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Research article

Problems concerning sharp coefficient functionals of bounded turning functions

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Abstract: The work presented in this article has been motivated by the recent research going on the Hankel determinant bounds and their related consequences, as well as the techniques used previously by many different authors. We aim to establish a new subfamily of holomorphic functions connected with the hyperbolic tangent function with bounded boundary rotation. We investigate the sharp estimate of the third Hankel determinant for this newly defined family of functions. Moreover, for the defined functions family, the Krushkal inequality, the first four initial sharp bounds of the logarithmic coefficients and the sharp second Hankel determinant of the logarithmic coefficients are given.

Keywords: holomorphic functions; Hankel determinant; subordination; hyperbolic function
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1. Introduction

By $\mathcal{A}$, we denote an analytic (regular) function's family and g defined in the following region:

$$\mathbf{D} = \{\varepsilon \in \mathbb{C} \text{ and } |\varepsilon| < 1\}, \tag{1.1}$$

with $g(0) = 0 = g'(0) - 1$. Thus, every function g of a family $\mathcal{A}$ is of the form:

$$g(\varepsilon) = \varepsilon + \sum_{k=2}^{\infty} a_k \varepsilon^k \quad \varepsilon \in \mathbf{D}. \tag{1.2}$$

Moreover, let $\mathcal{S}$ indicates a subfamily of $\mathcal{A}$, whose members are univalent in $\mathbf{D}$. Let $h_1, h_2 \in \mathcal{A}$; we state that the function h_1 is subordinate to h_2 (written as $h_1 \prec h_2$) if there exists a regular function u that satisfies

$$|u(\varepsilon)| \leq |\varepsilon| \quad \text{and} \quad u(0) = 0, \tag{1.3}$$

such that $h_1(\varepsilon) = h_2(u(\varepsilon))$ for $\varepsilon \in \mathbf{D}$. Moreover, if $h_2 \in \mathcal{S}$, then the above conditions imply the follwoing:

$$h_1 \prec h_2 \Leftrightarrow h_1(0) = h_2(0) \text{ and } h_1(\mathbf{D}) \subset h_2(\mathbf{D}). \tag{1.4}$$

In 1992, Ma and Minda [1] utilized the idea of subordination and initiated the family $\Lambda^*(\Omega)$ as follows:

$$\Lambda^*(\Omega) = \left\{ g \in \mathcal{A} : \frac{\varepsilon g'(\varepsilon)}{g(\varepsilon)} \prec \Omega(\varepsilon) \right\}, \tag{1.5}$$

where the image of Ω under $\mathbf{D}$ is a star-shaped functions satisfying that $\Omega(0) = 1$ and $\Omega'(0) > 0$. Also, they investigated various beautiful geometric results like growth, distortion and covering results. If we pick $\Omega(\varepsilon) = (1+\varepsilon)/(1-\varepsilon)$ categorically, then the family $\Lambda^*(\Omega)$ reduces to the family of functions whose image domain is star-shaped. For the numerous choices of $\Omega(\varepsilon)$ on the right hand side of (1.5), we get various subfamilies of $\mathcal{S}$ whose image domains have some beautiful geometrical interpretations. Among them some are recorded as follows:

1) If we take $\Omega(\varepsilon) = 1 + \sin\varepsilon$, then we obtain the family $\Lambda^*_{\sin} = \Lambda^*(1 + \sin\varepsilon)$, which is described by the functions bounded by the eight shaped region, and which was established and studied by Cho et al. [2].
2) By considering a function $\Omega(\varepsilon) = 1 + \varepsilon - \frac{1}{3}\varepsilon^3$, we get the recently investigated family $\Lambda^*_{nep} = \Lambda^*\left(1 + \varepsilon - \frac{1}{3}\varepsilon^3\right)$, introduced by Wani and Swaminathan [3]. The image of the function $\Omega(\varepsilon) = 1 + \varepsilon - \frac{1}{3}\varepsilon^3$ under an open unit disc is bounded by a nephroid shaped region.
3) The family $\Lambda^*_L = \Lambda^*\left(\sqrt{1+\varepsilon}\right)$, with $\Omega(\varepsilon) = \sqrt{1+\varepsilon}$, was established by Sokól et al. [4].
4) The family $\Lambda^*_{car} = \Lambda^*\left(1 + \frac{4}{3}\varepsilon + \frac{2}{3}\varepsilon^2\right)$ was recently investigated by Sharma et al. [5], and the image of $\Omega(\varepsilon) = 1 + \frac{4}{3}\varepsilon + \frac{2}{3}\varepsilon^2$ is cardioid shape under an open unit disc.
5) By choosing $\Omega(\varepsilon) = e^{\varepsilon}$, we get the family $\Lambda^*_{\exp} = \Lambda^*(e^{\varepsilon})$, which was established in [6]. On the other side, if we pick $\Omega(\varepsilon) = \varepsilon + \sqrt{1+\varepsilon^2}$, we get the family $\Lambda^*_{cre} = \Lambda^*\left(\varepsilon + \sqrt{1+\varepsilon^2}\right)$, which maps $\mathbf{D}$ to a crescent shaped region and was given by Raina [7].

For more particular subfamilies of the family of starlike functions, see the articles [8–10].

