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Research article

Analysis of COVID-19 outbreak in Democratic Republic of the Congo using fractional operators

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Abstract: The spread of COVID-19 in the Democratic Republic of the Congo is investigated in this work using fractional operators. To model the spread of the current COVID-19 variant among different age groups, we employ the epidemic scenario in the Democratic Republic of the Congo as a case study. In this study, the key characteristics of an epidemic problem such as COVID-19 are validated for existence and positivity, and unique solutions are demonstrated by applying certain findings from fixed-point theory. We also use the first derivative function to confirm the overall stability of the proposed system. The established methodology, which examines the impact of COVID-19 on various age groups, is highly sophisticated. Additionally, we use a method created by Atangana to solve the given model. This method stands as one of the most advanced approaches for addressing infectious problems; we also conduct an error analysis to identify and rectify any inaccuracies. Lastly, we assess the parameters to determine the effects of illness, and we provide numerical simulations implemented in MATLAB. These simulations illustrate the behavior of this infectious disease among various age groups in the Democratic Republic of the Congo.

Keywords: boundedness; global stability; uniqueness; positivity; Mittag-Leffler
Mathematics Subject Classification: 03C45, 33E12

1. Introduction

Mathematics has been applied in biology since the 12th century, when Fibonacci used the well-known Fibonacci series to describe a population's growth. Daniel Bernoulli applied mathematics to describe the impact of small forms. The term biomathematics was first coined by Johannes Reinke in 1901. Biomathematics primarily involves the theoretical analysis of mathematical models for the study of both the rules of the structural development and system behavior. It was developed to understand the curiosities of biological organisms. A case study in mathematical biology can be divided into various stages. The initial stage involves representing biological methods that can generate further biological queries, wherein mathematics could provide beneficial solutions. The second stage is explaining a mathematical process that can characterize an accurate biological model. The following stage involves implementing mathematical models and additional processes to derive mathematical rules from the model. The final step is drawing conclusions from the mathematical results regarding specific questions, considering biological methodologies.

COVID-19 (SARS-CoV-2) is a critical biological issue, representing a lethal pandemic that commenced global spread in the final quarter of 2019. There was an earlier wave in 2003, which was triggered by the coronavirus. This disease can be transmitted from one living organism to another and has an immediate impact. It spreads between individuals through actions like coughing, sneezing, talking, and breathing, making it airborne. Additionally, close contact with infected individuals or touching contaminated surfaces and then subsequently touching the eyes, nose, or mouth without proper hand hygiene can also lead to infection [1, 2].

Generally, disease symptoms begin to manifest 5 to 7 days after infection and peak at 2 to 12 days. To prevent disease transmission, individuals who have been affected must isolate themselves for 14 days. For a comprehensive understanding of this hidden transmission and the virus incubation period, refer to [3]. European researchers predicted that COVID-19 would likely spread in France around mid-January. However, this prediction turned out to be incorrect as the proportion of people infected by the virus was notably low in France and its neighboring regions [4]. Individuals aged over 70 who contracted COVID-19 while also having other conditions such as heart disease, lung disease, cancer and diabetes were at a higher risk of mortality [5]. However, the number of infected individuals was steadily increasing day by day [6, 7]. COVID-19 was so poorly publicized that a significant number of people were infected without being aware of its symptoms.

The original spread of this virus was disorderly and rapid. The rapid spread can be attributed to three primary factors: population density, the relatively short infection duration, and the virus's ease of transmission. A related study is presented in [8]. Another study aims to identify infected individuals and explore human-to-human transmission, as presented in [9]. The debatable aspects of identifying signs and symptoms in infected individuals, as well as determining the virus's transmission duration are discussed in [10] for USA and Chinese observers. The first case of COVID-19 was reported on March 10, 2020, in the Democratic Republic of the Congo (DRC). After six weeks, the virus had affected over 400 individuals in the USA; by three months, this number had surged to more than 5000 affected individuals [11]. These situations vividly illustrate the rapid spread of the disease, particularly in certain countries. The study highlights the spread of this disease in the DRC and underscores the parallels between its upward and downward slopes.

The index case in this context could impact contact instances; hence, discussions should remain open on this subject. The researcher formulated several mathematical models to illustrate the spread of infections, especially COVID-19 [12–19]. A mathematical model has been applied to demonstrate the fundamental role of the seafood market in the expansion of COVID-19 [20]. Mathematical models offer parameters for assessing COVID-19 and identify those parameters that aid in controlling the pandemic [21]. The proposed Atangana-Baleanu-Caputo (ABC) derivative model proves beneficial for both healthy and infected scenarios [22]. Fixed-point theory also provides support for the ABC derivative with fractional order [23]. The fractional ABC operator is a primary base for the mathematical version [24–26]. Another version features four elements: Vulnerable, uncovered, infected, and recovered [27].

In recent years, numerous definitions of fractional derivatives have been developed to create mathematical models for real-word systems. The primary objective of this study is to develop and analyze the Atangana-Baleanu fractional derivatives model of the COVID-19 pandemic. The existence and uniqueness of solutions for the fractional order system is established by using fixed-point theory and iterative methods. The effects of different parameters are shown graphically. The numerical results of the COVID-19 model, considering advanced fractional derivatives, are compared with the classical results for the COVID-19 model using various fractional parameters.

2. Preliminaries

This section covers the fundamental concepts of the Sumudu transform and fractional differential equations as described in references [27–29].

Definition 2.1. The fractional-order derivative of ABC operator in the Liouville-Caputo sense is defined as follows:

$$ {}^{ABC}_{\gamma_1}D_t^{\gamma_1}\{f(t)\} = \frac{AB(\gamma_1)}{m-\gamma_1}\int_{\gamma_1}^{t}\frac{d^m}{dw^m}f(w)E_{\gamma_1}[-\gamma_1\frac{(t-w)^{\gamma_1}}{m-\gamma_1}]dw, m-1<\gamma_1<m, $$

where E_{γ_1} is the Mittag-Leffler function and $AB(\gamma_1)$ is a normalization function with $AB(0) = AB(1) = 1$.

Definition 2.2. The Sumudu transform of the function $\psi(z)$ is defined as follows:

$$ S = \psi(z) : \exists\, \hbar,\ \chi_1, \chi_2 > 0,\ \ \psi(z) < \hbar\, exp(\frac{|\chi|}{\chi_1}), if\ \ z \in (-1)^j \times [0,\infty) $$

and

$$ F(v) = ST[\psi(z)] = \int_0^\infty exp(-\chi)\psi(v\chi)d\chi, v \in (-\chi_1, \chi_2. $$

Definition 2.3. The Atangana-Baleanu derivative is defined as follows:

$$ {}^{ABC}_{\alpha}D_t^{\alpha}(\psi(t)) = \frac{AB(\alpha)}{n-\alpha}\int_{\alpha}^{\chi}\frac{d^n}{dw^n}f(w)E_{\alpha}-\alpha\frac{(\chi-w)^{\alpha}}{n-\alpha}dw,\ n-1<\alpha<n. $$

Applying a Laplace transformation to the above equation, we obtain:

$$ L[{}^{ABC}_{\alpha}D_t^{\alpha}(\psi(t))](S) = \frac{AB(\alpha)}{1-\alpha}\frac{S^{\alpha}L[\psi(\tau)](S) - S^{\alpha-1}\psi(0)}{S^{\alpha}+\frac{\alpha}{1-\alpha}}. $$

Applying the Sumudu transform to the above equation yields:

$$ST[{}_0^{ABC}D_t^{\alpha}(\psi(t))](S) = \frac{B(\alpha)}{1-\alpha+\alpha S^{\alpha}} \times [ST\psi(t) - \psi(0)].$$

3. Materials and method

N is composed of seven compartments: S, E, I_1, I_2, I_3, R, and D which respectively represent susceptible, exposed, infectious begin from (young people), infectious respiration form (adults people), infectious reanimatory form (old and comorbidity people), recovered and COVID-19 deaths. The parameters a and μ represent the birth rate and the rate of natural mortality, respectively. Susceptible individuals (S) can become exposed when in contact with infected individuals I_1, I_2, and I_3 at a transmission rate β. Exposed individuals will progress to the infectious compartment I according to the rates ρ_1 to form I_1, ρ_2, to form I_2 and ρ_3 to form I_3. Infected individuals may be eliminated according to the rates γ_1 for I_1 form, γ_2 for I_2 form and γ_3 for I_3 form. The death rates due to COVID-19 are denoted as d_1, d_2, and d_3 for the individuals I_1, I_2, and I_3, respectively. Additionally, we assume that $I = I_1 + I_2 + I_3$. Consequently, the following set of seven differential equations constitutes the mathematical model [27]:

$$\begin{cases} \dfrac{dS}{dt} = aN - \beta\dfrac{S(t)I(t)}{N} - \mu S(t), \\ \dfrac{dE}{dt} = \beta\dfrac{S(t)I(t)}{N} - (\rho_1+\rho_2+\rho_3+\mu)E(t), \\ \dfrac{dI_1}{dt} = \rho_1 E(t) - (\mu+\gamma_1+d_1)I_1(t), \\ \dfrac{dI_2}{dt} = \rho_2 E(t) - (\mu+\gamma_2+d_2)I_2(t), \\ \dfrac{dI_3}{dt} = \rho_3 E(t) - (\mu+\gamma_3+d_3)I_3(t), \\ \dfrac{dR}{dt} = \sum_{i=1}^{3} \gamma_i I_i(t) - \mu R(t), \\ \dfrac{dD}{dt} = \sum_{i=1}^{3} d_i I_i(t), \\ I = \sum_{i=1}^{3} I_i(t), \\ N = S + E + I_1 + I_2 + I_3 + R + D, \end{cases} \tag{3.1}$$

with initial conditions $S(0) = S_0, E(0) = E_0, I_1(0) = I_{1(0)}, I_2(0) = I_{2(0)}, I_3(0) = I_{3(0)}, R(0) = R_0, D(0) = D_0$, and $I(0) = I_0$.

Utilizing the Atangana-Baleanu definition in the Caputo sense, we derive the mathematical model for COVID-19 as follows:

$$\begin{cases} {}_0^{ABC}D_t^{\rho}S = aN - \beta\frac{S(t)I(t)}{N} - \mu S(t), \\ {}_0^{ABC}D_t^{\rho}E = \beta\frac{S(t)I(t)}{N} - (\rho_1 + \rho_2 + \rho_3 + \mu)E(t), \\ {}_0^{ABC}D_t^{\rho}I_1 = \rho_1 E(t) - (\mu + \gamma_1 + d_1)I_1(t), \\ {}_0^{ABC}D_t^{\rho}I_2 = \rho_2 E(t) - (\mu + \gamma_2 + d_2)I_2(t), \\ {}_0^{ABC}D_t^{\rho}I_3 = \rho_3 E(t) - (\mu + \gamma_3 + d_3)I_3(t), \\ {}_0^{ABC}D_t^{\rho}R = \sum_{i=1}^{3}\gamma_i I_i(t) - \mu R(t), \\ {}_0^{ABC}D_t^{\rho}D = \sum_{i=1}^{3} d_i I_i(t), \\ {}_0^{ABC}D_t^{\rho}I = \sum_{i=1}^{3} I_i(t). \end{cases} \tag{3.2}$$

Here, ${}_0^{ABC}D_t^{\rho_i}$ represents the ABC fractional derivative, where $0 < \rho < 1$. The initial conditions for the system are given by: $S(0) = S_0$, $E(0) = E_0$, $I_1(0) = I_{1(0)}$, $I_2(0) = I_{2(0)}$, $I_3(0) = I_{3(0)}$, $R(0) = R_0$, $D(0) = D_0$, $I(0) = I_0$.

