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Research article

The Bijective N-soft set decision system

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Abstract: The characteristics of many decision-making problems in an N-soft set environment are that the parameters consist of condition and decision parameters of which each alternative is only related to one parameter, and the union of partition by parameter sets is a universe. We propose a new method called a Bijective N-soft set for handling such problems. The Bijective N-soft set is a particular case of an N-soft set. The complement of the Bijective N-soft set, the restricted AND operation on Bijective N-soft sets and the dependence between two Bijective N-soft sets are discussed. Then, the properties associated with existing operations are further examined in this paper; among other things, the complement and AND operations satisfy the closed properties. Furthermore, related to the decision system, the reduction of the Bijective N-soft set is also proposed, and an algorithm to group condition parameters that influence the decision parameter as an application in the decision-making process whose cases can be represented as a Bijective N-soft set is also given in this paper.

Keywords: N-soft sets; Bijective fuzzy soft sets; Bijective N-soft sets
Mathematics Subject Classification: 03E72, 94D05

1. Introduction

The problems of uncertainty often occur in various fields of life, such as economics, health sciences, engineering and others as a result of uncertain data. These problems can be resolved with problem-solving techniques and models using probability theory, fuzzy sets, rough sets, soft sets and other sciences.

In 1965 Prof. L. A. Zadeh [20] introduced the fuzzy set theory to solve the problem of making decisions about objects whose data contain an element of uncertainty. In fuzzy set theory, the membership values of objects are studied, provided that the membership values are in the interval $[0, 1]$. However, the fuzzy set theory also has limitations; thus, in 1999, Molodtsov [17] introduced the soft set (SS) theory to deal with decision-making problems involving more than one parameter or attribute. This soft set theory is used for grouping objects towards a certain parameter or attribute.

Parameters or attributes in the soft set can be numbers, sentences, functions and others. If an object is related to a certain parameter, it is given a value of 1; otherwise, the value is 0. However, the fact is that some problems encountered cannot always be evaluated with real numbers in the closed interval [0,1]. Consequently, some of them can also be evaluated with a rank (grade) in the form of a non-negative integer. Therefore, in 2018, Fatimah et al. [7] extended the soft set theory called N-soft set (NSS), in which objects studied are ranked according to a parameter or attribute.

At present, the soft set theory is progressing rapidly. Many real-life problems need decision-making, which can be represented as a SS, in which every alternative or object corresponds to one or more parameters. Some of the problems are called Condition and Decision of Decision Systems (C&DDS), in which there are two sets of parameters; Condition Parameter (CP) and Decision Parameter (DP). The CP relates to some SSs, while the DP is related to another. Gong et al. [9] proposed a method of decision-making problems of C&DDS, called the Bijective Soft Set (BSS), as a new type of SS. A BSS is a SS where every object is related to only one parameter, and the union of partition by sets of the parameter is a universe. On the other hand, as said before, some decision-making problems are represented as NSSs, and some may have characteristics of which every alternative can be mapped only into one parameter, and the union of partition by sets of the parameter is also a universe.

As an example, a government agency needs to know what parameters in the Condition Parameter affect the education index in the Decision Parameter of a village (alternative). Suppose there are three CPs; C_1, C_2 and C_3. C_1 states the set of parameters related to the level of health services at a_1 = clinic, a_2 = public health centre and a_3 = hospital. C_2 states the set of parameters related to the level of ease of access to education to b_1 = elementary school, b_2 = junior high school and b_3 = senior high school and C_3 states the set of parameters that categorizes the village economy, namely c_1 = family economy, c_2 = village budget and c_3 = village fund allocation. Moreover, DP is the education index in the village consisting of d_1 = high and d_2 = low. If every object corresponds to only one parameter and the union of partition by parameter sets is a universe, then Gong's method can be applied in the case of SS. However, in the case of NSS, Gong's method fails. Therefore, a method in the NSS environment should be extended in order to solve decision-making problems containing CP and DP.

Some studies on bijective soft sets and their extensions are found in [11] and [14], where the concept of bijective interval-valued fuzzy soft set and bijective soft matrix theory are introduced and some of their operations and properties are discussed and studied.

The applications of bijective soft set theory are also extended for decision-making problems. In [6], a new supervised hybrid bijective soft set neural network-based classification method is introduced to predict the Egyptian neonatal jaundice dataset. Gopal et al. [12] classifies disease data using a bijective soft set. Hong et al. [13] proposed the concept of information quantity of condition parameter set and parameter reduction of bijective soft sets based on the information quantity of every parameter. Kumar et al. [15] present a novel approach based on Bijective soft sets for generating classification rules from the data set. Kumar et al. [16] discuss the bijective-soft-set-based classification method for gene expression data of three different cancers, which are breast cancer, lung cancer and leukemia. Tiwari et al. [19] introduces a novel method to handle the qualitative description and subjective judgements of designers on design criteria to design concepts through the bijective soft sets-based method for concept selection.

Furthermore, Gong et al. in 2013, [10] also introduced a particular type of fuzzy soft set called the bijective fuzzy soft set. A bijective fuzzy soft set combines a bijective soft set and a fuzzy soft set. Moreover, research related to decision-making using the approach of the N-soft set continues to grow. Akram et al. [2] developed an application of parameter reduction of N-soft sets. Akram et al. [1] introduced a new hybrid model called fuzzy N-soft sets, investigated some useful properties of fuzzy N-soft sets, constructed fundamental operations on them and presented algorithms as potential applications of fuzzy N-soft sets in decision-making. Fatimah and Alcantud [8] introduced a novel hybrid model called the multi-fuzzy N-soft set and designed an adjustable decision-making methodology for solving problems where the inputs appear in this form. Zhang et al. [21] proposed a new hybrid model called N-soft rough sets, which can be seen as a combination of rough sets and N-soft sets.

We propose a new model in the NSS environment called Bijective N-soft set (BNSS) for solving decision-making problems containing CP and DP, of which every object corresponds to only one parameter and the union of partition by parameter sets is a universe. Based on BNSS, a complement of BNSS and the operation AND are constructed in order to study the relationship between BNSSs. Later, the Condition and Decision of the Decision Systems in the NSS environment are introduced by defining the dependent degree of two NSSs and the reduction of the Decision Systems. Furthermore, by the definitions, an algorithm to study the application of BNSS for decision-making problems are developed.

This paper is organized as follows. Section 2 recalls definitions of fuzzy sets, soft sets, fuzzy soft sets, N-soft sets, bijective soft sets and bijective fuzzy soft sets. Section 3 introduces our new model BNSSs, its complement, operations and properties. This section also presents some definitions corresponding to the Decision System in NSSs. In Section 4, an algorithm as an application of a BNSS is constructed, referring to the definitions related to the Decision System, and a numerical example illustrating a decision-making problem is presented in the BNSS information using the algorithm. In section 5, we present the comparative studies to show the potential benefit of our contribution. Lastly, section 6 concludes the paper.

2. Preliminaries

This section reviews some definitions, such as the fuzzy set (FS), soft set (SS), fuzzy soft set (FSS), N-soft set (NSS), bijective soft set (BSS) and bijective fuzzy soft set (BFSS).

2.1. Fuzzy set

The fuzzy set theory is generally employed to solve decision-making problems for objects whose data contains an element of uncertainty. The following is a definition of the fuzzy set introduced by L. A. Zadeh in 1965 [20].

Definition 2.1. [10] Suppose that U is a set of objects. A fuzzy set (FS) A over U is defined as

$$A = \{\langle u; \mu(u)\rangle \mid u \in U\},$$

where the function $\mu : U \to [0,\ 1]$ is called membership function, and $\mu(u)$ is membership value of $u \in U$ in FS A.

Example 1. Consider that $U = \{u_1, u_2, u_3\}$ is the set of several lamp brands in a shop. The fuzzy set that describes the "brightness of the lights in illuminating a room" can be defined as follows,

$$\begin{aligned} \mu(u_1) &= 0.3, \\ \mu(u_2) &= 0.7, \\ \mu(u_3) &= 0.5. \end{aligned}$$

Furthermore, it can be stated that the fuzzy set A over the set of several lamp brands in a store is

$$A = \{\langle u_1; 0.3\rangle, \langle u_2; 0.7\rangle, \langle u_3; 0.5\rangle\}.$$

2.2. Soft set

Soft set theory is used for grouping objects into a certain attribute or parameter which is expressed by the number 1 or 0. The following is the definition of soft set introduced by Molodtsov in 1999 [17].

Definition 2.2. [17] Let U be a set of objects, $P(U)$ is the power set of U and E is a set of parameters or attributes. A pair (F, E) is called a soft set (SS) over U if and only if $F\colon E \to P(U)$ is a function such that

$$(F, E) = \{(\varepsilon_i, F(\varepsilon_i)) \mid \varepsilon_i \in E, F(\varepsilon_i) \in P(U)\}.$$

For simplification, a soft set can be written as

$$(F, E) = \{(\varepsilon_i, \{(u_j, \mu_{\varepsilon_i}(u_j)) \mid u_j \in U\}) \mid \varepsilon_i \in E\}, \tag{2.1}$$

where $\mu_{\varepsilon_i}(u_j) := \begin{cases} 1, & if\ u_j \in F(\varepsilon_i), \\ 0, & others. \end{cases}$

Here, $\mu_{\varepsilon_i}(u_j)$ is called the grade or rating of the objek u_j with respect to the parameter ε_i.

By the above definition, a soft set can be represented by a table called the Representation Table of a soft set (see Table 1).

Table 1. Representation Table of Soft Set (F, E).