Pommerenke [11, 12] was the first to initiate the idea of a Hankel determinant $H_{q,n}(g)$ for a function $g \in \mathcal{S}$ of the form (1.2), where the parameters $q, n \in \mathbb{N} = \{1, 2, 3, \cdots\}$ are as follows:

$$H_{q,n}(g) = \begin{vmatrix} a_n & a_{n+1} & \ldots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \ldots & a_{n+q} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n+q-1} & a_{n+q} & \ldots & a_{n+2q-2} \end{vmatrix}. \tag{1.6}$$

For particular values, e.g., $q = 2$ and $n = 1$, we get the Hankel determinant

$$\left|H_{2,1}(g)\right| = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}$$

$$= \left|a_3 - a_2^2\right|, \text{ where } a_1 = 1.$$

And for $q = 2$ and $n = 2$, in (1.6) we get the second order Hankel determinant

$$\begin{aligned} H_{2,2}(g) &= \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} \\ &= a_2 a_4 - a_3^2. \end{aligned}$$

For the third order Hankel determinant we take $q = 3$ and $n = 1$, which yields the following form

$$\left|H_{3,1}(g)\right| = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

The functional

$$H_{2,1}(g) = a_3 - a_2^2$$

is known as the Fekete-Szego functional. For numerous subfamilies of a regular function's family $\mathcal{A}$, the best possible value of the upper bound for $\left|H_{2,1}(g)\right|$ has been evaluated by various authors (see [13–15]). Moreover, the second Hankel determinant and the extreme value has been studied and investigated by several authors from many different directions and perspectives. For instance, the readers may refer to see [16–21]. Furthermore, Babalola [22] described the Hankel determinant $H_{3,1}(g)$ for several subfamilies of regular functions. For some recent works on the third-order Hankel determinant, we refer the reader to [23–27] and the references therein.

Recently, Allah et al. [28] defined the family of starlike functions based on the trigonometric hyperbolic tangent function as follows:

$$\Lambda^*_{\tanh} = \left\{g \in \mathcal{A} : \frac{\varepsilon g'(\varepsilon)}{g(\varepsilon)} \prec 1 + \tanh(\varepsilon)\right\} \quad (\varepsilon \in \mathbf{D}).$$

Motivated by the work mentioned above, we now introduce the subfamily of analytic functions:

$$\mathcal{R}_{\tanh} = \{g \in \mathcal{A} : g'(\varepsilon) \prec 1 + \tanh(\varepsilon)\} \quad (\varepsilon \in \mathbf{D}). \tag{1.7}$$

In this paper, we evaluate first three initial sharp coefficient bounds, sharp Fekete-Szegö functional, the sharp second Hankel determinant, sharp third Hankel determinant and Krushkal inequality for functions belonging to this family. Further, the sharp initial four logarithmic coefficient bounds and the second Hankel determinant are investigated.

2. A collection of lemmas

We next indicate by $\mathcal{P}$ the family of all holomorphic functions p satisfying that $Re(p(\varepsilon)) > 0$, $\varepsilon \in \mathbf{D}$, and that also has series representation:

$$p(\varepsilon) = 1 + \sum_{k=1}^{\infty} p_k \varepsilon^k \quad \varepsilon \in \mathbf{D}. \tag{2.1}$$

Lemma 2.1. *[29] Suppose that $p \in \mathcal{P}$. Then, for x and δ with $|x| \leq 1$and $|\delta| \leq 1$, it follows that*

$$2p_2 = p_1^2 + x(4 - p_1^2), \tag{2.2}$$

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)\delta. \tag{2.3}$$

Lemma 2.2. *If $p \in \mathcal{P}$, then the following estimations hold*

$$|p_k| \leq 2, \; k \geq 1, \tag{2.4}$$

$$|p_{k+n} - \mu p_k p_n| < 2, \;\; 0 < \mu \leq 1, \tag{2.5}$$

$$\left|p_2 - \frac{p_1^2}{2}\right| \leq 2 - \frac{|p_1|^2}{2}, \tag{2.6}$$

and for $\eta \in \mathbb{C}$, we have

$$\left|p_2 - \eta p_1^2\right| < 2 \max\{1, |2\eta - 1|\}. \tag{2.7}$$

For the inequalities (2.4)–(2.6) see [30] and (2.7) is given in [31].

Lemma 2.3. *[32] If $p \in \mathcal{P}$ and it has the form* (2.1)*, then*

$$|\alpha_1 p_1^3 - \alpha_2 p_1 p_2 + \alpha_3 p_3| \leq 2|\alpha_1| + 2|\alpha_2 - 2\alpha_1| + 2|\alpha_1 - \alpha_2 + \alpha_3|, \tag{2.8}$$

where α_1, α_2 and α_3 are real numbers.

Lemma 2.4. *[33] Let m_1, n_1, l_1 and r_1 satisfy that $m_1, r_1 \in (0, 1)$ and*

$$\begin{aligned} &8r_1(1 - r_1)\left[(m_1 n_1 - 2l_1)^2 + (m_1(r_1 + m_1) - n_1)^2\right] \\ &+ m_1(1 - m_1)(n_1 - 2r_1 m_1)^2 \\ &\leq 4m_1^2(1 - m_1)^2 r_1(1 - r_1). \end{aligned}$$

If $h \in \mathcal{P}$ and it is of the form (2.1), then

$$\left|l_1 p_1^4 + r_1 p_2^2 + 2m_1 p_1 p_3 - \frac{3}{2} n_1 p_1^2 p_2 - p_4\right| \leq 2.$$

