AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(11): 27363–27375.
DOI: 10.3934/math.20231400
Received: 29 July 2023
Revised: 12 September 2023
Accepted: 19 September 2023
Published: 26 September 2023

Research article

Differences weighted composition operators in several variables between some spaces of analytic functions

Aydah Mohammed Ayed Al-Ahmadi*

Department of Mathematics, College of Science, Jouf University, Saudi Arabia

* **Correspondence:** Email: amahmadi@ju.edu.sa; aydahahmadi2011@gmail.com.

Abstract: The boundedness and compactness of weighted composition operators have been extensively investigated on various analytic function spaces. In this paper, we study the boundedness and compactness of two several variables differences weighted composition operators on some analytic function spaces.

Keywords: weighted composition operator; compact difference; bounded operator; unit disk; weighted analytic space
Mathematics Subject Classification: 47B91

1. Basic notation and auxiliary results

Let $\mathbb{D} = \{z \in \mathbb{C} \ / \ |z| < 1 \ \}$ be the unit disk in the complex space. $O(\mathbb{D})$ denotes the space of functions that are analytic in $\mathbb{D}$ and $\mathcal{H}^\infty(\mathbb{D})$ denotes the Banach space of bounded analytic functions on $\mathbb{D}$ with the norm $\|f\|_\infty = \sup\limits_{z\in\mathbb{D}} |f(z)|$. For analytic self-mapping $\varphi \in \mathcal{S}(\mathbb{D}) = \{\varphi \in O(\mathbb{D}) : \varphi(\mathbb{D}) \subset \mathbb{D})\}$ and analytic function $u : \mathbb{D} \longrightarrow \mathbb{C}$, the pair (u, φ) induces the linear map $W_{\varphi,\,u} : O(\mathbb{D}) \longrightarrow O(\mathbb{D})$ defined by

$$W_{\varphi,u}(f)(z) = u(z)(f \circ \varphi(z)), \quad f \in O(\mathbb{D}),\ z \in \mathbb{D}.$$

$W_{\varphi,\,u}$ which is called weighted composition operator with symbols u and φ. Observe that $W_{\varphi,\,u}(f) = M_u C_\varphi(f)$, where $M_u(f) = u.f$, is the multiplication operator with symbol u, and $C_\varphi(f) = f \circ \varphi$, is the composition operator with symbol φ.

If $u \equiv 1$, then $W_{\varphi,\,u} = C_\varphi$, and if φ is the identity ($\varphi(z) = z$), then $W_{\varphi,\,u} = M_u$.

During the past few decades, composition operators and weighted composition operators have been studied extensively on spaces of analytic functions on various domains in $\mathbb{C}$ or $\mathbb{C}^n$. We refer the readers to the monographs [1, 3, 5, 13, 18–20, 22, 25] for detailed information and the references therein.

For $a \in \mathbb{D}$ the Möbius transformation $\varphi_a(z)$ is defined by

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \text{ for } z \in \mathbb{D}.$$

For each $a \in \mathbb{D}$, the Green´s function with logarithmic singularity at $a \in \mathbb{D}$ is denoted by

$$g(z,a) = \log\Big(\frac{1}{|\varphi_a(z)|}\Big).$$

The pseudohyperbolic distance $\rho : \mathbb{D} \times \mathbb{D} \longrightarrow [0,1)$ is defined by

$$\rho(a,z) = |\varphi_a(z)| = \left|\frac{a-z}{1-\overline{a}z}\right| \text{ for } a, z \in \mathbb{D}.$$

We will denote by

$$\rho(\varphi(z), \psi(z)) = \left|\frac{\varphi(z)-\psi(z)}{1-\overline{\varphi(z)}\psi(z)}\right|.$$

It is easy to check that $\rho(a,z)$ satisfies the following inequalities:

$$\frac{1-\rho(a,z)}{1+\rho(a,z)} \leq \frac{1-|z|^2}{1-|a|^2} \leq \frac{1+\rho(a,z)}{1-\rho(a,z)}, \quad z, a \in \mathbb{D}.$$

For $0 < \alpha < \infty$, recall that an $f \in \mathcal{O}(\mathbb{D})$ is said to belong to the α-Bloch space (or Bloch-type space) $\mathcal{B}^\alpha$ if

$$\mathcal{B}_\alpha(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f'(z)| < \infty.$$

With the norm $||f|| = |f(0)| + \mathcal{B}_\alpha(f)$, $\mathcal{B}^\alpha$ is a Banach space. When $\alpha = 1$, $\mathcal{B}^1 = \mathcal{B}$ is the well-known Bloch space. For more information on Bloch spaces we refer the interested reader to [21]. Let $\mathcal{B}_0^\alpha$ be the space which consists of all $f \in \mathcal{B}^\alpha$ satisfying

$$\lim_{|z|\longrightarrow 0}(1-|z|^2)^\alpha |f'(z)| = 0.$$

This space is called the little Bloch-type space. See [5] for more information on Bloch spaces.

Let $\alpha \geq 0$. The Bers-type space, denoted by $\mathcal{H}_\alpha^\infty(\mathbb{D})$, is a Banach space defined by

$$\mathcal{H}_\alpha^\infty(\mathbb{D}) := \{ f \in \mathcal{O}(\mathbb{D}) \ / \sup_{z\in\mathbb{D}}((1-|z|^2)^\alpha |f(z)|) < \infty\},$$

$$\mathcal{H}_{(\alpha,\,0)}^\infty(\mathbb{D}) := \{ f \in \mathcal{O}(\mathbb{D}) \ / \lim_{|z|\longrightarrow 1^-}((1-|z|^2)^\alpha |f(z)|) = 0 \}$$

equipped with the norm

$$||f||_{\mathcal{H}_\alpha^\infty(\mathbb{D})} := \sup_{z\in\mathbb{D}}((1-|z|^2)^\alpha |f(z)|) \quad \text{for } f \in \mathcal{H}_\alpha^\infty(\mathbb{D}).$$

Note that, $\mathcal{H}_\alpha^\infty(\mathbb{D})$ is a Banach space with the norm $||.||_{\mathcal{H}_\alpha^\infty(\mathbb{D})}$.
When $\alpha = 0$, $\mathcal{H}_0^\infty(\mathbb{D})$ is just the bounded analytic function space $\mathcal{H}^\infty(\mathbb{D})$. For more information about several studied on Bers-type spaces we refer to [3, 22].