To approximate a solution of the system, we apply the Sumudu transform operator. The operator is utilized on both sides of Eq (3.2) as follows:

$$\frac{B(\rho)\rho\Gamma(\rho+1)}{1-\rho}E_\rho(-\frac{1}{1-\rho}\omega^\rho)ST[S(t)-S(0)] = ST[aN - \beta\frac{S(t)I(t)}{N} - \mu S(t)], \tag{3.3}$$

$$\frac{B(\rho)\rho\Gamma(\rho+1)}{1-\rho}E_\rho(-\frac{1}{1-\rho}\omega^\rho)ST[E(t)-E(0)] = ST[\beta\frac{S(t)I(t)}{N} - (\rho_1+\rho_2+\rho_3+\mu)E(t)], \tag{3.4}$$

$$\frac{B(\rho)\rho\Gamma(\rho+1)}{1-\rho}E_\rho(-\frac{1}{1-\rho}\omega^\rho)ST[I_1(t)-I_1(0)] = ST[\rho_1 E(t) - (\mu+\gamma_1+d_1)I_1(t)], \tag{3.5}$$

$$\frac{B(\rho)\rho\Gamma(\rho+1)}{1-\rho}E_\rho(-\frac{1}{1-\rho}\omega^\rho)ST[I_2(t)-I_2(0)] = ST[\rho_2 E(t) - (\mu+\gamma_2+d_2)I_2(t)], \tag{3.6}$$

$$\frac{B(\rho)\rho\Gamma(\rho+1)}{1-\rho}E_\rho(-\frac{1}{1-\rho}\omega^\rho)ST[I_3(t)-I_3(0)] = ST[\rho_3 E(t) - (\mu+\gamma_3+d_3)I_3(t)], \tag{3.7}$$

$$\frac{B(\rho)\rho\Gamma(\rho+1)}{1-\rho}E_\rho(-\frac{1}{1-\rho}\omega^\rho)ST[R(t)-R(0)] = ST[\sum_{i=1}^{3}\gamma_i I_i(t) - \mu R(t)], \tag{3.8}$$

$$\frac{B(\rho)\rho\Gamma(\rho+1)}{1-\rho}E_\rho(-\frac{1}{1-\rho}\omega^\rho)ST[D(t)-D(0)] = ST[\sum_{i=1}^{3} d_i I_i(t)], \tag{3.9}$$

$$\frac{B(\rho)\rho\Gamma(\rho+1)}{1-\rho}E_\rho(-\frac{1}{1-\rho}\omega^\rho)ST[I(t)-I(0)] = ST[\sum_{i=1}^{3} I_i(t)]. \tag{3.10}$$

Rearranging the equations above, we obtain

$$ST[S(t)] = S(0) + \frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \times ST[aN - \beta\frac{S(t)I(t)}{N} - \mu S(t)], \tag{3.11}$$

$$ST[E(t)] = E(0) + \frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \times ST[\beta\frac{S(t)I(t)}{N} - (\rho_1+\rho_2+\rho_3+\mu)E(t)], \tag{3.12}$$

$$ST[I_1(t)] = I_1(0) + \frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \times ST[\rho_1 E(t) - (\mu + \gamma_1 + d_1)I_1(t)], \tag{3.13}$$

$$ST[I_2(t)] = I_2(0) + \frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \times ST[\rho_2 E(t) - (\mu + \gamma_2 + d_2)I_2(t)], \tag{3.14}$$

$$ST[I_3(t)] = I_3(0) + \frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \times ST[\rho_3 E(t) - (\mu + \gamma_3 + d_3)I_3(t)], \tag{3.15}$$

$$ST[R(t)] = R(0) + \frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \times ST[\sum_{i=1}^{3} \gamma_i I_i(t) - \mu R(t)], \tag{3.16}$$

$$ST[D(t)] = D(0) + \frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \times ST[\sum_{i=1}^{3} d_i I_i(t)], \tag{3.17}$$

$$ST[I(t)] = I(0) + \frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \times ST[\sum_{i=1}^{3} I_i(t)]. \tag{3.18}$$

The following equations are derived by applying the inverse Sumudu transform to the system described by Eqs (3.11)–(3.18).

$$S(t) = S(0) + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \times ST[aN - \beta\frac{S(t)I(t)}{N} - \mu S(t)], \tag{3.19}$$

$$E(t) = E(0) + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \times ST[\beta\frac{S(t)I(t)}{N} - (\rho_1 + \rho_2 + \rho_3 + \mu)E(t)], \tag{3.20}$$

$$I_1(t) = I_1(0) + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \times ST[\rho_1 E(t) - (\mu + \gamma_1 + d_1)I_1(t)], \tag{3.21}$$

$$I_2(t) = I_2(0) + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \times ST[\rho_2 E(t) - (\mu + \gamma_2 + d_2)I_2(t)], \tag{3.22}$$

$$I_3(t) = I_3(0) + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \times ST[\rho_3 E(t) - (\mu + \gamma_3 + d_3)I_3(t)], \tag{3.23}$$

$$R(t) = R(0) + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \times ST[\sum_{i=1}^{3} \gamma_i I_i(t) - \mu R(t)], \tag{3.24}$$

$$D(t) = D(0) + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \times ST[\sum_{i=1}^{3} d_i I_i(t)], \tag{3.25}$$

$$I(t) = I(0) + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \times ST[\sum_{i=1}^{3} I_i(t)]. \tag{3.26}$$

Therefore, the following outcomes are achieved:

$$S_{n+1}(t) = S_n(0) + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \times ST[aN - \beta\frac{S_n(t)I_n(t)}{N} - \mu S_n(t)], \tag{3.27}$$

$$E_{n+1}(t) = E_n(0) + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)}\times ST[\beta\frac{S_n(t)I_n(t)}{N} - (\rho_1+\rho_2+\rho_3+\mu)E_n(t)], \tag{3.28}$$

$$I_{1(n+1)}(t) = I_{1(n)}(0) + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)}\times ST[\rho_1 E_n(t) - (\mu+\gamma_1+d_1)I_{1(n)}(t)], \tag{3.29}$$

$$I_{2(n+1)}(t) = I_{2(n)}(0) + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)}\times ST[\rho_2 E_n(t) - (\mu+\gamma_2+d_2)I_{2(n)}(t))], \tag{3.30}$$

$$I_{3(n+1)}(t) = I_{3(n)}(0) + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)}\times ST[\rho_3 E_n(t) - (\mu+\gamma_3+d_3)I_{3(n)}(t)], \tag{3.31}$$

$$R_{n+1}(t) = R_n(0) + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)}\times ST[\sum_{i=1}^{3}\gamma_i I_{i(n)}(t) - \mu R_n(t)], \tag{3.32}$$

$$D_{n+1}(t) = D_n(0) + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)}\times ST[\sum_{i=1}^{3} d_{i(n)}I_{i(n)}(t)], \tag{3.33}$$

$$I_{n+1}(t) = I_n(0) + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)}\times ST[\sum_{i=1}^{3} I_{i(n)}(t)]. \tag{3.34}$$

The solution obtained for Eqs (3.27)–(3.34) are presented as follows:

$$S(t) = \lim_{n\to\infty} S_n(t); \quad E(t) = \lim_{n\to\infty} E_n(t); \quad I(t) = \lim_{n\to\infty} I_n(t); \quad I_1(t) = \lim_{n\to\infty} I_{1(n)}(t); \quad I_2(t) = \lim_{n\to\infty} I_{2(n)}(t);$$
$$I_3(t) = \lim_{n\to\infty} I_{3(n)}(t); \quad R(t) = \lim_{n\to\infty} R_n(t); \quad D(t) = \lim_{n\to\infty} D_n(t); \quad I(t) = \lim_{n\to\infty} I_n(t).$$

4. Stability analysis by iterative method

Let $(X, |.|)$ be a Banach space and H a self-map of X. Consider a particular recursive procedure given by $z_{n+1} = g(H, z_n)$. We have the following conditions that are satisfied for $z_{n+1} = Hz_n$:

* At least one element exists in the fixed point set of H,
* z_n converges to $P \in F(H)$,
* $\lim_{n\to\infty} x_n(t) = P$.

Theorem 4.1. Let $(X, |.|)$ be a Banach space and H a self-map of X satisfying the inequality:

$$||H_x - H_r| \leq \theta||X - H_x| + \theta||x - r|$$

for all $x, r \in X$,where $0 \leq \theta \leq 1$. Assume that H is Picard H-stable.

Consider the following recursive formulas:

$$S_{n+1}(t) = S_n(0) + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)}\times ST[aN - \beta\frac{S_n(t)I_n(t)}{N} - \mu S_n(t)], \tag{4.1}$$

$$E_{n+1}(t) = E_n(0) + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)}\times ST[\beta\frac{S_n(t)I_n(t)}{N} - (\rho_1+\rho_2+\rho_3+\mu)E_n(t)], \tag{4.2}$$

$$I_{1(n+1)}(t)=I_{1(n)}(0)+ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_{\rho}(-\frac{1}{1-\rho}\omega^{\rho})}\times ST[\rho_1E_n(t)-(\mu+\gamma_1+d_1)I_{1(n)}(t)], \tag{4.3}$$

$$I_{2(n+1)}(t)=I_{2(n)}(0)+ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_{\rho}(-\frac{1}{1-\rho}\omega^{\rho})}\times ST[\rho_2E_n(t)-(\mu+\gamma_2+d_2)I_{2(n)}(t))], \tag{4.4}$$

$$I_{3(n+1)}(t)=I_{3(n)}(0)+ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_{\rho}(-\frac{1}{1-\rho}\omega^{\rho})}\times ST[\rho_3E_n(t)-(\mu+\gamma_3+d_3)I_{3(n)}(t)], \tag{4.5}$$

$$R_{n+1}(t)=R_n(0)+ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_{\rho}(-\frac{1}{1-\rho}\omega^{\rho})}\times ST[\sum_{i=1}^{3}\gamma_iI_{i(n)}(t)-\mu R_n(t)], \tag{4.6}$$

$$D_{n+1}(t)=D_n(0)+ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_{\rho}(-\frac{1}{1-\rho}\omega^{\rho})}\times ST[\sum_{i=1}^{3}d_{i(n)}I_{i(n)}(t)], \tag{4.7}$$

$$I_{n+1}(t)=I_n(0)+ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_{\rho}(-\frac{1}{1-\rho}\omega^{\rho})}\times ST[\sum_{i=1}^{3}I_{i(n)}(t)]. \tag{4.8}$$

where

$$\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_{\rho}(-\frac{1}{1-\rho}\omega^{\rho})} \tag{4.9}$$

corresponds to the fractional Lagrange multiplier.

Proof. We define a self-map K as follows:

$$K[S_{n+1}]=S_{n+1}(t)=S_n(0)+ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_{\rho}(-\frac{1}{1-\rho}\omega^{\rho})}\times ST[aN-\beta\frac{S_n(t)I_n(t)}{N}-\mu S_n(t)], \tag{4.10}$$

$$K[E_{n+1}]=E_{n+1}(t)=E_n(0)+ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_{\rho}(-\frac{1}{1-\rho}\omega^{\rho})}\times ST[\beta\frac{S_n(t)I_n(t)}{N}-(\rho_1+\rho_2+\rho_3+\mu)E_n(t)], \tag{4.11}$$

$$K[I_{1(n+1)}]=I_{1(n+1)}(t)=I_{1(n)}(0)+ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_{\rho}(-\frac{1}{1-\rho}\omega^{\rho})}\times ST[\rho_1E_n(t)-(\mu+\gamma_1+d_1)I_{1(n)}(t)], \tag{4.12}$$

$$K[I_{2(n+1)}]=I_{2(n+1)}(t)=I_{2(n)}(0)+ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_{\rho}(-\frac{1}{1-\rho}\omega^{\rho})}\times ST[\rho_2E_n(t)-(\mu+\gamma_2+d_2)I_{2(n)}(t))], \tag{4.13}$$

$$K[I_{3(n+1)}]=I_{3(n+1)}(t)=I_{3(n)}(0)+ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_{\rho}(-\frac{1}{1-\rho}\omega^{\rho})}\times ST[\rho_3E_n(t)-(\mu+\gamma_3+d_3)I_{3(n)}(t)], \tag{4.14}$$

$$K[R_{n+1}]=R_{n+1}(t)=R_n(0)+ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_{\rho}(-\frac{1}{1-\rho}\omega^{\rho})}\times ST[\sum_{i=1}^{3}\gamma_i(I_{i(n)}(t)\mu(R_n(t)], \tag{4.15}$$

$$K[D_{n+1}] = D_{n+1}(t) = D_n(0) + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)}\times ST[\sum_{i=1}^{3}(d_{i(n)}(t))(I_{i(n)}(t)], \tag{4.16}$$

$$K[I_{n+1}] = I_{n+1}(t) = I_n(0) + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)}\times ST[\sum_{i=1}^{3} I_{i(n)}(t)], \tag{4.17}$$

which is shown to be K-stable in $L^1(a,b)$.

