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Research article

Conformable finite element method for conformable fractional partial differential equations

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Abstract: The finite element (FE) method is a widely used numerical technique for approximating solutions to various problems in different fields such as thermal diffusion, mechanics of continuous media, electromagnetism and multi-physics problems. Recently, there has been growing interest among researchers in the application of fractional derivatives. In this paper, we present a generalization of the FE method known as the conformable finite element method, which is specifically designed to solve conformable fractional partial differential equations (CF-PDE). We introduce the basis functions that are used to approximate the solution of CF-PDE and provide error estimation techniques. Furthermore, we provide an illustrative example to demonstrate the effectiveness of the proposed method. This work serves as a starting point for tackling more complex problems involving fractional derivatives.

Keywords: CCF-PDE; Galerkin method; conformable finite element (CFE) method
Mathematics Subject Classification: 26A33, 34A12

1. Introduction

The FE method has proven to be a valuable tool in engineering and scientific applications, encompassing various fields such as fluid mechanics, weather prediction, petroleum reservoir simulation and ocean circulation modeling. It has been extensively utilized in research and practical studies, as evidenced by the works cited in [1–5]. The FE method provides a flexible and efficient approach to numerically solving complex problems in these domains, offering reliable and accurate results. Its versatility and wide range of applications make it a popular choice for researchers and practitioners in engineering and science.

The popularity of fractional derivatives (FDs) has grown significantly due to their ability to provide more accurate models for real-world phenomena compared to classical integer derivatives. Various definitions of FDs have been proposed by researchers to cater to specific applications. These include the Grunwald-Letnikow, Riemann-Liouville and Caputo derivatives [6], as well as the Caputo-Fabrizio [7], Atangana-Baleanu [8] and Caputo-Hadamard [9] derivatives. These FDs have found applications in diverse fields such as engineering [10], biology [11], economics [12], chemistry [13], psychology [14] and many others. However, it is worth noting that most of these FDs do not satisfy all the classical properties associated with integer derivatives, such as the chain rule, quotient rule and product rule.

The fractional order derivative has always been an interesting research topic in the theory of functional space for many years [15–23]. Various types of FDs were introduced, among which the following Riemann-Liouville and Caputo are the most widely used ones:

The Riemann-Liouville FD of order α is:

$$D^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}x(s)ds. \tag{1.1}$$

The Caputo FD of order α is:

$$^{C}D^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}x^{'}(s)ds. \tag{1.2}$$

Both the Riemann-Liouville definition and the Caputo definition are defined via fractional integrals. Therefore, these two FDs inherit some nonlocal behaviors including historical memory and future dependence. All definitions including (1.1) and (1.2) above satisfy the property that the FD is linear. This is the only property inherited from the 1st derivative. However, the existing FDs do not satisfy the following properties which the integral derivatives have. In order to address these challenges, Khalil et al. proposed an alternative FD known as the conformable derivative (CD) [24], as defined in Definition 1. Unlike the previously mentioned FDs, the CD exhibits compatibility with integer derivatives and shares a common set of fundamental properties with them. This unique characteristic of the CD distinguishes it from other FDs. Further properties and investigations related to the CD can be found in studies such as [25, 26]. The CD has attracted significant attention from numerous researchers, leading to diverse extensions and applications. For instance, Naifar et al. [27] investigated the stability of nonlinear conformable systems, K"utahyalioglu et al. [28] examined a class of Hopfield neural networks, and Hammouch et al. [29] studied the global stability of the equilibria in a mathematical model for the Ebola virus. Zhao et al. [30] explored the stability problem for conformable autonomous linear systems. Furthermore, the stability of conformable Lotka-Volterra systems was examined in our study, and Rebiai et al. [31] investigated stability properties for various classes of systems, including perturbed systems and nonlinear conformable equations with control. In the field of control theory, several researchers have focused on fundamental concepts and classical results related to dynamical systems described by the conformable derivative. These include studies on finite-dimensional systems, such as controllability and observability [32–34], stability analysis [35–40], the conformable linear-quadratic problem [32] and investigations on infinite-dimensional systems [41, 42]. Additionally, Pedram [43] provided numerical solutions to initial boundary value problems involving the perturbed conformable time Korteweg-de Vries equation.

The objective of this mathematical model is to find a function $x : [0, 1] \longrightarrow \mathbb{R}$ that satisfies the conformable problem stated as follows:

$$-C^{\alpha}(C^{\alpha}x)(t) + c(t)x(t) = f(t), \ \ \frac{1}{2} < \alpha \leq 1, \tag{1.3}$$

where $C^{\alpha}x(t)$ represents the conformable derivative (CD) of function x with order α. The problem is accompanied by the boundary conditions:

$$x(0) = x_0, \quad x(1) = x_f, \tag{1.4}$$

where c and f are given functions defined on the interval $[0, 1]$, which are associated with the material properties of the wire and external forces, respectively. This problem is commonly referred to as a "boundary problem" since it involves an equation (in this case, an ordinary differential equation) within the domain (in this case, the interval $]0, 1[$), along with conditions imposed at the boundaries of the domain (at 0 and 1), which are known as the "boundary conditions" of the problem. With this context in mind, we are motivated to investigate the generalization of the FE method to tackle this problem.

In this paper, we present a generalization of the FE method known as the conformable finite element method, which is specifically designed to solve CF-PDE. We introduce the basis functions that are used to approximate the solution of CF-PDE and provide error estimation techniques. The rest of the paper is organized as follows: In Section 2, some necessary definitions of conformable derivative and its proprieties are presented; the conformable variational formulation is introduced in Section 3; the conformable finite element method is introduced in Section 4; effective calculation of the approximate solution and basis functions in Section 5; some theorems for the error analysis of the method presented in Section 6; we give test example, to illustrate the application steps of the method in Section 7; finally, in Section 8, we present the discussion.

2. Conformable derivative and its proprieties

Definition 1. *[24] Let the function $x : [0, +\infty[\rightarrow \mathbb{R}$. The conformable derivative of function x order $\alpha \in]0, 1]$ is defined by:*

$$C^{\alpha}x(t) = \lim_{\theta \to 0} \frac{x\left(t + \theta t^{1-\alpha}\right) - x(t)}{\theta}, \tag{2.1}$$

for all $t > 0$. If $\lim\limits_{t \to 0^+} C^{\alpha}x(t)$ exists, then define $C^{\alpha}x(0) = \lim_{t \to 0^+} C^{\alpha}x(t)$.