(F, E)	ε_1	ε_2	...	ε_i	...	ε_n
u_1	$\mu_{\varepsilon_1}(u_1)$	$\mu_{\varepsilon_2}(u_1)$	...	$\mu_{\varepsilon_i}(u_1)$	...	$\mu_{\varepsilon_n}(u_1)$
u_2	$\mu_{\varepsilon_1}(u_2)$	$\mu_{\varepsilon_2}(u_2)$	...	$\mu_{\varepsilon_i}(u_2)$	...	$\mu_{\varepsilon_n}(u_2)$
$\vdots$						
u_j	$\mu_{\varepsilon_1}(u_j)$	$\mu_{\varepsilon_2}(u_j)$	...	$\mu_{\varepsilon_i}(u_j)$	...	$\mu_{\varepsilon_n}(u_j)$
$\vdots$						
u_m	$\mu_{\varepsilon_1}(u_m)$	$\mu_{\varepsilon_2}(u_m)$	...	$\mu_{\varepsilon_i}(u_m)$	...	$\mu_{\varepsilon_n}(u_m)$

Example 2. Consider that $U = \{u_1, u_2, u_3, u_4\}$ is the set of students enrolled in a mathematics department through the SNMPTN (State University National Entrance Exam). Assume $E = \{e_1, e_2, e_3\}$ is the set of attributes where e_1, e_2 and e_3 respectively state the required qualifications "School

Accreditation of A", "Minimum Math Score of 80" and "Have Achievement in Mathematics". After conducting an assessment that groups the students above based on the three parameters, the assessment results for each student are obtained as follows. Students u_1, u_2, u_3 include students with school accreditation of A, students u_1, u_2, u_4 include students with a minimum math score of 80 and students u_2, u_3 include students who have achievements in mathematics. These results can be expressed in a soft set as follows, $F(e_1) = \{u_1, u_2, u_3\}$, $F(e_2) = \{u_1, u_2, u_4\}$, $F(e_3) = \{u_2, u_3\}$. Therefore, $(F, E) = \{(e_1, \{u_1, u_2, u_3\}), (e_2, \{u_1, u_2, u_4\}), (e_3, \{u_2, u_3\})\}$.

Table 2. Soft set.

(F, E)	e_1	e_2	e_3
u_1	1	1	0
u_2	1	1	1
u_3	1	0	1
u_4	0	1	0

Table 2 is the Representation Table of the above SS (F, E), where the value of 1 states that an element is related to an attribute, while the value of 0 states that an element is not related to an attribute set. In Table 2 it can be seen that students u_1, u_2, u_3 include students with school accreditation of A, students u_1, u_2, u_4 include students with the minimum math score of 80, and students u_2, u_3 include students with good academic achievements in mathematics.

Definition 2.3. *[17]* Suppose that (F, A) and (G, B) are two soft sets over U. The operation AND between (F, A) and (G, B), denoted by (F, A) AND (G, B), is a soft set $(H, A \times B)$=$(F, A) \wedge (G, B)$ defined by $H(a, b) = F(a) \cap G(b)$.

Example 3. Consider (F, A) and (G, B) are two soft sets over U which are represented by Tables 3 and 4, respectively. (F, A) AND (G, B) are shown in Table 5.

Table 3. Soft set (F, A).

(F, A)	a_1	a_2	a_3
u_1	1	0	1
u_2	1	1	0
u_3	1	1	1

In Table 5, the grade of every object associated with the parameter $(a, b) \in A \times B$ is obtained from the minimum grade of the object associated with parameters a and b.

2.3. Fuzzy soft set

In the fuzzy soft set theory, each object associated with a particular attribute is assigned a membership value expressed in a real number between 0 and 1. The fuzzy soft set is defined as a combination of concepts between fuzzy sets and soft sets.

Table 4. Soft set (G, B).

(G, B)	b_1	b_2
u_1	1	1
u_2	1	0
u_3	0	1

Table 5. *(F,A) AND (G,B).*

$(H, A \times B)$	(a_1, b_1)	(a_1, b_2)	(a_2, b_1)	(a_2, b_2)	(a_3, b_1)	(a_3, b_2)
u_1	1	1	0	0	1	1
u_2	1	0	1	0	0	0
u_3	0	1	0	1	0	1

Definition 2.4. [18] Let U be the set of objects, E is a set of parameters/attribute and $A \subseteq E$. A fuzzy soft set (FSS) G_A over U is a set

$$G_A = \{(a, g_A(a)) \mid a \in A, g_A(a) \in I^U\},$$

where $g_A : A \to I^U$ is a function and I^U is a collection of all fuzzy sets over U. Here,

$$g_A(a) = \{(u; \mu_a(u)) \mid u \in U\},$$

with $\mu_a : U \to [0, 1]$ is a FS over U. Therefore, $G_A = \{(a, \{(u; \mu_a(u)) \mid u \in U\}) \mid a \in A\}$.

A fuzzy soft set can be presented by a Representation Table as in Table 6.

Table 6. Representation Table of a fuzzy soft set G_A.

G_A	a_1	a_2	$\ldots$	a_i	$\ldots$	a_n
u_1	$\mu_{a_1}(u_1)$	$\mu_{a_2}(u_1)$	$\ldots$	$\mu_{a_i}(u_1)$	$\ldots$	$\mu_{a_n}(u_1)$
u_2	$\mu_{a_1}(u_2)$	$\mu_{a_2}(u_2)$	$\ldots$	$\mu_{a_i}(u_2)$	$\ldots$	$\mu_{a_n}(u_2)$
$\vdots$						
u_j	$\mu_{a_1}(u_j)$	$\mu_{a_2}(u_j)$	$\ldots$	$\mu_{a_i}(u_j)$	$\ldots$	$\mu_{a_n}(u_j)$
$\vdots$						
u_m	$\mu_{a_1}(u_m)$	$\mu_{a_2}(u_m)$	$\ldots$	$\mu_{a_i}(u_m)$	$\ldots$	$\mu_{a_n}(u_m)$

Example 4. Consider that a cosmetics shop conducts a product feasibility test before being sold with $U = \{u_1, u_2, u_3, u_4\}$ being the set of products to be sold. Assume $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ is a set of attributes and $A = \{e_1, e_2, e_3\}$ is a set of selected attributes with e_1 denoting quality product ingredients, e_2 denoting low price, and e_3 denoting packaging that meets standards. After making observations that describe the level of quality of product materials, cheapness and packaging that meets the standards of each product above, the following results are obtained for each product:

$$G(e_1) \quad = \quad \{(u_1; 0.70), (u_2; 0.80), (u_3; 0.85), (u_4; 0.75)\},$$

$$G(e_2) = \{(u_1; 0.80), (u_2; 0.80), (u_3; 0.80), (u_4; 0.80)\},$$
$$G(e_3) = \{(u_1; 0.85), (u_2; 0.78), (u_3; 0.90), (u_4; 0.80)\}.$$

Based on the observations above, $G(e_1)$ states the level of material quality in which the first product (u_1) is given a value of 0.70, the second product (u_2) is given a value of 0.80, the third product (u_3) is given a value of 0.85 and the fourth product (u_4) is given a value of 0.75. The same interpretation applies to the level of cheapness and packaging that meets the standards of each product. Thus, the above observations can be expressed in a fuzzy soft set G_A over U as follows.

$$G_A = \{(e_1, \{(u_1; 0.70), (u_2; 0.80), (u_3; 0.85), (u_4; 0.75)\}), (e_2, \{(u_1; 0.80), (u_2; 0.80), (u_3; 0.80), (u_4; 0.80)\}), (e_3, \{(u_1; 0.85), (u_2; 0.78), (u_3; 0.90), (u_4; 0.80)\})\}.$$

The fuzzy soft set G_A can also be written in the form of Table 7.

Table 7. Fuzzy soft set G_A.

U\A	e_1	e_2	e_3
u_1	0.70	0.80	0.85
u_2	0.80	0.80	0.78
u_3	0.85	0.80	0.90
u_4	0.75	0.80	0.80

2.4. N-soft set

In the N-soft set theory, each object is given a rating or grade as a non-negative integer number that meets a particular attribute. The following is a definition of the N-soft set introduced by Fatimah, et al. in 2018 [7], which is an expansion of the soft set concept.

Definition 2.5. [7] Suppose that U is a set of objects, E is a set of parameters and $A \subseteq E$. Let $R = \{0, 1, 2, \ldots, N-1\}$ be the set of grades or ratings where $N \in \{2, 3, 4, ...\}$. Triple (F, A, N) is called an N-soft set (NSS) over U, if $F : A \to 2^{U \times R}$ is a function where for each $a_i \in A$, there is an unique $(u_j,\ r_{a_i}(u_j)) \in U \times R$ such that $(u_j,\ r_{a_i}(u_j)) \in F(a_i), u_j \in U, r_{a_i}(u_j) \in R$. An NSS can be expressed as

$$(F, A, N) = \{(a_i, F(a_i)) \mid a_i \in A\} = \{(a_i, \{(u_j, r_{a_i}(u_j)) \mid u_j \in U\}) \mid a_i \in A\}$$

with $F(a_i) = \{(u_j, r_{a_i}(u_j)) \mid u_j \in U\}$. For $(u_j, r_{a_i}(u_j)) \in F(a_i)$, we may write $F(a_i)(u_j) = r_{a_i}(u_j)$.

The set of attributes used in the N-soft set is a benchmark or reference in determining the rating of each object. N-soft sets can also be expressed in a table called the Representation Table of an N-soft set (Table 8).

Table 8. Representation Table of an N-soft set (F, A, N).

(F,A,N)	a_1	a_2	$\dots$	a_q
u_1	$r_{a_1}(u_1)$	$r_{a_2}(u_1)$	$\dots$	$r_{a_q}(u_1)$
u_2	$r_{a_1}(u_2)$	$r_{a_2}(u_2)$	$\dots$	$r_{a_q}(u_2)$
$\vdots$				
$\vdots$				
u_p	$r_{a_1}(u_p)$	$r_{a_2}(u_p)$	$\dots$	$r_{a_q}(u_p)$

Example 5. Consider $U = \{u_1, u_2, u_3, u_4\}$ is the set of objects of water that is suitable for drinking, where u_1, u_2, u_3, u_4 respectively represent well water, water from Municipal Waterworks (PDAM), rainwater and water from a spring. Then assume $A \subseteq E$ is a set of attributes in the form of institutions that test the quality of the four types of water above as fit for drinking, where $A = \{a_1 =$ Provincial Health Office, $a_2 =$ Department of Chemistry, $a_3 =$ Department of Public Health, $a_4 =$ Department of Environmental Engineering, $a_5 =$ Perum Jasa Tirta II, $a_6 =$ Sucofindo$\}$. Furthermore, a rating will be given to each water based on the assessment predicate carried out by each institution with conditions as follows:

(1) " 5 " for the very good predicate.
(2) " 4 " for the good predicate.
(3) " 3 " for a rather good predicate.
(4) " 2 " for fair predicate.
(5) " 1 " for rather bad predicate.
(6) " 0 " for the bad predicate.

