end{bmatrix}. \tag{2.6}$$

In other words, we can reinterpret the problem of computing the l_∞-induced norm $||\mathbf{G}||_\infty$ by that of computing the matrix ∞-norm of P (described by (2.6)), i.e.,

$$||\mathbf{G}||_\infty = |P|_\infty. \tag{2.7}$$

As clarified from the definition of P in (2.6), however, a direct and exact computation of the ∞-norm $|P|_\infty$ is quite difficult due to an infinite number of columns in P. To alleviate this difficulty, we treat P in a truncated fashion described by

$$P = P_N^- + P_N^+, \tag{2.8}$$

where

$$P_N^- := \begin{bmatrix} D & CB & CAB & \cdots & CA^NB & 0 & \cdots \end{bmatrix}, \tag{2.9}$$

$$P_N^+ := \begin{bmatrix} 0 & \cdots & 0 & CA^{N+1}B & CA^{N+2}B & \cdots \end{bmatrix}, \tag{2.10}$$

where $N \in \mathbb{N}_0$ is the truncation parameter.

It would be worthwhile to note that $|P_N^-|_\infty$ for any finite $N \in \mathbb{N}_0$ can be exactly obtained and this value converges to $|P|_\infty(= ||\mathbf{G}||_\infty)$ as the truncation parameter N becomes larger, since $|P_N^+|_\infty$ tends to 0 by taking N larger from the stability assumption on A (and the details will be provided in the following section). In the same line, the infinite-dimensional matrix P_N^- is redefined by ignoring the zero terms, i.e.,

$$P_N^- = \begin{bmatrix} D & CB & CAB & \cdots & CA^NB \end{bmatrix}, \tag{2.11}$$

because this replacement does not affect computing the l_∞-induced norm $|P|_\infty$ and is just for the simplicity of the argument. From the definition of P_N^- and P, we are led to the following result.

Theorem 2.1. *The matrix ∞-norm of the finite dimensional matrix P_N^- converges to that of the infinite dimensional matrix P, i.e.,*

$$|P_N^-|_\infty \to |P|_\infty \quad (N \to \infty). \tag{2.12}$$

Furthermore, the matrix ∞-norm $|P|_\infty$ coincides with the l_∞-induced norm $\|\mathbf{G}\|_\infty$.

This theorem implies that taking a sufficiently large N in $|P_N^-|_\infty$ is theoretically meaningful for computing the original l_∞-induced norm $\|\mathbf{G}\|_\infty = |P|_\infty$. However, $|P_N^-|_\infty$ corresponds to a just lower bound on $|P|_\infty$. More importantly, it is still unclear how close the approximate value $|P_N^-|_\infty$ for a given N is to the original value of the l_∞-induced norm $\|\mathbf{G}\|_\infty = |P|_\infty$. Hence, it is required to treat the tail part P_N^+ in a rigorous fashion to derive a more accurate computation of $|P|_\infty$. With respect to this, the following section develops an approximate approach to the tail part P_N^+, by which an upper bound and a lower bound on $|P|_\infty$ together with the corresponding convergence rate could be derived.

3. Upper and lower bounds on the l_∞-induced norm $\|\mathbf{G}\|_\infty$ with convergence analysis

This section introduces the main results of this paper, i.e., sophisticated arguments on dealing with P_N^+ tailored to computing an upper bound and a lower bound on $|P|_\infty$.

As a preliminary step to establishing such arguments, we first denote for $i = 1, \ldots, n_y$ and $j = 1, \ldots, n_u$ the i-th row vector of C, j-th column vector of B and (i, j)-th element of D by C_i, B_j and D_{ij}, respectively. Then, the matrix ∞-norms $|P|_\infty$, $|P_N^-|_\infty$ and $|P_N^+|_\infty$ can be described by

$$|P|_\infty = \max_{1 \le i \le n_y} \sum_{j=1}^{n_u} (|D_{ij}| + \sum_{k=0}^{\infty} |C_i A^k B_j|), \tag{3.1}$$

$$|P_N^-|_\infty = \max_{1 \le i \le n_y} \sum_{j=1}^{n_u} (|D_{ij}| + \sum_{k=0}^{N} |C_i A^k B_j|), \tag{3.2}$$

$$|P_N^+|_\infty = \max_{1 \le i \le n_y} \sum_{j=1}^{n_u} \sum_{k=N+1}^{\infty} |C_i A^k B_j|, \tag{3.3}$$

where $C_i A^k B_j$ and D_{ij} are scalars for all $i = 1, \ldots, n_y$ and $j = 1, \ldots, n_u$ from the definitions of C_i, B_j and D_{ij}.