Let $K:[0,\ \infty)\longrightarrow(0,\ \infty)$ be right continuous and nondecreasing function. The authors El-Sayed A. and Bakhit in [7] introduced the $\mathcal{N}_K(\mathbb{D})$ spaces as follows:
The analytic $\mathcal{N}_K(\mathbb{D})$-space is defined by

$$\mathcal{N}_K(\mathbb{D}):=\{\ f\in\mathcal{O}(\mathbb{D})\ /\ \int_{\mathbb{D}}|f(z)|^2K(g(z,a))dA(z)<\infty\},$$

$$\mathcal{N}_{(K,\,0)}(\mathbb{D}):=\{\ f\in\mathcal{O}(\mathbb{D})\ /\ \lim_{|a|\longrightarrow1^-}\int_{\mathbb{D}}|f(z)|^2K(g(z,a))dA(z)=0\},$$

equipped with the norm

$$||f||^2_{\mathcal{N}_K(\mathbb{D})}=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f(z)|^2K(g(z,a))dA(z),\ f\in\mathcal{N}_K(\mathbb{D}).$$

Remark 1.1. *We make the following observations:*
(1) *If $K(t)=t^p$, then $\mathcal{N}_K(\mathbb{D})=\mathcal{N}_p(\mathbb{D})$, since $g(z,a)\approx(1-|\varphi_a|^2)$.*
(2) *If $K(t)\equiv1$, then $\mathcal{N}_1(\mathbb{D})=\mathcal{A}^2$ (the Bergman space), where for $0<p<\infty$, the Bergman space $\mathcal{A}^p$ is the set of analytic functions f in the unit disk $\mathbb{D}$ with*

$$||f||^p_{\mathcal{A}^p}=\frac{1}{\pi}\int_{\mathbb{D}}|f(z)|^pdA(z)<\infty.$$

Remark 1.2. *In the study of the space $\mathcal{N}_K(\mathbb{D})$, the authors in [7] assumed that the following condition*

$$\sup_{0\le t\le1}\int_0^1\frac{(1-t)^2}{(1-tr^2)^3}K(\log(\frac{1}{r}))rdr<\infty \tag{1.1}$$

is satisfied, so that the $\mathcal{N}_K(\mathbb{D})$ space is not trivial.

Lemma 1.1. *[8, Lemma 2.2] Assume that the function K satisfies (1.1). For each $w\in\mathbb{D}$, let $h_w(z)=\dfrac{1-|w|^2}{(1-\overline{w}z)^2}$ for $z\in\mathbb{D}$. Then h_w satisfies the following conditions:*
(i) $h_w\in\mathcal{N}_K(\mathbb{D})$,
(ii) $||h_w||_{\mathcal{N}_K(\mathbb{D})}\lesssim1$,
(iii) $\sup_{\omega\in\mathbb{D}}||h_w||_{\mathcal{N}_K(\mathbb{D})}\le1$.

Several important properties of the $\mathcal{N}_K(\mathbb{D})$-spaces and $H^\infty_\alpha(\mathbb{D})$ spaces and also of weighted composition operators from $\mathcal{N}_K(\mathbb{D})$-spaces to the spaces $H^\infty_\alpha(\mathbb{D})$ and from $H^\infty_\alpha(\mathbb{D})$-spaces to $\mathcal{N}_K(\mathbb{D})$ have been characterized in [7, 8, 15].

We cite here main results from [15] for the reader's convenience.

Theorem 1.1. *[15, 24] Let $K:[0,\infty)\to[0,\infty)$ be a nondecreasing function and φ be a analytic self-map of $\mathbb{D}$. For $\alpha\in(0,\infty)$, $u\in\mathcal{O}(\mathbb{D})$ and $W_{\varphi,\,u}:=uC_\varphi:\mathcal{N}_K(\mathbb{D})\longrightarrow\mathcal{H}^\infty_\alpha(\mathbb{D})$ the weighted composition operator. Then we have:*
(1) *$W_{\varphi,\,u}$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}}\Big(\frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)}\Big)<\infty. \tag{1.2}$$

(2) *$W_{\varphi,\,u}$ is compact if and only if*

$$\lim_{r\longrightarrow 1}\sup_{|\varphi(z)|>r}\left(\frac{|u(z)|(1-|z|^2)^{\alpha}}{1-|\varphi(z)|^2}\right)=0. \tag{1.3}$$

Remark 1.3. *When $K(t)=t^p$, Theorem 1.1 coincides with [24, Thoerem 3, Corollary 2].*

Theorem 1.2. *[15, 24] Let $K:[0,\infty)\to[0,\infty)$ be a nondecreasing function and φ be a analytic self-map of $\mathbb{D}$. For $\alpha\in(0,\infty)$ and $u\in\mathcal{O}(\mathbb{D})$. Then the following properties hold:*
(1) *The weighted composition operator $W_{\varphi,\,u}=uC_{\varphi}:\mathcal{H}^{\infty}_{\alpha}(\mathbb{D})\longrightarrow\mathcal{N}_K(\mathbb{D})$ is bounded.*
(2) *u and φ satisfy*

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|u(z)|^2}{(1-|\varphi(z)|^2)^{2\alpha}}K(g(z,a))dA(z)<\infty. \tag{1.4}$$

Remark 1.4. *When $K(t)=t^p$, Theorem 1.2 coincides with [24, Thoerem 1].*

Lemma 1.2. *[7, Proposition 2.1] For each right continuous and nondecreasing function $K:[0,\infty)\to[0,\infty)$, the following inclusion holds:*

$$\mathcal{N}_K(\mathbb{D})\subset\mathcal{H}^{\infty}_{1}(\mathbb{D}).$$

Our goal here is to investigate the boundedness and compactness of multivariable difference of two weighted composition operators acting from $\prod_{j=1}^{n}\mathcal{N}_{K_j}(\mathbb{D})$-spaces to $\prod_{j=1}^{n}\mathcal{H}^{\infty}_{\alpha_j}(\mathbb{D})$-spaces and form $\prod_{j=1}^{n}\mathcal{H}^{\infty}_{\alpha_j}(\mathbb{D})$-spaces to $\prod_{j=1}^{n}\mathcal{N}_{K_j}(\mathbb{D})$-spaces.
We consider that each of the product spaces is equipped with the following norm.
For $f=(f_1,\cdots,f_n)\in X=\prod_{i=1}^{n}X_i$, we set $\|f\|_X=\sum_{i=1}^{n}\|f_i\|_{X_i}$.
To this end we introduce analytic maps $\varphi_k,\psi_k:\mathbb{D}\longrightarrow\mathbb{D}$ and $u_k,v_k:\mathbb{D}\longrightarrow\mathbb{C}$ for $k=1.\cdots,n$ and look at the operator

$$T_{\varphi_k,\psi_k}:=W_{\varphi_k,\,u_k}-W_{\psi_k,\,v_k}=u_kC_{\varphi_k}-v_kC_{\psi_k}.$$

Let

$$\varphi=(\varphi_1,\cdots,\varphi_n)\in\mathcal{S}(\mathbb{D})^n,\ \psi=(\psi_1,\cdots,\psi_n)\in\mathcal{S}(\mathbb{D})^n$$

and

$$u=(u_1,\cdots,u_n)\in\mathbb{O}(\mathbb{D})^n, v=(v_1,\cdots,v_n)\in\mathcal{O}(\mathbb{D})^n.$$

Set

$$\mathbf{K}=(K_1,\cdots,K_n)\text{ and }\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathbb{R}^n.$$