By using the properties of the norm and accounting for the triangular inequality, we obtain the following when K satisfies the conditions outlined in Theorem 4.1:

$$\begin{aligned}
||K[S_n] - K[S_m(t)]|| &\le ||S_n(t) - S_m(t)|| + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \\
&\times ST\left[aN - \beta\frac{(S_n(t) - S_m(t))(I_n(t) - I_m(t))}{N} - \mu(S_n(t) - S_m(t))\right], \\
||K[E_n] - K[E_m(t)]|| &\le ||E_n(t) - E_m(t)|| + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \\
&\times ST\left[\beta\frac{(S_n(t) - S_m(t))(I_n(t) - I_m(t))}{N} - (\rho_1 + \rho_2 + \rho_3 + \mu)(E_n(t) - E_m(t))\right], \\
||K[I_{1(n)}] - K[I_{1(m)}(t)]|| &\le ||I_{1(n)}(t) - I_{1(m)}(t)|| + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \\
&\times ST[\rho_1(E_n(t) - E_m(t)) - (\mu + \gamma_1 + d_1)(I_{1(n)}(t) - I_{1(m)}(t)], \\
||K[I_{2(n)}] - K[I_{2(m)}(t)]|| &\le ||I_{2(n)}(t) - I_{2(m)}(t)|| + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \\
&\times ST[\rho_2(E_n(t) - E_m(t)) - (\mu + \gamma_2 + d_2)(I_{2(n)}(t) - I_{2(m)}(t)], \\
||K[I_{3(n)}] - K[I_{3(m)}(t)]|| &\le ||I_{3(n)}(t) - I_{3(m)}(t)|| + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \\
&\times ST\left[\rho_3(E_n(t) - E_m(t)) - (\mu + \gamma_3 + d_3)(I_{3(n)}(t) - I_{3(m)}(t)\right], \\
||K[R_n] - K[R_m(t)]|| &\le ||R_n(t) - R_m(t)|| + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \\
&\times ST\left[\sum_{i=1}^{3}\gamma_i(I_{i(n)} - I_{i(m)})(t)\mu(R_n - R_m)(t)\right], \\
||K[D_n] - K[D_m(t)]|| &\le ||D_n(t) - D_m(t)|| + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \\
&\times ST\left[\sum_{i=1}^{3}(d_{i(n)} - d_{i(m)})(I_{i(n)} - I_{i(m)})(t)\right], \\
||K[I_n] - K[I_m(t)]|| &\le ||I_n(t) - I_m(t)|| + ST^{-1}\frac{1-\rho}{B(\rho)\rho\Gamma(\rho+1)E_\rho(-\frac{1}{1-\rho}\omega^\rho)} \\
&\times ST\left[\sum_{i=1}^{3}(I_{i(n)} - I_{i(m)})(t)\right].
\end{aligned}$$

□

4.1. Boundness and positivity of the model

Theorem 4.2. Under straight-line conditions, the suggested solution of the COVID-19 model with diabetes is distinct and constrained in R^8_+.

Proof. We have

$$\begin{aligned}
{}^{abc}D_t^{\theta}S(t) =& aN \geq 0;\\
{}^{abc}D_t^{\theta}E(t) =& \beta\frac{S(t)I(t)}{N} \geq 0;\\
{}^{abc}D_t^{\theta}I_1(t) =& \rho_1 E(t) \geq 0;\\
{}^{abc}D_t^{\theta}I_2(t) =& \rho_2 E(t) \geq 0;\\
{}^{abc}D_t^{\theta}I_3(t) =& \rho_3 E(t) \geq 0;\\
{}^{abc}D_t^{\theta}R(t) =& \sum_{i=1}^{3}\gamma_i I_i(t) \geq 0;\\
{}^{abc}D_t^{\theta}D(t) =& \sum_{i=1}^{3} d_i I_i(t) \geq 0;\\
{}^{abc}D_t^{\theta}I(t) =& \sum_{i=1}^{3} I_i(t) \geq 0.
\end{aligned}$$

If $(S(0), E(0), I_1(0), I_2(0), I_3(0), R(0), D(0), I(0)) \in R^8_+$, then the solution cannot escape from the hyperplane. The domain R^8_+ is a positivity invariant set, as the vector field on each hyperplane enclosing the non-negative orthant likewise points into it. □

4.2. Uniqueness of the special solution

Theorem 3. The special solution of Eq (3.2) implying the iteration method is unique singular solution.

Proof. The Hilbert space is taken into consideration. $H = L^2((c, d) \times (0, r))$ that can be defined as the set of the following functions:

$$f : (c, d) \times [0, T] \to R, \int\int\int gfdgdf < \infty.$$

In this regard, the following operator is considered

$$\theta = (0,0,0,0,0,0,0,0), \theta = \begin{bmatrix} aN - \beta\frac{S(t)I(t)}{N} - \mu S(t) \\ \beta\frac{S(t)I(t)}{N} - (\rho_1 + \rho_2 + \rho_3 + \mu)E(t) \\ \rho_1 E(t) - (\mu + \gamma_1 + d_1)I_1(t) \\ \rho_2 E(t) - (\mu + \gamma_2 + d_2)I_2(t) \\ \rho_3 E(t) - (\mu + \gamma_3 + d_3)I_3(t) \\ \sum_{i=1}^{3}\gamma_i I_i(t) - \mu R(t) \\ \sum_{i=1}^{3} d_i I_{i(t)} \\ \sum_{i=1}^{3} I_{i(t)} \end{bmatrix}.$$

We create the inner product of

$$T(S_{11}(t)-S_{12}(t), E_{21}(t)-E_{22}(t), I_{1(31)}(t)-I_{1(32)}, I_{2(41)}(t)-I_{2(42)}, I_{3(51)}(t)-I_{3(52)}(t), R_{61}(t)-R_{62}(t), D_{71}(t)-D_{72}(t), I_{81}(t)-I_{82}(t)(v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8)),$$

where $(S_{11}(t)-S_{12}(t), E_{21}(t)-E_{22}(t), I_{1(31)}(t)-I_{1(32)}, I_{2(41)}(t)-I_{2(42)}, I_{3(51)}(t)-I_{3(52)}(t)), R_{61}(t)-R_{62}(t), D_{71}(t)-D_{72}(t), I_{81}(t)-I_{82}(t))$ are special solutions of the system. We can achieve the following result by using the relationship between the norm and the inner function:

$$\begin{aligned}
&(aN-\beta\frac{(S_{11}(t)-S_{12}(t))(I_{81}(t)-I_{82}(t))}{N}-\mu(S_{11}(t)-S_{12}(t)))\\
&\qquad\le \|-aN\|\|v_1\|\|-\beta\frac{(S_{11}(t)-S_{12}(t))(I_{81}(t)-I_{82}(t))}{N}\|\|v_1\|-\|\mu(S_{11}(t)-S_{12}(t))\|\|v_1\|,\\
&(\beta\frac{(S_{11}(t)-S_{12}(t))(I_{81}(t)-I_{82}(t))}{N}-(\rho_1+\rho_2+\rho_3+\mu)(E_{21}(t)-E_{22}(t)))\\
&\qquad\le \|\beta\frac{(S_{11}(t)-S_{12}(t))(I_{81}(t)-I_{82}(t))}{N}\|\|v_2\|-\|(\rho_1+\rho_2+\rho_3+\mu)(E_{21}(t)-E_{22}(t))\|\|v_2\|,\\
&(\rho_1(E_{21}(t)-E_{22}(t))-(\mu+\gamma_1+d_1)(I_{1(31)}(t)-I_{1(32)}))\\
&\qquad\le \|\rho_1(E_{21}(t)-E_{22}(t)\|\|v_3\|-\|(\mu+\gamma_1+d_1)(I_{1(31)}(t)-I_{1(32)}\|\|v_3\|,
\end{aligned}$$

$$\begin{aligned}
&(\rho_2(E_{21}(t)-E_{22}(t))-(\mu+\gamma_2+d_2)(I_{2(41)}(t)-I_{2(42)}))\\
&\qquad\le \|\rho_2(E_{21}(t)-E_{22}(t)\|\|v_4\|-\|(\mu+\gamma_2+d_2)(I_{2(41)}(t)-I_{2(42)}\|\|v_4\|,\\
&(\rho_3(E_{21}(t)-E_{22}(t))-(\mu+\gamma_3+d_3)(I_{3(51)}(t)-I_{3(52)}))\\
&\qquad\le \|\rho_3(E_{21}(t)-E_{22}(t)\|\|v_5\|-\|(\mu+\gamma_3+d_3)(I_{3(51)}(t)-I_{3(52)}\|\|v_5\|,\\
&(\sum_{i=1}^{3}\gamma_i(I_{i(81)}(t)-I_{i(82)}(t))-\mu(R_{61}(t)-R_{62}(t))\le\|\sum_{i=1}^{3}\gamma_i(I_{i(81)}(t)-I_{i(82)}(t)\|\|v_6\|-\|\mu(R_{61}(t)-R_{62}(t)\|\|v_6\|,
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{3}d_i(I_{i(81)}(t)-I_{i(82)}(t)&\le\|\sum_{i=1}^{3}d_i(I_{i(81)}(t)-I_{i(82)}(t)\|\|v_7\|,\\
\sum_{i=1}^{3}(I_{i(81)}(t)-I_{i(82)}(t)&\le\|\sum_{i=1}^{3}(I_{i(81)}(t)-I_{i(82)}(t)\|\|v_8\|.
\end{aligned}$$

So,

$$\begin{aligned}
\|S(t)-S_{11}(t)\|, \|S(t)-S_{12}(t)\|&<\frac{xe_1}{\varpi};\\
\|E(t)-E_{21}(t)\|, \|E(t)-E_{22}(t)\|&<\frac{xe_2}{\zeta};\\
\|I_1(t)-I_{1(31)}(t)\|, \|I_1(t)-I_{1(32)}(t)\|&<\frac{xe_3}{\nu};\\
\|I_2(t)-I_{2(41)}(t)\|, \|I_2(t)-I_{2(42)}(t)\|&<\frac{xe_4}{\eta};
\end{aligned}$$

$$\|I_3(t) - I_{3(51)}(t)\|, \|I_3(t) - I_{3(52)}(t)\| < \frac{xe_5}{\psi};$$
$$\|R(t) - R_{61}(t)\|, \|R(t) - R_{62}(t)\| < \frac{xe_6}{\upsilon};$$
$$\|D(t) - D_{71}(t)\|, \|S(t) - S_{72}(t)\| < \frac{xe_7}{\omega};$$
$$\|I(t) - I_{81}(t)\|, \|I(t) - I_{82}(t)\| < \frac{xe_8}{\chi};$$

where

$$\varpi = 4(aN - \beta\|\frac{(S_{11}(t) - S_{12}(t))(I_{81}(t) - I_{82}(t))}{N})\| - \mu\|(S_{11}(t) - S_{12}(t))\|\|v_1\|,$$
$$\zeta = 4(\beta\|\frac{(S_{11}(t) - S_{12}(t))(I_{81}(t) - I_{82}(t))}{N}\| - (\rho_1 + \rho_2 + \rho_3 + \mu)\|(E_{21}(t) - E_{22}(t))\|\|v_2\|,$$
$$\nu = 4(\rho_1\|(E_{21}(t) - E_{22}(t))\| - (\mu + \gamma_1 + d_1)\|(I_{1(31)}(t) - I_{1(32)}\|\|v_3\|,$$
$$\eta = 4(\rho_2\|(E_{21}(t) - E_{22}(t))\| - (\mu + \gamma_2 + d_2)\|(I_{2(41)}(t) - I_{2(42)}\|\|v_4\|,$$
$$\psi = 4(\rho_3\|(E_{21}(t) - E_{22}(t))\| - (\mu + \gamma_3 + d_3)\|(I_{3(51)}(t) - I_{3(52)}\|\|v_5\|,$$
$$\upsilon = 4(\sum_{i=1}^{3} \gamma_i\|(I_{i(81)}(t) - I_{i(82)}(t)\| - \mu\|(R_{61}(t) - R_{62}(t)\|\|v_6\|,$$
$$\omega = 4(\sum_{i=1}^{3} d_i\|(I_{i(81)}(t) - I_{i(82)}(t)\|\|v_7\|,$$
$$\chi = 4(\sum_{i=1}^{3} \|(I_{i(81)}(t) - I_{i(82)}(t)\|\|v_8\|,$$

and

$$(aN - \beta\|\frac{(S_{11}(t) - S_{12}(t))(I_{81}(t) - I_{82}(t))}{N})\| - \mu\|(S_{11}(t) - S_{12}(t))\|) \neq 0,$$
$$(\beta\|\frac{(S_{11}(t) - S_{12}(t))(I_{81}(t) - I_{82}(t))}{N}\| - (\rho_1 + \rho_2 + \rho_3 + \mu)\|(E_{21}(t) - E_{22}(t))\|) \neq 0,$$

$$(\rho_1\|(E_{21}(t) - E_{22}(t))\| - (\mu + \gamma_1 + d_1)\|(I_{1(31)}(t) - I_{1(32)}\|) \neq 0,$$
$$(\rho_2\|(E_{21}(t) - E_{22}(t))\| - (\mu + \gamma_2 + d_2)\|(I_{2(41)}(t) - I_{2(42)}\|) \neq 0,$$
$$(\rho_3\|(E_{21}(t) - E_{22}(t))\| - (\mu + \gamma_3 + d_3)\|(I_{3(51)}(t) - I_{3(52)}\|) \neq 0,$$
$$(\sum_{i=1}^{3} \gamma_i\|(I_{i(81)}(t) - I_{i(82)}(t)\| - \mu\|(R_{61}(t) - R_{62}(t)\|) \neq 0,$$

$$(\sum_{i=1}^{3} d_i\|(I_{i(81)}(t) - I_{i(82)}(t)\|) \neq 0,$$
$$(\sum_{i=1}^{3} \|(I_{i(81)}(t) - I_{i(82)}(t)\|) \neq 0,$$

$$||v_1||, ||v_2||, ||v_3||, ||v_4||, ||v_5||, ||v_6||, ||v_7||, ||v_8|| \neq 0;$$

$$||(S_{12}(t) - S_{11}(t))||, ||(E_{22}(t) - E_{21}(t))||, ||(I_{1(32)}(t) - I_{1(31)}(t))||, ||(I_{2(42)}(t) - I_{2(41)}(t))||,$$
$$||(I_{3(52)}(t) - I_{3(51)}||, ||(R_{62}(t) - R_{61}(t))||, ||(D_{72}(t) - D_{71}(t))||, ||(I_{82}(t) - I_{81}(t))||;$$

$$S_{11} = S_{12}, E_{21} = E_{22}, , I_{1(31)} = I_{1(32)}, I_{2(41)} = I_{2(42)}, I_{3(51)} = I_{3(52)}, R_{61} = R_{62}, D_{71} = D_{72}, I_{81} = I_{82}.$$

This completes the proof of uniqueness. □

4.3. Global stability analysis by Lyapunov's first derivative

The global stability analysis is demonstrated by using Lyapunov's approach and LaSalle's invariance principle.