Definition 2. *Let $0 < \alpha \leq 1$ and an interval $[a, b]$ define*

$$[a, b]^{\alpha} = \left\{ x : [a, b] \rightarrow \mathbb{R} / \int_a^b \frac{x(\tau)}{\tau^{1-\alpha}} d\tau < \infty \right\}.$$

Theorem 1. *[24] Let $\alpha \in]0, 1]$ defined conformable integral by*

$$I_{\alpha}(x)(t) = \int_0^t x(\tau) d\alpha(\tau), \ \ x \in [0, t]^{\alpha},$$

with $d\alpha(\tau) = \frac{d\tau}{\tau^{1-\alpha}}$, *then*

$$C^{\alpha}(I_{\alpha}(x)(t)) = x(t). \tag{2.2}$$

Theorem 2. *[25] Let* $x, y : [a,b] \to \mathbb{R}$ *be two functions such that* xy *is differentiable and* $0 < \alpha \leq 1$. *Then*

$$\int_a^b (C_a^{\alpha} x)(t) y(t) d\alpha(t,a) = x(t)y(t)|_a^b - \int_a^b x(t) (C_a^{\alpha} y)(t) d\alpha(t,a),$$

with $d\alpha(\tau, a) = \frac{d\tau}{(\tau-a)^{1-\alpha}}$.

Let $1 \leq p < \infty$, we denote

$$L_{\alpha}^{p}(]0,1[) := \{x :]0,1[\to \mathbb{R} \backslash \int_0^1 |x(t)|^p d\alpha(t) < \infty\},$$

this space is a Banach space, i.e. a complete normed vector space, endowed with the norm:

$$\|x\|_{L_{\alpha}^{p}(]0,1[)} = \left(\int_0^1 |x(t)|^p d\alpha(t) \right)^{1/p}.$$

For the particular case of $p = 2$, the space $L_{\alpha}^{2}(]0,1[)$ is a Hilbert space [44, 45] for the scalar product defined by:

$$(x, y)_{L_{\alpha}^{2}(]0,1[)} = \int_0^1 x(t) y(t) d\alpha(t).$$

Definition 3. *Let the space* $H^{1,\alpha}(]0,1[)$ *defined by:*

$$H^{1,\alpha}(]0,1[) = \left\{ x \in L_{\alpha}^{2}(]0,1[) / \ C^{\alpha}(x) \in L_{\alpha}^{2}(]0,1[) \right\}.$$

We endow this space with the following scalar product:

$$(x, y)_{H^{1,\alpha}(]0,1[)} = (x, y)_{L_{\alpha}^{2}(]0,1[)} + (C^{\alpha}(x), C^{\alpha}(y))_{L_{\alpha}^{2}(]0,1[)}.$$

The corresponding standard is:

$$\|x\|_{H^{1,\alpha}(]0,1[)} = \sqrt{(x, x)_{H^{1,\alpha}(]0,1[)}} = \sqrt{\|x\|^2_{L_{\alpha}^{2}(]0,1[)} + \|C^{\alpha}(x)\|^2_{L_{\alpha}^{2}(]0,1[)}}.$$

Theorem 3. *The space* $H^{1,\alpha}(]0,1[)$ *is a Hilbert space.*

Proof. It suffices to show that the space $H^{1,\alpha}(]0,1[)$ is complete. Let $(y_m)_{m\in\mathbb{N}}$ be a Cauchy sequence in the space $H^{1,\alpha}(]0,1[)$; so: the sequence $(y_m)_{m\in\mathbb{N}}$ is Cauchy in the space $L_{\alpha}^{2}(]0,1[)$ and the sequence $\left(C^{\alpha}(y_m)\right)_{m\in\mathbb{N}}$ is Cauchy in $L_{\alpha}^{2}(]0,1[)$.

The space $L_{\alpha}^{2}(]0,1[)$ being complete in [44], we deduce that there exists a function $w \in L_{\alpha}^{2}(]0,1[)$ and functions z such that:

(1) $(y_m)_{m\in\mathbb{N}}$ converges to w in the space $L_{\alpha}^{2}(]0,1[)$;

(2) the sequence $\left(C^{\alpha}(y_m)\right)_{m\in\mathbb{N}}$ converges to z in $L_{\alpha}^{2}(]0,1[)$.

Let us first show that $z = C^{\alpha}(w)$, we deduce from point 1) above that $(y_m)_{m\in\mathbb{N}}$ converges to w in $\mathcal{D}'(]0,1[)$ and also that $\left(\frac{\partial y_m}{\partial x_i}\right)_{m\in\mathbb{N}}$ converges to $\frac{\partial w}{\partial x_i}$ in $\mathcal{D}'(]0,1[)$. Similarly, we deduce from point 2) above that $\left(\frac{\partial y_m}{\partial x_i}\right)_{m\in\mathbb{N}}$ converges to w_i in $\mathcal{D}'(]0,1[)$. By uniqueness of the limit in this space, we then have that $\frac{\partial w}{\partial x_i} = w_i$. This result is established, we deduce that each of the first partial derivatives $\frac{\partial w}{\partial x_i}$ is in $L^2_{\alpha}(]0,1[)$, therefore w is in space $H^{1,\alpha}(]0,1[)$. Moreover, y_m converges to w in the sense of the norm $\|\cdot\|_{H^{1,\alpha}(]0,1[)}$, which ends the proof. □

3. Conformable variational formulation

Let's quickly give an idea of the strategy corresponding to the first step (this will be detailed later). The function c in Eq (1.3) is assumed to be continuous on $[0,1]$ as well as the data f. Suppose the problem has a solution x, with $x \in C^{2\alpha}([0,1])$ and $y \in C^{\alpha}([0,1])$ another function verifying the boundary conditions, i.e. $y(0) = y(1) = 0$. Multiply Eq (1.3) by y and integrate over $[0,1]$ the relation obtained; we have:

$$\int_0^1 -C^{\alpha}(C^{\alpha}x)(t)y(t)d\alpha(t) + \int_0^1 c(t)x(t)y(t)d\alpha(t) = \int_0^1 f(t)y(t)d\alpha(t).$$

Next, use integration by parts (Theorem 2) to transform the first term; with the conditions $x(0) = x(1) = 0$, we obtain a "variational form of the conformable problem" which is written:

$$\int_0^1 C^{\alpha}x(t)C^{\alpha}y(t)d\alpha(t) + \int_0^1 c(t)x(t)y(t)d\alpha(t) = \int_0^1 f(t)y(t)d\alpha(t).$$