Each of these institutions fully determines the definition of rating standards. Based on the assessment results from each of these institutions, rating data for each object can be obtained, which can be expressed as the following 6-soft set.

$I(a_1) = \{(u_1, 3), (u_2, 5), (u_3, 2), (u_4, 4)\}$,
$I(a_2) = \{(u_1, 4), (u_2, 5), (u_3, 4), (u_4, 2)\}$,
$I(a_3) = \{(u_1, 4), (u_2, 3), (u_3, 3), (u_4, 3)\}$,
$I(a_4) = \{(u_1, 3), (u_2, 3), (u_3, 3), (u_4, 3)\}$,
$I(a_5) = \{(u_1, 5), (u_2, 4), (u_3, 4), (u_4, 4)\}$,
$I(a_6) = \{(u_1, 4), (u_2, 4), (u_3, 4), (u_4, 3)\}$.

The above example can be stated in the form of Table 9.

Table 9. 6-soft set.

$(I,A,6)$	a_1	a_2	a_3	a_4	a_5	a_6
u_1	3	4	4	3	5	4
u_2	5	5	3	3	4	4
u_3	2	4	3	3	4	4
u_4	4	2	3	3	4	3

In Table 9, the assessment carried out shows that by a_1 = Provincial Health Office, the first water (u_1) gets a rating of 3, the second water (u_2) gets a rating of 5, the third water (u_3) gets a rating of 2 and the fourth water (u_4) obtain a rating of 4. The same interpretation applies to the second to sixth institutions for each type of water.

Definition 2.6. [7] Let (F, A, N_1) and (G, B, N_2) be two N-soft sets over U. The operation AND between (F, A, N_1) and (G, B, N_2), denote by (F, A, N_1) AND (G, B, N_2)=$(F, A, N_1) \wedge (G, B, N_2)$ is an NSS $(H, A \times B, min(N_1, N_2))$ where, for each $c = (a, b) \in A \times B$ and $u \in U$,

$$(u, r_c(u)) \in H(c) \Leftrightarrow r_c(u) := min(r_a(u), r_b(u)), \text{for } (u, r_a(u)) \in F(a) \text{ and } (u, r_b(u)) \in G(b).$$

Example 6. Consider that $(F, A, 7)$ and $(G, B, 6)$ are two N-soft sets over U which are represented by Tables 10 and 11 respectively. $(F, A, 7)$ *AND* $(G, B, 6)$) are shown in Table 12.

Table 10. 7-soft set.

$(F, A, 7)$	a_1	a_2	a_3
u_1	4	3	1
u_2	2	6	0
u_3	1	0	5

Table 11. 6-soft set.

$(G, B, 6)$	b_1	b_2
u_1	3	5
u_2	1	4
u_3	2	0

Table 12. *(F,A,7) AND (G,B,6).*

$(H, A \times B, 6)$	(a_1, b_1)	(a_1, b_2)	(a_2, b_1)	(a_2, b_2)	(a_3, b_1)	(a_3, b_2)
u_1	3	3	1	4	3	1
u_2	1	1	0	2	4	0
u_3	1	0	2	0	0	0

In Table 12, the rating of each object associated with attribute $(a, b) \in A \times B$ is obtained from the minimum rating of the object associated with attributes a and b.

Definition 2.7. [2] Let (F_1, A, N) and (F_2, B, N) be two N-soft sets over U. The NSS (F_1, A, N) is called a subset of the N-soft set (F_2, B, N), denoted by $(F_1, A, N) \Subset (F_2, B, N)$ if

(1) $A \subseteq B$,

(2) $\forall\, a_i \in A$, and $u_j \in U, r^1_{a_i}(u_j) \leq r^2_{b_i}(u_j)$, where $(u_j, r^1_{a_i}(u_j)) \in F_1(a_i)$ and $(u_j, r^2_{b_i}(u_j)) \in F_2(b_i)$.

Definition 2.8. [7] Let (F_1, A, N) and (F_2, B, N) be two N-soft sets over U. The NSSs (F_1, A, N_1) and (F_2, B, N_2) are equal if

(1) $F_1 = F_2$,
(2) $A = B$,
(3) $N_1 = N_2$.

2.5. Bijective soft set

In the bijective soft set theory, each object is only related to one particular parameter or attribute. The following is the definition of a bijective soft set, a particular type of soft set, introduced by Gong, et al. in 2010 [9].

Definition 2.9. *[9]* Let E be a set of attributes and (F, E) is a SS over U where F is a map $F : E \to P(U)$. The SS (F, E) is called a bijective soft set (BSS) if

(1) $\bigcup_{e\in E} F(e) = U$,
(2) For two attributes $e_i, e_j \in E, e_i \neq e_j$, then $F(e_i) \cap F(e_j) = \varnothing$.

Table 13. Bijective soft set $(F, \{e_1, e_2, e_3\})$.

(F, E)	e_1	e_2	e_3
u_1	1	0	0
u_2	1	0	0
u_3	1	0	0
u_4	0	1	0
u_5	0	1	0
u_6	0	1	0
u_7	0	0	1

Example 7. Consider that $U = \{x_1, x_2, \ldots.., x_7\}$ is the set of several job applicants in a company. Assume that $E = \{e_1, e_2, e_3, e_4\}$ is a set of attributes where e_1, e_2, e_3 and e_4, respectively, state the required qualifications, namely "Responsible", "Hard Working", "Work Experience" and "Soft skills". After conducting an assessment, the job applicants are grouped based on these four parameters, and the assessment results for each job applicant are obtained as follows. Applicants x_1, x_2, x_3 include responsible applicants, applicants x_4, x_5, x_6 include hard-working applicants, the applicant x_7 includes applicants with work experience and applicants x_4, x_5, x_6, x_7 include applicants with soft skills. These assessments are represented as follows:

$$\begin{aligned} F(e_1) &= \{x_1, x_2, x_3\}, \\ F(e_2) &= \{x_4, x_5, x_6\}, \\ F(e_3) &= \{x_7\}, \\ F(e_4) &= \{x_4, x_5, x_6, x_7\}. \end{aligned}$$

Based on Definition 2.9, $(F, \{e_1, e_2, e_3\})$ and $(F, \{e_1, e_4\})$ are bijective soft sets, while $(F, \{e_1, e_2\})$ and $(F, \{e_1, e_3\})$ are not. The Representation Table of BSS $(F, \{e_1, e_2, e_3\})$ is in Table 13.

Now, we recall a theorem that the operation AND between two bijective soft sets is a bijective soft set.

Theorem 2.10. [9] *Suppose that (F, A) and (G, B) are two bijective soft sets. The operation AND between (F, A) and (G, B), denoted by $(F, A) \wedge (G, B) = (H, A \times B)$, is also a bijective soft set.*

2.6. Bijective fuzzy soft set

The following describes the definition of bijective fuzzy soft sets introduced by Gong et al. in 2013 [10]. First, the definition of λ-level soft set is given where $\lambda \in [0, 1]$ is a threshold of the membership value used to transform the fuzzy soft set into a soft set. The threshold or $\lambda \in [0, 1]$ chosen depends on the decision maker. If λ chosen results in a bijective soft set, it is included in this study; otherwise, it is not.

Definition 2.11. [10] Given a fuzzy soft set

$$F_A = \{(a, \{(u; \mu_a(u)) \mid u \in U\}) \mid a \in A\},$$

over U. Let $\lambda \in [0, 1]$. A λ-level soft set of the fuzzy soft set F_A is a soft set

$$(G, A) = \{(a, \{(u, \xi_{u_\lambda}(a)) \mid u \in U\}) \mid a \in A\},$$

where $\xi_{u_\lambda}(a) := \begin{cases} 1, & if\ \mu_a(u) \geqslant \lambda, \\ 0, & elsewhere. \end{cases}$

Here, $\xi_{u_\lambda}(a)$ is called the λ-level soft set characteristic function of the fuzzy soft set F_A.

Example 8. Given a FSS F_A as in Table 14. Suppose a decision maker wants to consider each object whose membership value is at least 0.75. Then, by choosing $\lambda = 0.75$ as a threshold, it is obtained a 0.75-level soft set of the fuzzy soft set F_A as follows (see also Table 15):

$$\begin{aligned}(G, A) =\{&(a_1, \{(u_1, 1), (u_2, 0), (u_3, 0), (u_4, 0), (u_5, 0), (u_6, 1)\}),\\ &(a_2, \{(u_1, 0), (u_2, 1), (u_3, 1), (u_4, 0), (u_5, 1), (u_6, 0)\}),\\ &(a_3, \{(u_1, 0), (u_2, 0), (u_3, 0), (u_4, 1), (u_5, 0), (u_6, 0)\})\}.\end{aligned}$$

Table 14. Fuzzy soft set F_A.

F_A	a_1	a_2	a_3
u_1	0.75	0.30	0.30
u_2	0.55	0.80	0.70
u_3	0.60	0.85	0.60
u_4	0.30	0.55	0.80
u_5	0.70	0.80	0.55
u_6	0.80	0.55	0.25

Table 15. *0.75-level soft set of F_A.*

(G,A)	a_1	a_2	a_3
u_1	1	0	0
u_2	0	1	0
u_3	0	1	0
u_4	0	0	1
u_5	0	1	0
u_6	1	0	0

The following is a definition of a fuzzy soft set called a bijective fuzzy soft set.