We define

$$\mathbf{T}_{\varphi,\psi}:=(T_{\varphi_1,\psi_1},\cdots,T_{\varphi_n,\psi_n}):\prod_{i=1}^{n}\mathcal{N}_{K_i}(\mathcal{D})\longrightarrow\prod_{i=1}^{n}\mathcal{H}_{\alpha_i}(\mathcal{D}).$$

$$
\begin{aligned}
\mathbf{T}_{\varphi,\psi}(f) &= ((T_{\varphi_1,\psi_1})f_1, \cdots, (T_{\varphi_n,\psi_n})f_n) \\
&= \Big((u_1C_{\varphi_1} - v_1C_{\psi_1})f_1,\ \cdots, (u_nC_{\varphi_n} - v_nC_{\psi_n})f_n\Big) \\
&= \Big(u_1f_1 \circ \varphi_1 - v_1f_1 \circ \psi_1, \cdots, u_nf_n \circ \varphi_n - v_nf_n \circ \psi_n\Big).
\end{aligned}
$$

2. Main results

2.1. *Multivariable differences of Weighted composition operators from $\prod_{k=1}^{n} \mathcal{N}_{K_k}(\mathbb{D})$ into $\prod_{k=1}^{n} \mathcal{H}^{\infty}_{\alpha_k}(\mathbb{D})$*

In this section we study the boundedness and compactness of differences of two weighted composition operators $\mathbf{T}_{\varphi,\,\psi} := W_{\varphi,\,u} - W_{\psi,\,v} : \prod_{k=1}^{n} \mathcal{N}_{K_k}\mathbb{D}) \longrightarrow \prod_{k=1}^{n} \mathcal{H}^{\infty}_{\alpha_k}(\mathbb{D})$. In fact, the following results corresponds to the results obtained in [2, 4, 6, 9–12, 23].

We are now ready to prove a necessary and sufficient condition for the boundedness of $\mathbf{T}_{\varphi,\psi}$: $\prod_{1\le i\le n} \mathcal{N}_{K_i}(\mathbb{D}) \longrightarrow \prod_{1\le i\le n} \mathcal{H}^{\infty}_{\alpha_i}(\mathbb{D})$.
For that purpose, consider the following conditions:

$$\sup_{1\le k\le n}\left(\sup_{z\in\mathbb{D}}\left(\frac{|u_k(z)|(1-|z|^2)^{\alpha_k}}{(1-|\varphi_k(z)|^2)}\rho(\varphi_k(z),\psi_k(z)\right)\right) < \infty. \tag{2.1}$$

$$\sup_{1\le k\le n}\left(\sup_{z\in\mathbb{D}}\left(\frac{|v_k(z)|(1-|z|^2)^{\alpha_k}}{(1-|\psi_k(z)|^2)}\rho(\varphi_k(z),\psi_k(z))\right)\right) < \infty. \tag{2.2}$$

$$\sup_{1\le k\le n}\left(\sup_{z\in\mathbb{D}}\left|\frac{|u_k(z)|(1-|z|^2)^{\alpha_k}}{(1-|\varphi_k(z)|^2)} - \frac{|v_k(z)|(1-|z|^2)^{\alpha_k}}{(1-|\psi_k(z)|^2)}\right|\right) < \infty. \tag{2.3}$$

Lemma 2.1. *[16, 17] Let $f \in \mathcal{H}^{\infty}_{\beta}(\mathbb{D})$, $\beta > 0$. Then*
$\left|(1-|z|^2)^{\beta} f(z) - (1-|w|^2)^{\beta} f(w)\right| \le \|f\|_{\mathcal{H}^{\infty}_{\beta}(\mathbb{D})}\rho(z,w)$ *for all $z, w \in \mathbb{D}$.*

Theorem 2.1. *Let $K_j : [0,\ \infty) \longrightarrow [0,\ \infty)$ for $j = 1, \cdots, n$ be a nondecreasing function, φ_j and ψ_j are analytic self-maps from $\mathbb{D}$ to $\mathbb{D}$. For $u = (u_1, \cdots, u_n), v = (v_1, \cdots, v_n) \in \mathcal{O}(\mathbb{D})^n, \varphi = (\varphi_1, \cdots, \varphi_n),\ \psi = (\psi_1, \cdots, \psi_n)$ and $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n,\ \alpha_i > 0$. Then the following statements are equivalent:*

(1) $\mathbf{T}_{\varphi,\psi} : \prod_{i=1}^{n} \mathcal{N}_{K_i}(\mathbb{D}) \longrightarrow \prod_{i=1}^{n} \mathcal{H}^{\infty}_{\alpha}(\mathbb{D})$ *is bounded.*
(2) φ, ψ *and* u, v *satisfy the conditions* (2.1) *and* (2.3).
(3) φ, ψ *and* u, v *satisfy the conditions* (2.2) *and* (2.3).

Proof. (3) $\Rightarrow$ (1). Assume that the functions φ, ψ and u, v satisfy the conditions (2.2) and (2.3). We need to prove that $\mathbf{T}_{\varphi,\psi}$ is a bounded operator. Indeed,
let $f = (f_1, \cdots, f_n) \in \prod_{i=1}^{n} \mathcal{N}_{K_i}(\mathbb{D})$, then we have