The natural functional space to solve this "variational conformable problem" is then the following:

$$H_0^{1,\alpha}(]0,1[) = \{y :]0,1[\longrightarrow \mathbb{R}, y \in L^2_{\alpha}(]0,1[), C^{\alpha}y \in L^2_{\alpha}(]0,1[), y(0) = y(1) = 0\}.$$

We will also show that this space $H_0^{1,\alpha}(]0,1[)$ is a Hilbert for the scalar product defined by:

$$(x,y)_{H_0^{1,\alpha}(]0,1[)} = (x,y)_{L^2_{\alpha}(]0,1[)} + (C^{\alpha}x, C^{\alpha}(y))_{L^2_{\alpha}(]0,1[)}\,.$$

Let's pose

$$\mathcal{L}(x) = \int_0^1 f(t)x(t)d\alpha(t), \tag{3.1}$$

and

$$\mathcal{A}(x,y) = \int_0^1 C^{\alpha}x(t)C^{\alpha}y(t)d\alpha(t) + \int_0^1 c(t)x(t)y(t)d\alpha(t). \tag{3.2}$$

L is a continuous linear form on $H_0^{1,\alpha}(]0,1[)$. According to conformable Riesz's theorem [25], $H_0^{1,\alpha}(]0,1[)$ is identified with its topological dual $(H_0^{1,\alpha}(]0,1[))'$, which means that there exists a unique $x \in H_0^{1,\alpha}(]0,1[)$ such that, for any function y in $H_0^{1,\alpha}(]0,1[)$ we have:

$$\mathcal{A}(x,y) = \mathcal{L}(y).$$

4. Conformable finite element method

In this section, we present the principle of the CFE method: to seek the solution of an approximate variational conformable problem solved in a finite-dimensional space. Then, we describe the method in dimension one and estimate the error between the solution of the initial problem and that of the discrete problem. Consider the following general variational conformable problem:

$$\text{Find } x \in H_0^{1,\alpha}(]0,1[) \text{ such that for all } y \in H_0^{1,\alpha}(]0,1[) \text{ we have : } \mathcal{A}(x,y) = \mathcal{L}(y), \tag{4.1}$$

where $H_0^{1,\alpha}(]0,1[)$ is a Hilbert space, $\mathcal{L}$ a continuous linear form on $H_0^{1,\alpha}(]0,1[)$ and $\mathcal{A}$ a bilinear.

The basic idea consists in solving this variational conformable problem, not in the whole space $H_0^{1,\alpha}(]0,1[)$ (which is not accessible in general), but in a subspace of finite dimension, denoted V_h^α, of $H_0^{1,\alpha}(]0,1[)$ (h is a strictly positive parameter intended to tend towards 0) "approaching" the space $H_0^{1,\alpha}(]0,1[)$ in a sense to be defined: this is the principle of the "Galerkin method".

Why V_h^α of finite dimension? To have only a finite number of unknowns (or "degrees of freedom") to evaluate (which will be the components of the approximate solution in a basis of the space V_h^α) and that we can calculate easily by solving a linear system.

From a theoretical point of view, it is necessary that this number of degrees of freedom can be as large as possible, so as to approach the exact solution in the most precise way possible. In other words, we want the dimension, denoted n, of the space V_h^α to tend to $+\infty$ when h tends to 0 (for example, n is inversely proportional to h). More precisely, we will make the following assumptions on the spaces V_h^α:

Definition 4. *We say that the spaces $V_h^\alpha, h > 0$, form an internal approximation of $H_0^{1,\alpha}(]0,1[)$ if:*

(1) for all $h > 0$, $\quad V_h^\alpha \subset H_0^{1,\alpha}(]0,1[)$.
(2) for all $y \in H_0^{1,\alpha}(]0,1[)$, there exists $y_h \in V_h^\alpha$ such that

$$\|y - y_h\|_{H_0^{1,\alpha}(]0,1[)} \to 0, \text{ when } h \to 0.$$

From a practical point of view, it is also desirable that this space V_h^α be easy to construct: one can choose a space formed by proper functions of the operator associated with the form $\mathcal{A}$ (in this case, the linear system is particularly easy to solve because the matrix is diagonal), or polynomials, or piecewise polynomial functions, etc. Another important concern in the choice of this space is that of the computer storage of the matrix of the linear system: the more the matrix is hollow (i.e. contains many null elements), the less it occupies memory space.

The choice of these spaces V_h^α being made, we propose to solve the following approximate variational conformable problem:

$$\text{Find } x_h \in V_h^\alpha \text{ such that for all } y_h \in V_h^\alpha \text{ we have : } \mathcal{A}(x_h, y_h) = \mathcal{L}(y_h). \tag{4.2}$$

Note that we could also consider the case of forms $\mathcal{A}_h$ and $\mathcal{L}_h$ approaching respectively $\mathcal{A}$ and $\mathcal{L}$ in this discrete problem.

We first have the following result:

Proposition 1. *Let $H_0^{1,\alpha}(]0,1[)$ be a Hilbert space and V_h^α a finite dimensional subspace of $H_0^{1,\alpha}(]0,1[)$. We suppose the linear form $\mathcal{L}$ Eq (3.1) continuous on $H_0^{1,\alpha}(]0,1[)$, the bilinear form $\mathcal{A}$*

Eq (3.2) continuous and $H_0^{1,\alpha}(]0,1[)$-elliptic, i.e. (there exists a constant $\lambda > 0$ such that for everything $y \in H_0^{1,\alpha}(]0,1[)$, $\mathcal{A}(y,y) \geq \lambda \|y\|^2_{H_0^{1,\alpha}(]0,1[)}$), so that the variational conformable problem (4.1) admits a unique solution $x \in H_0^{1,\alpha}(]0,1[)$. The approximate variational problem (4.2) admits also a unique solution x_h in V_h^α and we also have a first estimate of the error between x and x_h in the form:

$$\|x - x_h\|_{H_0^{1,\alpha}(]0,1[)} \leq \frac{N}{\lambda} \inf_{y_h \in V_h^\alpha} \|x - y_h\|_{H_0^{1,\alpha}(]0,1[)} . \tag{4.3}$$

Proof. As V_h^α is a finite dimensional subspace of $H_0^{1,\alpha}(]0,1[)$, it is a closed subspace of $H_0^{1,\alpha}(]0,1[)$ and $H_0^{1,\alpha}(]0,1[)$ being a Hilbert, V_h^α is also a Hilbert for the space-induced norm $H_0^{1,\alpha}(]0,1[)$. From $\mathcal{A}$ est elliptique and $\mathcal{L}$ continuous on $H_0^{1,\alpha}(]0,1[)$ then the Lax-Milgram Theorem being unchanged, we deduce that the problem (4.2) admits a unique solution $x_h \in V_h^\alpha$. Moreover, like $V_h^\alpha \subset H_0^{1,\alpha}(]0,1[)$, we obtain by difference: for all $y_h \in V_h^\alpha$, $\quad \mathcal{A}(x - x_h, y_h) = 0$.