Definition 2.12. [10] Let A be a set of parameters or attributes. Suppose that (G, A) is a λ-level soft set of a fuzzy soft set F_A over U. F_A is called a λ-level bijective fuzzy soft set if (G, A) is a bijective soft set.

For simplicity, a λ-level bijective fuzzy soft set will be called a bijective fuzzy soft set when λ is already known.

Example 9. Based on Definition 2.9, in Example 8 (Table 15), the 0.75-level soft set of fuzzy soft set F_A qualifies as a bijective soft set because,

i. $\bigcup_{a\in A} G(a) = U$,
ii. For $a_1, a_2, a_3 \in A$,, $G(a_1) \cap G(a_2) = \emptyset$, $G(a_1) \cap G(a_3) = \emptyset$ and $G(a_2) \cap G(a_3) = \emptyset$.

Therefore, a bijective soft set (G, A) is obtained. By Definition 2.12, F_A is the 0.75-level bijective fuzzy soft set.

Definition 2.13. [10] Let F_A be a λ-level bijective fuzzy soft set over U with

$$F_A = \{(a, \{(u; \mu_a(u)) \mid u \in U\} \mid a \in A\}.$$

Suppose that (G, A) is a λ-level soft set of the fuzzy soft set F_A with

$$(G, A) = \{(a, \{(u, \xi_{u_\lambda}(a)) \mid u \in U\}) \mid a \in A\}.$$

The combined λ-level bijective fuzzy soft set of F_A denoted by H_A is defined as

$$H_A = \{(a, H(a)) \mid a \in A\},$$

with $H(a) = \{(u, \mu_a(u) \cdot \xi_{u_\lambda}(a)) \mid u \in U\}$. Clearly by definition, H_A is a fuzzy soft set.

Example 10. Referring to Example 8, then based on Definition 2.13, the combined 0.75-level bijective fuzzy soft set of F_A can be stated in Table 16.

Table 16 is the only combined 0.75-level bijective fuzzy soft set of F_A, in which the membership values for objects $u_1, u_2, u_3, u_4, u_5, u_6$ are obtained from the initial membership value owned by each object multiplied by the characteristic function on each of these objects.

Table 16. The combined 0.75-level bijective fuzzy soft set of F_A.

H_A	a_1	a_2	a_3
u_1	0.75	0	0
u_2	0	0.80	0
u_3	0	0.85	0
u_4	0	0	0.80
u_5	0	0.80	0
u_6	0.80	0	0

3. Bijective N-soft set

Considering the rating of the N-soft set as a membership value in terms of the fuzzy soft set, the idea in Definition 2.12 can be adopted and applied to N-soft sets. On this basis, the study of the bijective N-soft set is developed, including the properties derived from the bijective N-soft set concept. The following describes in detail the definition of a bijective N-soft set. First, a T-level soft set is defined where $T \in \{1, 2, ..., N-1\}$ is a threshold of the rating used to transform the N-soft set into a soft set. The threshold or $T \in \{1, 2, ..., N-1\}$ chosen depends on the decision maker. If T chosen results in a bijective soft set, it is included in this study; otherwise, it is not included.

Definition 3.1. Let $R = \{0, 1, 2, ..., N-1\}$ be a set of ratings or grades, with $N \in \{2, 3, 4, ...\}$. Given an N-soft set

$$(F, A, N) = \{(a, \{(u, r_a(u)) \mid u \in U\}) \mid a \in A\},$$

over U with $r_a(u) \in R$, and $T \in \{1, 2, ..., N-1\}$. A T-level soft set of the N-soft set (F, A, N) is defined as a soft set (G,A) with

$$(G, A) = \{(a, \{(u, \xi_{u_T}(a)) \mid u \in U\}) \mid a \in A\},$$

where $\xi_{u_T}(a) := \begin{cases} 1, & if\ r_a(u) \geqslant T, \\ 0, & elsewhere. \end{cases}$

Here, $\xi_{u_T}(a)$ is called the T-level soft set characteristic function of the N-soft set (F, A, N).

Example 11. Given a 6-soft set $(F, A, 6)$ as in Table 17.

Table 17. 6-soft set $(F, A, 6)$.

$(F, A, 6)$	a_1	a_2	a_3
u_1	5	2	1
u_2	0	3	2
u_3	1	2	4
u_4	4	2	0
u_5	3	1	1
u_6	0	5	1

Suppose a decision maker wants to consider objects with a minimum rating of three; then the threshold $T = 3$ is chosen. The 3-level soft set (G, A) of 6-soft set $(F, A, 6)$ is obtained as follows:

$$\begin{aligned}(G, A) =&\{(a_1, \{(u_1, 1), (u_2, 0), (u_3, 0), (u_4, 1), (u_5, 1), (u_6, 0)\}), (a_2, \{(u_1, 0), (u_2, 1),\\ &(u_3, 0), (u_4, 0), (u_5, 0), (u_6, 1)\}), (a_3, \{(u_1, 0), (u_2, 0), (u_3, 1), (u_4, 0), (u_5, 0),\\ &(u_6, 0)\})\}.\end{aligned}$$

Table 18 is the Representation Table of (G, A).

Table 18. 3-level soft set of $(F, A, 6)$.

(G, A)	a_1	a_2	a_3
u_1	1	0	0
u_2	0	1	0
u_3	0	0	1
u_4	1	0	0
u_5	1	0	0
u_6	0	1	0

In Table 18, for an object and particular parameter, the rating on (G, A) will have a value of 1 if the rating on $(F, A, 6)$ is greater than or equal to T. Furthermore, the rating on (G, A) will have a value of 0 if the rating on $(F, A, 6)$ is less than T.

Now, a definition of when an N-soft set is called a bijective N-soft set is introduced.

Definition 3.2. Let A be a set of parameter or attributes and (G, A) is a T-level soft set of the N-soft set (F, A, N) over U. (F, A, N) is called a T-level bijective N-soft set if (G, A) is a bijective soft set.

For simplicity, a T-level bijective N-soft set is said a bijective N-soft set if T is already known.

Example 12. Based on the Definition 2.9, 3-level soft set (G, A) of $(F, A, 6)$, as shown in Table 18, satisfies the condition as bijective soft set, because

i. $\bigcup_{a\in A} G(a) = U$,
ii. For $a_1, a_2, a_3 \in A,$, $G(a_1) \cap G(a_2) = \emptyset$, $G(a_1) \cap G(a_3) = \emptyset$ and $G(a_2) \cap G(a_3) = \emptyset$.

Therefore, (G, A) is a bijective soft set. Then, based on Definition 3.2, $(F, A, 6)$ is a 3-level bijective 6-soft set.

Definition 3.3. Let (F, A, N) be a T-level bijective N-soft set over U with

$$(F, A, N) = \{(a, \{(u, r_a(u)) \mid u \in U\}) \mid a \in A\}.$$

Let (G, A) be a T-level soft set of the N-soft set (F, A, N) with

$$(G, A) = \{(a, \{(u, \xi_{u_T}(a)) \mid u \in U\}) \mid a \in A\}.$$

The combined T-level bijective N-soft set of (F, A, N) denoted by (H, A, N) is defined as

$$(H, A, N) = \{(a, H(a)) \mid a \in A\},$$

with $H(a) = \{(u, r_a(u) \cdot \xi_{u_T}(a)) \mid u_j \in U\}$. It is clear that (H, A, N) is an N-soft set.

Example 13. For Example 11, by Definition 3.3, the combined 3-level bijective-soft set of $(F, A, 6)$ can be stated in Table 19.

Table 19. The combined 3-level bijective 6-soft set of $(F, A, 6)$.

$(H, A, 6)$	a_1	a_2	a_3
u_1	5	0	0
u_2	0	3	0
u_3	0	0	4
u_4	4	0	0
u_5	3	0	0
u_6	0	5	0

The combined 3-level bijective 6-soft set of $(F, A, 6)$ as in Table 19 is unique, as the rating of each object $u_1, u_2, u_3, u_4, u_5, u_6$ is obtained from the initial rank owned by each object multiplied by the characteristic function of each of these objects.

Definition 3.4. Let (F, A, N) be a T-level bijective N-soft set over U with

$$(F, A, N) = \{(a, \{(u, r_a(u)) \mid u \in U\}) \mid a \in A\}.$$

Let (H, A, N) be the combined T-level bijective N-soft set of (F, A, N) with

$$(H, A, N) = \{(a, \{(u, r_a(u) \cdot \xi_{u_T}(a)) \mid u \in U\}) \mid a \in A\}.$$

The rating total of (H, A, N) is defined as

$$H = \sum_{u \in U} \sum_{a \in A} r'_a(u),$$

where $r'_a(u) = r_a(u) \cdot \xi_{u_T}(a), \forall\, u \in U,\ a \in A$.

Example 14. For Table 19, the rating total of $(H, A, 6)$ is $H = \sum_{u \in U} \sum_{a \in A} r'_a(u) = 5+3+4+4+3+5 = 24$.

Definition 3.5. Let (F, A, N) be a T-level bijective N-soft set. If the T-level soft set of (F, A, N) has one attribute column in which all members have values of 1 and the other columns have values of 0, then (F, A, N) is called the vertical T-level bijective N-soft set.

Example 15. Let $(J, A, 6)$ be the 3-level bijective 6-soft set as in Table 20. Since the 3-level soft set of $(J, A, 6)$ is as in Table 21, given that each object has a value of 1 in attribute column a_1 and a value of 0 in attribute columns a_2 and a_3, thus $(J, A, 6)$ is the vertical 3-level bijective 6-soft set.

Definition 3.6. Let (F, A, N) be the T-level bijective N-soft set. Suppose that $|U|$ and $|A|$ denote the number of elements of U and A, respectively. For $|U| = |A|$, if the T-level soft set of (F, A, N) has values of 1 in the diagonal, and in the others are zero, then (F, A, N) is called the diagonal T-level bijective N-soft set.

Table 20. 3-level bijective 6-soft set.