We next define the scalar-valued constants ρ_i, ρ_{iN}^-, and ρ_{iN}^+ for $i = 1, \ldots, n_y$, respectively as

$$\rho_i := \sum_{j=1}^{n_u} \left(|D_{ij}| + \sum_{k=0}^{\infty} |C_i A^k B_j| \right), \tag{3.4}$$

$$\rho_{iN}^- := \sum_{j=1}^{n_u} \left(|D_{ij}| + \sum_{k=0}^{N} |C_i A^k B_j| \right), \tag{3.5}$$

$$\rho_{iN}^+ := \sum_{j=1}^{n_u} \sum_{k=N+1}^{\infty} |C_i A^k B_j|. \tag{3.6}$$

These scalar-value constants ρ_i, ρ^-_{iN} and ρ^+_{iN} coincide with the matrix ∞-norms of the ith row vectors of P, P^-_N and P^+_N, respectively. Then, it immediately follows from the definition of the matrix ∞-norm that

$$|P|_\infty = |\rho|_\infty = |\rho^-_N + \rho^+_N|_\infty, \tag{3.7}$$

with

$$\rho := \begin{bmatrix}\rho_1 & \rho_2 & \cdots \rho_{n_y}\end{bmatrix}^T, \tag{3.8}$$

$$\rho^-_N := \begin{bmatrix}\rho^-_{1N} & \rho^-_{2N} & \cdots \rho^-_{n_yN}\end{bmatrix}^T, \tag{3.9}$$

$$\rho^+_N := \begin{bmatrix}\rho^+_{1N} & \rho^+_{2N} & \cdots \rho^+_{n_yN}\end{bmatrix}^T. \tag{3.10}$$

For any fixed $N \in \mathbb{N}$, the computation of ρ^-_{iN} can be conducted immediately, while that of ρ^+_{iN} is still quite difficult due to the infinite-dimensional property of P^+_N. With respect to this issue, we first note that all the elements of ρ, ρ^-_N and ρ^+_N are nonnegative. Hence, if we can derive an upper bound and a (positive) lower bound on ρ^+_{iN} for each $i = 1, \ldots, n_y$, then replacing ρ^+_{iN} with these values in (3.7) leads to an upper bound and a lower bound on $|P|_\infty$. In this sense, we develop a method for computing such upper and lower bounds and establish the corresponding convergence property in the following subsections.

3.1. Upper and lower bounds on ρ^+_N

Regarding deriving an upper bound and a lower bound on ρ^+_{iN}, we first note that ρ^+_{iN} defined as (3.6) coincides with the l_∞-induced norm of the following multi-input/single-output (MISO) discrete-time system G^+_{iN}.

$$G^+_{iN} : \begin{cases} x_{k+1} = Ax_k + Bu_k, \\ y_k = C_iA^{N+1}x_k. \end{cases} \tag{3.11}$$

Equivalently to the case of $\mathbf{G}$, let us denote the input/output operator of G^+_{iN} by $\mathbf{G}^+_{iN}$. Based on the fact that $\rho^+_{iN} = \|\mathbf{G}^+_{iN}\|_\infty$, we provide the following lemma associated with an upper bound on $\|\mathbf{G}^+_{iN}\|_\infty (= \rho^+_{iN})$.

Lemma 3.1. *With an $L \in \mathbb{N}$ such that $|A^L|_\infty < 1$, the inequality*

$$\|\mathbf{G}^+_{iN}\|_\infty \leq \frac{|C_iA_{NL}|_\infty \cdot |B|_\infty}{1 - |A^L|_\infty} =: \overline{\rho}^+_{iNL}, \tag{3.12}$$

holds, where

$$A_{NL} := \begin{bmatrix}A^{N+1} & A^{N+2} & \cdots & A^{N+L}\end{bmatrix}. \tag{3.13}$$

Remark 3.1. *There should exist a finite $L \in \mathbb{N}$ ensuring $|A^L|_\infty < 1$ because of the Schur stability assumption on A.*

Proof. It readily follows from (3.6) together with $\|\mathbf{G}^+_{iN}\|_\infty = \rho^+_{iN}$ that

$$\|\mathbf{G}^+_{iN}\|_\infty = \rho^+_{iN} = \sum_{j=1}^{n_u} \sum_{k=N+1}^{\infty} |C_iA^kB_j| = \sum_{k=N+1}^{\infty} \sum_{j=1}^{n_u} |C_iA^kB_j| = \sum_{k=N+1}^{\infty} |C_iA^kB|_\infty, \tag{3.14}$$

where the last equality follows by the fact that

$$|C_iA^kB|_\infty = \left|C_iA^k\begin{bmatrix}B_1 & \cdots & B_{n_u}\end{bmatrix}\right|_\infty = \left|\begin{bmatrix}C_iA^kB_1 & \cdots & C_iA^kB_{n_u}\end{bmatrix}\right|_\infty = \sum_{j=1}^{n_u}|C_iA^kB_j|. \tag{3.15}$$

From $|C_iA^kB|_\infty \leq |C_iA^k|_\infty \cdot |B|_\infty$, (3.14) further admits for an $L \in \mathbb{N}_0$ such that $|A^L|_\infty < 1$ the representation

$$\|\mathbf{G}_{iN}^+\|_\infty \leq \sum_{k=N+1}^{\infty}|C_iA^k|_\infty \cdot |B|_\infty = \sum_{k=0}^{\infty}\left(|C_iA^{N+1+Lk}|_\infty + \cdots + |C_iA^{N+L+Lk}|_\infty\right)\cdot |B|_\infty. \tag{3.16}$$

Here, note that

$$\begin{aligned}\left(|C_iA^{N+1+Lk}|_\infty + \cdots + |C_iA^{N+L+Lk}|_\infty\right) &\leq \left(|C_iA^{N+1}|_\infty + \cdots + |C_iA^{N+L}|_\infty\right)\cdot |A^{Lk}|_\infty \\ &\leq \left(|C_iA^{N+1}|_\infty + \cdots + |C_iA^{N+L}|_\infty\right)\cdot |A^L|_\infty^k.\end{aligned} \tag{3.17}$$