The equilibrium points and reproductive number for the proposed system [27] are respectively given as follows:

$$E_q = (\frac{\alpha}{\mu}, 0, 0, 0, 0, 0, 0),$$
$$E_q^* = (S^*, (E)^*, (I_1)^*, (I_2)^*, (I_3)^*, (R)^*, (D)^*)$$
$$= \Bigg(\frac{\alpha}{\mu}[\frac{1}{(R_0-1)+\beta}], \frac{\alpha}{\mu+\rho_1+\rho_2+\rho_3}[1-\frac{1}{R_0}], \frac{\alpha\rho_1}{(\mu+\rho_1+\rho_2+\rho_3)(\mu+\gamma_1+d_1)}[1-\frac{1}{R_0}],$$
$$\frac{\alpha\rho_2}{(\mu+\rho_1+\rho_2+\rho_3)(\mu+\gamma_2+d_2)}[1-\frac{1}{R_0}], \quad \frac{\alpha\rho_3}{(\mu+\rho_1+\rho_2+\rho_3)(\mu+\gamma_3+d_3)}[1-\frac{1}{R_0}],$$
$$\frac{\alpha}{\mu(\mu+\rho_1+\rho_2+\rho_3)}[1-\frac{1}{R_0}]\Bigg(\frac{\rho_1}{\mu+\gamma_1+d_1} + \frac{\rho_2}{\mu+\gamma_2+d_2} + \frac{\rho_3}{\mu+\gamma_3+d_3}\Bigg),$$
$$1-(S^*, (E)^*, (I_1)^*, (I_2)^*, (I_3)^*, (R)^*)\Bigg),$$

and

$$R_0 = \frac{\alpha\beta}{\mu(\rho_1+\rho_2+\rho_3+\mu)}(\frac{\rho_1}{\mu+\gamma_1+d_1} + \frac{\rho_2}{\mu+\gamma_2+d_2} + \frac{\rho_3}{\mu+\gamma_3+d_3}).$$

Theorem 4.4. The endemic equilibrium points of this model are globally asymptotically stable when the reproductive number $R_0 > 1$.

Proof. The Lyapunov function is defined and can be written in the form as follows:

$$L = L(S, E, I_1, I_2, I_3, R, D) = \left(S - S^* - S^* log\frac{S^*}{S}\right) + \left(E - (E)^* - (E)^* log\frac{(E)^*}{E}\right)$$
$$+ \left(I_1 - (I_1)^* - (I_1)^* log\frac{(I_1)^*}{I_1}\right) + \left(I_2 - (I_2)^* - (I_2)^* log\frac{(I_2)^*}{I_2}\right)$$
$$+ \left(I_3 - (I_3)^* - (I_3)^* log\frac{(I_3)^*}{I_3}\right) + \left(R - (R)^* - (R)^* log\frac{(R)^*}{R}\right)$$
$$+ \left(D - (D)^* - (D)^* log\frac{(D)^*}{D}\right).$$

Clearly, $L(S, E, I_1, I_2, I_3, R, D)$ for all $S, E, I_1, I_2, I_3, R, D > 0$, and

$$L(S^*, (E)^*, (I_1)^*, (I_2)^*, (I_3)^*, (R)^*, (D)^*) = 0.$$

By applying the derivative with respect to t on both sides, we get

$$\frac{dL}{dt} = \dot{L} = \left(\frac{S-S^*}{S}\right)\dot{S} + \left(\frac{E-(E)^*}{E}\right)\dot{E} + \left(\frac{I_1-(I_1)^*}{I_1}\right)\dot{I}_1 + \left(\frac{I_2-(I_2)^*}{I_2}\right)\dot{I}_2 + \left(\frac{I_3-(I_3)^*}{I_3}\right)\dot{I}_3 + \left(\frac{R-(R)^*}{R}\right)\dot{R} + \left(\frac{D-(D)^*}{D}\right)\dot{D}.$$

Now, it may be written as follows

$$\begin{aligned}\dot{L} = &\left(\frac{S-S^*}{S}\right)\left(aN - \beta\frac{SI}{N} - \mu S\right) + \left(\frac{(E)-(E)^*}{(E)}\right)\left(\beta\frac{SI}{N} - (\rho_1+\rho_2+\rho_3+\mu)E\right)\\ &+ \left(\frac{(I_1)-(I_1)^*}{(I_1)}\right)(\rho_1 E - (\mu+\rho_1+d_1)I_1) + \left(\frac{(I_2)-(I_2)^*}{(I_2)}\right)(\rho_2 E - (\mu+\rho_2+d_2)I_2)\\ &+ \left(\frac{(I_3)-(I_3)^*}{(I_3)}\right)(\rho_3 E - (\mu+\rho_3+d_3)I_3) + \left(\frac{(R)-(R)^*}{(R)}\right)(\gamma_1 I_1 + \gamma_2 I_2 + \gamma_3 I_3 - \mu R)\\ &+ \left(\frac{(D)-(D)^*}{(D)}\right)(d_1 I_1 + d_2 I_2 + d_3 I_3).\end{aligned}$$

Putting $S = S - S^*$, $E = E - (E)^*$, $I_1 = I_1 - (I_1)^*$, $I_2 = I_2 - (I_2)^*$, $I_3 = I_3 - (I_3)^*$, $R = R - (R)^*$, $D = D - (D)^*$, we get

$$\begin{aligned}\dot{L} = &\left(\frac{S-S^*}{S}\right)\left(aN - \beta\frac{(S-S^*)(I-(I)^*)}{N} - \mu(S-S^*)\right)\\ &+ \left(\frac{E-(E)^*}{E}\right)\left(\beta\frac{(S-S^*)(I-(I)^*)}{N} - (\rho_1+\rho_2+\rho_3+\mu)(E-(E)^*)\right)\\ &+ \left(\frac{I_1-(I_1)^*}{I_1}\right)(\rho_1(E-(E)^*) - (\mu+\rho_1+d_1)(I_1-(I_1)^*))\\ &+ \left(\frac{I_2-(I_2)^*}{I_2}\right)(\rho_2(E-(E)^*) - (\mu+\rho_2+d_2)(I_2-(I_2)^*))\\ &+ \left(\frac{I_3-(I_3)^*}{I_3}\right)(\rho_3(E-(E)^*) - (\mu+\rho_3+d_3)(I_3-(I_3)^*))\\ &+ \left(\frac{R-(R)^*}{R}\right)(\gamma_1(I_1-(I_1)^* + \gamma_2(I_2-(I_2)^* + \gamma_3(I_3-(I_3)^*) - \mu(R-(R)^*))\\ &+ \left(\frac{D-(D)^*}{D}\right)(d_1(I_1-(I_1)^* + d_2(I_2-(I_2)^* + d_3(I_3-(I_3)^*)).\end{aligned}$$

$$
\begin{aligned}
\dot{L} =& aN - aN\frac{S^*}{S} - \beta\left(\frac{(S-S^*)^2 I}{NS}\right) + \beta\left(\frac{(S-S^*)^2 I^*}{NS}\right) + \mu\frac{(S-S^*)^2}{S} + \frac{\beta}{N}(SI) - \frac{\beta}{N}(SI^*) - \frac{\beta}{N}(S^*I) + \frac{\beta}{N}(S^*I^*) \\
& - \frac{\beta E^*}{NE}(SI) + \frac{\beta E^*}{NE}(SI^*) + \frac{\beta E^*}{NE}(S^*I) - \frac{\beta E^*}{NE}(S^*I^*) - (\rho_1+\rho_2+\rho_3+\mu)\left(\frac{(E-E^*)^2}{E}\right) \\
& + \rho_1 E - \rho_1(E)^* - \rho_1 E(\frac{I_1^*}{I_1}) + \rho_1(E)^*(\frac{I_1^*}{I_1}) - (\mu+\rho_1+d_1)\left(\frac{(I_1-I_1^*)^2}{I_1}\right) + \rho_2 E - \rho_2(E)^* - \rho_2 E(\frac{I_2^*}{I_2}) + \rho_2(E)^*(\frac{I_2^*}{I_2}) \\
& - (\mu+\rho_2+d_2)\left(\frac{(I_2-I_2^*)^2}{I_2}\right) + \rho_3 E - \rho_3(E)^* - \rho_3 E(\frac{I_3^*}{I_3}) + \rho_3(E)^*(\frac{I_3^*}{I_3}) - (\mu+\rho_3+d_3)\left(\frac{(I_3-I_3^*)^2}{I_3}\right) \\
& + \gamma_1(I_1) - \gamma_1(I_1)^* + \gamma_2(I_2) - \gamma_2(I_2)^* + \gamma_3(I_3) - \gamma_3(I_3)^* - \gamma_1(I_1)(\frac{R^*}{R}) + \gamma_1(I_1)^*(\frac{R^*}{R}) - \gamma_2(I_2)(\frac{R^*}{R}) + \gamma_2(I_2)^*(\frac{R^*}{R}) \\
& - \gamma_3(I_3)(\frac{R^*}{R}) + \gamma_3(I_3)^*(\frac{R^*}{R}) - \mu\left(\frac{(R-R^*)^2}{R}\right) + d_1(I_1) - d_1(I_1)^* + d_2(I_2) - d_2(I_2)^* + d_3(I_3) - d_3(I_3)^* \\
& - d_1(I_1)(\frac{D^*}{D}) + d_1(I_1)^*(\frac{D^*}{D}) - d_2(I_2)(\frac{D^*}{D}) + d_2(I_2)^*(\frac{D^*}{D}) - d_3(I_3)(\frac{D^*}{D}) + d_3(I_3)^*(\frac{D^*}{D}),
\end{aligned}
$$

which can be written as $\dot{L} = \Lambda - \Sigma$, where

$$
\begin{aligned}
\Lambda =& aN + \beta\left(\frac{(S-S^*)^2 I^*}{NS}\right) + \mu\frac{(S-S^*)^2}{S} + \frac{\beta}{N}(SI) + \frac{\beta}{N}(S^*I^*) + \frac{\beta E^*}{NE}(SI^*) + \frac{\beta E^*}{NE}(S^*I) \\
& + \rho_1 E + \rho_1(E)^*(\frac{I_1^*}{I_1}) + \rho_2 E + \rho_2(E)^*(\frac{I_2^*}{I_2}) + \rho_3 E + \rho_3(E)^*(\frac{I_3^*}{I_3}) + \gamma_1(I_1) + \gamma_2(I_2) + \gamma_3(I_3) \\
& + \gamma_2(I_2)^*(\frac{R^*}{R}) + \gamma_3(I_3)^*(\frac{R^*}{R}) + d_1(I_1) + d_2(I_2) + d_3(I_3) + d_1(I_1)^*(\frac{D^*}{D}) + d_2(I_2)^*(\frac{D^*}{D}) + d_3(I_3)^*(\frac{D^*}{D}),
\end{aligned}
$$

and

$$
\begin{aligned}
\Sigma =& aN\frac{S^*}{S} + \beta\left(\frac{(S-S^*)^2 I}{NS}\right) + \frac{\beta}{N}(SI^*) + \frac{\beta}{N}(S^*I) + \frac{\beta E^*}{NE}(SI) + \frac{\beta E^*}{NE}(S^*I^*) + (\rho_1+\rho_2+\rho_3+\mu)\left(\frac{(E-E^*)^2}{E}\right) \\
& + \rho_1(E)^* + \rho_1 E(\frac{I_1^*}{I_1}) + (\mu+\rho_1+d_1)\left(\frac{(I_1-I_1^*)^2}{I_1}\right) + \rho_2(E)^* + \rho_2 E(\frac{I_2^*}{I_2}) + (\mu+\rho_2+d_2)\left(\frac{(I_2-I_2^*)^2}{I_2}\right) \\
& + \rho_3(E)^* + \rho_3 E(\frac{I_3^*}{I_3}) + (\mu+\rho_3+d_3)\left(\frac{(I_3-I_3^*)^2}{I_3}\right) + \gamma_1(I_1)^* + \gamma_2(I_2)^* + \gamma_3(I_3)^* + \gamma_1(I_1)(\frac{R^*}{R}) + \gamma_2(I_2)(\frac{R^*}{R}) \\
& + \gamma_3(I_3)(\frac{R^*}{R}) + \mu\left(\frac{(R-R^*)^2}{R}\right) + d_1(I_1)^* + d_2(I_2)^* + d_3(I_3)^* + d_1(I_1)(\frac{D^*}{D}) + d_2(I_2)(\frac{D^*}{D}) + d_3(I_3)(\frac{D^*}{D}).
\end{aligned}
$$