3. Main results

Theorem 3.1. *If $g(\varepsilon) \in \mathcal{R}_{\tanh}$ and it has the form given by (1.2), then*

$$|a_2| \leq \frac{1}{2}, \tag{3.1}$$

$$|a_3| \leq \frac{1}{3}, \tag{3.2}$$

$$|a_4| \leq \frac{1}{4}. \tag{3.3}$$

Equalities of these inequalities are obtained for functions as follows:

$$g_1(\varepsilon) = \int_0^{\varepsilon} (1 + \tanh(t))\, dt = \varepsilon + \frac{1}{2}\varepsilon^2 + \cdots, \tag{3.4}$$

$$g_2(\varepsilon) = \int_0^{\varepsilon} \left(1 + \tanh\left(t^2\right)\right) dt = \varepsilon + \frac{1}{3}\varepsilon^3 + \cdots, \tag{3.5}$$

$$g_3(\varepsilon) = \int_0^{\varepsilon} \left(1 + \tanh\left(t^3\right)\right) dt = \varepsilon + \frac{1}{4}\varepsilon^4 + \cdots, \tag{3.6}$$

respectively.

Proof. Let $g(\varepsilon) \in \mathcal{R}_{\tanh}$ then by the definitions of subordinations there exists a Schwarz function $u(\varepsilon)$ with the properties given in (1.3), such that

$$g'(\varepsilon) = 1 + \tanh(u(\varepsilon)). \tag{3.7}$$

Let $p \in \mathcal{P}$; then, it can be written in terms of Schwarz functions as

$$p(\varepsilon) = \frac{1 + u(\varepsilon)}{1 - u(\varepsilon)} = 1 + p_1\varepsilon + p_2\varepsilon^2 + p_3\varepsilon^3 + \cdots. \tag{3.8}$$

Or

$$\begin{aligned} u(\varepsilon) &= \frac{p(\varepsilon) - 1}{p(\varepsilon) + 1} = \frac{p_1\varepsilon + p_2\varepsilon^2 + p_3\varepsilon^3 + \cdots}{2 + p_1\varepsilon + p_2\varepsilon^2 + p_3\varepsilon^3 + \cdots} \\ &= \frac{1}{2}p_1\varepsilon + \left(\frac{1}{2}p_2 - \frac{1}{4}p_1\right)\varepsilon^2 + \left(\frac{1}{8}p_1 - \frac{1}{2}p_1p_2 + \frac{1}{2}p_3\right)\varepsilon^3 + \cdots. \end{aligned}$$

Now, from (3.7) , we have

$$g'(\varepsilon) = 1 + 2a_2\varepsilon + 3a_3\varepsilon^2 + 4a_4\varepsilon^3 + 5a_5\varepsilon^4 + \cdots. \tag{3.9}$$

And

$$\begin{aligned} 1 + \tanh(u(\varepsilon)) &= 1 + \frac{1}{2}p_1\varepsilon + \left(\frac{1}{2}p_2 - \frac{1}{4}p_1^2\right)\varepsilon^2 + \left(\frac{1}{12}p_1^3 - \frac{1}{2}p_1p_2 + \frac{1}{2}p_3\right)\varepsilon^3 \\ &\quad + \left(\frac{1}{2}p_4 - \frac{1}{2}p_1p_3 + \frac{1}{4}p_1^2p_2 - \frac{1}{4}p_2^2\right)\varepsilon^4 + \cdots. \end{aligned} \tag{3.10}$$

Comparing (3.9) and (3.10) , we get

$$a_2 = \frac{1}{4}p_1, \tag{3.11}$$

$$a_3 = \frac{1}{6}\left(p_2 - \frac{1}{2}p_1^2\right), \tag{3.12}$$

$$a_4 = \frac{1}{48}p_1^3 - \frac{1}{8}p_1p_2 + \frac{1}{8}p_3, \tag{3.13}$$

$$a_5 = \frac{1}{10}p_4 - \frac{1}{10}p_1p_3 + \frac{1}{20}p_1^2p_2 - \frac{1}{20}p_2^2. \tag{3.14}$$

Applying (2.4) to (3.11), we get

$$|a_2| \le \frac{1}{2}.$$

From (3.12), and by using (2.6), we have

$$\begin{aligned} |a_3| &= \frac{1}{6}\left|p_2 - \frac{1}{2}p_1^2\right| \\ &\le \frac{1}{6}\left(2 - \frac{|p_1|^2}{2}\right) = H(p_1). \end{aligned}$$

Clearly, $H(p_1)$ is a decreasing function with the maximum attained at $p_1 = 0$; hence,

$$|a_3| \le \frac{1}{3}.$$

Applying Lemma 2.3 to (3.13), we get

$$|a_4| \le \frac{1}{4}.$$

□

Theorem 3.2. *If $g(\varepsilon) \in \mathcal{R}_{\tanh}$ and it has the form given by (1.2), then*

$$\left|a_3 - \lambda a_2^2\right| \le \frac{1}{3}\max\left\{1, \frac{3|\lambda|}{4}\right\}. \tag{3.15}$$

Equalities of this inequality can be obtained for the function g_2 defined in (3.5) for $|\lambda| \le \frac{4}{3}$ and for the function g_1 defined by (3.4) for $|\lambda| \ge \frac{4}{3}$.

Proof. From (3.11) and (3.12), we get

$$\left|a_3 - \lambda a_2^2\right| = \frac{1}{6}\left|p_2 - \frac{4-3\lambda}{8}p_1^2\right|.$$

Applying (2.7) to the above equation we get the required results. □

Corollary 3.3. *If $g(\varepsilon) \in \mathcal{R}_{\tanh}$ and it has the form given by (1.2), then*

$$\left|a_3 - a_2^2\right| \le \frac{1}{3}. \tag{3.16}$$

The equality of this inequality can be obtained for the function g_2 defined in (3.5).