$(J, A, 6)$	a_1	a_2	a_3
u_1	5	1	2
u_2	5	0	1
u_3	3	2	1
u_4	4	1	2
u_5	3	2	1
u_6	5	2	1

Table 21. 3-level soft set dari $(J, A, 6)$.

(J_1, A)	a_1	a_2	a_3
u_1	1	0	0
u_2	1	0	0
u_3	1	0	0
u_4	1	0	0
u_5	1	0	0
u_6	1	0	0

Example 16. Let $(F, A, 6)$ be the 3-level bijective 6-soft set as in Table 22, with $|U| = |A|$. Since 3-level soft set of $(F, A, 6)$ is as in Table 23 thus $(F, A, 6)$ is the diagonal 3-level bijective 6-soft set.

Definition 3.7. Let (F, A, N) be the T-level bijective N-soft set with

$$(F, A, N) = \{(a, \{(u, r_a(u)) \mid u \in U\}) \mid a \in A\}.$$

The complement T-level bijective N-soft set of (F, A, N) for $|A| = 2$ is the NSS (F^c, A, N) where

$$(F^c, A, N) = \{(a, F^c(a)) \mid a \in A\},$$

with $F^c(a) := \{(u, ((N - 1) - r_a(u)) \mid u \in U\}$.

Table 22. 3-level bijective 6-Soft Set.

$(F, A, 6)$	a_1	a_2	a_3	a_4
u_1	5	1	2	1
u_2	0	5	2	1
u_3	1	2	3	0
u_4	1	2	1	4

Table 23. 3-level soft set of $(F, A, 6)$.

(G, A)	a_1	a_2	a_3	a_4
u_1	1	0	0	0
u_2	0	1	0	0
u_3	0	0	1	0
u_4	0	0	0	1

Example 17. Suppose that $(F, A, 8)$ is the 4-level bijective 8-soft set with $|A| = 2$, as in Table 24. The complement 4-level bijective 8-soft set of $(F, A, 8)$ is as in Table 25. Table 25 is the only complement 4-level bijective 8-soft set of $(F, A, 8)$ given that the rating of each object is obtained from the maximum rating of 7 minus the initial rating of the object.

Table 24. The 4-level bijective 8-soft set.

$(F, A, 8)$	a_1	a_2
u_1	7	2
u_2	3	4
u_3	2	5
u_4	0	6
u_5	4	1
u_6	2	4

Table 25. The complement 4-level bijective 8-soft set of $(F, A, 8)$.

$(F^c, A, 8)$	a_1	a_2
u_1	0	5
u_2	4	3
u_3	5	2
u_4	7	1
u_5	3	6
u_6	5	3

Now, we propose a proposition, given that the complement T-level bijective N-soft set is the T-level bijective N-soft set.

Proposition 1. *Let (F, A, N) be the T-level bijective N-soft set, $|A| = 2$, $N = \{4, 6, 8, ...\}$, and $T = \frac{1}{2}N$. The complement T-level bijective N-soft set of (F, A, N), namely (F^c, A, N) is the T-level bijective N-soft set.*

Proof. Let (F, A, N) be the T-level bijective N-soft, $|A| = 2$, $N = \{4, 6, 8, ...\}$ and $T = \frac{1}{2}N$. Suppose that (F, A, N) and (F^c, A, N) are represented as in Table 26 and Table 27, respectively.

(F^c, A, N) will be the T-level bijective N-soft set, $\forall\, u_j \in U$, if $(N-1) - r_{a_1}(u_j) \geqslant T$ then $(N-1) - r_{a_2}(u_j) < T$, or conversely, if $(N-1) - r_{a_1}(u_j) < T$ then $(N-1) - r_{a_2}(u_j) \geqslant T$.

Since (F, A, N) is the T-level bijective N-soft set, if $r_{a_1}(u_j) \geqslant T$ then $r_{a_2}(u_j) < T$ or conversely, if $r_{a_1}(u_j) < T$ then $r_{a_2}(u_j) \geqslant T$.

The proposition now can be confirmed, if $r_{a_1}(u_j) \geqslant T$ then $(N-1) - r_{a_1}(u_j) < T$ and if $r_{a_2}(u_j) < T$ then $(N-1) - r_{a_2}(u_j) \geqslant T$ or conversely, if $r_{a_1}(u_j) < T$ then $(N-1) - r_{a_1}(u_j) \geqslant T$ and if $r_{a_2}(u_j) \geqslant T$ then $(N-1) - r_{a_2}(u_j) < T$.

Without of loss generality, if $r_{a_1}(u_j) \geq T$ then it will be proven that $(N-1) - r_{a_1}(u_j) < T$, and if $r_{a_2}(u_j) < T$ then it will be proved that $(N-1) - r_{a_2}(u_j) \geq T$, as follows, respectively:

(1) By the contradiction, assume that $(N-1) - r_{a_1}(u_j) \geq T$. Consider that,

$$\begin{aligned}
(N-1) - r_{a_1}(u_j) &\geq T, \\
(N-1) - r_{a_1}(u_j) &\geq \frac{1}{2}N, \\
-r_{a_1}(u_j) &\geq \frac{1}{2}N - N + 1, \\
-r_{a_1}(u_j) &\geq -\frac{1}{2}N + 1, \\
r_{a_1}(u_j) &\leq \frac{1}{2}N - 1, \\
r_{a_1}(u_j) &\leq \frac{1}{2}N - 1 < \frac{1}{2}N, \\
r_{a_1}(u_j) &< \frac{1}{2}N, \\
r_{a_1}(u_j) &< T.
\end{aligned}$$

This contradicts the hypothesis $r_{a_1}(u_j) \geq T$.

(2) Similar to (1), assume that $(N-1) - r_{a_2}(u_j) < T$. Consider that

$$\begin{aligned}
(N-1) - r_{a_2}(u_j) &< T, \\
(N-1) - r_{a_2}(u_j) &< \frac{1}{2}N, \\
-r_{a_2}(u_j) &< \frac{1}{2}N - N + 1, \\
-r_{a_2}(u_j) &< -\frac{1}{2}N + 1, \\
r_{a_2}(u_j) &> \frac{1}{2}N - 1, \\
r_{a_2}(u_j) &> T - 1.
\end{aligned}$$

Hence $T - 1 < r_{a_2}(u_j) < T$. This is impossible because T and $r_{a_2}(u_j)$ are natural numbers.

Therefore, it is proved that, $\forall\, u_j \in U$, if $(N-1) - r_{a_1}(u_j) \geqslant T$ then $(N-1) - r_{a_2}(u_j) < T$, or conversely, if $(N-1) - r_{a_1}(u_j) < T$ then $(N-1) - r_{a_2}(u_j) \geqslant T$. This means that (F^c, A, N) is the T-level bijective N-soft set. $\square$

Table 26. T-level bijective N-soft set.

(F, A, N)	a_1	a_2
u_1	$r_{a_1}(u_1)$	$r_{a_2}(u_1)$
u_2	$r_{a_1}(u_2)$	$r_{a_2}(u_2)$
$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$
u_j	$r_{a_1}(u_j)$	$r_{a_2}(u_j)$

Table 27. The complement T-level bijective N-soft set of (F, A, N).

(F^c, A, N)	a_1	a_2
u_1	$(N-1) - r_{a_1}(u_1)$	$(N-1) - r_{a_2}(u_1)$
u_2	$(N-1) - r_{a_1}(u_2)$	$(N-1) - r_{a_2}(u_2)$
$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$
u_j	$(N-1) - r_{a_1}(u_j)$	$(N-1) - r_{a_2}(u_j)$

Proposition 2. *Let* (F, A, N) *be the* T*-level bijective N-soft set,* $|A| = 2$*,* $N = \{3, 5, 7, ...\}$ *and* $T = \frac{1}{2}(N + 1)$*. The complement* T*-level bijective N-soft set of* (F, A, N)*, namely,* (F^c, A, N) *is the* $(T - 1)$*-level bijective N-soft set.*

Proof. Let (F, A, N) be the T-level bijective N-soft set, $|A| = 2$, $N = \{3, 5, 7, ...\}$ and $T = \frac{1}{2}(N + 1)$. Suppose that (F, A, N) and (F^c, A, N) are represented as in Table 26 and Table 27, respectively. The (F^c, A, N) will be the $(T - 1)$-level bijective N-soft set, $\forall\, u_j \in U$, if $(N - 1) - r_{a_1}(u_j) \geqslant T - 1$ then $(N-1)-r_{a_2}(u_j) < T-1$, or conversely, if $(N-1)-r_{a_1}(u_j) < T-1$ then $(N-1)-r_{a_2}(u_j) \geqslant T-1$. Since (F, A, N) is the T-level bijective N-soft set, if $r_{a_1}(u_j) \geqslant T$ then $r_{a_2}(u_j) < T$, or conversely, if $r_{a_1}(u_j) < T$ then $r_{a_2}(u_j) \geqslant T$.

The proposition can be proved, if $r_{a_1}(u_j) \geqslant T$ then $(N - 1) - r_{a_1}(u_j) < T - 1$ and if $r_{a_2}(u_j) < T$ then $(N - 1) - r_{a_2}(u_j) \geqslant T - 1$, or conversely, if $r_{a_1}(u_j) < T$ then $(N - 1) - r_{a_1}(u_j) \geqslant T - 1$ and if $r_{a_2}(u_j) \geqslant T$ then $(N - 1) - r_{a_2}(u_j) < T - 1$.

Without of loss generality, if $r_{a_1}(u_j) \geq T$ then it will prove that $(N - 1) - r_{a_1}(u_j) < T - 1$ and if $r_{a_2}(u_j) < T$ then it must hold that $(N - 1) - r_{a_2}(u_j) \geq T - 1$, as follows, respectively.