In particular, we therefore have, for any function $y_h \in V_h^\alpha$:

$$\begin{aligned}\lambda \|x - x_h\|^2_{H_0^{1,\alpha}(]0,1[)} &\leq \mathcal{A}(x - x_h, x - x_h) = \mathcal{A}(x - x_h, x - y_h) \\ &\leq N \|x - x_h\|_{H_0^{1,\alpha}(]0,1[)} \|x - y_h\|_{H_0^{1,\alpha}(]0,1[)} ,\end{aligned}$$

so we deduce trivially (4.3). □

The bilinear form $\mathcal{A}$ continuous and elliptic in the space $H_0^{1,\alpha}(]0,1[)$ is the similarity of problem in [46].

5. Effective calculation of the approximate solution and basis functions

Let us now specify the effective calculation of this approximate solution x_h. The space V_h^α being of finite dimension n, it admits a basis, denoted by $\{\phi_1^\alpha(t), ..., \phi_n^\alpha(t)\}$. Our approach involves expressing x_h as a linear combination of the elements in this basis. Thus, we look for x_h in the following form:

$$x_h(t) = \sum_{j=1}^{n} x_j \phi_j^\alpha(t). \tag{5.1}$$

We then have the following result:

Proposition 2. *The function defend in Eq (5.1) to V_h^α is a solution of the approximate variational conformable problem (4.2) if and only the vector $X \in \mathbb{R}^n$ of components x_i is a solution of the following linear system:*

$$AX = B, \tag{5.2}$$

where A is the matrix of size $n \times n$, of elements

$$A_{i,j} = \mathcal{A}(\phi_i^\alpha, \phi_j^\alpha), \quad (i,j) \in \{1, ..., n\}^2, \tag{5.3}$$

and where B is the n dimension vector of components:

$$B_i = \mathcal{L}(\phi_i^\alpha), \quad i \in \{1, ..., n\}. \tag{5.4}$$

Moreover, the matrix A is positive definite and the linear system (5.2) admits a unique solution.

Proof. The equation is linear with respect to x_h, it holds for any function $x_h \in V_h^\alpha$ if and only if it holds for each of the elements ϕ_i^α of the basis of V_h^α, which gives us, using the decomposition (5.1) of the unknown x_h and the linearity of $\mathcal{A}$ with respect to its first argument:

$$\forall i \in \{1, ..., n\}, \quad \sum_{j=1}^{n} x_j \mathcal{A}(\phi_j^\alpha, \phi_i^\alpha) = \mathcal{L}(\phi_i^\alpha).$$

The linear system thus obtained has precisely for matrix writing the Eq (5.2). Let us show that the matrix A is positive definite. Let x be a vector of $\mathbb{R}^n$ with components x_i; using the bilinearity of $\mathcal{A}$ then its $H_0^{1,\alpha}(]0,1[)$-ellipticity, it comes, noting a the constant of $(H_0^{1,\alpha}(]0,1[))'$-ellipticity of $\mathcal{A}$:

$$x^T A x = \sum_i \sum_j A_{i,j} x_i x_j = \mathcal{A}(\sum_{i=1}^{n} x_i \phi_i^\alpha, \sum_{j=1}^{n} x_j \phi_j^\alpha) \geq \lambda \parallel \sum_{i=1}^{n} x_i \phi_i^\alpha \parallel^2_{H_0^{1,\alpha}(]0,1[)} .$$

This inequality shows that $x^T A x > 0$ and that if $x^T A x = 0$, then for all i, $x_i = 0$, i.e. $x = 0$, which ends the proof. □

We partition the segment $[0, 1]$ into $n + 1$ intervals of length $h = \frac{1}{n+1}$ with n given natural integer; we have: $h \neq 0$. We denote by $t_i = ih$, for $i \in \{0, ..., n + 1\}$ the $n + 2$ points of the mesh thus defined (Figure 1). We have in particular: $t_0 = 0$ and $t_{n+1} = 1$.

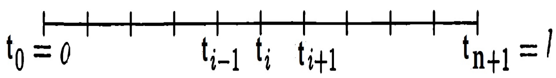

Figure 1. Uniform mesh of the segment $[0, 1]$.

Let the $\phi_i(t)$ is a conformable finite-element basis function (see Figure 2) defined on a grid of time points t_i by

$$\phi_i^\alpha(t) = \begin{cases} \frac{t^\alpha - t_{i-1}^\alpha}{t_i^\alpha - t_{i-1}^\alpha}, & t_{i-1} \leq t \leq t_i \\ \frac{t_{i+1}^\alpha - t^\alpha}{t_{i+1}^\alpha - t_i^\alpha}, & t_i \leq t \leq t_{i+1} \\ 0, & \text{otherwise} . \end{cases}$$

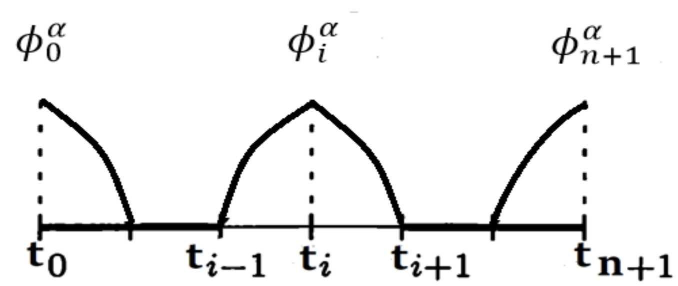

Figure 2. Function graph of ϕ_i^α.

We introduce the discrete variational space V_h^α defined by:

$$V_h^\alpha = \{\phi_0^\alpha(t), ..., \phi_{n+1}^\alpha(t)\}. \tag{5.5}$$

Proposition 3. *(1) The dimension of the space V_h^α is $n+2$ and a basis is formed of the following functions ϕ_i^α, $i \in \{0, ..., n+1\}$:*

$$\phi_i^\alpha(t_j) = \delta_{ij}, \tag{5.6}$$

with δ_{ij} is Kronecker symbol. And we have for everything $x_h \in V_h^\alpha$

$$x_h = \sum_{i=0}^{n+1} x_h(t_i)\phi_i^\alpha.$$

(2) $V_h^\alpha \subset H_0^{1,\alpha}(]0,1[)$.