Theorem 3.4. *If $g(\varepsilon) \in \mathcal{R}_{\tanh}$ and it has the form given by (1.2), then*

$$|a_2 a_3 - a_4| \le \frac{1}{4}. \tag{3.17}$$

The equality of this inequality can be obtained for the function g_3 defined in (3.6).

Proof. From (3.11)–(3.13), we get

$$|a_2a_3 - a_4| = \left|\frac{1}{24}p_1^3 - \frac{1}{6}p_2p_1 + \frac{1}{8}p_3\right|.$$

Applications of Lemma 2.3 lead us to the required results. □

Theorem 3.5. *If $g(\varepsilon) \in \mathcal{R}_{\tanh}$ and it has the form given by (1.2), then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{9}. \tag{3.18}$$

The equality of this inequality can be obtained for the function g_2 defined in (3.5).

Proof. From (3.11)–(3.13), we get

$$a_2a_4 - a_3^2 = -\frac{1}{576}p_1^4 - \frac{1}{288}p_1^2p_2 + \frac{1}{32}p_3p_1 - \frac{1}{36}p_2^2.$$

Applying (2.2) and (2.3) to write p_2 and p_3 in terms of $p_1 = p \in [0, 2]$, we get

$$\begin{aligned} a_2a_4 - a_3^2 &= -\frac{1}{384}p^4 - \frac{1}{144}\left(4-p^2\right)^2 x^2 \\ &\quad -\frac{1}{128}\left(4-p^2\right)x^2p^2 + \frac{1}{64}\left(4-p^2\right)p\left(1-|x|^2\right)\delta. \end{aligned}$$

Implementing the triangle inequality and using $|\delta| \leq 1$ and $|x| = y \leq 1$, we have

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| &\leq \frac{1}{384}p^4 + \frac{1}{144}\left(4-p^2\right)^2 y^2 + \frac{1}{128}p^2\left(4-p^2\right)y^2 \\ &\quad +\frac{1}{64}p\left(4-p^2\right)\left(1-y^2\right) = G(p,y). \end{aligned}$$

Now, differentiating partially with respect to y, we get

$$\frac{\partial R(p,y)}{\partial y} = \frac{1}{72}\left(4-p^2\right)^2 y + \frac{1}{64}p^2\left(4-p^2\right)y - \frac{1}{32}p\left(4-p^2\right)y.$$

Obviously, $\frac{\partial R(p,y)}{\partial y} > 0$ is an increasing function, so it has a maximum at $y = 1$; thus

$$\begin{aligned} R(p,y) &\leq Y(p,1) = \frac{1}{384}p^4 + \frac{1}{144}\left(4-p^2\right)^2 + \frac{1}{128}p^2\left(4-p^2\right) \\ &= \frac{1}{576}p^4 - \frac{7}{288}p^2 + \frac{1}{9}. \end{aligned}$$

Now, differentiating with respect to p, we get

$$G'(p,1) = \frac{1}{144}p^3 - \frac{7}{144}p.$$

Clearly, $G'(p,1) = 0$ only has the root $p = 0 \in [0,2]$. Hence, $G''(p,1) < 0$ at $p = 0$, so the maximum value is attained, that is

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{9}.$$

□

Theorem 3.6. *If $g(\varepsilon) \in \mathcal{R}_{\tanh}$ and it has the form given by (1.2), then,*

$$\left|H_{3,1}(g)\right| \leq \frac{1}{16}.$$

The equality of this inequality can be obtained for the function given by (3.6).

Proof. We know that

$$H_{3,1}(g) = a_3 a_2 a_4 - a_3^3 + a_4 a_2 a_3 - a_4^2 + a_5 a_3 - a_5 a_2^2.$$

Setting the values of (3.11)–(3.14), and putting $p_1 = p$, we get

$$H_{3,1}(f) = \frac{1}{34\,560}\begin{pmatrix} -504p_1p_4 + 144p_1p_3 + 48p_1p_2 - 448p_2 - 576p_3 \\ -25p_1 - 120p_1p_2 + 576p_2p_4 + 864p_1p_2p_3 \end{pmatrix}. \tag{3.19}$$

Now, supposing that $p_1 = p$ and $t = \left(4 - p_1^2\right)$ in (2.2), (2.3) and (3.19), we get

$$p_2 = \frac{1}{2}\left[p^2 + xt\right], \tag{3.20}$$

$$p_3 = \frac{1}{4}\left[p_1^3 + 2tpx - ptx^2 + 2t(1 - |x|^2)\delta\right], \tag{3.21}$$

$$p_4 = \frac{1}{8}\left[\begin{array}{c} \left(4x + \left(x^2 - 3x + 3\right)p^2\right)tx - 4m(1 - |x|^2) \\ -\rho\left(1 - |\delta|^2\right) + (x-1)\delta p + \delta\overline{x} + p^4 \end{array}\right].$$