(1) Assume that $(N - 1) - r_{a_1}(u_j) \geq T - 1$. Let us consider

$$\begin{aligned}
(N-1) - r_{a_1}(u_j) &\geq T - 1,\\
(N-1) - r_{a_1}(u_j) &\geq \frac{1}{2}(N+1) - 1,\\
-r_{a_1}(u_j) &\geq \frac{1}{2}N + \frac{1}{2} - 1 - N + 1,\\
-r_{a_1}(u_j) &\geq -\frac{1}{2}N + \frac{1}{2},
\end{aligned}$$

$$\begin{aligned}
-r_{a_1}(u_j) &\geq \frac{1}{2}(-N+1),\\
r_{a_1}(u_j) &\leq \frac{1}{2}(N-1),\\
r_{a_1}(u_j) &\leq \frac{1}{2}(N-1) < \frac{1}{2}(N+1),\\
r_{a_1}(u_j) &< \frac{1}{2}(N+1),\\
r_{a_1}(u_j) &< T.
\end{aligned}$$

This contradicts to the hypothesis $r_{a_1}(u_j) \geq T$.

(2) Assume that $(N-1) - r_{a_2}(u_j) < T - 1$. Consider that

$$\begin{aligned}
(N-1) - r_{a_2}(u_j) &< T-1,\\
(N-1) - r_{a_2}(u_j) &< \frac{1}{2}(N+1) - 1,\\
-r_{a_2}(u_j) &< \frac{1}{2}N + \frac{1}{2} - 1 - N + 1,\\
-r_{a_2}(u_j) &< -\frac{1}{2}N + \frac{1}{2},\\
r_{a_2}(u_j) &> \frac{1}{2}N - \frac{1}{2},\\
r_{a_2}(u_j) &> \frac{1}{2}(N-1),\\
r_{a_2}(u_j) &> \frac{1}{2}(N+1-2),\\
r_{a_2}(u_j) &> \frac{1}{2}(N+1) - 1,\\
r_{a_2}(u_j) &> T-1.
\end{aligned}$$

Therefore, $T - 1 < r_{a_2}(u_j) < T$. This is a contradiction.

Hence, it is proved that (F^c, A, N) is $(T-1)$-level bijective N-soft set, because $\forall\ u_j \in U$, if $(N-1) - r_{a_1}(u_j) \geqslant T-1$ then $(N-1) - r_{a_2}(u_j) < T-1$, or conversely, if $(N-1) - r_{a_1}(u_j) < T-1$ then $(N-1) - r_{a_2}(u_j) \geqslant T-1$. □

Afterward, a theorem is given, that the operation AND between two T-level bijective N-soft sets is the T-level bijective N-soft set.

Theorem 3.8. *Suppose that (F, A, N_1) and (G, B, N_2) are two T-level bijective N_i-soft sets. Then $(F, A, N_1) \wedge (G, B, N_2) = (H, A \times B, min(N_1, N_2))$ is also the T-level bijective $min(N_1, N_2)$-soft set.*

Proof. Let (F, A, N_1) and (G, B, N_2) be two bijective N-soft sets as in Table 28 and Table 29, respectively. In these tables, each cell or entry has rating of $0, 1, ...,$ or $N-1$. Let T be a threshold. The following are several cases of each possible rating for each object:

(i) If $r_{a_i}(u_j) \geq T$ and $s_{b_i}(u_j) \geq T$, then by Definition 3.1 $\xi_{u_T}(a_i) = \xi_{u_T}(b_i) = 1$. Therefore, $min(\xi_{u_T}(a_i), \xi_{u_T}(b_i)) = min(1, 1) = 1$.

(ii) If $r_{a_i}(u_j) \geq T$ and $s_{b_i}(u_j) < T$, then $\xi_{u_T}(a_i) = 1$ and $\xi_{u_T}(b_i) = 0$. Hence, $min(\xi_{u_T}(a_i), \xi_{u_T}(b_i)) = min(1, 0) = 0$.

(iii) if $r_{a_i}(u_j) < T$ and $s_{b_i}(u_j) \geq T$, then $\xi_{u_T}(a_i) = 0$ and $\xi_{u_T}(b_i) = 1$. Hence, $min(\xi_{u_T}(a_i), \xi_{u_T}(b_i)) = min(0, 1) = 0$.

(iv) If $r_{a_i}(u_j) < T$ and $s_{b_i}(u_j) < T$, then $\xi_{u_T}(a_i) = \xi_{u_T}(b_i) = 0$, $min(\xi_{u_T}(a_i), \xi_{u_T}(b_i)) = min(0, 0) = 0$.

Based on Definition 3.2, $(F, A, N_1) \wedge (G, B, N_2) = (H, A \times B, min(N_1, N_2))$ will be the T-level bijective N-soft set, if T-level soft of $(H, A \times B, min(N_1, N_2))$, namely $(H_1, A \times B)$, is the bijective soft set (i.e., for each object associated with all parameters in $A \times B$ has exactly one parameter having a value of 1 and the others are 0).

Let (F_1, A) and (G_1, B) are two T-level soft sets of (F, A, N_1) and (G, B, N_2), respectively. Since (F, A, N_1) and (G, B, N_2) are two T-level bijective N-soft set, then by Definition 3.2 T-level soft set of (F, A, N_1) and (G, B, N_2), namely (F_1, A) and (G_1, B), respectively, are bijective soft sets. Hence, by Theorem 2.10, $(F_1, A) \wedge (G_1, B) = (H_1, A \times B)$ is the bijective soft set. Since $(H_1, A \times B)$ is the bijective soft set and T-level soft set of $(H, A \times B, min(N_1, N_2))$, then by Definition 3.2, $(H, A \times B, min(N_1, N_2))$ is the T-level bijective $min(N_1, N_2)$-soft set. □

Table 28. Representation of the bijective N-soft set (F, A, N_1).

(F, A, N_1)	a_1	a_2	...	a_i
u_1	$r_{a_1}(u_1)$	$r_{a_2}(u_1)$	...	$r_{a_i}(u_1)$
u_2	$r_{a_1}(u_2)$	$r_{a_2}(u_2)$	...	$r_{a_i}(u_2)$
:				
:				
u_j	$r_{a_1}(u_j)$	$r_{a_2}(u_j)$	...	$r_{a_i}(u_j)$

Table 29. Representation of the Bijective N-soft set (G, B, N_2).

(G, B, N_2)	b_1	b_2	...	b_i
u_1	$s_{b_1}(u_1)$	$s_{b_2}(u_1)$	...	$s_{b_i}(u_1)$
u_2	$s_{b_1}(u_2)$	$s_{b_2}(u_2)$	...	$s_{b_i}(u_2)$
:				
:				
u_j	$s_{b_1}(u_j)$	$s_{b_2}(u_j)$	...	$s_{b_i}(u_j)$

Definition 3.9. Let U be a set of objects and $2^{U \times R}$ is the set of all subsets of $U \times R$. Suppose that (G, B, N_2) and (F, A, N_1) are two T-level bijective N-soft sets over U, and (H, A, N_1) is the combined T-level bijective N-soft set of (F, A, N_1). The operation Restricted AND between (H, A, N_1) and $G(b)$ for some $b \in B$, denoted by $(H, A, N_1) \wedge_R G(b)$, is defined as

$$\bigcup_{a \in A} \{H(a) \mid H(a) \Subset G(b)\}$$

with $H(a) = \{(u, r_a(u) \cdot \xi_{u_T}(a)) \mid u \in U\}$.

Example 18. From Example 13 (see Table 19), $(H, A, 6)$ is the combined 4-level bijective 6-soft set of $(F, A, 6)$, with $H(a_1) = \{(u_1, 5), (u_4, 4), (u_5, 3)\}$, $H(a_2) = \{(u_2, 3), (u_6, 5)\}$, $H(a_3) = \{(u_3, 4)\}$, (here, the objects with a rating of 0 are hidden). Let $G(b) = \{(u_1, 6), (u_3, 5), (u_4, 6), (u_5, 4)\}$. Based on Definition 2.7, $H(a_1)$ and $H(a_3)$ are subsets of $G(b)$. Meanwhile, $H(a_2)$ is not a subset of $G(b)$. Hence, by Definition 3.9, $(H, A, 6) \wedge_R G(b) = \{(u_1, 5), (u_3, 4), (u_4, 4), (u_5, 3)\}$.

Definition 3.10. Let U be a set of objects. Suppose that (F, A, N) is the collections of n T-level bijective N-soft sets over U (F_i, A_i, N_i), denoted by $\bigwedge_{i=1}^{n}(F_i, A_i, N_i)$, $i = 1, ..., n$, where $\forall\ i \neq j$, $A_i \cap A_j = \emptyset, j = 1, 2, ...n$. Given the T-level bijective N-soft set over U, (G, B, N), with $B \cap A_i = \emptyset$. $((F, A, N), (G, B, N), U)_T$ is called a Decision System of T-level bijective N-soft set over U, where (F, A, N) and (G, B, N) is called the condition and decision of Decision System of T-level bijective N-soft set over U, respectively.

Definition 3.11. Suppose that (F, A, N_1) and (G, B, N_2) are two combined T-level bijective N-soft sets over U of (F', A, N_1) and (G', B, N_2),respectively, with $A \cap B = \emptyset$. Let $(J, C, N_3) = \bigcup_{b \in B}((F, A, N_1) \wedge_R G(b))$ with $C = B$. (F, A, N_1) is said to depend on (G, B, N_2) by a degree α, $0 \leq \alpha \leq 1$, denoted by $\gamma((F, A, N_1), (G, B, N_2))$ if

$$\alpha = \tfrac{H_C}{H_A},$$

where $H_C = \sum_{u \in U} \sum_{c \in C} r_c(u)$ and $H_A = \sum_{u \in U} \sum_{a \in A} r_a(u)$, for $(u, r_c(u)) \in J(c)$ and $(u, r_a(u)) \in F(a)$.

If $\alpha = 1$ then (F, A, N_1) is called to be completely dependent on (G, B, N_2). If $\alpha = 0$ then (F, A, N_1) is called to be independent of (G, B, N_2).