Substituting this into (3.16) leads to

$$\|\mathbf{G}_{iN}^+\|_\infty \leq \sum_{k=0}^{\infty}\left(|C_iA^{N+1}|_\infty + \cdots + |C_iA^{N+L}|_\infty\right)\cdot |B|_\infty \cdot |A^L|_\infty^k = \frac{\left(|C_iA^{N+1}|_\infty + \cdots + |C_iA^{N+L}|_\infty\right)\cdot |B|_\infty}{1-|A^L|_\infty}. \tag{3.18}$$

Because C_iA^k is a row vector for all $k \in \mathbb{N}_0$, it immediately follows from the definition of the matrix ∞-norm that

$$|C_iA^{N+1}|_\infty + \cdots + |C_iA^{N+L}|_\infty = \left|\begin{bmatrix}C_iA^{N+1} & \cdots & C_iA^{N+L}\end{bmatrix}\right|_\infty = |C_iA_{NL}|_\infty. \tag{3.19}$$

By combining (3.18) and (3.19), we can easily see that

$$\|\mathbf{G}_{iN}^+\|_\infty \leq \frac{|C_iA_{NL}|_\infty \cdot |B|_\infty}{1-|A^L|_\infty} = \overline{\rho}_{iNL}^+. \tag{3.20}$$

This completes the proof.

This lemma can be regarded as a discrete-time version of Proposition 2 in [20] for the L_∞-induced norm analysis of sampled-data systems. We can see from Lemma 3.1 that an upper bound on ρ_{iN}^+ can be readily obtained by $\overline{\rho}_{iNL}^+$ defined as (3.12), and this upper bound obviously converges to 0 by taking N larger since $|A_{NL}|_\infty$ tends to 0 as N increases.

For the purpose of deriving a tighter bound for $|\rho|_\infty$, we next consider a lower bound on $\|\mathbf{G}_{iN}^+\|_\infty(= \rho_{iN}^+)$. Such a lower bound can be derived by using the relationship between the generalized H_2 norm (i.e., the induced norm from l_2 to l_∞) and the l_∞-induced norm for MISO discrete-time systems. In connection with this, we induce the following result.

Lemma 3.2. *The inequality*

$$\underline{\rho}_{iN}^+ := \left(C_iA^{N+1}X(A^T)^{N+1}C_i^T\right)^{\frac{1}{2}} \leq \rho_{iN}^+ = \|\mathbf{G}_{iN}^+\|_\infty, \tag{3.21}$$

holds, where X is the solution of the discrete-time Lyapunov equation given by

$$AXA^T - X + BB^T = 0. \tag{3.22}$$

Proof. We first note that

$$\{u \mid \|u\|_2 \le \alpha\} \subseteq \{u \mid \|u\|_\infty \le \alpha\}, \quad \forall \alpha > 0. \tag{3.23}$$

This implies for any discrete-time linear system that the generalized H_2 norm (i.e., the induced norm from l_2 to l_∞) is not larger than the l_∞-induced norm since they take the same performance measure (i.e., the l_∞ norm) for the regulated output, although the domain of the former is involved in that of the latter. In other words, it immediately follows that

$$\|\mathbf{G}_{iN}^{+}\|_{GH_2} \le \|\mathbf{G}_{iN}^{+}\|_\infty, \tag{3.24}$$

where $\|\cdot\|_{GH_2}$ means the generalized H_2 norm of $(\cdot)$, and thus $\|\mathbf{G}_{iN}^{+}\|_{GH_2}$ corresponds to a lower bound on $\|\mathbf{G}_{iN}^{+}\|_\infty = \rho_{iN}^{+}$. In regard to computing the generalized H_2 norm of G_{iN}^{+}, we next note that the corresponding input/output relation is given by

$$y_k = \sum_{j=0}^{k-1} C_i A^{N+k-j} B u_j. \tag{3.25}$$

Applying the discrete-time Cauchy-Schwartz inequality to (3.25) leads to

$$y_k^2 \le \left(\sum_{j=0}^{k-1} (C_i A^{N+k-j} B)(C_i A^{N+k-j} B)^T\right) \cdot \left(\sum_{j=0}^{k-1} u_j^T u_j\right), \tag{3.26}$$

where the equality holds when $u_i = \lambda (C_i A^{N+k-i-1} B)^T$ for a constant λ. This further establishes that

$$\sup_{\|u\|_2=1} y_k^2 = \left(\sum_{j=0}^{k-1} (C_i A^{N+k-j} B)(C_i A^{N+k-j} B)^T\right). \tag{3.27}$$

By taking the supreumum of (3.27) in terms of $k \in \mathbb{N}_0$, we can obtain that

$$\sup_{k\in\mathbb{N}_0} \sup_{\|u\|_2=1} y_k^2 = \left(\sum_{j=0}^{\infty} C_i A^{N+j+1} B B^T (A^{N+j+1})^T C_i^T\right) = C_i A^{N+1} X (A^T)^{N+1} C_i^T, \tag{3.28}$$

because

$$\sum_{j=0}^{\infty} A^j B B^T (A^T)^j = X, \tag{3.29}$$

where X is the solution to the discrete-time Lyapunov equation described by

$$A X A^T - X + B B^T = 0. \tag{3.30}$$

This completes the proof.