It is deduced that $\frac{dL}{dt} < 0$ if $\Lambda < \Sigma$ and $\frac{dL}{dt} = 0$ if $\Lambda - \Sigma = 0$, when $S = S^*$, $E = (E)^*$, $I_1 = (I_1)^*$, $I_2 = (I_2)^*$, $I_3 = (I_3)^*$, $R = (R)^*$ and $D = (D)^*$.
The largest compact invariant set for the proposed system in

$$
(\{S^*, (E)^*, (I_1)^*, (I_2)^*, (I_3)^*, (R)^*, (D)^*)\varepsilon\Gamma : \frac{dL}{dt} = 0\}
$$

is the point E_q^*, i.e., the endemic equilibrium for the proposed model. With the help of this Lasalle's invariance principle, it follows that E^* is globally asymptotically stable in Γ if $\Lambda < \Sigma$. □

5. Analysis via the Atangana Toufik technique

We consider

$$\begin{cases} {}_0^{ABC}DS = f_1(t, j), \\ {}_0^{ABC}DE = f_2(t, j), \\ {}_0^{ABC}DI_1 = f_3(t, j), \\ {}_0^{ABC}DI_2 = f_4(t, j), \\ {}_0^{ABC}DI_3 = f_6(t, j), \\ {}_0^{ABC}DR = f_6(t, j), \\ {}_0^{ABC}DD = f_7(t, j), \\ {}_0^{ABC}DI = f_8(t, j), \end{cases} \tag{5.1}$$

where $j = S, E, I_1, I_2, I_3, R, D, I$ and $S(0) = S_0$, $E(0) = E_0$, $I_1(0) = I_{1(0)}$, $I_2(0) = I_{2(0)}$, $I_3(0) = I_{3(0)}$, $R(0) = R_0$, $D(0) = D_0$, $I(0) = I_0$. The fundamental theorem of fractional calculus can be used to convert the preceding equations to corresponding fractional integral equations:

$$S(t) - S(0) = \frac{1-\rho}{ABC(\rho)} f_1(t, j) + \frac{\rho}{\Gamma(\rho) \times ABC(\rho)} \int_0^t f_1(\vartheta, S(\vartheta), F)(t-\vartheta)^{\rho-1} d\vartheta,$$

$$E(t) - E(0) = \frac{1-\rho}{ABC(\rho)} f_2(t, j) + \frac{\rho}{\Gamma(\rho) \times ABC(\rho)} \int_0^t f_2(\vartheta, S(\vartheta), F)(t-\vartheta)^{\rho-1} d\vartheta,$$

$$I_1(t) - I_1(0) = \frac{1-\rho}{ABC(\rho)} f_3(t, j) + \frac{\rho}{\Gamma(\rho) \times ABC(\rho)} \int_0^t f_4(\vartheta, S(\vartheta), F)(t-\vartheta)^{\rho-1} d\vartheta,$$

$$I_2(t) - I_2(0) = \frac{1-\rho}{ABC(\rho)} f_4(t, j) + \frac{\rho}{\Gamma(\rho) \times ABC(\rho)} \int_0^t f_5(\vartheta, S(\vartheta), F)(t-\vartheta)^{\rho-1} d\vartheta,$$

$$I_3(t) - I_3(0) = \frac{1-\rho}{ABC(\rho)} f_5(t, j) + \frac{\rho}{\Gamma(\rho) \times ABC(\rho)} \int_0^t f_6(\vartheta, S(\vartheta), F)(t-\vartheta)^{\rho-1} d\vartheta,$$

$$R(t) - R(0) = \frac{1-\rho}{ABC(\rho)} f_6(t, j) + \frac{\rho}{\Gamma(\rho) \times ABC(\rho)} \int_0^t f_7(\vartheta, S(\vartheta), F)(t-\vartheta)^{\rho-1} d\vartheta,$$

$$D(t) - D(0) = \frac{1-\rho}{ABC(\rho)} f_7(t, j) + \frac{\rho}{\Gamma(\rho) \times ABC(\rho)} \int_0^t f_7(\vartheta, S(\vartheta), F)(t-\vartheta)^{\rho-1} d\vartheta,$$

$$I(t) - I(0) = \frac{1-\rho}{ABC(\rho)} f_8(t, j) + \frac{\rho}{\Gamma(\rho) \times ABC(\rho)} \int_0^t f_8(\vartheta, S(\vartheta), F)(t-\vartheta)^{\rho-1} d\vartheta,$$

where $F = E(\vartheta), I_1(\vartheta)), I_2(\vartheta), I_3(\vartheta), R(\vartheta), D(\vartheta), I(\vartheta)$. At a certain point, $t = t_{n+1}$, $n = 0, 1, 2, ...$, the above set of equations are rewritten as follows:

$$S(t_{n+1}) - S(0) = \frac{1-\rho}{ABC(\rho)} f_1(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \int_0^{t_{n+1}} f_1(\vartheta, S(\vartheta), F), I(\vartheta)))(t_{n+1}-\vartheta)^{\rho-1} d\vartheta,$$

$$E(t_{n+1}) - E(0) = \frac{1-\rho}{ABC(\rho)} f_2(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \int_0^{t_{n+1}} f_2(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta,$$

$$I_1(t_{n+1}) - I_1(0) = \frac{1-\rho}{ABC(\rho)} f_3(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \int_0^{t_{n+1}} f_3(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta,$$

$$I_2(t_{n+1}) - I_2(0) = \frac{1-\rho}{ABC(\rho)} f_4(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \int_0^{t_{n+1}} f_4(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta,$$

$$I_3(t_{n+1}) - I_3(0) = \frac{1-\rho}{ABC(\rho)} f_5(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \int_0^{t_{n+1}} f_5(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta,$$

$$R(t_{n+1}) - R(0) = \frac{1-\rho}{ABC(\rho)} f_6(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \int_0^{t_{n+1}} f_6(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta,$$

$$D(t_{n+1}) - D(0) = \frac{1-\rho}{ABC(\rho)} f_7(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \int_0^{t_{n+1}} f_7(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta,$$

$$I(t_{n+1}) - I(0) = \frac{1-\rho}{ABC(\rho)} f_8(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \int_0^{t_{n+1}} f_8(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta,$$

where $Q = E(t_n), I_1(t_n), I_2(t_n), I_3(t_n), R(t_n), D(t_n), I(t_n)$, and

$$S(t_{n+1}) - S(0) = \frac{1-\rho}{ABC(\rho)} f_1(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} f_1(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta, \tag{5.2}$$

$$E(t_{n+1}) - E(0) = \frac{1-\rho}{ABC(\rho)} f_2(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} f_1(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta, \tag{5.3}$$

$$I_1(t_{n+1}) - I_1(0) = \frac{1-\rho}{ABC(\rho)} f_3(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} f_3(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta, \tag{5.4}$$

$$I_2(t_{n+1}) - I_2(0) = \frac{1-\rho}{ABC(\rho)} f_4(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} f_4(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta, \tag{5.5}$$

$$I_3(t_{n+1}) - I_3(0) = \frac{1-\rho}{ABC(\rho)} f_5(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} f_5(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta, \tag{5.6}$$

$$R(t_{n+1}) - R(0) = \frac{1-\rho}{ABC(\rho)} f_6(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho) \times ABC(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} f_6(\vartheta, S(\vartheta), F)(t_{n+1} - \vartheta)^{\rho-1} d\vartheta, \tag{5.7}$$

$$D(t_{n+1}) - D(0) = \frac{1-\rho}{ABC(\rho)} f_7(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho) \times ABC(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} f_7(\vartheta, S(\vartheta), F)(t_{n+1} - \vartheta)^{\rho-1} d\vartheta, \tag{5.8}$$

$$I(t_{n+1}) - I(0) = \frac{1-\rho}{ABC(\rho)} f_8(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho) \times ABC(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} f_8(\vartheta, S(\vartheta), F)(t_{n+1} - \vartheta)^{\rho-1} d\vartheta. \tag{5.9}$$

Within the interval $[t_k, t_{k+1}]$, using the two-step Lagrange polynomial interpolation, the function $f(\vartheta, y(\vartheta))$ can be approximated as follows:

$$\begin{aligned}
P_k(\vartheta) &= \frac{\vartheta - t_{k-1}}{t_k - t_{k-1}} f_1(t_k, J) - \frac{\vartheta - t_{k-1}}{t_k - t_{k-1}} f_1(t_{k-1}, K),\\
P_k(\vartheta) &= \frac{\vartheta - t_{k-1}}{t_k - t_{k-1}} f_2(t_k, J) - \frac{\vartheta - t_{k-1}}{t_k - t_{k-1}} f_2(t_{k-1}, K),\\
P_k(\vartheta) &= \frac{\vartheta - t_{k-1}}{t_k - t_{k-1}} f_3(t_k, J) - \frac{\vartheta - t_{k-1}}{t_k - t_{k-1}} f_3(t_{k-1}, K),\\
P_k(\vartheta) &= \frac{\vartheta - t_{k-1}}{t_k - t_{k-1}} f_4(t_k, J) - \frac{\vartheta - t_{k-1}}{t_k - t_{k-1}} f_4(t_{k-1}, K),\\
P_k(\vartheta) &= \frac{\vartheta - t_{k-1}}{t_k - t_{k-1}} f_5(t_k, J) - \frac{\vartheta - t_{k-1}}{t_k - t_{k-1}} f_5(t_{k-1}, K),\\
P_k(\vartheta) &= \frac{\vartheta - t_{k-1}}{t_k - t_{k-1}} f_6(t_k, J) - \frac{\vartheta - t_{k-1}}{t_k - t_{k-1}} f_6(t_{k-1}, K),\\
P_k(\vartheta) &= \frac{\vartheta - t_{k-1}}{t_k - t_{k-1}} f_7(t_k, J) - \frac{\vartheta - t_{k-1}}{t_k - t_{k-1}} f_7(t_{k-1}, K),\\
P_k(\vartheta) &= \frac{\vartheta - t_{k-1}}{t_k - t_{k-1}} f_8(t_k, J) - \frac{\vartheta - t_{k-1}}{t_k - t_{k-1}} f_8(t_{k-1}, K),
\end{aligned}$$

where

$$\begin{aligned}
J &= S(t_k), E(t_k), I_1(t_k), I_2(t_k), I_3(t_k), R(t_k), D(t_k), I(t_k),\\
K &= S(t_{k-1}), E(t_{k-1}), I_1(t_{k-1}), I_2(t_{k-1}), I_3(t_{k-1}), R(t_{k-1}), D(t_{k-1}), I(t_{k-1}),
\end{aligned}$$

and

$$\begin{aligned}
P_k(\vartheta) &= \frac{f_1(t_k, J)}{h}(\vartheta - t_{k-1}) - \frac{f_1(t_{k-1}, K)}{h}(\vartheta - t_k)\\
&\simeq \frac{f_1(t_k, J)}{h}(\vartheta - t_{k-1}) - \frac{f_1(t_{k-1}, K)}{h}(\vartheta - t_k) \qquad (5.10)\\
P_k(\vartheta) &= \frac{f_2(t_k, J)}{h}(\vartheta - t_{k-1}) - \frac{f_2(t_{k-1}, K)}{h}(\vartheta - t_k)
\end{aligned}$$

$$\simeq \frac{f_2(t_k, J)}{h}(\vartheta - t_{k-1}) - \frac{f_2(t_{k-1}, K)}{h}(\vartheta - t_k) \tag{5.11}$$