Example 19. Suppose that $(\bigwedge_{i=1}^{2}(F_i, A_i, N_i), (F_3, A_3, N_3), U)_4$ is a Decision System of 4-level bijective N-soft set. Subsequently, given the combined 4-level bijective N-soft sets $(F_1, A_1, 7)$,$(F_2, A_2, 8)$ and $(F_3, A_3, 8)$, respectively, as follows:

$$\begin{aligned}
(F_1, A_1, 7) &= \{(a_1, \{(u_1, 5), (u_4, 4)\}), (a_2, \{(u_2, 6), (u_5, 4), (u_6, 4)\}), (a_3, \{(u_3, 5)\})\},\\
(F_2, A_2, 8) &= \{(b_1, \{(u_1, 6), (u_3, 5), (u_4, 7), (u_5, 5)\}), (b_2, \{(u_2, 7), (u_6, 6)\})\},\\
(F_3, A_3, 8) &= \{(c_1, \{(u_2, 7), (u_3, 6), (u_5, 5), (u_6, 6)\}), (c_2, \{(u_1, 6), (u_4, 5)\})\}.
\end{aligned}$$

All bijective N-soft sets above are represented in Table 30, Table 31 and Table 32, respectively.

Table 30. Combined 4-level bijective 7-soft set.

$(F_1, A_1, 7)$	a_1	a_2	a_3
u_1	5	0	0
u_2	0	6	0
u_3	0	0	5
u_4	4	0	0
u_5	0	4	0
u_6	0	4	0

Table 31. Combined 4-level bijective 8-soft set.

$(F_2, A_2, 8)$	b_1	b_2
u_1	6	0
u_2	0	7
u_3	5	0
u_4	7	0
u_5	5	0
u_6	0	6

Table 32. Combined 4-level bijective 8-soft set.

$(F_3, A_3, 8)$	c_1	c_2
u_1	0	6
u_2	7	0
u_3	6	0
u_4	0	5
u_5	5	0
u_6	6	0

The dependent degree α of a condition 4-level bijective N-soft set on a decision 4-level bijective N-soft set now can be determined.

(1) $\gamma((F_1, A_1, 7), (F_3, A_3, 8))$.
By Definition 2.7, $F_1(a_2)$ and $F_1(a_3)$ are subsets of $F_3(c_1)$. Therefore, $(F_1, A_1, 7) \wedge_R F_3(c_1) = \{(u_2, 6), (u_3, 5), (u_5, 4), (u_6, 4)\}$. $F_1(a_1)$ is the subset of $F_3(c_2)$. So, $(F_1, A_1, 7) \wedge_R F_3(c_2) = \{(u_1, 5), (u_4, 4)\}$. It is obtained

$$(J, C, 7) = \bigcup_{c \in A_3} (F_1, A_1, 7) \wedge_R F_3(c) = \{(u_1, 5), (u_2, 6), (u_3, 5), (u_4, 4), (u_5, 4), (u_6, 4)\}.$$

By Definition 3.11, $\gamma((F_1, A_1, 7), (F_3, A_3, 8))$ is

$$\alpha = \frac{(5 + 6 + 5 + 4 + 4 + 4)}{(5 + 4 + 6 + 4 + 4 + 5)} = \frac{28}{28} = 1.$$

(2) $\gamma((F_2, A_2, 8), (F_3, A_3, 8))$.
$F_2(b_1)$ and $F_2(b_2)$ are not subsets of $F_3(c_1)$. So, $(F_2, A_2, 8) \wedge_R F_3(c_1) = \varnothing$. $F_2(b_1)$ and $F_2(b_2)$ are not also subsets of $F_3(c_2)$, such that $(F_2, A_2, 8) \wedge_R F_3(c_2) = \varnothing$. It is obtained

$$(J, C, N_4) = \bigcup_{c \in A_3} (F_2, A_2, 8) \wedge_R F_3(c) = \varnothing.$$

$\gamma((F_2, A_2, 8), (F_3, A_3, 8))$ is obtained as follows

$$\alpha = \frac{(0)}{(6 + 5 + 7 + 5 + 7 + 6)} = \frac{0}{36} = 0.$$

(3) $\gamma((F_1, A_1, 7) \wedge (F_2, A_2, 8), (F_3, A_3, 8))$.
Suppose that $(F, A_1 \times A_2, 7)$ is a $(F_1, A_1, 7)$ AND $(F_2, A_2, 8)$, as in Table 33.

Table 33. $(F_1, A_1, 7)$ *AND* $(F_2, A_2, 8)$.

$(F, A_1 \times A_2, 7)$	(a_1, b_1)	(a_1, b_2)	(a_2, b_1)	(a_2, b_2)	(a_3, b_1)	(a_3, b_2)
u_1	5	0	0	0	0	0
u_2	0	0	0	6	0	0
u_3	0	0	0	0	5	0
u_4	4	0	0	0	0	0
u_5	0	0	4	0	0	0
u_6	0	0	0	4	0	0

From Table 33 it is obtained
$(F, A_1 \times A_2, 7) = \{((a_1, b_1), \{(u_1, 5), (u_4, 4)\}), ((a_2, b_1), \{(u_5, 4)\}), ((a_2, b_2), \{(u_2, 6), (u_6, 4)\}), ((a_3, b_1), \{(u_3, 5)\})\}$.
Since $(F_3, A_3, 8) = \{(c_1, \{(u_2, 7), (u_3, 6), (u_5, 5), (u_6, 6)\}), (c_2, \{(u_1, 6), (u_4, 5)\})\}$, we obtain
$(F, A_1 \times A_2, 7) \wedge_R F_3(c_1) = \{(u_5, 4), (u_2, 6), (u_6, 4), (u_3, 5)\}$, and
$(F, A_1 \times A_2, 7) \wedge_R F_3(c_2) = \{(u_1, 5), (u_4, 4)\}$.
We get

$$(J, C, 7) = \bigcup_{c \in A_3} (F, A_1 \times A_2, 7) \wedge_R F_3(c) = \{(u_1, 5), (u_2, 6), (u_3, 5), (u_4, 4), (u_5, 4), (u_6, 4)\}.$$

Finally, $\gamma((F_1, A_1, 7) \wedge (F_2, A_2, 8), (F_3, A_3, 8))$ is obtained as follows

$$\alpha = \frac{(5 + 6 + 5 + 4 + 4 + 4)}{(5 + 4 + 4 + 6 + 4 + 5)} = \frac{28}{28} = 1.$$

Definition 3.12. Suppose that $((F, A, N), (G, B, N), U))_T$ is the Decision System of T-level bijective N-soft set, in which (F, A, N) is a collection of n T-level bijective N-soft sets over U, denoted by $\bigwedge_{i=1}^{n} (F_i, A_i, N_i)$, $i = 1, ..., n$. $\gamma(\bigwedge_{i=1}^{n} (F_i, A_i, N_i), (G, B, N))$ is the dependent degree of $(F_1, A_1, N_1) \wedge ... \wedge (F_n, A_n, N_n)$ on (G, B, N). If

$$\gamma(\bigwedge_{i=1}^{m} (F_{\sigma(i)}, A_{\sigma(i)}, N_{\sigma(i)}), (G, B, N)) = \gamma(\bigwedge_{i=1}^{n} (F_i, A_i, N_i), (G, B, N)),$$

with σ is the permutation of $\{1, ..., n\}$, then $\bigwedge_{i=1}^{m} (F_{\sigma(i)}, A_{\sigma(i)}, N_{\sigma(i)})$ is called the reduction of Decision System of T-level bijective N-soft set $((F, A, N), (G, B, N), U))_T$.

Example 20. From Example 19, we obtained that $\gamma((F_1, A_1, 7), (F_3, A_3, 8)) = \gamma((F_1, A_1, 7) \wedge (F_2, A_2, 8), (F_3, A_3, 8)) = 1$. By Definition 3.12, we get that $(F_1, A_1, 8)$ is the reduction of Decision System of 4-level bijective N-soft set $(\bigwedge_{i=1}^{2} (F_i, A_i, N_i), (F_3, A_3, N_3), U)_4$.

4. An application of Bijective N-soft set

In this section, a decision-making problem algorithm will be given in order to apply the concept of a T-level bijective N-soft set using the previously given definitions.

Algorithm:

(1) Define a Decision System of T-level bijective N-soft set $(\bigwedge_{i=1}^{n}(F_i, A_i, N_i), (F_{n+1}, A_{n+1}, N_{n+1}), U)_T$.
(2) Calculate the dependent degree of each operation AND between $\bigwedge_{i=1}^{n}(F_i, A_i, N_i)$ and $(F_{n+1}, A_{n+1}, N_{n+1})$, $0 < i \leq n$, using Definition 3.11.
(3) By Definition 3.12, find the reduction of Decision System of T-level bijective N-soft set $(\bigwedge_{i=1}^{n}(F_i, A_i, N_i), (F_{n+1}, A_{n+1}, N_{n+1}), U)_T$.
(4) Refer to the reduction of Decision System of T-level bijective N-soft set $(\bigwedge_{i=1}^{n}(F_i, A_i, N_i), (F_{n+1}, A_{n+1}, N_{n+1}), U)_T$ above, we get a grouping of attributes that produce a decision.

The following is an illustrative example of a case that can be presented as a Bijective N-soft set, which is the decision-making process using the above algorithm. In this example, the technicalities and accuracy of obtaining data are not explained as this requires independent and in-depth research involving experts in the studied field.

Example 21. A government agency researches villages to determine the attributes affecting the education index of a village. Suppose $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ is the set of villages in an area. Suppose $A = A_1 \cup A_2 \cup A_3 \cup A_4$ is a union of four sets of attributes A_1, A_2, A_3 and A_4. A_1 states the set of parameters related to the level of health services at a_1 = clinic, a_2 = a public health center, and a_3 = hospital. A_2 states the set of parameters related to ease of access to education to b_1 = elementary school, b_2 = junior high school, and b_3 = senior high school. A_3 which categorizes the village economy states the set of parameters c_1 = family economy, c_2= village budget, and c_3 = village fund allocation. A_4 which categorizes the education index in the village represents the set of parameters d_1 = high and d_2 = low. A rating later is given to each village and parameter at A_1, A_2 and A_3 based on the assessment predicate defined by the expert with the following conditions:

(1) “ 5 ” for the very good predicate.
(2) “ 4 ” for the good predicate.
(3) “ 3 ” for the not good predicate.
(4) “ 2 ” for the bad predicate.
(5) “ 1 ” for the very bad predicate.