This lemma provides a readily computable lower bound on the l_∞-induced norm of the tail system G_{iN}^{+}, i.e., ρ_{iN}^{+}. It is also obvious from (3.21) that the lower bound $\underline{\rho}_{iN}^{+}$ converges to 0 as N becomes larger since A is Schur stable.

From the point of view of (3.7) based on Lemmas 3.1 and 3.2, we can obtain the following theorem relevant to an upper bound and a lower bound on $\|\mathbf{G}\|_\infty = |P|_\infty$.

Theorem 3.1. *Consider the column vectors $\overline{\rho}^+_{NL}$ and $\underline{\rho}^+_N$ defined respectively as*

$$\overline{\rho}^+_{NL} := \left[\overline{\rho}^+_{1NL} \quad \cdots \overline{\rho}^+_{n_yNL}\right], \tag{3.31}$$

$$\underline{\rho}^+_N := \left[\underline{\rho}^+_{1N} \quad \cdots \underline{\rho}^+_{n_yN}\right]. \tag{3.32}$$

Then, the following inequality holds:

$$|\rho^-_N + \underline{\rho}^+_N|_\infty \le |P|_\infty \le |\rho^-_N + \overline{\rho}^+_{NL}|_\infty. \tag{3.33}$$

Furthermore, the gap between the upper and lower bounds in (3.33) *converges to* 0 *within an exponential order of N regardless of the choice of L.*

Proof. The first assertion is readily established by the definition of the matrix ∞-norm. On the other hand, if we note that

$$\overline{\rho}^+_{iNL} - \underline{\rho}^+_{iN} \le \overline{\rho}^+_{iNL}, \tag{3.34}$$

then the second assertion also immediately follows because $\overline{\rho}^+_{iNL}$ converges to 0 in the exponential order of γ^N for some $0 < \gamma < 1$ from the Schur stability assumption on A.

From Theorem 3.1, we can compute an upper bound and a lower bound on the l_∞-induced norm $\|\mathbf{G}\|_\infty = |P|_\infty$ and ensure that they converge to each other within an exponential order of N regardless of the choice of L.

3.2. Convergence analysis of truncation method

Beyond the naive analysis of the convergence property discussed in the preceding subsection, this subsection aims at establishing more rigorous arguments. More precisely, we show that the gap between the upper bound $\overline{\rho}^+_{iNL}$ and lower bound $\underline{\rho}^+_{iN}$ is monotonically decreasing in terms of N, and clarifies that this gap should converge to the zero. To this end, we first take the notation e^+_{iNL} to denote such a gap described by

$$e^+_{iNL} := \overline{\rho}^+_{iNL} - \underline{\rho}^+_{iN} = \frac{|C_iA_{NL}|_\infty \cdot |B|_\infty}{1 - |A^L|_\infty} - \left(C_iA^{N+1}X(A^T)^{N+1}C_i^T\right)^{\frac{1}{2}}. \tag{3.35}$$

With this, we provide the following two lemmas associated with the convergence property of e^+_{iNL}.

Lemma 3.3. *For each $i = 1, 2, \ldots, n_y$, e^+_{iNL} is a monotonically decreasing sequence in terms of $N \in \mathbb{N}_0$.*

Proof. The objective of this proof is to show that

$$e^+_{iN+1L} \le e^+_{iNL}, \quad (\forall N \in \mathbb{N}_0). \tag{3.36}$$

According to the definition of e^+_{iNL} in (3.35), we can see that

$$\begin{aligned} &e^+_{iN+1L} - e^+_{iNL} \\ &= \left(\frac{|C_iA_{N+1L}|_\infty \cdot |B|_\infty}{1 - |A^L|_\infty} - \left(C_iA^{N+2}X(A^T)^{N+2}C_i^T\right)^{\frac{1}{2}}\right) - \left(\frac{|C_iA_{NL}|_\infty \cdot |B|_\infty}{1 - |A^L|_\infty} - \left(C_iA^{N+1}X(A^T)^{N+1}C_i^T\right)^{\frac{1}{2}}\right) \end{aligned}$$

$$= \left(\frac{|C_i A_{N+1L}|_\infty \cdot |B|_\infty}{1 - |A^L|_\infty} - \frac{|C_i A_{NL}|_\infty \cdot |B|_\infty}{1 - |A^L|_\infty} \right) + \left(\left(C_i A^{N+1} X (A^T)^{N+1} C_i^T \right)^{\frac{1}{2}} - \left(C_i A^{N+2} X (A^T)^{N+2} C_i^T \right)^{\frac{1}{2}} \right). \quad (3.37)$$

Substituting the triangular inequality

$$\sqrt{x} - \sqrt{y} \leq \sqrt{x - y} \quad (x \geq y \geq 0) \quad (3.38)$$

into the second term of the right-hand-side (RHS) of (3.37) leads to

$$e^+_{iN+1L} - e^+_{iNL} \leq \frac{|C_i A_{N+1L}|_\infty \cdot |B|_\infty}{1 - |A^L|_\infty} - \frac{|C_i A_{NL}|_\infty \cdot |B|_\infty}{1 - |A^L|_\infty} + \left(C_i A^{N+1} X (A^T)^{N+1} C_i^T - C_i A^{N+2} X (A^T)^{N+2} C_i^T \right)^{\frac{1}{2}}, \quad (3.39)$$