$$\left\|\mathbf{T}_{\varphi,\psi}(f)\right\|_{\prod_{i=1}^{n}\mathcal{H}^{\infty}_{\alpha_i}(\mathbb{D})} = \sum_{i=1}^{n} \|T_{\varphi_i,\psi_i} f_i\|_{\mathcal{H}^{\infty}_{\alpha_i}(\mathbb{D})}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \sup_{z\in\mathbb{D}} \Big((1-|z|^2)^{\alpha_i} \left| T_{\varphi_i,\psi_i} f_i(z) \right| \Big) \\
&= \sum_{i=1}^{n} \sup_{z\in\mathbb{D}} \Big((1-|z|^2)^{\alpha_i} \Big| (u_i C_{\varphi_i} - v_i C_{\psi_i}) f_i(z) \Big| \Big) \\
&= \sum_{i=1}^{n} \sup_{z\in\mathbb{D}} \Big| u_i(z)(1-|z|^2)^{\alpha_i} f_i(\varphi_i(z)) - v_i(z)(1-|z|^2)^{\alpha_i} f_i(\psi_i(z)) \Big| \\
&= \sum_{i=1}^{n} \sup_{z\in\mathbb{D}} \Big| (1-|\varphi_i(z)|^2) f_i(\varphi_i(z)) \Big[\frac{u_i(z)(1-|z|^2)^{\alpha_i}}{(1-|\varphi_i(z)|^2)} - \frac{v_i(z)(1-|z|^2)^{\alpha_i}}{(1-|\psi(z)|^2)} \Big] \\
&\quad + \frac{v_i(z)(1-|z|^2)^{\alpha_i}}{(1-|\psi_i(z)|^2)} \Big[(1-|\varphi_i(z)|^2) f_i(\varphi_i(z)) - (1-|\psi_i(z)|^2) f_i(\psi_i(z)) \Big] \Big| \\
&\le \sum_{i=1}^{n} \sup_{z\in\mathbb{D}} \Big\{ \Big| (1-|\varphi_i(z)|^2) f_i(\varphi_i(z)) \Big| \Big| \Big[\frac{u_i(z)(1-|z|^2)^{\alpha_i}}{(1-|\varphi_i(z)|^2)} - \frac{v_i(z)(1-|z|^2)^{\alpha_i}}{(1-|\psi_i(z)|^2)} \Big] \\
&\quad + \Big| \frac{v_i(z)(1-|z|^2)^{\alpha_i}}{(1-|\psi_i(z)|^2)} \Big| \Big| \Big[(1-|\varphi_i(z)|^2) f_i(\varphi_i(z)) - (1-|\psi_i(z)|^2) f_i(\psi_i(z)) \Big] \Big| \Big\} \\
&\le \sum_{i=1}^{n} \Big\{ \|f_i\|_{\mathcal{N}_{K_i}(\mathbb{D})} \sup_{z\in\mathbb{D}} \Big| \frac{|u_i(z)|(1-|z|^2)^{\alpha_i}}{(1-|\varphi_i(z)|^2)} - \frac{|v_i(z)|(1-|z|^2)^{\alpha_i}}{(1-|\psi_i(z)|^2)} \Big| \\
&\quad + \sup_{z\in\mathbb{D}} \Big(\frac{|v_i(z)|(1-|z|^2)^{\alpha_i}}{(1-|\psi_i(z)|^2))} \rho(\varphi_i(z),\psi_i(z)) \Big) \|f_i\|_{\mathcal{N}_{K_i}(\mathbb{D})} \Big\} \\
&\le \Big\{ \|f\|_{\prod_{i=1}^{n} \mathcal{N}_{K_i}(\mathbb{D})} \sup_{1\le i\le n} \sup_{z\in\mathbb{D}} \Big| \frac{|u_i(z)|(1-|z|^2)^{\alpha_i}}{(1-|\varphi_i(z)|^2)} - \frac{|v_i(z)|(1-|z|^2)^{\alpha_i}}{(1-|\psi_i(z)|^2)} \Big| \\
&\quad + \sup_{1\le i\le n} \sup_{z\in\mathbb{D}} \Big(\frac{|v_i(z)|(1-|z|^2)^{\alpha_i}}{(1-|\psi_i(z)|^2))} \rho(\varphi_i(z),\psi_i(z)) \Big) \|f\|_{\prod_{i=1}^{n} \mathcal{N}_{K_i}(\mathbb{D})} \Big\}.
\end{aligned}
$$

The last inequality is obtained by taking in to account that $\mathcal{N}_{K_i}(\mathbb{D}) \subset \mathcal{H}_1^\infty(\mathbb{D})$ ([7, Proposition 2.1]) for $i = 1, \cdots, n$ and Lemma 2.1. This means that, under the conditions (2.2) and (2.3) we have

$$
\left\| \mathbf{T}_{\varphi,\,\psi}(f) \right\|_{\prod_{i=1}^{n} \mathcal{H}_{\alpha_i}^\infty(\mathbb{D})} \le C \, \|f\|_{\prod_{i=1}^{n} \mathcal{N}_{K_i}(\mathbb{D})} \quad \text{for all} \quad f = (f_1, \cdots, f_n) \in \prod_{i=1}^{n} \mathcal{N}_{K_i}(\mathbb{D}),
$$

where C is a positive constant.

Therefore $\mathbf{T}_{\varphi,\psi}$ is bounded form $\prod_{i=1}^{n} \mathcal{N}_{K_i}(\mathbb{D})$ to $\prod_{i=1}^{n} \mathcal{H}_{\alpha_i}^\infty(\mathbb{D})$ as required.

(2) $\Rightarrow$ (3). Observe that for $i = 1, \cdots, n$ we have

$$
\begin{aligned}
\Big(\frac{|v_i(z)|(1-|z|^2)^{\alpha_i}}{(1-|\psi(z)|^2)} \rho(\varphi_i(z),\psi_i(z)) \Big) \;\le\; & \Big(\frac{|u_i(z)|(1-|z|^2)^{\alpha}}{(1-|\varphi_i(z)|^2)} \rho(\varphi_i(z),\psi_i(z)) \Big) \\
& + \Big| \frac{|u_i(z)|(1-|z|^2)^{\alpha_i}}{(1-|\varphi_i(z)|^2)} - \frac{|v_i(z)|(1-|z|^2)^{\alpha_i}}{(1-|\psi_i(z)|^2)} \Big| \rho(\varphi_i(z),\psi_i(z)),
\end{aligned}
$$

which implies that (2.3) holds.

Finally we show the implication (1) $\Rightarrow$ (2).
Assume that $\mathbf{T}_{\varphi,\,\psi}$ is bounded from $\prod_{i=1}^{n} \mathcal{N}_{K_i}(\mathbb{D})$ to $\prod_{i=1}^{n} \mathcal{H}_{\alpha_i}^{\infty}(\mathbb{D})$ and prove that (2.1) and (2.3) hold.
Since $\mathbf{T}_{\varphi,\psi}$ is bounded, we have for all $f = (f_1, \cdots, f_n) \in \prod_{i=1}^{n} \mathcal{N}_{K_i}(\mathbb{D})$

$$\left\|\mathbf{T}_{\varphi,\psi}(f)\right\|_{\prod_{i=1}^{n} \mathcal{H}_{\alpha_i}^{\infty}(\mathbb{D})} \lesssim \|f\|_{\prod_{i=1}^{n} \mathcal{N}_{K_i}(\mathbb{D})}.$$

For each $z \in \mathbb{D}$, set for $i = 1, \cdots, n$, $h_{\omega}^{i}(z) = \dfrac{1 - |\varphi_i(\omega)|^2}{\left(1 - \overline{\varphi_i(w)}z\right)^2}$ be the function test in Lemma 1.1.

By taking into account Lemma 1.1, we have $h_w = (h_{\omega}^{1}, \cdots, h_{\omega}^{n}) \in \prod_{i=1}^{n} \mathcal{N}_{K_i}(\mathbb{D})$ and

$$\|h_w\|_{\mathcal{N}_K(\mathbb{D})} \lesssim 1.$$

Furthermore

$$\begin{aligned} \infty > \|\mathbf{T}_{\varphi,\,\psi}(h_{\omega})\|_{\prod_{i=1}^{n} \mathcal{H}_{\alpha_i}^{\infty}(\mathbb{D})} &\geq (1 - |w|^2)^{\alpha_i}\left|u_i(\omega)h_{\omega}(\varphi_i(\omega)) - v_i(\omega)h_{\omega}^{i}(\psi_i(\omega))\right| \\ &\geq |A(\omega) + B(\omega)|, \end{aligned}$$

where

$$A(\omega) = \frac{(1 - |\omega|^2)u_i(\omega)}{\left(1 - |\varphi_i(\omega)|^2\right)^2} - \frac{(1 - |\omega|^2)v_i(\omega)}{\left(1 - |\psi_i(\omega)|^2\right)^2}$$

and

$$B(\omega) = \frac{(1 - |\omega|^2)u_i(\omega)}{\left(1 - |\varphi_i(\omega)|^2\right)^2}\Big[(1 - |w|^2)^{\alpha_i}u_i(\omega)h_{\omega}^{i}(\varphi_i(\omega)) - (1 - |w|^2)^{\alpha_i}v_i(\omega)h_{\omega}^{i}(\psi_i(\omega))\Big].$$