by putting the above values of p_2, p_3 and p_4 in (3.19), we get

$$H_{3,1}(f) = \frac{1}{34\,560}\left[\begin{array}{c} -\frac{15}{4}p^6 - 27p^4tx^3 + \frac{9}{2}p^4tx^2 + 12p^4tx + 108p^3(1 - |x|^2)tx\delta + 45p^3(1 - |x|^2)t\delta \\ +108p^2(1 - |x|^2)tx\delta^2 - 108(1 - |\delta|^2)\rho p^2(1 - |x|^2)t + \frac{9}{4}p^2t^2x^4 \\ -81p^2t^2x^3 - 9p^2t^2x^2 - 108p^2tx^2 - 9p(1 - |x|^2)t^2x^2\delta \\ +90p(1 - |x|^2)t^2x\delta - 135(1 - |x|^2)^2t^2\delta^2 - 144(1 - |x|^2)t^2x^2\delta^2 \\ +144(1 - |\delta|^2)\rho(1 - |x|^2)t^2x - 56t^3x^3 + 144t^2x^3 \end{array}\right],$$

where $t = \left(4 - p^2\right)$; then, we have

$$H_{3,1}(f) = \frac{1}{34\,560}\left[v_1(p,x) + v_2(p,x)\delta + v_3(p,x)\delta^2 + \phi(p,x,\delta)\rho\right],$$

where

$$\begin{aligned} v_1(p,x) &= -\frac{1}{4}(4 - p^2)x\left[\left(\begin{array}{c} (4 - p^2)x\left(100p^2x - 9p^2x^2 + 36p^2 + 320x\right) \\ +\left(432p^2x - 18p^4x + 108p^4x^2 - 48p^4\right) \end{array}\right)\right] - \frac{15}{4}p^6, \\ v_2(p,x) &= -9p(4 - p^2)\left(1 - |x|^2\right)\left((4 - p^2)\left(x^2 - 10x\right) - 5p^2 - 12p^2x\right), \\ v_3(p,x) &= -9(4 - p^2)\left(1 - |x|^2\right)\left((4 - p^2)\left(15 + x^2\right) - 12p^2x\right), \end{aligned}$$

and

$$\phi(p,x,\delta) = 36(4 - p^2)\left(1 - |x|^2\right)\left(4x(4 - p^2) - 3p^2\right)\left(1 - |\delta|^2\right).$$

Now, let $|\delta| = y$, $|x| = x$ *and* $|\rho| \leq 1$. Then we have

$$\begin{aligned} \left|H_{3,1}(f)\right| &\leq \frac{1}{34\,560}\left(|v_1(p,x)| + |v_2(p,x)|\,y + |v_3(p,x)|\,y^2 + |\phi(p,x,\delta)|\right) \\ &\leq \frac{1}{34\,560}\left(H(p,x,y)\right), \end{aligned}$$

where

$$H(p,x,y) = h_1(p,x) + h_2(p,x)y + h_3(p,x)y^2 + h_4(p,x)(1-y^2). \tag{3.22}$$

Then,

$$\begin{aligned} h_1(p,x) &= \frac{1}{4}(4-p^2)x\left(\begin{array}{c} (4-p^2)x\left(100p^2x + 9p^2x^2 + 36p^2 + 320x\right) \\ +\left(432p^2x + 18p^4x + 108p^4x^2 + 48p^4\right) \end{array}\right) + \frac{15}{4}p^6, \\ h_2(p,x) &= 9p(4-p^2)\left(1-x^2\right)\left((4-p^2)\left(x^2+10x\right) + 5p^2 + 12p^2x\right), \\ h_3(p,x) &= 9(4-p^2)\left(1-x^2\right)\left((4-p^2)\left(15+x^2\right) + 12p^2x\right), \\ h_4(p,x) &= 36(4-p^2)\left(1-x^2\right)\left(4x(4-p^2) + 3p^2\right). \end{aligned}$$

Now, we need to attain the maxima of $H(p,x,y)$ in the interior of the closed cuboid $\Delta : [0,2] \times [0,1] \times [0,1]$. In order to do this, we have to maximize $H(p,x,y)$ on all six internal faces and at the 12 edges of the cuboid Δ.

1) First, we will check for the maximum of the function H in the interior of Δ. Let $(p,x,y) \in (0,2) \times (0,1) \times (0,1)$. Then differentiating (3.22) partially with rspect to y, we get

$$\frac{\partial H(p,x,y)}{\partial y} = 9(4-p^2)\left(1-x^2\right)\left[\begin{array}{c} p\left((4-p^2)\left(x^2+10x\right) + 5p^2 + 12p^2x\right) \\ +2\left((4-p^2)\left(15+x^2\right) + 12p^2x\right)y - 8\left(4x(4-p^2) + 3p^2\right)y \end{array}\right];$$

by setting $\frac{\partial H(p,x,y)}{\partial y} = 0$, we get

$$y = \frac{p\left((4-p^2)\left(x^2+10x\right) + p^2(12x+5)\right)}{2\left((4-p^2)(15-x) - 12p^2\right)(x-1)} = y_0 \in (0,1),$$

which is possible only if

$$\begin{aligned} & p^3(12x+5) + px(4-p^2)(x+10) + 2\left(4-p^2\right)(15-x)(1-x) \\ < \;& 24p^2(1-x) \end{aligned} \tag{3.23}$$

and

$$p^2 > \frac{4(15-x)}{27-x}.$$

Let $g(x) = \frac{4(15-x)}{27-x}$; it follows that

$$\frac{\partial}{\partial x}\left(\frac{4(15-x)}{27-x}\right) = -\frac{48}{(x-27)^2} < 0 \text{ in } (0,1).$$

This shows that $g(x)$ is a decreasing function. Hence, $p^2 > \frac{28}{13}$; a simple calculation show that the above inequality (3.23) does not hold true for the given values of $x \in (0,1)$, so the function L is no critical point in the interior of the cuboid.