where the relation '$C_i A^{N+1} X (A^T)^{N+1} C_i^T \geq C_i A^{N+2} X (A^T)^{N+2} C_i^T$' is established from

$$C_i A^{N+1} X (A^T)^{N+1} C_i^T - C_i A^{N+2} X (A^T)^{N+2} C_i^T = C_i A^{N+1} B B^T (A^T)^{N+1} C_i^T = |B^T (A^T)^{N+1} C_i^T|_2^2 \quad (3.40)$$

because X is the solution to the discrete-time Lyapunov equation $AXA^T - X + BB^T = 0$. Then, it immediately follows from (3.39) and (3.40) that

$$\begin{aligned}
e^+_{iN+1L} - e^+_{iNL} &\leq \frac{|C_i A_{N+1L}|_\infty \cdot |B|_\infty}{1 - |A^L|_\infty} - \frac{|C_i A_{NL}|_\infty \cdot |B|_\infty}{1 - |A^L|_\infty} + |B^T (A^T)^{N+1} C_i^T|_2 \\
&= \frac{\left| \begin{bmatrix} C_i A^{N+2} & \cdots & C_i A^{N+L+1} \end{bmatrix} \right|_\infty \cdot |B|_\infty}{1 - |A^L|_\infty} - \frac{\left| \begin{bmatrix} C_i A^{N+1} & \cdots & C_i A^{N+L} \end{bmatrix} \right|_\infty \cdot |B|_\infty}{1 - |A^L|_\infty} + |B^T (A^T)^{N+1} C_i^T|_2 \\
&= \frac{|C_i A^{N+L+1}|_\infty \cdot |B|_\infty}{1 - |A^L|_\infty} - \frac{|C_i A^{N+1}|_\infty \cdot |B|_\infty}{1 - |A^L|_\infty} + |B^T (A^T)^{N+1} C_i^T|_2 \\
&\leq \frac{|C_i A^{N+1}|_\infty \cdot |A^L|_\infty \cdot |B|_\infty}{1 - |A^L|_\infty} - \frac{|C_i A^{N+1}|_\infty \cdot |B|_\infty}{1 - |A^L|_\infty} + |B^T (A^T)^{N+1} C_i^T|_2 \\
&= -|C_i A^{N+1}|_\infty |B|_\infty + |B^T (A^T)^{N+1} C_i^T|_2. \quad (3.41)
\end{aligned}$$

Because $B^T (A^T)^{N+1} C_i^T$ is a column vector, the following inequality holds.

$$|B^T (A^T)^{N+1} C_i^T|_2 \leq |B^T (A^T)^{N+1} C_i^T|_1 \leq |B^T|_1 |(A^T)^{N+1} C_i^T|_1 = |C_i A^{N+1}|_\infty |B|_\infty. \quad (3.42)$$

Substituting (3.42) into (3.41) derives

$$e^+_{iN+1L} - e^+_{iNL} \leq -|C_i A^{N+1}|_\infty |B|_\infty + |B^T (A^T)^{N+1} C_i^T|_2 \leq -|C_i A^{N+1}|_\infty |B|_\infty + |C_i A^{N+1}|_\infty |B|_\infty = 0. \quad (3.43)$$

This completes the proof.

This lemma clarifies that the gap between the upper bound $\overline{\rho}^+_{iNL}$ and the lower bound $\underline{\rho}^+_{iN}$ on ρ^+_{iN} is monotonically decreasing as the truncation parameter N becomes larger. Furthermore, it might be expected from Lemma 3.3 that this gap converges to a specific value. In connection with this, we are led to the following lemma.

Lemma 3.4. *For each $i = 1, 2, \ldots, n_y$, e^+_{iNL} converges to the zero in terms of $N \in \mathbb{N}_0$.*

Proof. The objective of this proof is to show that there exists an $N_0 \in \mathbb{N}_0$ such that

$$|e^+_{iNL}| < \epsilon, \quad \forall N \geq N_0, \tag{3.44}$$

for an arbitrary $\epsilon > 0$. Based on the definition of e^+_{iNL} in (3.35), we note that

$$|e^+_{iNL}| = \frac{|C_i A_{NL}|_\infty \cdot |B|_\infty}{1 - |A^L|_\infty} - \left(C_i A^{N+1} X (A^T)^{N+1} C_i^T\right)^{\frac{1}{2}} < \frac{|C_i A_{NL}|_\infty \cdot |B|_\infty}{1 - |A^L|_\infty} \tag{3.45}$$

$$\leq \frac{|C_i \begin{bmatrix} A^{N+1} & A^{N+2} & \cdots & A^{N+L} \end{bmatrix}|_\infty \cdot |B|_\infty}{1 - |A^L|_\infty} \leq \frac{|C_i|_\infty \cdot |B|_\infty \cdot |\begin{bmatrix} A & A^2 & \cdots & A^L \end{bmatrix}|_\infty}{1 - |A^L|_\infty} |A^N|_\infty. \tag{3.46}$$