In view of Lemma 2.1 and (2.1) we deduce that $|B(\omega)| < \infty$ for all $w \in \mathbb{D}$, which implies that $|A(\omega)| < \infty$ for all $w \in \mathbb{D}$. Thus, the condition (2.3) is proved.
Fix $\omega \in \mathbb{D}$ and consider the function g_{ω} defined by

$$g_{\omega}(z) = (g_{\omega}^{1}(z), \cdots, g_{\omega}^{n}(z)),$$

where

$$g_{\omega}^{i}(z) = \frac{1 - |\varphi_i(\omega)|^2}{\left(1 - \overline{\varphi_i(\omega)}z\right)^2} \times \frac{\varphi_{\psi_i(\omega)}(z)}{\varphi_{\psi_i(\omega)}(\varphi_i(\omega))},\ i = 1, \cdots, n$$

for $z \in \mathbb{D}$. We have

$$\|g_{\omega}\|_{\prod_{i=1}^{n} \mathcal{N}_K(\mathbb{D})} \leq C\|h_{\omega}\|_{\prod_{i=1}^{n} \mathcal{N}_{K_i}(\mathbb{D})}.$$

Thus $g_{\omega} \in \prod_{i=1}^{n} \mathcal{N}_{K_i}(\mathbb{D})$ and moreover

$$g_{\omega}(\varphi(\omega)) = h_{\omega}(\varphi(\omega)),\ g_{\omega}(\psi(\omega)) = 0. \tag{2.4}$$

By the boundedness of

$$\mathbf{T}_{\varphi,\,\psi} = W_{\varphi,\,u} - W_{\psi,\,v} : \prod_{i=1}^{n} \mathcal{N}_{K_i}(\mathbb{D}) \longrightarrow \prod_{i=1}^{n} \mathcal{H}_{\alpha_i}^{\infty}(\mathbb{D}),$$

then it follows that

$$\begin{aligned}
\infty > \|\mathbf{T}_{\varphi,\,\psi} g_\omega\|_{\mathcal{H}^\infty_\alpha(\mathbb{D})} &= \sum_{i=1}^{n} \sup_{z\in\mathbb{D}} \Big((1-|z|^2)^{\alpha_i}\Big|u_i(z)g^i_\omega(\varphi_i(z) - v_i(z)g^i_\omega(\psi_i(z))\Big|\Big)\\
&\geq \Big((1-|\omega|^2)^{\alpha_i}\Big|u_i(\omega)g^i_\omega(\varphi-i(\omega) - v_i(\omega)g^i_\omega(\psi_i(\omega))\Big|\Big),\ \forall\ i=1,\cdots,n\\
&\geq \frac{(1-|\omega|^2)^{\alpha_i}|u_i(\omega)|(1-|\varphi_i(\omega)|^2)}{(1-|\varphi_i(\omega)|^2)^2}\ \ \forall\ i=1,\cdots,n\\
&\geq \frac{(1-|\omega|^2)^{\alpha_i}|u_i(\omega)|}{1-|\varphi_i(\omega)|^2}\\
&\geq \frac{(1-|\omega|^2)^{\alpha_i}|u_i(\omega)|}{1-|\varphi_i(\omega)|^2}\rho(\varphi_i(z),\psi_i(z))\ \ \forall\ i=1,\cdots,n.
\end{aligned}$$

Hence the condition (2.2) is satisfied. □

Remark 2.1. *the statement* (1) *of Theorem 1.1 follows easily for the simple case* $n = 1$ *and* $v_i \equiv 0, i = 1,\cdots,n$ *of Theorem 2.1.*

Corollary 2.1. *Let* $K_i : [0, \infty) \longrightarrow [0, \infty)$ *be a nondecreasing function for* $i = 1,\cdots,n$, $\varphi = (\varphi_1,\cdots,\varphi_n)$ *and* $\psi = (\psi_1,\cdots,\psi_n)$ *are in* $\mathcal{O}(\mathbb{D})^n$*. For* $u = (u_1,\cdots,u_n) \in \mathcal{O}(\mathbb{D})^n$ *and* $\alpha = (\alpha_1,\cdots,\alpha_n)$, $\alpha_i > 0$*, then* $uC_\varphi - uC_\psi : \prod_{i=1}^n \mathcal{N}_K(\mathbb{D}) \longrightarrow \prod_{i=1}^n \mathcal{H}^\infty_{\alpha-i}(\mathbb{D})$ *is bounded if and only if the following two conditions hold:*

$$\sup_{1\leq i\leq n}\left(\sup_{z\in\mathbb{D}}\left(\frac{(1-|z|^2)^{\alpha_i}|u_i(z)|}{1-|\varphi_i(z)|^2}\rho(\varphi_i(z),\psi_i(z))\right)\right) < \infty, \tag{2.5}$$

and

$$\sup_{1\leq i\leq n}\left(\sup_{z\in\mathbb{D}}\left(\frac{(1-|z|^2)^{\alpha_i}|u_i(z)|}{1-|\psi_i(z)|^2}\rho(\varphi_i(z),\psi_i(z))\right)\right) < \infty. \tag{2.6}$$

Proof. Assume that $\mathbf{T}_{\varphi,\,\psi}$ is bounded. Then by letting $v = u$ in Theorem 2.1 it follows that the conditions (2.5) and (2.6) hold.

Conversely, assume that the conditions (2.5) and (2.6) hold. To prove that $\mathbf{T}_{\varphi,\,\psi}$ is bounded, it suffices in view of Theorem 2.1 to prove that

$$\sup_{1\leq i\leq n}\left(\sup_{z\in\mathbb{D}}\left(\frac{(1-|z|^2)^{\alpha_i}|u_i(z)|}{1-|\varphi_i(z)|^2} - \frac{(1-|z|^2)^{\alpha_i}|u_i(z)|}{1-|\psi-i(z)|^2}\right)\right) < \infty.$$

In fact, we have for $i = 1,\cdots,n$

$$\begin{aligned}
&\left|\frac{(1-|z|^2)^{\alpha_i}|u_i(z)|}{1-|\varphi_i(z)|^2} - \frac{(1-|z|^2)^{\alpha_i}|u_i(z)|}{1-|\psi_i(z)|^2}\right|\\
=\ &\frac{(1-|z|^2)^{\alpha_i}|u_i(z)|}{1-|\varphi(z)|^2}\left|1-\frac{(1-|\varphi_i(z)|^2)}{1-|\psi_i(z)|^2}\right|\\
\leq\ &\frac{(1-|z|^2)^{\alpha_i}|u_i(z)|}{1-|\varphi_i(z)|^2}\left|1-\frac{1+\rho(\varphi_i(z),\psi_i(z))}{1-\rho(\varphi_i(z),\psi_i(z))}\right|
\end{aligned}$$

$$\leq \quad \frac{(1-|z|^2)^{\alpha_i}|u_i(z)|}{1-|\varphi(z)|^2}\frac{2\rho(\varphi_i(z),\psi_i(z))}{1-\rho(\varphi_i(z),\psi_i(z))} < \infty.$$