2) We will check the maximum value on the six faces. First on $p = 0$, we have

$$H(0,x,y) = m_1(x,y) = 1280x^3 + 144\left(1-x^2\right)\left(15+x^2\right)y^2 + 2304x\left(1-x^2\right)\left(1-y^2\right).$$

Now,

$$\frac{\partial m_1(x,y)}{\partial y} = -288y\,(x-1)^2\left(x^2-14x-15\right) \neq 0, \quad (x,y \in (0,1))$$

Hence, $H(0,x,y)$ is no optimal point in $(0,1)\times(0,1)$.

At $p = 2$,

$$H(2,x,y) = \frac{15}{4}(2)^6 = 240.$$

At $x = 0$,

$$\begin{aligned} H(p,0,y) &= m_2(p,y) = \frac{15}{4}p^6 + 45p^3(4-p^2)y + 135(4-p^2)^2y^2 \\ &\quad +108p^2(4-p^2)\left(1-y^2\right) \end{aligned}$$

and

$$\frac{\partial m_2(p,y)}{\partial y} = -9\left(p^2-4\right)\left(5p^3-54yp^2+120y\right),$$

by taking $\frac{\partial m_2(p,y)}{\partial y} = 0$, we get that $y = \frac{5p^3}{\left(54p^2-120\right)} =: y_0$. For the provided range of y, $p > p_0 = 1.490711984999$.

Further,

$$\frac{\partial m_2}{\partial p} = \frac{9}{2}p\left(5p^4 - 50p^3y + 216p^2y^2 - 96p^2 + 120py - 672y^2 + 192\right).$$

Taking $\frac{\partial m_2}{\partial p} = 0$, we get

$$\left(5p^4 - 50p^3y + 216p^2y^2 - 96p^2 + 120py - 672y^2 + 192\right) = 0.$$

By putting $y = \frac{5p^3}{\left(54p^2-120\right)}$, we get

$$135p^8 - 6232p^6 + 37584p^4 - 80640p^2 + 57600 = 0.$$

When solving for $p \in (0,2)$, the solution is $p = 1.2751$; upon checking we conclude that there is no optimal solution for $H(p,0,y) = m_2(p,y)$ in $(0,2)\times(0,1)$.

At $x = 1$, we have

$$H(p,1,y) = m_3(p,y) = -\frac{7}{2}p^6 - 144p^4 + 372p^2 + 1280,$$

and

$$\frac{\partial m_3}{\partial p} = -21p^5 - 576p^3 + 744p.$$

Now, for the critical point put $\frac{\partial m_3}{\partial p}=0$; we obtain the solution to be $p=1.1117$, at which m_3 yields the maximum value, which is

$$m_3(p,y)\leq 1513.2.$$

At $y=0$, we obtain

$$\begin{aligned} H(p,x,0) &= m_4(p,x)=\frac{9}{4}p^6x^4-2p^6x^3+\frac{9}{2}p^6x^2-12p^6x+\frac{15}{4}p^6-18p^4x^4-156p^4x^3 \\ &\quad -54p^4x^2+192p^4x-108p^4+36p^2x^4+912p^2x^3 \\ &\quad +144p^2x^2-1152p^2x+432p^2-1024x^3+2304x. \end{aligned}$$

From the computation it is clear that the system of equations has no solution for $(0,2)\times(0,1)$.

At $y=1$,

$$\begin{aligned} H(p,x,1) &= m_5(p,x)=\frac{9}{4}p^6x^4-2p^6x^3+\frac{9}{2}p^6x^2-12p^6x+\frac{15}{4}p^6-9p^5x^4 \\ &\quad +18p^5x^3+54p^5x^2-18p^5x-45p^5-27p^4x^4 \\ &\quad +96p^4x^3-288p^4x^2-60p^4x+135p^4+72p^3x^4 \\ &\quad +288p^3x^3-252p^3x^2-288p^3x+180p^3+108p^2x^4 \\ &\quad -672p^2x^3+1584p^2x^2+432p^2x-1080p^2-144px^4 \\ &\quad -1440px^3+144px^2+1440px-144x^4+1280x^3 \\ &\quad -2016x^2+2160. \end{aligned}$$

Computation indicates that the solution for the system of equations associated with $\frac{\partial m_5}{\partial x}=0$ and $\frac{\partial m_5}{\partial p}=0$ in the region $(0,2)\times(0,1)$ does not exist.

3) Now, we will check the maximum of $H(p,x,y)$ at the 12 edges of cuboid.

By putting $x=0$ and $y=0$, we have

$$H(p,0,0)=m_6(p)=\frac{15}{4}p^6-108p^4+432p^2.$$

For the critical points put $\frac{\partial m_6}{\partial p}=0$; its critical point is $p=1.5059$, at which the maximum value of $m_6(p)$ is

$$m_6(p)\leq 467.99.$$

By putting $x=0$ and $y=1$,

$$H(p,0,1)=m_7(p)=\frac{15}{4}p^6-45p^5+135p^4+180p^3-1080p^2+2160.$$

For critical point

$$\frac{\partial m_7}{\partial p}=\frac{45}{2}p^5-225p^4+540p^3+540p^2-2160p.$$

As $\frac{\partial m_7}{\partial p}<0$, for $p\in[0,2]$, $\frac{\partial m_7}{\partial p}$ is a decreasing function that achieves its maximum value at $p=0$, which is