If we take $p, r \in \mathbb{N}_0$ such that $0 \leq r \leq L - 1$ and $pL + r = N$, we can further obtain that

$$|e^+_{iNL}| < \frac{|C_i|_\infty \cdot |B|_\infty \cdot |\begin{bmatrix} A & A^2 & \cdots & A^L \end{bmatrix}|_\infty}{1 - |A^L|_\infty} |A^{pL+r}|_\infty \leq \frac{|C_i|_\infty \cdot |B|_\infty \cdot |\begin{bmatrix} A & A^2 & \cdots & A^L \end{bmatrix}|_\infty}{1 - |A^L|_\infty} |A^r|_\infty \cdot |A^L|^p_\infty \tag{3.47}$$

$$\leq \frac{|C_i|_\infty \cdot |B|_\infty \cdot |\begin{bmatrix} A & A^2 & \cdots & A^L \end{bmatrix}|_\infty}{1 - |A^L|_\infty} \left(\max_{0 \leq r \leq L-1} |A^r|_\infty\right) \cdot |A^L|^p_\infty =: \gamma |A^L|^p_\infty. \tag{3.48}$$

Take p by

$$p = \max(\mathrm{ceil}(\log_{|A^L|_\infty}(\frac{\epsilon}{\gamma})), 1). \tag{3.49}$$

With this p and an arbitrary $0 \leq r \leq L - 1$, let $N_0 = pL + r$. Then, it readily follows from (3.48) that

$$|e^+_{iN_0L}|_\infty < \gamma |A^L|^p_\infty \leq \epsilon. \tag{3.50}$$

This together with the fact that e^+_{iNL} is monotonically decreasing in terms of n from Lemma 3.3 completes the proof.

This lemma implies that the gap between the upper bound $\overline{\rho}^+_{iNL}$ and the lower bound $\underline{\rho}^+_{iN}$ on ρ^+_{iN} converges to the zero by taking the truncation parameter N larger.

From Lemmas 3.3 and 3.4, we obtain the following theorem, i.e., the main results in this subsection.

Theorem 3.2. *The gap e^+_{iNL} between the upper bound $\overline{\rho}^+_{iNL}$ and the lower bound $\underline{\rho}^+_{iN}$ on ρ^+_{iN} is monotonically decreasing and converges to the zero as the truncation parameter N becomes larger.*

4. Bisection-based l_∞-induced norm computation

For the practical applicability of the results in the preceding section, we develop a bisection-based method for computing the l_∞-induced norm. More precisely, we aim at finding an $N \in \mathbb{N}_0$ ensuring $e^+_{iNL} < \epsilon$, $\forall i = 1, \ldots, n_y$ for a given tolerance $\epsilon(> 0)$ via the bisection approach [28]. The details can be represented by the following pseudo-code based Algorithm 1.

Algorithm 1 Bisection-based method for computing the l_∞-induced norm.

Input: System matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_u}$, $C \in \mathbb{R}^{n_y\times n}$, $D \in \mathbb{R}^{n_y\times n_u}$, parameter $L \in \mathbb{N}_0$ such that $|A^L|_\infty < 1$, and tolerance $\epsilon > 0$

Solve $AXA^T - A + BB^T = 0$ and store X
Split the matrix C into the row vectors C_i $(i = 1, 2, \cdots, n_y)$ and store it
$A_L \leftarrow \begin{bmatrix} A & A^2 & \cdots A^L \end{bmatrix}$
Init upper bound $\leftarrow 0$, lower bound $\leftarrow 0$, $\rho_N^- \leftarrow 0$, $\overline{\rho}_{NL}^+ \leftarrow 0$, and $\underline{\rho}_N^+ \leftarrow 0$
for all $i = 1$ to n_y **do**
  Init $N \leftarrow 0$
  for all $m = 1$ to ∞ **do**
    $A_{2^m L} \leftarrow A^{2^m} \times A_L$
    if $\dfrac{|C_i A_{NL}|_\infty \cdot |B|_\infty}{1 - |A^L|_\infty} - \left(C_i A^{N+1} X (A^T)^{N+1} C_i^T\right)^{\frac{1}{2}} < \epsilon$ **then**
      break
    else
      continue
    end if
  end for
  $x \leftarrow 2^{m-1}$, $y \leftarrow 2^m$, and $z \leftarrow \dfrac{x+y}{2}$
  while $|x - y| \neq 1$ **do**
    $\tilde{A} \leftarrow A^z$ and $\tilde{A}_L \leftarrow \tilde{A} \times A_L$
    if $\dfrac{|C_i \tilde{A}_L|_\infty \cdot |B|_\infty}{1 - |A^L|_\infty} - \left(C_i \tilde{A} A X A^T \tilde{A}^T C_i^T\right)^{\frac{1}{2}} < \epsilon$ **then**
      $y \leftarrow z$ and $z \leftarrow \dfrac{x+y}{2}$
    else
      $x \leftarrow z$ and $z \leftarrow \dfrac{x+y}{2}$
    end if
  end while
  if $N < \max(x, y)$ **then**
    $N \leftarrow \max(x, y)$
  end if
  $\tilde{A} \leftarrow A^N$ and $\tilde{A}_L \leftarrow \tilde{A} \times A_L$
  $\rho_N^-[i] \leftarrow \sum_{j=1}^{n_u} (|D_{ij}| + \sum_{k=0}^{N} |C_i A^k B_j|)$
  $\overline{\rho}_{NL}^+[i] \leftarrow \dfrac{|C_i \tilde{A}_L|_\infty \cdot |B|_\infty}{1 - |A^L|_\infty}$
  $\underline{\rho}_N^+[i] \leftarrow \left(C_i \tilde{A} A X A^T \tilde{A}^T C_i^T\right)^{\frac{1}{2}}$
end for
upper bound $\leftarrow \max_{1\le i\le n_y} \left(\rho_N^-[i] + \overline{\rho}_{NL}^+[i]\right)$
lower bound $\leftarrow \max_{1\le i\le n_y} \left(\rho_N^-[i] + \underline{\rho}_N^+[i]\right)$
return upper bound, lower bound

5. Numerical Examples

This section demonstrates the theoretical validity of the developed l_∞-induced norm computation method through a numerical example for a multivariable discrete-time linear system. Furthermore, the practical effectiveness of the developed computation method is verified in a comparative fashion to the method in [26] through a numerical example for a SISO discrete-time linear system.