Using Theorem 2.1, we obtain the boundedness of $uC_\varphi - uC_\psi : \prod_{i=1}^n \mathcal{N}_{K_i}(\mathbb{D}) \longrightarrow \prod_{i=1}^n \mathcal{H}_{\alpha_i}^\infty(\mathbb{D})$. The proof of the corollary is complete. □

Remark 2.2. *There exist non-bounded multivariable weighted composition operators such that their difference is bounded.*

In the following example we give operators such that neither $W_{\varphi,\,u}$, $W_{\psi,\,v}$ and $T_{\varphi,\,\psi} = W_{\varphi,\,u} - W_{\psi,\,v}$ are bounded from $\prod_{i=1}^n \mathcal{N}_{K_i}(\mathbb{D})$ to $\prod_{i=1}^n \mathcal{H}_{\alpha_i}^\infty(\mathbb{D})$.

Example 2.1. *By choosing the maps u, v, φ and ψ as follows:*

$$u_i(z) = v_i(z) \equiv 1, \text{ for } i = 1, \cdots, n \quad \text{and} \quad \varphi_i(z) = z,\ \psi_i(z) = -z,\ i = 1, \cdots, n \ \ 0 < \alpha_i < \frac{1}{2}; i = 1, \cdots, n.$$

A direct calculation shows

$$\sup_{z\in\mathbb{D}}\Big(\frac{|u_i(z)|(1-|z|^2)^{\alpha_i}}{(1-|\varphi_i(z)|^2))}\rho(\varphi_i(z),\psi_i(z))\Big) = \sup_{z\in\mathbb{D}}\Big(\frac{|v_i(z)|(1-|z|^2)^{\alpha_i}}{(1-|\psi_i(z)|^2))}\rho(\varphi_i(z),\psi_i(z))\Big) = \infty,\ i = 1, \cdots, n.$$

In view of Theorem 2.1, it follows that neither $W_{\varphi,\,u} : \prod_{i=1}^n \mathcal{N}_K(\mathbb{D}) \longrightarrow \prod_{i=1}^n \mathcal{H}_{\alpha_i}^\infty(\mathbb{D})$ nor $W_{\psi,\,v} : \prod_{i=1}^n \mathcal{N}_{K_i}(\mathbb{D}) \longrightarrow H_{\alpha_i}^\infty(\mathbb{D})$ is bounded. However from condition (2.1) *or* (2.2) *it is clear that the difference operator $W_{\varphi,u} - W_{\psi,\,v} : \prod_{i=1}^n \mathcal{N}_{K_i}(\mathbb{D}) \longrightarrow \prod_{i=1}^n \mathcal{H}_{\alpha_i}^\infty(\mathbb{D})$ is not bounded.*

The following theorem characterize when the difference weighted composition operators $\mathbf{T}_{\varphi,\,\psi}$ acting between weighted analytic type spaces $\prod_{i=1}^n \mathcal{N}_K(\mathbb{D})$ and $\prod_{i=1}^n \mathcal{H}_{\alpha_i}^\infty(\mathbb{D})$ are compact.

Theorem 2.2. *Let $\varphi = (\varphi_1, \cdots, \varphi_n), \psi = (\psi_1, \cdots, \psi_n) \in \mathcal{S}(\mathbb{D})^n$ and $u = (u_1, \cdots, u_n), v = (v_1, \cdots, v_n) \in \mathcal{O}(\mathbb{D})^n$. Let further $W_{\varphi,u}$ and $W_{\psi,v}$ be two weighted composition operators acting from $\prod_{i=1}^n \mathcal{N}_{K_i}(\mathbb{D})$ into $\prod_{i=1}^n H_{\alpha_i}^\infty(\mathbb{D})$. Then the operators $T_{\varphi,\psi} = W_{\varphi,u} - W_{\psi,v}$ is compact if and only if the following conditions hold.*

$$\lim_{r\longrightarrow 1^-}\Big(\sup_{1\le i\le n}\sup_{|\varphi_i(z)|>r}\Big(\frac{|u_i(z)|(1-|z|^2)^{\alpha_i}}{(1-|\varphi_i(z)|^2)}\rho(\varphi_i(z),\psi_i(z))\Big)\Big) = 0, \tag{2.7}$$

$$\lim_{r\longrightarrow 1^-}\Big(\sup_{1\le i\le n}\sup_{|\psi_i(z)|>r}\Big(\frac{|v_i(z)|(1-|z|^2)^{\alpha_i}}{(1-|\psi_i(z)|^2)}\rho(\varphi_i(z),\psi_i(z))\Big)\Big) = 0, \tag{2.8}$$

$$\lim_{r\longrightarrow 1^-}\Big(\sup_{1\le i\le n}\sup_{\min\{|\varphi_i(z)|,\ |\psi_i(z)|\ \}>r}(\Lambda_i(z))\Big) = 0, \tag{2.9}$$

where

$$\Lambda_i(z) = |u_i(z) - v_i(z)|\, min\Big[\frac{(1-|z|^2)^{\alpha_i}}{(1-|\varphi_i(z)|^2}, \frac{(1-|z|^2)^{\alpha_i}}{(1-|\psi_i(z)|^2)}\Big].$$

Proof. We omit the proof, since the techniques are similar to those of [14, Theorem 2.4]. □

Corollary 2.2. *Let $K_i : [0, \infty) \longrightarrow [0, \infty)$ be a nondecreasing function for $i = 1, \cdots, n$, $\varphi = (\varphi_1, \cdots, \varphi_n)$ and $\psi = (\psi_1, \cdots, \psi_n)$ where φ_i and ψ_i are analytic self-maps from $\mathbb{D}$ to $\mathbb{D}$ for $i = 1, \cdots, n$. For $u = (u_1, \cdots, u_n) \in \mathcal{O}(\mathbb{D})^n$ and $\alpha = (\alpha_1, \cdots, \alpha_n)$, where $\alpha_i > 0, i = 1, \cdots, n$ then*

$uC_\varphi - uC_\psi : \prod_{i=1}^n \mathcal{N}_{K_i}(\mathbb{D}) \longrightarrow \prod_{i=1}^n \mathcal{H}_{\alpha_i}^\infty(\mathbb{D})$ is compact if and only if the following two conditions hold:

$$\lim_{r\longrightarrow 1^-}\left(\sup_{1\le i\le n}\sup_{|\varphi_i(z)|>r}\left(\frac{(1-|z|^2)^{\alpha_i}|u_i(z)|}{1-|\varphi_i(z)|^2}\rho(\varphi_i(z),\psi_i(z))\right)\right)=0 \tag{2.10}$$

and

$$\lim_{r\longrightarrow 1^-}\left(\sup_{1\le i\le n}\sup_{|\psi_i(z)|>r}\left(\frac{(1-|z|^2)^{\alpha_i}|u_i(z)|}{1-|\psi_i(z)|^2}\rho(\varphi_i(z),\psi_i(z))\right)\right)=0. \tag{2.11}$$