$$\begin{aligned} P_k(\vartheta) &= \frac{f_3(t_k, J)}{h}(\vartheta - t_{k-1}) - \frac{f_3(t_{k-1}, K)}{h}(\vartheta - t_k) \\ &\simeq \frac{f_3(t_k, J)}{h}(\vartheta - t_{k-1}) - \frac{f_3(t_{k-1}, K)}{h}(\vartheta - t_k) \end{aligned} \tag{5.12}$$

$$\begin{aligned} P_k(\vartheta) &= \frac{f_4(t_k, J)}{h}(\vartheta - t_{k-1}) - \frac{f_4(t_{k-1}, K)}{h}(\vartheta - t_k) \\ &\simeq \frac{f_4(t_k, J)}{h}(\vartheta - t_{k-1}) - \frac{f_4(t_{k-1}, K)}{h}(\vartheta - t_k) \end{aligned} \tag{5.13}$$

$$\begin{aligned} P_k(\vartheta) &= \frac{f_5(t_k, J)}{h}(\vartheta - t_{k-1}) - \frac{f_5(t_{k-1}, K)}{h}(\vartheta - t_k) \\ &\simeq \frac{f_5(t_k, J)}{h}(\vartheta - t_{k-1}) - \frac{f_5(t_{k-1}, K)}{h}(\vartheta - t_k) \end{aligned} \tag{5.14}$$

$$\begin{aligned} P_k(\vartheta) &= \frac{f_6(t_k, J)}{h}(\vartheta - t_{k-1}) - \frac{f_6(t_{k-1}, K)}{h}(\vartheta - t_k) \\ &\simeq \frac{f_6(t_k, J)}{h}(\vartheta - t_{k-1}) - \frac{f_6(t_{k-1}, K)}{h}(\vartheta - t_k) \end{aligned} \tag{5.15}$$

$$\begin{aligned} P_k(\vartheta) &= \frac{f_7(t_k, J)}{h}(\vartheta - t_{k-1}) - \frac{f_7(t_{k-1}, K)}{h}(\vartheta - t_k) \\ &\simeq \frac{f_7(t_k, J)}{h}(\vartheta - t_{k-1}) - \frac{f_7(t_{k-1}, K)}{h}(\vartheta - t_k) \end{aligned} \tag{5.16}$$

$$\begin{aligned} P_k(\vartheta) &= \frac{f_8(t_k, J)}{h}(\vartheta - t_{k-1}) - \frac{f_8(t_{k-1}, K)}{h}(\vartheta - t_k) \\ &\simeq \frac{f_8(t_k, J)}{h}(\vartheta - t_{k-1}) - \frac{f_8(t_{k-1}, K)}{h}(\vartheta - t_k), \end{aligned} \tag{5.17}$$

where $h = t_k - t_{k-1}$. Therefore, we have:

$$\begin{aligned} S(t_{n+1}) = S(0) + \frac{1-\rho}{ABC(\rho)} f_1(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho) \times ABC(\rho)} \sum_{k=0}^{n} \Bigg(\frac{f_1(t_k, J)}{h} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta \\ - \frac{f_1(t_{k-1}, K)}{h} \int_{t_k}^{t_{k+1}} (\vartheta - t_k)(t_{n+1} - \vartheta)^{\rho-1} d\vartheta \Bigg), \end{aligned} \tag{5.18}$$

$$\begin{aligned} E(t_{n+1}) = E(0) + \frac{1-\rho}{ABC(\rho)} f_2(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho) \times ABC(\rho)} \sum_{k=0}^{n} \Bigg(\frac{f_2(t_k, J)}{h} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta \\ - \frac{f_2(t_{k-1}, K)}{h} \int_{t_k}^{t_{k+1}} (\vartheta - t_k)(t_{n+1} - \vartheta)^{\rho-1} d\vartheta \Bigg), \end{aligned} \tag{5.19}$$

$$I_1(t_{n+1}) = I_1(0) + \frac{1-\rho}{ABC(\rho)} f_3(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho) \times ABC(\rho)} \sum_{k=0}^{n} \Bigg(\frac{f_3(t_k, J)}{h} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta$$

$$-\frac{f_3(t_{k-1},K)}{h}\int_{t_k}^{t_{k+1}}(\vartheta-t_k)(t_{n+1}-\vartheta)^{\rho-1}d\vartheta\Big), \quad (5.20)$$

$$I_2(t_{n+1})=I_2(0)+\frac{1-\rho}{ABC(\rho)}f_4(t_n,S(t_n),Q)+\frac{\rho}{\Gamma(\rho)\times ABC(\rho)}\sum_{k=0}^{n}\Big(\frac{f_4(t_k,J)}{h}\int_{t_k}^{t_{k+1}}(\vartheta-t_{k-1})(t_{n+1}-\vartheta)^{\rho-1}d\vartheta$$
$$-\frac{f_4(t_{k-1},K)}{h}\int_{t_k}^{t_{k+1}}(\vartheta-t_k)(t_{n+1}-\vartheta)^{\rho-1}d\vartheta\Big), \quad (5.21)$$

$$I_3(t_{n+1})=I_3(0)+\frac{1-\rho}{ABC(\rho)}f_5(t_n,S(t_n),Q)+\frac{\rho}{\Gamma(\rho)\times ABC(\rho)}\sum_{k=0}^{n}\Big(\frac{f_5(t_k,J)}{h}\int_{t_k}^{t_{k+1}}(\vartheta-t_{k-1})(t_{n+1}-\vartheta)^{\rho-1}d\vartheta$$
$$-\frac{f_5(t_{k-1},K)}{h}\int_{t_k}^{t_{k+1}}(\vartheta-t_k)(t_{n+1}-\vartheta)^{\rho-1}d\vartheta\Big), \quad (5.22)$$

$$R(t_{n+1})=R(0)+\frac{1-\rho}{ABC(\rho)}f_6(t_n,S(t_n),Q)+\frac{\rho}{\Gamma(\rho)\times ABC(\rho)}\sum_{k=0}^{n}\Big(\frac{f_6(t_k,J)}{h}\int_{t_k}^{t_{k+1}}(\vartheta-t_{k-1})(t_{n+1}-\vartheta)^{\rho-1}d\vartheta$$
$$-\frac{f_6(t_{k-1},K)}{h}\int_{t_k}^{t_{k+1}}(\vartheta-t_k)(t_{n+1}-\vartheta)^{\rho-1}d\vartheta\Big), \quad (5.23)$$

$$D(t_{n+1})=D(0)+\frac{1-\rho}{ABC(\rho)}f_7(t_n,S(t_n),Q)+\frac{\rho}{\Gamma(\rho)\times ABC(\rho)}\sum_{k=0}^{n}\Big(\frac{f_7(t_k,J)}{h}\int_{t_k}^{t_{k+1}}(\vartheta-t_{k-1})(t_{n+1}-\vartheta)^{\rho-1}d\vartheta$$
$$-\frac{f_7(t_{k-1},K)}{h}\int_{t_k}^{t_{k+1}}(\vartheta-t_k)(t_{n+1}-\vartheta)^{\rho-1}d\vartheta\Big), \quad (5.24)$$

$$I(t_{n+1})=I(0)+\frac{1-\rho}{ABC(\rho)}f_8(t_n,S(t_n),Q)+\frac{\rho}{\Gamma(\rho)\times ABC(\rho)}\sum_{k=0}^{n}\Big(\frac{f_8(t_k,J)}{h}\int_{t_k}^{t_{k+1}}(\vartheta-t_{k-1})(t_{n+1}-\vartheta)^{\rho-1}d\vartheta$$
$$-\frac{f_8(t_{k-1},K)}{h}\int_{t_k}^{t_{k+1}}(\vartheta-t_k)(t_{n+1}-\vartheta)^{\rho-1}d\vartheta\Big). \quad (5.25)$$

For simplicity, we set $A_{\rho,k,1}$ and $A_{\rho,k,2}$ based on the above equations

$$A_{\rho,k,1}=\int_{t_k}^{t_{k+1}}(\vartheta-t_{k-1})(t_{n+1}-\vartheta)^{\rho-1}d\vartheta, \quad (5.26)$$

$$A_{\rho,k,2}=\int_{t_k}^{t_{k+1}}(\vartheta-t_k)(t_{n+1}-\vartheta)^{\rho-1}d\vartheta. \quad (5.27)$$

Integrating the above equations, we have

$$A_{\rho,k,1}=h^{\rho+1}\frac{(n+1-k)^{\rho}(n-k+2+\rho)-(n-k)^{\rho}(n-k+2+2\rho)}{\rho(\rho+1)},$$
$$A_{\rho,k,2}=h^{\rho+1}\frac{(n+1-k)^{\rho+1}-(n-k)^{\rho}(n-k+1+\rho)}{\rho(\rho+1)}.$$

In the next section, we will look at the numerical error of the above estimates.

6. Error analysis

Theorem 6.1. Let Eqs (5.2)–(5.9) be a system of non-linear fractional differential equations with non local and non-singular kernel fractional derivatives such that the feature has a bounded second by-product; as a consequence, the error is predicted to satisfy the following:

$$
\begin{aligned}
|R_n^\rho| &\leq \frac{\rho(h^{\rho+2})}{2ABC(\rho)\Gamma(\rho+2)} \max_{[0,t_{n+1}]} |\frac{\partial^2}{\partial\vartheta^2}[f_1(\vartheta, S(\vartheta), F)]|((n+1)^n - \rho(n^\rho))\frac{n(n+4+2\rho)}{2},\\
|R_n^\rho| &\leq \frac{\rho(h^{\rho+2})}{2ABC(\rho)\Gamma(\rho+2)} \max_{[0,t_{n+1}]} |\frac{\partial^2}{\partial\vartheta^2}[f_2(\vartheta, S(\vartheta), F)]|((n+1)^n - \rho(n^\rho))\frac{n(n+4+2\rho)}{2},\\
|R_n^\rho| &\leq \frac{\rho(h^{\rho+2})}{2ABC(\rho)\Gamma(\rho+2)} \max_{[0,t_{n+1}]} |\frac{\partial^2}{\partial\vartheta^2}[f_3(\vartheta, S(\vartheta), F)]|((n+1)^n - \rho(n^\rho))\frac{n(n+4+2\rho)}{2},\\
|R_n^\rho| &\leq \frac{\rho(h^{\rho+2})}{2ABC(\rho)\Gamma(\rho+2)} \max_{[0,t_{n+1}]} |\frac{\partial^2}{\partial\vartheta^2}[f_4(\vartheta, S(\vartheta), F)]|((n+1)^n - \rho(n^\rho))\frac{n(n+4+2\rho)}{2},\\
|R_n^\rho| &\leq \frac{\rho(h^{\rho+2})}{2ABC(\rho)\Gamma(\rho+2)} \max_{[0,t_{n+1}]} |\frac{\partial^2}{\partial\vartheta^2}[f_5(\vartheta, S(\vartheta), F)]|((n+1)^n - \rho(n^\rho))\frac{n(n+4+2\rho)}{2},\\
|R_n^\rho| &\leq \frac{\rho(h^{\rho+2})}{2ABC(\rho)\Gamma(\rho+2)} \max_{[0,t_{n+1}]} |\frac{\partial^2}{\partial\vartheta^2}[f_6(\vartheta, S(\vartheta), F)]|((n+1)^n - \rho(n^\rho))\frac{n(n+4+2\rho)}{2},\\
|R_n^\rho| &\leq \frac{\rho(h^{\rho+2})}{2ABC(\rho)\Gamma(\rho+2)} \max_{[0,t_{n+1}]} |\frac{\partial^2}{\partial\vartheta^2}[f_7(\vartheta, S(\vartheta), F)]|((n+1)^n - \rho(n^\rho))\frac{n(n+4+2\rho)}{2},\\
|R_n^\rho| &\leq \frac{\rho(h^{\rho+2})}{2ABC(\rho)\Gamma(\rho+2)} \max_{[0,t_{n+1}]} |\frac{\partial^2}{\partial\vartheta^2}[f_8(\vartheta, S(\vartheta), F)]|((n+1)^n - \rho(n^\rho))\frac{n(n+4+2\rho)}{2}.
\end{aligned}
$$