5.1. Demonstration of Theoretical Validity with Multivariable Case

Let us consider the two-mass-spring-damper system as shown in Figure 1, where (m_1, m_2), (c_1, c_2), (k_1, k_2), (d_1, d_2) and (f_1, f_2) denote the masses, damping constants, spring constants, displacements from the equilibrium points and external disturbances applied to the masses, respectively.

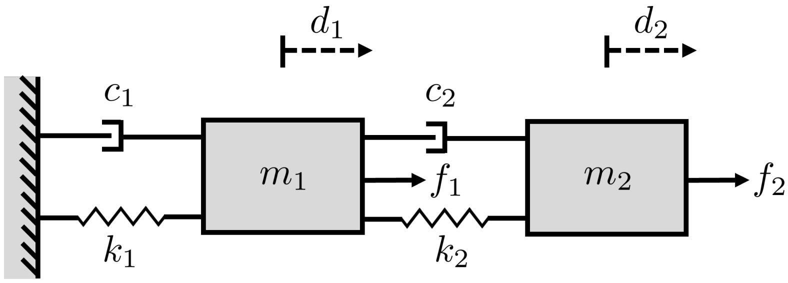

Figure 1. Two-mass-spring-damper system.

Taking $x := \begin{bmatrix} d_1 & \dot{d}_1 & d_2 & \dot{d}_2 \end{bmatrix}^T$, $u := \begin{bmatrix} f_1 & f_2 \end{bmatrix}^T$ and $y := \begin{bmatrix} d_1 & d_2 \end{bmatrix}^T$ leads to the continuous-time linear time-invariant (LTI) differential equation given by

$$G : \begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{k_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} u, \\ y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x. \end{cases} \tag{5.1}$$

In this 2-input/2-output (i.e., MIMO) case, let $m_1 = 2.0$, $m_2 = 1.0$, $k_1 = 4.0$, $k_2 = 2.0$, $c_1 = 1.5$ and $c_2 = 0.5$. For this continuous-time LTI system, we consider its discretization model through the zero-order-hold (ZOH) method [29] with the sampling time $T = 0.1$, i.e.,

$$G_d : \begin{cases} x_{k+1} = \begin{bmatrix} 0.9856 & 0.0947 & 0.0047 & 0.0013 \\ -0.2814 & 0.8916 & 0.0920 & 0.0277 \\ 0.0096 & 0.0027 & 0.9903 & 0.0972 \\ 0.1864 & 0.0555 & -0.1918 & 0.9423 \end{bmatrix} x_k + \begin{bmatrix} 0.0024 & 0.0000 \\ 0.0474 & 0.0013 \\ 0.0000 & 0.0049 \\ 0.0013 & 0.0972 \end{bmatrix} u_k, \\ y_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x_k, \end{cases} \tag{5.2}$$

where $x_k := x(kT)$, $u_k := u(t)$ $(kT \leq t < (k+1)T)$, and $y_k := y(kT)$.

With $L = 100$ leading to $(|A^L|_\infty = 0.3641)$, we conduct Algorithm 1 by taking the tolerance ϵ ranging from 5 to 0.001. The results for the truncation parameter N completing this algorithm, the upper and lower bounds provided in (3.33) of Theorem 3.1 (i.e., $|\rho_N^- + \overline{\rho}_{NL}^+|_\infty$ and $|\rho_N^- + \underline{\rho}_N^+|_\infty$), the gap between these bounds and the required CPU time are given in Table 2.

Table 2. Computation results for truncation parameter, upper bound, lower bound, and gap.

ϵ	5	1	0.1	0.01	0.001
N	56	153	292	428	567
Upper bound	7.557185	4.621345	3.964013	3.901031	3.894637
Lower bound	2.576044	3.631531	3.865266	3.891087	3.893644
Gap	4.98×10^{0}	9.90×10^{-1}	9.87×10^{-1}	9.94×10^{-3}	9.92×10^{-4}
CPU time [s]	2.43×10^{-3}	4.68×10^{-3}	6.23×10^{-3}	7.38×10^{-3}	1.03×10^{-2}

We can observe from this table that the gap between the upper and lower bounds is monotonically decreasing and converges to 0 by taking the truncation parameter N larger, as clarified in Theorem 3.2 (as well as Theorem 3.1), although the CPU time with respect to completing Algorithm 1 is naturally increasing as the truncation parameter N becomes larger.

To put it another way, this observation clearly demonstrates that taking the arguments in Theorems 3.1 and 3.2 together with Algorithm 1 is valid for computing the l_∞-induced norm of multivariable discrete-time linear systems.