Proof. Assume that $\mathbf{T}_{\varphi,\psi}$ is compact. Then by letting $v = u$ in Theorem 2.2 it follows that the conditions (2.10) and (2.11) hold.
Conversely, assume that the conditions (2.10) and (2.11) hold. To prove that $\mathbf{T}_{\varphi,\psi}$ is compact, it suffices in view of Theorem 2.2 to prove that the condition (2.9) is holds. Since $u \equiv v$, then $\lim_{r\longrightarrow 1^-}\left(\sup_{1\le i\le n}\sup_{\min\{|\varphi_i(z)|,\,|\psi_i(z)|\}>r}(\Lambda_i(z))\right) = 0$. Using Theorem 2.2, we obtain the compactness of $uC_\varphi - uC_\psi : \prod_{i=1}^n \mathcal{N}_{K_i}(\mathbb{D}) \longrightarrow \prod_{i=1}^n \mathcal{H}_{\alpha_i}^\infty(\mathbb{D})$. The proof of the corollary is complete. □

2.2. *Differences of weighted composition operators from $\prod_{i=1}^n \mathcal{H}_{\alpha_i}^\infty(\mathbb{D})$ into $\prod_{i=1}^n \mathcal{N}_{K_i}(\mathbb{D})$*

In this section, we investigate the boundedness of differences weighted composition operators $T_{\varphi,\psi} := W_{\varphi,u} - W_{\psi,v} : \prod_{i=1}^n \mathcal{H}_{\rangle\alpha_i}^\infty(\mathbb{D}) \longrightarrow \prod_{i=1}^n \mathcal{N}_{K_i}(\mathbb{D})$.

Theorem 2.3. *Let $K_i : [0, \infty) \longrightarrow [0, \infty)$ be a nondecreasing function for $i = 1, \cdots, n$, $\varphi = (\varphi_1. \cdots, \varphi_n)$ and $\psi = (\psi_1, \cdots, \psi_n)$ where φ_i and ψ_i for $i = 1, \cdots, n$ are analytic self-maps from $\mathbb{D}$ to $\mathbb{D}$ for $i = 1, \cdots, n$. let $u = (u_1, \cdots, u_n) \in \mathcal{O}(\mathbb{D})^n$, $v = (v_1, \cdots, v_n) \in \mathcal{O}(\mathbb{D})^n$ and $\alpha = (\alpha_1, \cdots, \alpha_n)$, with $\alpha_i > 0, i = 1, \cdots, n$. Then the operator $\mathbf{T}_{\varphi,\psi} : \prod_{i=1}^n \mathcal{H}_{\alpha_i}^\infty(\mathbb{D}) \longrightarrow \prod_{i=1}^n \mathcal{N}_{K_i}(\mathbb{D})$ is bounded if the following condition is satisfies* $\max(I, J) < \infty$,*where*

$$I = \sup_{a\in\mathbb{D}}\left(\sup_{1\le i\le n}\int_{\mathbb{D}}\frac{|u_i(z)|^2}{(1-|\varphi_i(z)|^2)^{2\alpha_i}}K_i(g(z,a))A(z)\right)$$

and

$$J = \sup_{a\in\mathbb{D}}\left(\sup_{1\le i\le n}\int_{\mathbb{D}}\frac{|v_i(z)|^2}{(1-|\psi(z)|^2)^{2\alpha_i}}K_i(g(z,a))dA(z)\right).$$

Proof. Assume that the condition in the statement (2) is holds and let $f = (f_1, \cdots, f_n) \in H_\alpha^\infty(\mathbb{D})$. We have

$$\begin{aligned}&\left\|\mathbf{T}_{\varphi,\psi}(f)\right\|_{\mathcal{N}_K(\mathbb{D})}\\ =\;& \sum_{k=1}^n \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left|T_{\varphi_k,\psi_k}(f_k)(z)\right|^2 K_k(g(z,a))dz\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|u_k T_{\varphi_k}(f_k)(z) - v_k C_{\psi} f_k(z)\right|^2 K_k(g(z,a))dz \\
&= \sum_{k=1}^{n} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |u_k(z) f_k(\varphi_k(z)) - v_k(z) f_k(\psi_k(z))|^2 K_k(g(z,a))dz \\
&\leq \sum_{k=1}^{n} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \Big(|u_k(z) f_k(\varphi(z))| + |(z) f_k(\psi_k(z))|\Big)^2 K_k(g(z,a))dz \\
&\leq 2\sum_{k=1}^{n} \int_{\mathbb{D}} \Big(|u_k(z) f_k(\varphi(z))|^2 + |v_k(z) f_k(\psi_k(z))|^2\Big) K_k(g(z,a))dA(z) \\
&= 2\sum_{k=1}^{n} \sup \int_{\mathbb{D}} |u_k(z) f_k(\varphi(z))|^2 K_k(g(z,a))dz + 2\sum_{k=1}^{n} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |v_k(z) f_k(\psi(z))|^2 K_k(g(z,a))dA(z) \\
&= 2\sum_{k=1}^{n} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|u_k(z)|^2}{(1-|\varphi_k(z)|^2)^{2\alpha_k}} (1-|\varphi(z)|^2)^{2\alpha_k} |f_k(\varphi_k(z))|^2 |K_k(g(z,a))dA(z) \\
&\quad +2\sum_{k=1}^{n} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|v_k(z)|^2}{(1-|\psi_k(z)|^2)^{2\alpha_k}} (1-|\psi(z)|^2)^{2\alpha_k} |f_k(\psi_k(z))|^2 K_k(g(z,a))dz \\
&\leq 2\sum_{k=1}^{n} \|f\|_{\mathcal{H}^{\infty}_{\alpha_k}(\mathbb{D})} \left(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|u_k(z)|^2}{(1-|\varphi(z)|^2)^{2\alpha_k}} K_k(g(z,a))dz\right) \\
&\quad +2\sum_{k=1}^{n} \|f_k\|_{\mathcal{H}^{\infty}_{\alpha_k}(\mathbb{D})} \left(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|v_k(z)|^2}{(1-|\psi(z)|^2)^{2\alpha_k}} K_k(g(z,a))dA(z)\right) \\
&\leq 2\,\|f\|_{\mathcal{H}^{\infty}_{\alpha,\,\omega}(\mathbb{D})}\, I + 2\,\|f\|_{H^{\infty}_{\alpha}(\mathbb{D})}\, J \\
&\leq C\mathcal{H}^{\infty}_{\alpha,\,\omega}(\mathbb{D}).
\end{aligned}
$$

This means that $\mathbf{T}_{\varphi,\,\psi}$ is bounded form $\prod_{k=1}^{n} \mathcal{H}^{\infty}_{\alpha_k}(\mathbb{D})$ to $\prod_{k=1}^{n} \mathcal{N}_K(\mathbb{D})$. □

Finally, it seems to be natural to enquire a necessary and sufficient conditions for the boundedness and compactness of difference weighted composition operator

$$\mathbf{T}_{\varphi,\,\psi} : \prod_{k=1}^{n} H^{\infty}_{\alpha_k}(\mathbb{D}) \longrightarrow \prod_{k=1}^{n} \mathcal{N}_{K_k}(\mathbb{D}).$$

This problem will be addressed in a forthcoming paper.