Proof. From the above Eq (5.6) defines the functions $\phi_i^\alpha, i \in \{0, ..., n+1\}$ uniquely. Let us show that these functions form a free family of V_h^α. Indeed, suppose that there exist $n+2$ scalars $\mu_0, ..., \mu_{n+1}$ such that the function

$$x_h = \sum_{j=0}^{n+1} \mu_i\phi_j^\alpha,$$

is zero. We deduce a fortiori that it is zero at each of the $n+2$ points x_i, i.e. that, for all $i \in \{0, ..., n+1\}$ we have:

$$0 = x_h(t_i) = \sum_{j=0}^{n+1} \mu_j\phi_j^\alpha(t_i) = \mu_i.$$

Each of the coefficients a i is, therefore, zero, and the family is free.

Let us show that this family is generating. Let x_h be any function of V_h^α, and let ϕ be the function defined by

$$\phi = \sum_{j=0}^{n+1} x_h(t_j)\phi_j^\alpha.$$

This function is in the space V_h^α and it also coincides with x_h at each of the points t_i, since we have:

$$\phi(t_i) = \sum_{j=0}^{n+1} x_h(t_j)\phi_j^\alpha(t_i) = x_h(t_i).$$

According to the first point, it is equal to x, which shows (5.6) and the family is generating. The $n+2$ functions ϕ_i^α therefore form a basis of the space V_h^α, and this space is of dimension $n+2$. □

From Proposition 1, there exists a unique solution x_h to the discrete variational conformable problem (4.2) with V_h^α defined by (5.5), $\mathcal{A}$ and $\mathcal{L}$ defined by (3.2) and (3.1) respectively. According to the Proposition 3, this solution is of the form:

$$x_h = \sum_{j=1}^{n} x_j\phi_j^\alpha,$$

where the vector of $\mathbb{R}^n$ of components x_j is a solution of the linear systems (5.2)–(5.4). The knowledge of x_h is therefore reduced to the resolution of the linear system (5.2), of unknown $X = x_i(1 < j < n)$, with:

$$A_{i,j} = \int_0^1 C^\alpha \phi_i^\alpha(t) C^\alpha \phi_j^\alpha(t) d\alpha(t) + \int_0^1 c(t)\phi_i^\alpha(t)\phi_j^\alpha(t) d\alpha(t), \tag{5.7}$$

and

$$B_i = \int_0^1 f(t)\phi_i^\alpha(t) d\alpha(t). \tag{5.8}$$

To know x_h, it is therefore sufficient to calculate the matrix A (which is symmetric, since $\mathcal{A}$ is) and the second member B of this system, then to solve it. Let us start by calculating the matrix A, and for that let's explain the basic functions.

For $i \in \{1, ..., n\}$, the function ϕ_i^α has support in the interval $[t_i, t_{i+i}]$ (Figure 2); on its support, it is:

$$\phi_i^\alpha(t) = \begin{cases} \frac{t^\alpha - t_{i-1}^\alpha}{t_i^\alpha - t_{i-1}^\alpha}, & t_{i-1} \le t \le t_i \\ \frac{t_{i+1}^\alpha - t^\alpha}{t_{i+1}^\alpha - t_i^\alpha}, & t_i \le t \le t_{i+1} \\ 0, & \text{otherwise} . \end{cases}$$

The function ϕ_{n+1}^α has support in the interval $[t_n, 1]$ and we have:

$$\phi_{n+1}^\alpha(t) = \frac{t^\alpha - t_n^\alpha}{t_{n+1}^\alpha - t_n^\alpha}. \tag{5.9}$$

The function ϕ_0^α has support in the interval $[0, t_1]$ and we have:

$$\phi_0^\alpha(t) = \frac{-t^\alpha}{t_1^\alpha}. \tag{5.10}$$

Note that these functions can all be expressed using the two functions w_0 and w_1 defined on the interval $[0, 1]$, which is called "reference element", by:

$$w_0(t) = 1 - t, w_1(t) = t. \tag{5.11}$$

We have indeed, for each index $i \in \{1, ..., n\}$:

$$\phi_i^\alpha(t) = \begin{cases} w_1(\frac{t^\alpha - t_{i-1}^\alpha}{h_i}), & t_{i-1} \le t \le t_i \\ w_0(\frac{t^\alpha - t_i^\alpha}{h_{i+1}}), & t_i \le t \le t_{i+1}, \end{cases}$$

where $h_i = t_i^\alpha - t_{i-1}^\alpha$. Pour $i = n + 1, \phi_{n+1}(t) = w_1(\frac{t-t_n}{h_{n+1}})$ and $\phi_0(t) = w_0(\frac{t-t_0}{h_1})$.

Let us calculate the coefficients of the matrix A given by (5.7). Note that, for a given index i, there are at most three values of j for which the coefficient $A_{i,j}$ is a priori non-zero, which are: $j = i - 1$, i and $i + 1$ if $i \le n$ and $j = i - 1$ and i if $i = n + 1$; for these values indeed, the functions ϕ_i^α and ϕ_j^α have supports whose intersection is not of measure zero. The matrix is therefore tridiagonal: with the exception of the coefficients of the main diagonal (i.e. that formed by the elements and those of the two diagonals located on either side of the main diagonal, all the coefficients are zero. We also have, for $i \neq n+1$:

$$A_{i,i} = \int_{t_{i-1}}^{t_{i+1}} \frac{1}{t^{1-\alpha}} \left([C^\alpha \phi_i^\alpha]^2 + c\,[\phi_i^\alpha]^2 \right)(t)dt$$

$$A_{i,i-1} = \int_{t_{i-1}}^{t_i} \frac{1}{t^{1-\alpha}} \left(C^\alpha \phi_i^\alpha C^\alpha \phi_{i-1}^\alpha + c\phi_i^\alpha \phi_{i-1}^\alpha \right)(t)dt, \quad (i \geq 2)$$

$$A_{i,i+1} = \int_{t_i}^{t_{i+1}} \frac{1}{t^{1-\alpha}} \left(C^\alpha \phi_i^\alpha C^\alpha \phi_{i+1}^\alpha + c\phi_i^\alpha \phi_{i+1}^\alpha \right)(t)dt$$

and for $i = n+1$:

$$A_{n+1,n+1} = \int_{t_n}^{b} \frac{1}{t^{1-\alpha}} \left([C^\alpha \phi_{n+1}^\alpha]^2 + c\,[\phi_{n+1}^\alpha]^2 \right)(t)dt$$

$$A_{n+1,n} = \int_{t_n}^{b} \frac{1}{t^{1-\alpha}} \left(C^\alpha \phi_{n+1}^\alpha C^\alpha \phi_n^\alpha + c\phi_{n+1}^\alpha \phi_n^\alpha \right)(t)dt.$$