Proof. We have developed a numerical algorithm:

$$
\begin{aligned}
S(t_{n+1}) - S(0) &= \frac{1-\rho}{ABC(\rho)} f_1(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \int_0^{t_{n+1}} f_1(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta,\\
E(t_{n+1}) - E(0) &= \frac{1-\rho}{ABC(\rho)} f_2(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \int_0^{t_{n+1}} f_2(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta,\\
I_1(t_{n+1}) - I_1(0) &= \frac{1-\rho}{ABC(\rho)} f_3(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \int_0^{t_{n+1}} f_3(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta,\\
I_2(t_{n+1}) - I_2(0) &= \frac{1-\rho}{ABC(\rho)} f_4(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \int_0^{t_{n+1}} f_4(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta,\\
I_3(t_{n+1}) - I_3(0) &= \frac{1-\rho}{ABC(\rho)} f_5(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \int_0^{t_{n+1}} f_5(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta,\\
R(t_{n+1}) - R(0) &= \frac{1-\rho}{ABC(\rho)} f_6(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \int_0^{t_{n+1}} f_6(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta,\\
D(t_{n+1}) - D(0) &= \frac{1-\rho}{ABC(\rho)} f_7(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \int_0^{t_{n+1}} f_7(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta,\\
I(t_{n+1}) - I(0) &= \frac{1-\rho}{ABC(\rho)} f_8(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \int_0^{t_{n+1}} f_8(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta,
\end{aligned}
$$

$$S(t_{n+1}) - S(0) = \frac{1-\rho}{ABC(\rho)} f_1(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} f_1(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta,$$

$$E(t_{n+1}) - E(0) = \frac{1-\rho}{ABC(\rho)} f_2(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} f_2(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta,$$

$$I_1(t_{n+1}) - I_1(0) = \frac{1-\rho}{ABC(\rho)} f_3(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} f_3(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta,$$

$$I_2(t_{n+1}) - I_2(0) = \frac{1-\rho}{ABC(\rho)} f_4(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} f_4(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta,$$

$$I_3(t_{n+1}) - I_3(0) = \frac{1-\rho}{ABC(\rho)} f_5(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} f_5(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta,$$

$$R(t_{n+1}) - R(0) = \frac{1-\rho}{ABC(\rho)} f_6(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} f_6(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta,$$

$$D(t_{n+1}) - D(0) = \frac{1-\rho}{ABC(\rho)} f_7(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} f_7(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta,$$

$$I(t_{n+1}) - I(0) = \frac{1-\rho}{ABC(\rho)} f_8(t_n, S(t_n), Q) + \frac{\rho}{\Gamma(\rho)\times ABC(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} f_8(\vartheta, S(\vartheta), F)(t_{n+1}-\vartheta)^{\rho-1} d\vartheta.$$

For the function $f(\vartheta, S(\vartheta), E(\vartheta), I_1(\vartheta), I_2(\vartheta), I_3(\vartheta), R(\vartheta), D(\vartheta), I(\vartheta))$, we use the Lagrange polynomial interpolation:

$$\begin{aligned}
S(t_{n+1}) - S(0) &= \frac{1-\rho}{ABC(\rho)} f_1(t_n, S(t_n), Q) \\
&+ \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} (P_k(\vartheta) + \frac{(\vartheta-t_k)(\vartheta-t_{k-1})}{2!} \frac{\partial^1}{\partial\vartheta^2} | f_1(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta})(t_{n+1}-\vartheta)^{\rho-1} d\vartheta \\
&= \frac{1-\rho}{ABC(\rho)} f_1(t_n, S(t_n), Q) + \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \Big(\frac{f_2(t_k, j)}{h} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1}-\vartheta)^{\rho-1} d\vartheta \\
&- \frac{f_1(t_{k-1}, k)}{h} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1}-\vartheta)^{\rho-1} d\vartheta \Big) \\
&+ \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \frac{(\vartheta-t_k)(\vartheta-t_{k-1})}{2!} \frac{\partial^2}{\partial\vartheta^2} | f_1(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} (t_{n+1}-\vartheta)^{\rho-1} d\vartheta.
\end{aligned}$$

$$\begin{aligned}
E(t_{n+1}) - E(0) &= \frac{1-\rho}{ABC(\rho)} f_2(t_n, S(t_n), Q) \\
&+ \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} (P_k(\vartheta) + \frac{(\vartheta-t_k)(\vartheta-t_{k-1})}{2!} \frac{\partial^2}{\partial\vartheta^2} | f_2(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta})(t_{n+1}-\vartheta)^{\rho-1} d\vartheta \\
&= \frac{1-\rho}{ABC(\rho)} f_2(t_n, S(t_n), Q) + \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \Big(\frac{f_2(t_k, J)}{h} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1}-\vartheta)^{\rho-1} d\vartheta
\end{aligned}$$

$$- \frac{f_2(t_{k-1}, K)}{h} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta \Big)$$
$$+ \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \frac{(\vartheta - t_k)(\vartheta - t_{k-1})}{2!} \frac{\partial^2}{\partial\vartheta^2} |f_2(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} (t_{n+1} - \vartheta)^{\rho-1} d\vartheta.$$

$$\begin{aligned}
I_1(t_{n+1}) - I_1(0) &= \frac{1-\rho}{ABC(\rho)} f_1(t_n, S(t_n), Q) \\
&+ \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} (P_k(\vartheta) + \frac{(\vartheta - t_k)(\vartheta - t_{k-1})}{2!} \frac{\partial^2}{\partial\vartheta^2} |f_3(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta}) (t_{n+1} - \vartheta)^{\rho-1} d\vartheta \\
&= \frac{1-\rho}{ABC(\rho)} f_3(t_n, S(t_n), Q) + \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \Big(\frac{f_3(t_k, J))}{h} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta \\
&- \frac{f_3(t_{k-1}, K)}{h} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta \Big) \\
&+ \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \frac{(\vartheta - t_k)(\vartheta - t_{k-1})}{2!} \frac{\partial^2}{\partial\vartheta^2} |f_3(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} (t_{n+1} - \vartheta)^{\rho-1} d\vartheta.
\end{aligned}$$

$$\begin{aligned}
I_2(t_{n+1}) - I_2(0) &= \frac{1-\rho}{ABC(\rho)} f_4(t_n, S(t_n), Q) \\
&+ \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} (P_k(\vartheta) + \frac{(\vartheta - t_k)(\vartheta - t_{k-1})}{2!} \frac{\partial^2}{\partial\vartheta^2} |f_4(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta}) (t_{n+1} - \vartheta)^{\rho-1} d\vartheta \\
&= \frac{1-\rho}{ABC(\rho)} f_4(t_n, S(t_n), Q) + \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \Big(\frac{f_4(t_k, J)}{h} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta \\
&- \frac{f_4(t_{k-1}, K)}{h} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta \Big) \\
&+ \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \frac{(\vartheta - t_k)(\vartheta - t_{k-1})}{2!} \frac{\partial^2}{\partial\vartheta^2} |f_4(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} (t_{n+1} - \vartheta)^{\rho-1} d\vartheta.
\end{aligned}$$

$$\begin{aligned}
I_3(t_{n+1}) - I_3(0) &= \frac{1-\rho}{ABC(\rho)} f_5(t_n, S(t_n), Q) \\
&+ \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} (P_k(\vartheta) + \frac{(\vartheta - t_k)(\vartheta - t_{k-1})}{2!} \frac{\partial^2}{\partial\vartheta^2} |f_5(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta}) (t_{n+1} - \vartheta)^{\rho-1} d\vartheta \\
&= \frac{1-\rho}{ABC(\rho)} f_5(t_n, S(t_n), Q) + \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \Big(\frac{f_5(t_k, J)}{h} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta \\
&- \frac{f_5(t_{k-1}, K)}{h} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta \Big) \\
&+ \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \frac{(\vartheta - t_k)(\vartheta - t_{k-1})}{2!} \frac{\partial^2}{\partial\vartheta^2} |f_5(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} (t_{n+1} - \vartheta)^{\rho-1} d\vartheta.
\end{aligned}$$

$$
\begin{aligned}
R(t_{n+1}) - R(0) = & \frac{1-\rho}{ABC(\rho)} f_6(t_n, S(t_n), Q) \\
& + \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} (P_k(\vartheta) + \frac{(\vartheta - t_k)(\vartheta - t_{k-1})}{2!} \frac{\partial^2}{\partial\vartheta^2} |f_6(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta \\
& = \frac{1-\rho}{ABC(\rho)} f_6(t_n, S(t_n), Q) + \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \Big(\frac{f_6(t_k, J)}{h} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta \\
& - \frac{f_6(t_{k-1}, K)}{h} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta \Big) \\
& + \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \frac{(\vartheta - t_k)(\vartheta - t_{k-1})}{2!} \frac{\partial^2}{\partial\vartheta^2} |f_6(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} (t_{n+1} - \vartheta)^{\rho-1} d\vartheta.
\end{aligned}
$$

$$
\begin{aligned}
D(t_{n+1}) - D(0) = & \frac{1-\rho}{ABC(\rho)} f_7(t_n, S(t_n), Q) \\
& + \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} (P_k(\vartheta) + \frac{(\vartheta - t_k)(\vartheta - t_{k-1})}{2!} \frac{\partial^2}{\partial\vartheta^2} |f_7(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta \\
& = \frac{1-\rho}{ABC(\rho)} f_7(t_n, S(t_n), Q) + \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \Big(\frac{f_7(t_k, J)}{h} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta \\
& - \frac{f_7(t_{k-1}, K)}{h} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta \Big) \\
& + \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \frac{(\vartheta - t_k)(\vartheta - t_{k-1})}{2!} \frac{\partial^2}{\partial\vartheta^2} |f_7(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} (t_{n+1} - \vartheta)^{\rho-1} d\vartheta.
\end{aligned}
$$

$$
\begin{aligned}
I(t_{n+1}) - I(0) = & \frac{1-\rho}{ABC(\rho)} f_1(t_n, S(t_n), Q) \\
& + \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} (P_k(\vartheta) + \frac{(\vartheta - t_k)(\vartheta - t_{k-1})}{2!} \frac{\partial^2}{\partial\vartheta^2} |f_8(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta \\
& = \frac{1-\rho}{ABC(\rho)} f_8(t_n, S(t_n), Q) + \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \Big(\frac{f_8(t_k, J)}{h} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta \\
& - \frac{f_8(t_{k-1}, K)}{h} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta \Big) \\
& + \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \frac{(\vartheta - t_k)(\vartheta - t_{k-1})}{2!} \frac{\partial^2}{\partial\vartheta^2} |f_8(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} (t_{n+1} - \vartheta)^{\rho-1} d\vartheta.
\end{aligned}
$$

where the remainder is given as

$$
R_n^{\rho_1} = \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \frac{(\vartheta - t_k)(\vartheta - t_{k-1})}{2!} \frac{\partial^2}{\partial\vartheta^2} |f_1(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} (t_{n+1} - \vartheta)^{\rho-1} d\vartheta,
$$

$$R_n^{\rho_2} = \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \frac{(\vartheta - t_k)(\vartheta - t_{k-1})}{2!} \frac{\partial^2}{\partial \vartheta^2} |f_2(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} (t_{n+1} - \vartheta)^{\rho-1} d\vartheta,$$

$$R_n^{\rho_3} = \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \frac{(\vartheta - t_k)(\vartheta - t_{k-1})}{2!} \frac{\partial^2}{\partial \vartheta^2} |f_3(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} (t_{n+1} - \vartheta)^{\rho-1} d\vartheta,$$

$$R_n^{\rho_4} = \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \frac{(\vartheta - t_k)(\vartheta - t_{k-1})}{2!} \frac{\partial^2}{\partial \vartheta^2} |f_4(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} (t_{n+1} - \vartheta)^{\rho-1} d\vartheta,$$

$$R_n^{\rho_5} = \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \frac{(\vartheta - t_k)(\vartheta - t_{k-1})}{2!} \frac{\partial^2}{\partial \vartheta^2} |f_5(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} (t_{n+1} - \vartheta)^{\rho-1} d\vartheta,$$

$$R_n^{\rho_6} = \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \frac{(\vartheta - t_k)(\vartheta - t_{k-1})}{2!} \frac{\partial^2}{\partial \vartheta^2} |f_6(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} (t_{n+1} - \vartheta)^{\rho-1} d\vartheta,$$

$$R_n^{\rho_7} = \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \frac{(\vartheta - t_k)(\vartheta - t_{k-1})}{2!} \frac{\partial^2}{\partial \vartheta^2} |f_7(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} (t_{n+1} - \vartheta)^{\rho-1} d\vartheta,$$

$$R_n^{\rho_8} = \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \frac{(\vartheta - t_k)(\vartheta - t_{k-1})}{2!} \frac{\partial^2}{\partial \vartheta^2} |f_8(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} (t_{n+1} - \vartheta)^{\rho-1} d\vartheta.$$