3. Conclusions

We have extended the characterizations of compactness of differences of two weighted composition for single operator between weighted-type spaces of analytic functions to several variables differences of Weighted composition operators. Namely, $\mathbf{T}_{\varphi,\,\psi} := W_{\varphi,\,u} - W_{\psi,\,v} : \prod_{k=1}^{n} \mathcal{N}_{K_k}\mathbb{D}) \longrightarrow \prod_{k=1}^{n} \mathcal{H}^{\infty}_{\alpha_k}(\mathbb{D})$ where

$$\mathbf{T}_{\varphi,\psi}(f) = \Big(u_1 f_1 \circ \varphi_1 - v_1 f_1 \circ \psi_1, \cdots, u_n f_n \circ \varphi_n - v_n f_n \circ \psi_n\Big).$$

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was funded by the Deanship of Scientific Research at Jouf University under grant No (DSR-2021-03-0231).

We would like to thank reviewers for taking the necessary time and effort to review the manuscript. We sincerely appreciate all valuable comments and suggestions, which helped us to improve the quality of the article.

Conflict of interest

The author declares that he has no competing interests.

References

1. R. F. Allen, F. Colonna, Weighted composition operators from $\mathbf{H}_\alpha$ to the Bloch space of a bounded homogeneous domain, *Integr. Equat. Oper. Th.*, **66** (2010), 21–40. https://doi.org/10.1007/s00020-009-1736-4
2. J. Bonet, M. Lindstrom, E. Wolf, Differences of composition operators between weighted Banach spaces of holomorphic functions, *J. Aust. Math. Soc.*, **84** (2008), 9–20. https://doi.org/10.1017/S144678870800013X
3. B. S. Bourdon, J. A. Cima, A. L. Matheson, Compact composition operators on BMOA, *T. Am. Math. Soc.* **351** (1999), 2183–2169.
4. B. Choe, H. Koo, I. Park, Compact differences of composition operators over polydisks, *Integr. Equat. Oper. Th.*, **73** (2013), 57–91. https://doi.org/10.1007/s00020-012-1962-z
5. C. Cowen, B. Maccluer, *Composition operators on spaces of analytic functions*, Boca Raton: CRC Press, 1995.
6. X. Fu, Differences of weighetd composition operator from weighted Bergman spaces to weighted-type spaces, *Bull. Math. Anal. Appl.*, **5** (2013), 65–70.
7. A. E. S. Ahmed, M. A. Bakhit, *Holomorphic $\mathcal{N}_K$ and Bergman-type spaces*, Birkhuser Series on Operator Theory: Advances and Applications, Basel: BirkhuserVerlag Publisher, **195** (2009), 121–138. https://doi.org/10.1007/978-3-0346-0174-0-5
8. A. E. S. Ahmed, M. A. Bakhit, Hadamard products and $\mathcal{N}_K$ spaces, *Math. Comput. Model.*, **51** (2010), 33–43. https://doi.org/10.1016/j.mcm.2009.08.037
9. K. Heller, B. D. Maccluer, R. J. Weir, Compact differences of composition operators in several variables, *Integr. Equat. Oper. Th.*, **69** (2011), 419–428. https://doi.org/10.1007/s00020-010-1840-5

10. T. Hosokawa, S. Ohno, Differences of weighted composition operators from H^{∞} to Bloch space, *Taiwan. J. Math.*, **16** (2012), 2093–2105. https://doi.org/10.11650/twjm/1500406842

11. T. Hosokawa, Differences of weighted composition operators on the Bloch spaces, *Complex Anal. Oper. Th.*, **3** (2009), 847–866. https://doi.org/10.1007/s11785-008-0062-1

12. T. Hosokawa, S. Ohno, Differences of composition operators on the Bloch spaces, *J. Operat. Theor.*, **57** (2007), 229–242.

13. L. Jiang, C. Ouyang, Compact differences of composition operators on holomorphic function spaces in the unit ball, *Acta Math. Sci.*, **31** (2011), 1679–1693. https://doi.org/10.1016/S0252-9602(11)60291-9

14. B. Hu, L. H. Khoi, Compact difference of weighted composition operators on $\mathcal{N}_p$ spaces in the ball, *Roum. J. Pure Appl. Math.*, **60** (2015), 101–116.

15. A. E. Shammahy, Weighted composition operators acting between kind of weighted Bergman-type spaces and the Bers-type space, *Inter. J. Math. Comput. Phys. Electr. Comput. Eng.*, **8** (2014), 496–499.

16. M. Lindstrom, E. Wolf, Essential norm of the difference of weighted composition operators, *Monatsh. Math.*, **153** (2000), 133–143. https://doi.org/10.1007/s00605-007-0493-1

17. P. J. Nieminen, Compact differences of composition operators on Bloch and Lipschitz spaces, *Comput. Meth. Funct. Th.*, **7** (2007), 325–344. https://doi.org/10.1007/BF03321648

18. C. Shaolin, H. Hidetaka, Z. J. Feng, Composition operators on Bloch and Hardy type spaces, *Math. Z.*, **301** (2022), 3939–3957. https://doi.org/10.1007/s00209-022-03046-z

19. C. Shaolin, H. Hidetaka, Harmonic Lipschitz type spaces and composition operators meet majorants, *J. Geom. Anal.*, **33** (2023), 181. https://doi.org/10.1007/s12220-023-01232-x

20. J. H. Shapiro, *Composition operators and classical function theory*, New York: Springer-Verlag, 1993. https://doi.org/10.1007/978-1-4612-0887-7

21. R. Zhao, On α-Bloch functions and VMOA, *Acta Math. Sci.*, **3** (1996), 349–360. https://doi.org/10.1016/S0252-9602(17)30811-1

22. K. Zhu, *Operator theory in function spaces*, New York: Marcel Dekker, 2007. https://doi.org/10.1090/surv/138

23. X. Zhua, W. Yang, Differences of composition operators from weighted Bergman spaces to Bloch spaces, *Filomat*, **28** (2014), 1935–1941. https://doi.org/10.2298/FIL1409935Z

24. S. Ueki, Weighted composition operators acting between the $\mathcal{N}_p$-space and the Weighted-type space $\mathcal{H}_{\alpha}^{\infty}$, *Indagat. Math.*, **23** (2012), 243–255. https://doi.org/10.1016/j.indag.2011.11.006

25. E. Wolf, Weighted composition operators between weighted Bloch type spaces, *B. Soc. Royale Sci. Liège*, **80** (2011), 806–816.