Since

$$\int_{t_{j-1}}^{t_j} \frac{1}{t^{1-\alpha}} \phi_j^\alpha(t)\phi_{j-1}^\alpha(t)dt = \int_{t_{j-1}}^{t_j} \frac{1}{t^{1-\alpha}} w_1\left(\frac{t^\alpha - t_{j-1}^\alpha}{h_j}\right) w_0\left(\frac{t^\alpha - t_{j-1}^\alpha}{h_j}\right)dt = \frac{h_j}{\alpha}\int_0^1 w_1(y)w_0(y)dy,$$

where we put $y = \frac{t^\alpha - t_{j-1}^\alpha}{h}$, $dy = \frac{\alpha}{h} t^{\alpha-1}dt$. It then comes:

$$\int_{t_{j-1}}^{t_j} \frac{1}{t^{1-\alpha}} \phi_j^\alpha(t)\phi_{j-1}^\alpha(t)dt = \frac{h_j}{\alpha}\int_0^1 (1-y)ydy = \frac{h_j}{6\alpha}.$$

$$\begin{aligned} \int_{t_j}^{t_{j+1}} \frac{1}{t^{1-\alpha}} C^\alpha \phi_j(t) C^\alpha \phi_{j+1}(t)dt &= \int_{t_j}^{t_{j+1}} \frac{1}{t^{1-\alpha}} \left(-\frac{\alpha}{h_{j+1}}\right)\left(\frac{\alpha}{h_j}\right)dt \\ &= \int_{t_j}^{t_{j+1}} \frac{1}{t^{1-\alpha}} \left(-\frac{\alpha^2}{h_j h_{j+1}}\right)dt = -\frac{\alpha}{h_j}, \end{aligned}$$

$$\begin{aligned} \int_{t_j}^{t_{j+1}} \frac{1}{t^{1-\alpha}} \phi_j(t)\phi_{j+1}(t)dt &= \int_{t_j}^{t_{j+1}} \frac{1}{t^{1-\alpha}} w_0\left(\frac{t-t_j}{h_{j+1}}\right) w_1\left(\frac{t-t_j}{h_{j+1}}\right)dt \\ &= \frac{h_{j+1}}{\alpha}\int_0^1 w_0(y)w_1(y)dy = \frac{h_{j+1}}{6\alpha}, \end{aligned}$$

$$\begin{aligned} \int_{t_{j-1}}^{t_{j+1}} \frac{1}{t^{1-\alpha}} [C^\alpha \phi_i^\alpha]^2(t)dt &= \int_{t_{j-1}}^{t_j} \frac{1}{t^{1-\alpha}} [C^\alpha \phi_i^\alpha]^2(t)dt + \int_{t_j}^{t_{j+1}} \frac{1}{t^{1-\alpha}} [C^\alpha \phi_i^\alpha]^2(t)dt \\ &= \frac{\alpha}{h_j} + \frac{\alpha}{h_{j+1}}, \end{aligned}$$

$$\int_{t_{j-1}}^{t_{j+1}} \frac{1}{t^{1-\alpha}} \left[\phi_j^\alpha\right]^2(t)dt = \int_{t_{j-1}}^{t_j} \frac{1}{t^{1-\alpha}} \left[\phi_j^\alpha\right]^2(t)dt + \int_{t_j}^{t_{j+1}} \frac{1}{t^{1-\alpha}} \left[\phi_j^\alpha\right]^2(t)dt$$

$$
\begin{aligned}
&= \int_{t_{j-1}}^{t_j} \frac{1}{t^{1-\alpha}} w_1\Big(\frac{t^\alpha - t_{j-1}^\alpha}{h_j}\Big)^2 dt + \int_{t_j}^{t_{j+1}} \frac{1}{t^{1-\alpha}} w_0\Big(\frac{t^\alpha - t_j^\alpha}{h_{j+1}}\Big)^2 dt \\
&= \frac{h_j}{\alpha}\int_0^1 w_1(y)^2 dy + \frac{h_{j+1}}{\alpha}\int_0^1 w_0(y)^2 dy \\
&= \frac{h_j}{3\alpha} + \frac{h_{j+1}}{3\alpha},
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{t_{j-1}}^{t_j} \frac{1}{t^{1-\alpha}} C^\alpha \phi_j^\alpha(t) C^\alpha \phi_{j-1}^\alpha(t) dt &= \int_{t_{j-1}}^{t_j} \frac{1}{t^{1-\alpha}} \Big(\frac{\alpha}{h_j}\Big)\Big(-\frac{\alpha}{h_{j+1}}\Big) dt \\
&= \int_{t_{j-1}}^{t_j} \frac{1}{t^{1-\alpha}} \Big(-\frac{\alpha^2}{h_j h_{j+1}}\Big) dt \\
&= -\frac{\alpha}{h_{j+1}}.
\end{aligned}
$$

Similarly, we obtain, for $i \neq n+1$:

$$
A_{i,i} = \frac{\alpha}{h_j} + \frac{\alpha}{h_{j+1}} + c\Big(\frac{h_j}{3\alpha} + \frac{h_{j+1}}{3\alpha}\Big), \quad A_{i,i-1} = -\frac{\alpha}{h_{j+1}} + c\Big(\frac{h_j}{6\alpha}\Big),
$$

$$
A_{i,i+1} = -\frac{\alpha}{h_j} + c\frac{h_{j+1}}{6\alpha}
$$

and for $i = n+1$:

$$
A_{n+1,n+1} = \frac{\alpha}{h_{n+1}} + c\frac{h_{n+1}}{3\alpha}, \quad A_{n,n+1} = -\frac{\alpha}{h_n} + c\frac{h_{n+1}}{6\alpha},
$$

all other coefficients being zero.

6. Error estimate

We wish to specify the calculation of the error $\| x - x_h \|_{H_0^{1,\alpha}(]0,1[)}$ between the exact solution x of the conformable problem Eq (4.1) and the solution x_h of the variational conformable problem approximated (4.2) with V_h^α defined by (5.5). For simplicity, we will assume c and f continue on $[0,1]$, so that $x \in C^{2\alpha}([0,1])$.