Related to the A_4, a rating is given to each village for each parameter in A_4 based on the rating predicate defined by the expert with the conditions:
a) The village education index (x) is categorized as high if the village education index is greater than 0.5 by category:

(1) “ 5 ” for index $0.9 < x \leqslant 1.0$.
(2) “ 4 ” for index $0.8 < x \leqslant 0.9$.
(3) “ 3 ” for index $0.7 < x \leqslant 0.8$.
(4) “ 2 ” for index $0.6 < x \leqslant 0.7$.
(5) “ 1 ” for index $0.5 < x \leqslant 0.6$.

b) The village education index (x) is categorized as low if the village education index is less than 0.5 with the category:

(1) " 5 " for index $0.0 < x \leqslant 0.1$.
(2) " 4 " for index $0.1 < x \leqslant 0.2$.
(3) " 3 " for index $0.2 < x \leqslant 0.3$.
(4) " 2 " for index $0.3 < x \leqslant 0.4$.
(5) " 1 " for index $0.4 < x \leqslant 0.5$.

Based on the expert's assessment results, rating data for each object for each of the existing parameters are obtained and be stated as a 3-level bijective N-soft set presented in Tables 34–37.

Next, the combined 3-level bijective N-soft set of $(F_1, A_1, 6), (F_2, A_2, 6), (F_3, A_3, 6)$ and $(F_4, A_4, 5)$ are as in Table 38, Table 39, Table 40 and Table 41, respectively.

Table 34. 3-level bijective 6-soft set $(F_1, A_1, 6)$.

$(F_1, A_1, 6)$	a_1	a_2	a_3
u_1	2	5	2
u_2	1	2	3
u_3	5	2	2
u_4	2	1	3
u_5	2	4	2
u_6	2	2	3

Table 35. 3-level bijective 6-Soft Set $(F_2, A_2, 6)$.

$(F_2, A_2, 6)$	b_1	b_2	b_3
u_1	3	2	1
u_2	2	2	5
u_3	2	4	2
u_4	2	3	2
u_5	2	2	4
u_6	3	2	2

Table 36. 3-level bijective 6-Soft Set $(F_3, A_3, 6)$.

$(F_3, A_3, 6)$	c_1	c_2	c_3
u_1	3	1	2
u_2	1	4	2
u_3	3	2	2
u_4	2	1	5
u_5	4	2	2
u_6	3	1	2

Table 37. 3-level bijective 5-soft set $(F_4, A_4, 5)$.

$(F_4, A_4, 5)$	d_1	d_2
u_1	0	3
u_2	4	0
u_3	0	4
u_4	4	0
u_5	3	0
u_6	0	3

Table 38. The combined 3-level bijective 6-soft set $(F_1, A_1, 6)$.

$(F_1, A_1, 6)$	a_1	a_2	a_3
u_1	0	5	0
u_2	0	0	3
u_3	5	0	0
u_4	0	0	3
u_5	0	4	0
u_6	0	0	3

Table 39. The combined 3-level bijective 6-Soft Set $(F_2, A_2, 6)$.

$(F_2, A_2, 6)$	b_1	b_2	b_3
u_1	3	0	0
u_2	0	0	5
u_3	0	4	0
u_4	0	3	0
u_5	0	0	4
u_6	3	0	0

Table 40. The combined 3-level bijective 6-Soft Set $(F_3, A_3, 6)$.

$(F_3, A_3, 6)$	c_1	c_2	c_3
u_1	3	0	0
u_2	0	4	0
u_3	3	0	0
u_4	0	0	5
u_5	4	0	0
u_6	3	0	0

Table 41. The combined 3-level bijective 5-soft set $(F_4, A_4, 5)$.

$(F_4, A_4, 5)$	d_1	d_2
u_1	0	3
u_2	4	0
u_3	0	4
u_4	4	0
u_5	3	0
u_6	0	3

By the Algorithm (step 1), $(\bigwedge_{i=1}^{3}(F_i, A_i, N_i), (F_4, A_4, 5), U)_3$ is constructed as a Decision System 3-level bijective N-soft set, in which $\bigwedge_{i=1}^{3}(F_i, A_i, N_i)$ is the condition of the Decision System 3-level bijective N-soft set and $(F_4, A_4, 5)$ is the decision of Decision System 3-level bijective 5-soft set. By the second step, the dependent degree of each the operation AND between the combination of $\bigwedge_{i=1}^{3}(F_i, A_i, N_i)$ and $(F_4, A_4, 5)$ is calculated as in Table 42.

Table 42. The dependent degree of each the operation AND between the combination of $\bigwedge_{i=1}^{3}(F_i, A_i, N_i)$ and $(F_4, A_4, 5)$.

$\gamma(\wedge_{j=0}^{3}(F_j, A_j, N_j), (F_4, A_4, 5))$	α
$\gamma((F_1, A_1, 6), (F_4, A_4, 5))$	0
$\gamma((F_2, A_2, 6), (F_4, A_4, 5))$	0.2727
$\gamma((F_3, A_3, 6), (F_4, A_4, 5))$	0.1904
$\gamma((F_1, A_1, 6) \wedge (F_2, A_2, 6), (F_4, A_4, 5))$	0.8
$\gamma((F_1, A_1, 6) \wedge (F_3, A_3, 6), (F_4, A_4, 5))$	0.7058
$\gamma((F_2, A_2, 6) \wedge (F_3, A_3, 6), (F_4, A_4, 5))$	1
$\gamma((F_1, A_1, 6) \wedge (F_2, A_2, 6) \wedge (F_3, A_3, 6), (F_4, A_4, 5))$	1

From Table 42,

$$\gamma((F_2, A_2, 6) \wedge (F_3, A_3, 6), (F_4, A_4, 5)) = \gamma((F_1, A_1, 6) \wedge (F_2, A_2, 6) \wedge (F_3, A_3, 6), (F_4, A_4, 5)) = 1$$

is obtained. Therefore, $\{(F_2, A_2, 6),(F_3, A_3, 6)\}$ is the reduction of Decision System 3-level bijective 5-soft set $(\bigwedge_{i=1}^{3}(F_i, A_i, N_i), (F_4, A_4, 5), U)_3$. Thus the attributes affecting the village education index are the ease of education access to schools and the village economy.

Moreover, from the reduced attributes, the grouping of attributes that produces a decision is obtained, as shown in Tables 39–41. The objects corresponding to the parameters in A_2, A_3 and A_4, which have a non-zero rating, are later noted. In this case, based on existing data, it is found that the parameters influencing the decision parameter A_4 are parameters A_2 and A_3. This is based on the interpretation of the results of the last step of the algorithm above, as obtained from an overview of the conditions of each village, with the following details:

(1) If the ease of access to Elementary School and the family economy are not good; therefore, the education index is low.

(2) If the ease of access to Senior High School is very good and the village budget is good; therefore, the education index is high.
(3) If the ease of access to Junior High School is good and the family economy is not good; therefore, the education index is low.
(4) If the ease of access to Junior High School is not good and the allocation of village funds is very good; therefore, the education index is high.
(5) If the ease of access to Senior High School and the family economy are good; therefore, the education index is high.
(6) If the ease of access to Elementary school is not good and the family economy is not good; therefore, the education index is low.

From the illustrative example in the case above, an illustration obtained is that the algorithm defined in the last section can be used as an alternative method to determine which parameters from a set of conditional parameters influence the decision parameters in a Decision System bijective N-soft set. Therefore, this algorithm can be used in cases of decision-making problems requiring decisions like in the illustration above.

5. Comparative studies

By 2023, there have been many studies on N-Soft sets theory and their application for decision-making problems, such as in [2–5]. However, the scope of applications of such research is without considering condition and decision NSSs. This means that the decision-making problems in the NSS environment to identify which parameters in condition NSSs impact parameters in decision NSS are inadequate to be handled only by existing studies. The conditions imposed in Bijective N-soft sets are vital in enhancing their suitability. Bijective N-soft sets are designed to manage situations where non-binary evaluations are expected, making them adept at capturing and handling complex evaluations in decision-making scenarios, especially for solving decision-making problems containing Condition Parameters and Decision Parameters, of which every object corresponds to only one parameter and the union of partition by parameter sets is a universe. The success of the research on Bijective NSS indicates the potential benefits of our contribution. On the other hand, BNSS is the generalization of BSS, where for N=2, the concept of BNSS (F, A, N) is the same as that of BSS.

6. Conclusions

Bijective N-Soft set (BNSS) is a concept that can be applied to solve decision-making problems consisting of Condition N-Soft Sets and a Decision N-Soft Set, in which existing studies on N-Soft Set fail to be applied. We found that the Bijective N-soft set is a new model to handle such decision-making problems. Then, the complement of the Bijective N-soft set is also obtained in the concept of the Bijective N-soft set. We have defined the restricted AND operation on the Bijective N-soft sets, the dependence between two Bijective N-soft sets and the reduction of the Bijective N-soft set associated with the Decision System Bijective N-soft set. In addition, in the Bijective N-soft set, it is proved that the properties associated with the previously defined operations include the closed property of the complement and the AND operation. In the final part of this article, an algorithm that can be used as an alternative method in solving decision-making problems in Bijective N-soft sets environment has been defined. Using the algorithm, we can determine which parameters from a set of conditional parameters influence the decision parameters in a Decision System bijective N-soft set. Despite the fact that not all Condition and Decision N-Soft Sets (CDNSS) can be solved with BNSS, by defining a threshold T that makes CDNSS a BNSS, the concept of BNSS can be applied. As the concepts of NSS such as Fuzzy NSS, Hesitant NSS, Hesitant Fuzzy NSS and Intuitionistic Fuzzy NSS have been progressing rapidly, research studies on the issue of BNSS combined with them are yet to be explored.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of Interest

The authors declare that there are no conflicts of interest.

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