$$H(p,0,1)\leq 2160.$$

For $x = 0$ and $p = 0$, we have

$$H(0,0,y) = m_8(y) = 9(4)\,((4)\,(15))\,y^2 = 2160y^2.$$

As $m_8(y)$ is an increasing function, its maximum occurs at $y = 1$, that is

$$H(0,0,y) \leq 2160.$$

Now, the equation

$$H(p,1,y) = m_3(p,y) = -\frac{7}{2}p^6 - 144p^4 + 372p^2 + 1280$$

is free from y. So,

$$H(p,1,0) = H(p,1,1) = -\frac{7}{2}p^6 - 144p^4 + 372p^2 + 1280.$$

Then, $m_9(p) = -\frac{7}{2}p^6 - 144p^4 + 372p^2 + 1280$ has its maximum value at $p = 1.1117$, which corresponds to

$$m_9(p) \leq 1513.2.$$

For $p = 0$ and $x = 1$,

$$H(0,1,y) = m_{10}(y) = \frac{1}{4}(4)\,[((4)\,(320))] = 1280$$

For $p = 2$, all of the terms of $H(p,x,y)$ are free from p, x and y. So,

$$H(2,0,y) = H(2,1,y) = H(2,x,1) = H(2,x,0) = -\frac{7}{2}p^6 = -\frac{7}{2}\,(2)^6 = -224.$$

At $p = 0$ and $y = 0$,

$$H(0,x,0) = m_{11}(x) = 2304x - 1024x^3.$$

To find the critical point at which $m_{11}(x)$ gives the maximum value, put $\frac{\partial m_{11}}{\partial x} = 0$ which gives $x_0 = 1.5$ at which the maximum value of $m_{11}(x)$ is given by

$$H(0,x,0) = m_{10}(x) \leq 0.$$

For $p = 0$ and $y = 1$,

$$H(0,x,1) = m_{12}(x) = -144x^4 + 1280x^3 - 2016x^2 + 2160.$$

As $\frac{\partial m_{12}}{\partial x} = -576x^3 + 3840x^2 - 4032x < 0$, for $x \in (0,1)$ which shows that it is decreasing function then its maximum occurs at $x = 0$, that is

$$m_{12}(x) \leq 2160.$$

Here from all of the calculations, we conclude that

$$L(p,x,y) \leq 2160.$$

For $\triangle : [0,2] \times [0,1] \times [0,1]$, it follows that

$$\left|H_{3,1}(f)\right| \leq \frac{1}{34560}\,(L(p,x,y) \leq \frac{1}{16}.$$

□

4. Krushkal inequality

In this section for the particular choice of $n = 4$ and $p = 1$, we will give a direct proof of the inequality

$$\left|a_n^p - a_2^{p(n-1)}\right| \leq 2^{p(n-1)} - n^p$$

over the family $\mathcal{R}_{\tanh}$. For the whole family of univalent functions Krushkal [34] introduced and proved this inequality.

Theorem 4.1. *Let $g \in \mathcal{A}$ belong to $\mathcal{R}_{\tanh}$. Then,*

$$\left|a_4 - a_2^3\right| \leq \frac{1}{4}.$$

The equality associated with this inequality can be obtained for the function defined by (3.4).

Proof. From Eqs (3.11) and (3.13), we get

$$\left|a_4 - a_2^3\right| = \left|\frac{1}{192}p_1^3 - \frac{1}{8}p_2p_1 + \frac{1}{8}p_3\right|.$$

By applying Lemma 2.3 to the above equation, we get the required result. □

5. Logarithmic coefficients for the family $\mathcal{R}_{\tanh}$

The logarithmic coefficients of $g \in \mathcal{S}$ denoted by $\gamma_n = \gamma_n(g)$, are defined by the following series expansion:

$$\log \frac{g(\varepsilon)}{\varepsilon} = 2\sum_{n=1}^{\infty} \gamma_n \varepsilon^n.$$

For the functions g given by (1.2), the logarithmic coefficients are as follows

$$\gamma_1 = \frac{1}{2}a_2, \tag{5.1}$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right), \tag{5.2}$$

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right), \tag{5.3}$$

$$\gamma_4 = \frac{1}{2}\left(a_5 - a_2a_4 + a_2^2a_3 - \frac{1}{2}a_3^2 - \frac{1}{4}a_2^4\right), \tag{5.4}$$

$$\gamma_5 = \frac{1}{2}\left(a_6 - a_2a_5 - a_3a_4 + a_2a_3^2 + a_2^2a_4 - a_2^3a_3 + \frac{1}{5}a_2^5\right). \tag{5.5}$$

Theorem 5.1. *If $g(\varepsilon) \in \mathcal{R}_{\tanh}$ and it has the form given by (1.2), then*

$$|\gamma_1| \leq \frac{1}{4},$$

$$\begin{aligned}|\gamma_2| &\le \frac{1}{6},\\ |\gamma_3| &\le \frac{1}{8},\\ |\gamma_4| &\le \frac{1}{10}.\end{aligned}$$

The equality associated with these inequalities can be obtained for the function

$$g_n(\varepsilon) = \int_0^{\varepsilon} (1 + \tanh(t^n))\, dt = \varepsilon + \frac{1}{n+1}\varepsilon^{n+1} + \cdots \text{ for } n = 1, 2, 3, 4. \tag{5.6}$$