Proposition 4. *Let x be the solution of the variational conformable problems (4.1), (3.2), (3.1) and x_h that of the approximate variational conformable problem (4.2) with V_h^α defined by (5.5). We suppose c and f continuous on $[0,1]$, so that $x \in C^{2\alpha}([0,1])$. Then there exists a constant $K > 0$ (depending on the second CD of x) such that*

$$
\| x - x_h \|_{H_0^{1,\alpha}(]0,1[)} \leq Kh. \tag{6.1}
$$

Proof. By Proposition 1, it suffices to evaluate the error $\| x - x_h \|_{H_0^{1,\alpha}(]0,1[)}$ for a particular y_h of V_h^α. Let y_h choose for element the interpolation of x at each of the nodes of the mesh, i.e. y_h is the function of V_h^α which coincides with x at each of the t_i for $i \in \{1, ..., n+1\}$, and also naturally at $t_0 = 0$, since $y_h(0) = x(0)$ and $y_h(1) = x(1)$; this function is usually denoted $\check{x}_h$. We have:

$$
\| x - \check{x}_h \|^2_{H_0^{1,\alpha}(]0,1[)} = \| C^\alpha x - C^\alpha \check{x}_h \|^2_{L^2_\alpha(]0,1[)} = \sum_{i=0}^{n} \int_{t_i}^{t_{i+1}} |x - \check{x}_h|^2(t) d\alpha(t). \tag{6.2}
$$

Let $w = (x - \breve{x}_h)_{[t_i,t_{i+1}]}$; we have $w \in C^{2\alpha}(]t_i, t_{i+1}[)$ and by construction: $w(t_i) = w(t_{i+1}) = 0$; according to conformable Rolle's theorem [25], we deduce that there exists $\eta \in]t_i, t_{i+1}[$ such that $C^\alpha w(\eta) = 0$. We therefore have, on the interval $]t_i, t_{i+1}[$:

$$C^\alpha w(t) = \int_c^t C^{2\alpha} w(s) d\alpha(s) = \int_c^t C^{2\alpha} x(s) d\alpha(s),$$

so that: $|C^\alpha w(t)| < h \sup_{s\in[0,1]} |C^{2\alpha} x(s)|$. We then deduce:

$$\| C^\alpha w \|^2_{L^2_\alpha(]t_i,t_{i+1}[)} \leq h^3 (\sup_{s\in[0,1]} |C^{2\alpha} x(s)|)^2.$$

Transferring this estimate to (6.2), it comes:

$$\| x - \breve{x}_h \|^2_{H_0^{1,\alpha}(]0,1[)} \leq (n+1) h^3 (\sup_{s\in[0,1]} |C^{2\alpha} x(s)|)^2 \leq h^2 (\sup_{s\in[0,1]} |C^{2\alpha} x(s)|)^2,$$

since $h(n+1) = 1$. Finally using (4.3), we deduce:

$$\| x - x_h \|_{H_0^{1,\alpha}(]0,1[)} \leq \frac{N}{\lambda} h \sup_{s\in[0,1]} |C^{2\alpha} x(s)|,$$

therefore

$$\| x - x_h \|_{H_0^{1,\alpha}(]0,1[)} \leq Kh,$$

with $K = \frac{N}{\lambda} \sup_{s\in[0,1]} |C^{2\alpha} x(s)|$. □

7. Example

In this section, we present the numerical example illustrating the CFF method with MATLAB R2020b. We consider the CF-PDE:

$$-C^\alpha (C^\alpha x)(t) + 6x(t) = f(t), \quad \frac{1}{2} < \alpha \leq 1, \tag{7.1}$$

where $f(t) = -2\alpha + 6(\frac{t^{2\alpha}}{\alpha} - \frac{t^\alpha}{\alpha})$ and initial conditions

$$x(0) = 0, x(1) = 0. \tag{7.2}$$

The solution exact is:

$$x(t) = \frac{t^{2\alpha}}{\alpha} - \frac{t^\alpha}{\alpha}.$$

Let $n = 10$, then $V_h^\alpha = \{\phi_0^\alpha(t), \phi_1^\alpha(t), ..., \phi_{10}^\alpha(t), \phi_{11}^\alpha(t)\}$, from Eq (4.2) so the solution of Eq (7.1) is:

$$x_h(t) = \sum_{j=0}^{11} x_j \phi_j^\alpha(t),$$

such that for all $k \in \{1, ..., 11\}$ we have:

$$\mathcal{A}(x_h, \phi_k^\alpha) = L(\phi_k^\alpha). \tag{7.3}$$

Then

$$\sum_{j=0}^{11} x_j \mathcal{A}(\phi_j^{\alpha}(t), \phi_k^{\alpha}) = L(\phi_k^{\alpha}), \tag{7.4}$$

from Eq (7.2), we have $x_0 = x_{11} = 0$, we suppose $A = [\mathcal{A}(\phi_j^{\alpha}(t), \phi_k^{\alpha})]_{i,k}$, $B = [L(\phi_k^{\alpha})]$ and $X = [x_j]_{1\leq j\leq 10}$ so

$$AX = B, \tag{7.5}$$

so for $\alpha = 0.7$ we have

$$X = \begin{pmatrix} -0.2162 \\ -0.3014 \\ -0.3434 \\ -0.3569 \\ -0.3489 \\ -0.3232 \\ -0.2824 \\ -0.2285 \\ -0.1627 \\ -0.0863 \end{pmatrix},$$

for $\alpha = 0.8$ we have

$$X = \begin{pmatrix} -0.1559 \\ -0.2376 \\ -0.2858 \\ -0.3091 \\ -0.3117 \\ -0.2963 \\ -0.2648 \\ -0.2184 \\ -0.1583 \\ -0.0853 \end{pmatrix},$$

for $\alpha = 0.9$ we have

$$X = \begin{pmatrix} -0.1136 \\ -0.1882 \\ -0.2383 \\ -0.2677 \\ -0.2783 \\ -0.2714 \\ -0.2479 \\ -0.2084 \\ -0.1537 \\ -0.0841 \end{pmatrix},$$

for $\alpha = 1$ we have

$$X = \begin{pmatrix} -0.0829 \\ -0.1492 \\ -0.1989 \\ -0.2320 \\ -0.2486 \\ -0.2486 \\ -0.2320 \\ -0.1989 \\ -0.1492 \\ -0.0829 \end{pmatrix}.$$

In Figure 3, we plot the exact solution and the approximation solution for $n = 10$ and different values of α.