Furthermore, it would be worthwhile to note that the developed method for computing an l_∞-induced norm can be readily applied to more complex systems consisting of interconnections between a number of mass-spring-damper elements, i.e., multiple mass-spring-damper systems. These large-scale systems can be also described by multivariable discrete-time linear systems with a number of system variables. Thus, the developed computation method can be used for multiple mass-spring-damper systems regardless of their sizes, although the computational cost might increase as the number of the sizes becomes larger.

5.2. *Comparative study with SISO case*

Let us consider the single mass-spring-damper system as shown in Figure 2, where $m = 1.0$, $k = 1.5$ and $c = 0.5$. With the sampling time $T = 0.1$, its discrete-time state space model is given by

$$G_d : \begin{cases} x_{k+1} = \begin{bmatrix} 0.4343 & 0.6052 \\ -0.9078 & 0.1317 \end{bmatrix} x_k + \begin{bmatrix} 0.3772 \\ 0.6052 \end{bmatrix} u_k =: Ax_k + bu_k \\ y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k =: cx_k \end{cases} \tag{5.3}$$

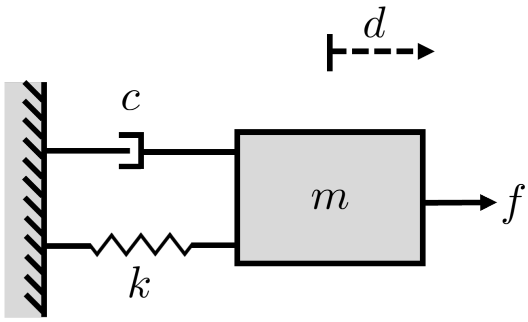

Figure 2. Single mass-spring-damper system.

For this example, we evaluate the practical effectiveness of the developed l_∞-induced norm computation method by comparing it with the conventional computation method in [26], since the structure of the conventional method is similar to that of the developed method. Simply put, the gap between an upper bound and a lower bound in [26] is described by

$$2\sum_{i=1}^{n}\sigma_i(W_o^{1/2}A^N W_c^{1/2}) - \sigma_1(W_o^{1/2}A^N W_c^{1/2}), \tag{5.4}$$

where $\sigma_i(\cdot)$ is the ith largest singular value of $(\cdot)$ and W_c and W_o are the solutions to the following discrete-time Lyapunov equations.

$$AW_cA^T - W_c + bb^T = 0, \quad A^T W_oA - W_o + c^T c = 0. \tag{5.5}$$

The results for the upper bound, the lower bound, the gap between these bounds and the required CPU time relevant to the arguments in [26] with the truncation parameter N ranging from 10 to 80 are given in Table 3, while those relevant to the developed arguments with the same truncation parameter and $L = 10$ are provided in Table 4.

Table 3. Computation results with the conventional method [26].

N	10	20	40	80
Upper bound	2.1381590837	2.0111416154	1.9987362622	1.9986440193
Lower bound	1.9329088063	1.9943273164	1.9986265009	1.9986440142
Gap	2.05250×10^{-1}	1.68143×10^{-2}	1.09761×10^{-4}	5.05627×10^{-9}
CPU time [s]	3.50×10^{-4}	1.04×10^{-3}	3.03×10^{-3}	1.24×10^{-2}

Table 4. Computation results with the developed method.

N	10	20	40	80
Upper bound	2.0772671476	2.0059689593	1.9986925232	1.9986440178
Lower bound	1.9132465024	1.9921825468	1.9986010354	1.9986440133
Gap	1.64021×10^{-1}	1.37864×10^{-2}	9.14878×10^{-5}	4.44698×10^{-9}
CPU time [s]	1.29×10^{-4}	2.02×10^{-4}	3.71×10^{-4}	7.27×10^{-4}

It can be observed from these tables that the gap in the developed method is always smaller than that in the conventional method [26] under the same truncation parameter N. More interestingly, the

CPU time required for completing the developed method is smaller than that for completing the conventional method under the same truncation parameter. To put it another way, these observations clearly demonstrate that the developed method is superior to the conventional method [26] in computing the l_∞-induced norm of SISO discrete-time systems, by showing that the former is more accurate and faster than the latter in this example. This clearly implies that the developed method is superior to the conventional method in computing the discrete-time l_∞-induced norm for this example, and thus the former could be an effective alternative to the latter even for the SISO case.

6. Conclusions

This paper developed a new approximation method for computing the l_∞-induced norm of a *multivariable* discrete-time linear system. This computation problem was reinterpreted as finding the ∞-norm of the infinite-dimensional matrix P corresponding to the input/output relation of the system. To alleviate difficulties occurring from the infinite-dimensional property, we applied a truncation idea to P, by which P is divided into the finite-dimensional truncated part P_N^- and the infinite-dimensional tail part P_N^+ with the truncation parameter $N \in \mathbb{N}_0$. An upper bound and a lower bound on the ∞-norm of the latter part (i.e., $\|P_N^+\|_\infty$) were derived in terms of the sum of a geometric sequence and the discrete-time generalized H_2 norm, respectively, while the former part (i.e., $\|P_N^-\|_\infty$) can be treated rigorously. They led to an upper bound and a lower bound on the l_∞-induced norm of the original multivariable discrete-time linear system, and the gap between these bounds was also shown to tend as 0 within an exponential order of N. Based on these arguments, we further introduced a bisection-based numerical algorithm for computing the l_∞-induced norm. Finally, some numerical studies were provided to demonstrate the overall arguments with respect to both the theoretical validity and the practical effectiveness.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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