Proof. Now from (5.1) to (5.5) and (3.11) to (3.14), we get

$$\gamma_1 = \frac{1}{8}p_1, \tag{5.7}$$

$$\gamma_2 = \frac{1}{12}\left(p_2 - \frac{11}{16}p_1^2\right), \tag{5.8}$$

$$\gamma_3 = \frac{3}{128}p_1^3 - \frac{1}{12}p_2p_1 + \frac{1}{16}p_3, \tag{5.9}$$

$$\gamma_4 = -\frac{137}{18\,432}p_1^4 + \frac{19}{360}p_1^2p_2 - \frac{21}{320}p_3p_1 - \frac{23}{720}p_2^2 + \frac{1}{20}p_4. \tag{5.10}$$

Applying (2.4) to (5.7), we get

$$|\gamma_1| \le \frac{1}{4}.$$

From (5.8) and by using (2.5), we get

$$|\gamma_2| \le \frac{1}{6}.$$

Applying Lemma 2.3 to (5.9), we get

$$|\gamma_3| \le \frac{1}{8}.$$

Also, applying Lemma 2.4 to (5.10), we get

$$|\gamma_4| \le \frac{1}{10}.$$

Proof of sharpness. Since

$$\begin{aligned}\log\frac{g_1(\varepsilon)}{\varepsilon} &= 2\sum_{n=2}^{\infty}\gamma(g_1)\varepsilon^n = \frac{1}{2}\varepsilon + \cdots,\\ \log\frac{g_2(\varepsilon)}{\varepsilon} &= 2\sum_{n=2}^{\infty}\gamma(g_2)\varepsilon^n = \frac{1}{3}\varepsilon^2 + \cdots,\\ \log\frac{g_3(\varepsilon)}{\varepsilon} &= 2\sum_{n=2}^{\infty}\gamma(g_2)\varepsilon^n = \frac{1}{4}\varepsilon^3 + \cdots,\\ \log\frac{g_4(\varepsilon)}{\varepsilon} &= 2\sum_{n=2}^{\infty}\gamma(g_2)\varepsilon^n = \frac{1}{5}\varepsilon^4 + \cdots,\end{aligned}$$

it follows that these inequalities are obtained for the functions $g_n(\varepsilon)$ for $n = 1, 2, 3, 4$ as defined in (5.6).
□

Theorem 5.2. *If $g(\varepsilon) \in \mathcal{R}_{\tanh}$ and it has the form given by (1.2), then*

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \le \frac{1}{36}.$$

The equality in this inequality can be obtained for the function g_2 in (5.6).

Proof. From (5.7)–(5.9), we have

$$\gamma_1\gamma_3 - \gamma_2^2 = -\frac{13}{36\,864}p_1^4 - \frac{1}{1152}p_1^2p_2 + \frac{1}{128}p_3p_1 - \frac{1}{144}p_2^2.$$

Applying (2.2) and (2.3) to write p_2 and p_3 in terms of $p_1 = p \in [0, 2]$, we get

$$\gamma_1\gamma_3 - \gamma_2^2 = -\frac{7}{12\,288}p^4 - \frac{1}{576}\left(4 - p^2\right)^2 x^2 - \frac{1}{512}p^2\left(4 - p^2\right)x^2p_1 + \frac{1}{256}p\left(4 - p^2\right)\left(1 - |x|^2\right)\delta.$$

By the triangle inequality, and by using $|\delta| \le 1$ and $|x| = y \le 1$, we get

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \le \frac{7}{12\,288}p^4 + \frac{1}{576}\left(4 - p^2\right)^2 y^2 + \frac{1}{512}p^2\left(4 - p^2\right)y^2 + \frac{1}{256}p\left(4 - p^2\right)\left(1 - y^2\right). \quad (5.11)$$

Now, differentiating Eq (5.11) partially with respect to y, we have

$$\frac{\partial G(p, y)}{\partial y} = \frac{1}{2304}y(p - 2)^2\left(-p^2 + 14p + 32\right).$$

It is easy to observe that $\frac{\partial G(p,y)}{\partial y} \ge 0$ in the interval $[0, 1]$, so the maximum is attained at $y = 1$; thus

$$\begin{aligned} G(p, y) &\le G(p, 1) = \frac{7}{12\,288}p^4 + \frac{1}{576}\left(4 - p^2\right)^2 + \frac{1}{512}p^2\left(4 - p^2\right) \\ &= \frac{13}{36\,864}p^4 - \frac{7}{1152}p^2 + \frac{1}{36}. \end{aligned}$$

Now, differentiating with respect to p, we get

$$G'(p, 1) = \frac{1}{9216}p^3 - \frac{1}{576}p.$$

Clearly, $G'(p, 1) = 0$, has three roots namely $0, \pm 4$, and the only root that lies in the interval $[0, 2]$ is 0, so

$$G''(p, 1) = \frac{1}{3072}p^2 - \frac{1}{576}.$$

Thus, $G''(0, 1) \le 0$, so the function has its maximum at $p = 0$, that is

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \le \frac{1}{36}.$$

□

6. Conclusions

Recently, the investigations of the Hankel determinant have attracted the attention of many researchers due to their applications in many diverse areas of mathematics and other sciences. In this paper, we have defined a new subfamily of analytic functions connected with the hyperbolic tangent function with bounded boundary rotation. We have also investigated the upper bound of the third Hankel determinant for this newly defined family of functions. On the other hand, we have obtained the Krushkal inequality and investigated the first four initial sharp bounds of the logarithmic coefficients and the sharp second Hankel determinant of the logarithmic coefficients for this defined family of functions.

Here, we want to remark on the fact that one can extend the suggested results investigated in this article to some other subclasses of analytic functions, and also that those interested scholars can use the D_q derivative operator and generalize the work presented here.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of Interest

The authors declare that they have no competing interest.

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