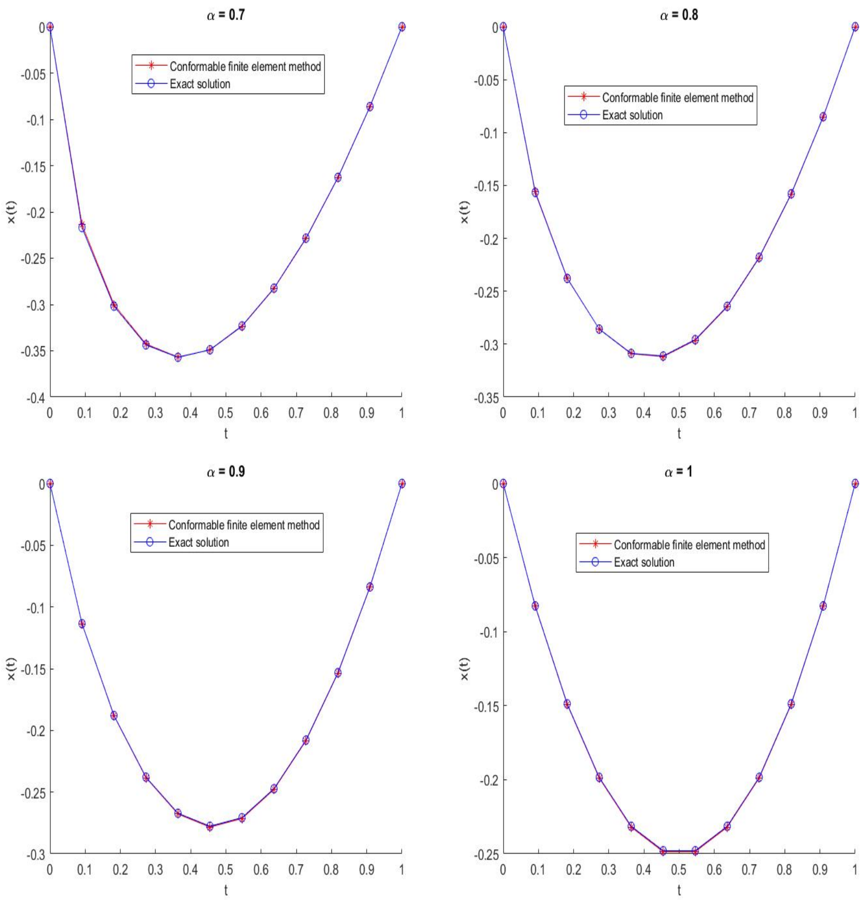

Figure 3. The exact solution and the approximation solution for $n = 10$ and different values of α.

In Figure 4, we plot the absolute error for $n = 10$ and different values of α.

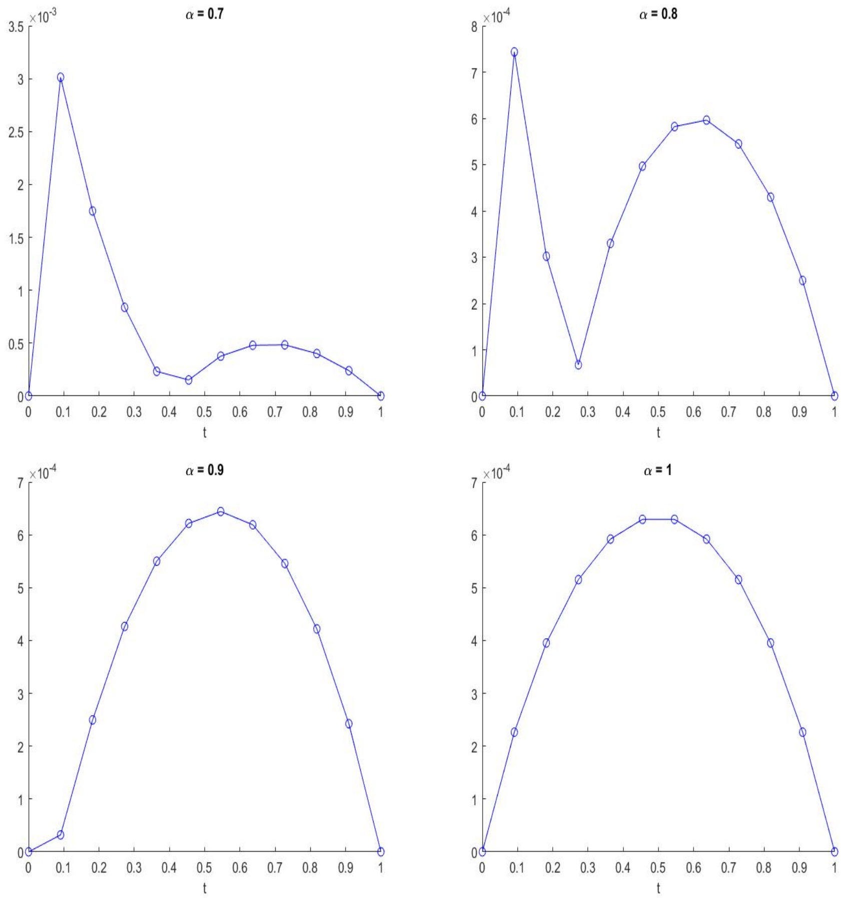

Figure 4. The absolute error for $n = 10$ and different values of α.

8. Discussion

- The approximate solutions of (7.1) and (7.1) are identical to the exact solution for $\alpha = 0.7$ $\alpha = 0.8$ and $\alpha = 0.9$ with $n = 10$.
- If the boundary conditions are non-zero, we put the variable change, and we get a problem with zero boundary conditions.

9. Conclusions

We focus on extending the FE method to handle CF-PDEs and introduce the conformable finite element method as a generalized approach to solving CF-PDEs. Furthermore, we provide the basis functions used for approximating the solution of CF-PDEs and discuss methods for estimating the error.

This study marks the starting point for further exploration of more intricate fractional differential problems, building upon the foundation established by the generalized FE method. This study also serves as a fundamental stepping stone towards the exploration of more intricate fractional differential problems. The introduction of the conformable finite element Method lays the groundwork for tackling a broader spectrum of fractional differential equations that exhibit complex behavior. As the understanding and utilization of fractional calculus continue to grow, our generalized FE method stands poised to foster further advancements in this domain. Through this research, we aim to catalyze the exploration of diverse applications and foster innovative solutions to challenging fractional differential problems. In our future work, we intend to solve conformable fractional partial differential equations extended to high-dimensional cases such as the advection–diffusion–reaction problem [47], Stokes Problems [48] and elliptic boundary value problems with mixed boundary conditions [49] when generalized to FD.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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