It is unquestionably true that the function $\vartheta \rightarrow^{yield} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1}$ is positive within the interval $[t_k, t_{k+1}]$; therefore, there exists $\xi_k \in [t_k, t_{k+1}]$ such that

$$\begin{aligned} R_n^{\rho_1} &= \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \frac{\partial^2}{\partial \vartheta^2} |f_1(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} \frac{\xi_k - t_k}{2} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta \\ &= \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \frac{\partial^2}{\partial \vartheta^2} |f_1(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} \frac{\xi_k - t_k}{2} \times (A_{\rho,k,1}) h^{\rho+1}, \end{aligned} \tag{6.1}$$

$$\begin{aligned} R_n^{\rho_2} &= \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \frac{\partial^2}{\partial \vartheta^2} |f_2(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} \frac{\xi_k - t_k}{2} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta \\ &= \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \frac{\partial^2}{\partial \vartheta^2} |f_2(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} \frac{\xi_k - t_k}{2} \times (A_{\rho,k,1}) h^{\rho+1}, \end{aligned} \tag{6.2}$$

$$\begin{aligned} R_n^{\rho_3} &= \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \frac{\partial^2}{\partial \vartheta^2} |f_3(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} \frac{\xi_k - t_k}{2} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta \\ &= \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \frac{\partial^2}{\partial \vartheta^2} |f_3(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} \frac{\xi_k - t_k}{2} \times (A_{\rho,k,1}) h^{\rho+1}, \end{aligned} \tag{6.3}$$

$$R_n^{\rho_4} = \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \frac{\partial^2}{\partial \vartheta^2} |f_4(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} \frac{\xi_k - t_k}{2} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta$$

$$= \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \frac{\partial^2}{\partial\vartheta^2} |f_4(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} \frac{\xi_k - t_k}{2} \times (A_{\rho,k,1}) h^{\rho+1}, \tag{6.4}$$

$$R_n^{\rho_5} = \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \frac{\partial^2}{\partial\vartheta^2} |f_5(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} \frac{\xi_k - t_k}{2} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta$$

$$= \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \frac{\partial^2}{\partial\vartheta^2} |f_5(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} \frac{\xi_k - t_k}{2} \times (A_{\rho,k,1}) h^{\rho+1}, \tag{6.5}$$

$$R_n^{\rho_6} = \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \frac{\partial^2}{\partial\vartheta^2} |f_6(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} \frac{\xi_k - t_k}{2} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta$$

$$= \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \frac{\partial^2}{\partial\vartheta^2} |f_6(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} \frac{\xi_k - t_k}{2} \times (A_{\rho,k,1}) h^{\rho+1}, \tag{6.6}$$

$$R_n^{\rho_7} = \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \frac{\partial^2}{\partial\vartheta^2} |f_7(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} \frac{\xi_k - t_k}{2} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta$$

$$= \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \frac{\partial^2}{\partial\vartheta^2} |f_7(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} \frac{\xi_k - t_k}{2} \times (A_{\rho,k,1}) h^{\rho+1}, \tag{6.7}$$

$$R_n^{\rho_8} = \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \frac{\partial^2}{\partial\vartheta^2} |f_8(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} \frac{\xi_k - t_k}{2} \int_{t_k}^{t_{k+1}} (\vartheta - t_{k-1})(t_{n+1} - \vartheta)^{\rho-1} d\vartheta$$

$$= \frac{\rho}{ABC(\rho)\Gamma(\rho)} \sum_{k=0}^{n} \frac{\partial^2}{\partial\vartheta^2} |f_8(\vartheta, S(\vartheta), F)|_{\vartheta=\xi_\vartheta} \frac{\xi_k - t_k}{2} \times (A_{\rho,k,1}) h^{\rho+1}, \tag{6.8}$$

where

$$A_{\rho,k,1} = \frac{(n+1-k)^\rho (n-k+2+\rho) - (n-k)^\rho (n-k+2+2\rho)}{\rho(\rho+1)}. \tag{6.9}$$

Now put the value of $A_{\rho,k,1}$ in Eqs (6.1)–(6.8),

$$R_n^{\rho_1} = \frac{\rho (h)^{\rho+2}}{2ABC(\rho)\Gamma(\rho+2)} \frac{\partial^2}{\partial\vartheta^2} |f_1(\vartheta, S(\vartheta), F)| \times \sum_{k=0}^{n} (G - (n-k)^\rho (n-k+2+2\rho)), \tag{6.10}$$

$$R_n^{\rho_2} = \frac{\rho (h)^{\rho+2}}{2ABC(\rho)\Gamma(\rho+2)} \frac{\partial^2}{\partial\vartheta^2} |f_2(\vartheta, S(\vartheta), F)| \times \sum_{k=0}^{n} (G - (n-k)^\rho (n-k+2+2\rho)), \tag{6.11}$$

$$R_n^{\rho_3} = \frac{\rho (h)^{\rho+2}}{2ABC(\rho)\Gamma(\rho+2)} \frac{\partial^2}{\partial\vartheta^2} |f_3(\vartheta, S(\vartheta), F)| \times \sum_{k=0}^{n} (G - (n-k)^\rho (n-k+2+2\rho)), \tag{6.12}$$

$$R_n^{\rho_4} = \frac{\rho(h)^{\rho+2}}{2ABC(\rho)\Gamma(\rho+2)} \frac{\partial^2}{\partial\vartheta^2}|f_4(\vartheta, S(\vartheta), F)| \times \sum_{k=0}^{n}(G-(n-k)^{\rho}(n-k+2+2\rho)), \tag{6.13}$$

$$R_n^{\rho_5} = \frac{\rho(h)^{\rho+2}}{2ABC(\rho)\Gamma(\rho+2)} \frac{\partial^2}{\partial\vartheta^2}|f_5(\vartheta, S(\vartheta), F)| \times \sum_{k=0}^{n}(G-(n-k)^{\rho}(n-k+2+2\rho)), \tag{6.14}$$

$$R_n^{\rho_6} = \frac{\rho(h)^{\rho+2}}{2ABC(\rho)\Gamma(\rho+2)} \frac{\partial^2}{\partial\vartheta^2}|f_6(\vartheta, S(\vartheta), F)| \times \sum_{k=0}^{n}(G-(n-k)^{\rho}(n-k+2+2\rho)), \tag{6.15}$$

$$R_n^{\rho_7} = \frac{\rho(h)^{\rho+2}}{2ABC(\rho)\Gamma(\rho+2)} \frac{\partial^2}{\partial\vartheta^2}|f_7(\vartheta, S(\vartheta), F)| \times \sum_{k=0}^{n}(G-(n-k)^{\rho}(n-k+2+2\rho)), \tag{6.16}$$

$$R_n^{\rho_8} = \frac{\rho(h)^{\rho+2}}{2ABC(\rho)\Gamma(\rho+2)} \frac{\partial^2}{\partial\vartheta^2}|f_8(\vartheta, S(\vartheta), F)| \times \sum_{k=0}^{n}(G-(n-k)^{\rho}(n-k+2+2\rho)). \tag{6.17}$$

where $G = (n+1-k)^{\rho}(n-k+2+\rho)$. As a result, we apply the norm to both sides of the Eqs (6.10)–(6.17) as well as take advantage of the norm, as follows:

$$|R_n^{\rho_1}| \leq \frac{\rho(h)^{\rho+2}}{2ABC(\rho)\Gamma(\rho+2)} \max_{[0,t_{n+1}]} |\frac{\partial^2}{\partial\vartheta^2}|f_1(\vartheta, S(\vartheta), F)|| \sum_{k=0}^{n}(G-(n-k)^{\rho}(n-k+2+2\rho))|, \tag{6.18}$$

$$|R_n^{\rho_2}| \leq \frac{\rho(h)^{\rho+2}}{2ABC(\rho)\Gamma(\rho+2)} \max_{[0,t_{n+1}]} |\frac{\partial^2}{\partial\vartheta^2}|f_2(\vartheta, S(\vartheta),)|| \sum_{k=0}^{n}(G-(n-k)^{\rho}(n-k+2+2\rho))|, \tag{6.19}$$

$$|R_n^{\rho_3}| \leq \frac{\rho(h)^{\rho+2}}{2ABC(\rho)\Gamma(\rho+2)} \max_{[0,t_{n+1}]} |\frac{\partial^2}{\partial\vartheta^2}|f_3(\vartheta, S(\vartheta), F)|| \sum_{k=0}^{n}(G-(n-k)^{\rho}(n-k+2+2\rho))|, \tag{6.20}$$

$$|R_n^{\rho_4}| \leq \frac{\rho(h)^{\rho+2}}{2ABC(\rho)\Gamma(\rho+2)} \max_{[0,t_{n+1}]} |\frac{\partial^2}{\partial\vartheta^2}|f_4(\vartheta, S(\vartheta), F)|| \sum_{k=0}^{n}(G-(n-k)^{\rho}(n-k+2+2\rho))|, \tag{6.21}$$

$$|R_n^{\rho_5}| \leq \frac{\rho(h)^{\rho+2}}{2ABC(\rho)\Gamma(\rho+2)} \max_{[0,t_{n+1}]} |\frac{\partial^2}{\partial\vartheta^2}|f_5(\vartheta, S(\vartheta), F)|| \sum_{k=0}^{n}(G-(n-k)^{\rho}(n-k+2+2\rho))|, \tag{6.22}$$

$$|R_n^{\rho_6}| \leq \frac{\rho(h)^{\rho+2}}{2ABC(\rho)\Gamma(\rho+2)} \max_{[0,t_{n+1}]} |\frac{\partial^2}{\partial\vartheta^2}|f_6(\vartheta, S(\vartheta), F)|| \sum_{k=0}^{n}(G-(n-k)^{\rho}(n-k+2+2\rho))|, \tag{6.23}$$

$$|R_n^{\rho_7}| \leq \frac{\rho(h)^{\rho+2}}{2ABC(\rho)\Gamma(\rho+2)} \max_{[0,t_{n+1}]} |\frac{\partial^2}{\partial\vartheta^2}|f_7(\vartheta, S(\vartheta), F)|| \sum_{k=0}^{n}(G-(n-k)^{\rho}(n-k+2+2\rho))|, \tag{6.24}$$

$$|R_n^{\rho_8}| \leq \frac{\rho(h)^{\rho+2}}{2ABC(\rho)\Gamma(\rho+2)} \max_{[0,t_{n+1}]} |\frac{\partial^2}{\partial\vartheta^2}|f_8(\vartheta, S(\vartheta), F)|| \sum_{k=0}^{n}(G-(n-k)^{\rho}(n-k+2+2\rho))|. \tag{6.25}$$

The summation of the right-hand side of the Eqs (6.18)–(6.25) converges as follows:

$$\begin{aligned}((n+1-k)^{\rho}(n-k+2+\rho)-(n-k)^{\rho}(n-k+2+2\rho)) &= ((n+1-k)^{\rho}(n-k+2+\rho)-(n-k)^{\rho}(n-k+2+\rho+\rho)) \\ &= ((n-k+2+\rho)((n+1-k)^{\rho}-(n-k)^{\rho}(\rho)(n+1-k)^{\rho}-\rho(n-k)^{\rho}) \\ &\leq ((n+1)^{\rho}-\rho(n)^{\rho})\sum_{k=0}^{n}(n-k+2+\rho) = ((n+1)^{\rho}-\rho(n)^{\rho})\frac{n(n+4+2\rho)}{2}.\end{aligned}$$

Thus

$$|R_n^{\rho_1}| \leq \frac{\rho(h)^{\rho+2}}{2ABC(\rho)\Gamma(\rho+2)} \max_{[0,t_{n+1}]} |\frac{\partial^2}{\partial\vartheta^2}|f_1(\vartheta, S(\vartheta), F)|((n+1)^{\rho} - \rho(n)^{\rho})\frac{n(n+4+2\rho)}{2}, \tag{6.26}$$

$$|R_n^{\rho_2}| \leq \frac{\rho(h)^{\rho+2}}{2ABC(\rho)\Gamma(\rho+2)} \max_{[0,t_{n+1}]} |\frac{\partial^2}{\partial\vartheta^2}|f_2(\vartheta, S(\vartheta), F)|((n+1)^{\rho} - \rho(n)^{\rho})\frac{n(n+4+2\rho)}{2}, \tag{6.27}$$

$$|R_n^{\rho_3}| \leq \frac{\rho(h)^{\rho+2}}{2ABC(\rho)\Gamma(\rho+2)} \max_{[0,t_{n+1}]} |\frac{\partial^2}{\partial\vartheta^2}|f_3(\vartheta, S(\vartheta), F)|((n+1)^{\rho} - \rho(n)^{\rho})\frac{n(n+4+2\rho)}{2}, \tag{6.28}$$

$$|R_n^{\rho_4}| \leq \frac{\rho(h)^{\rho+2}}{2ABC(\rho)\Gamma(\rho+2)} \max_{[0,t_{n+1}]} |\frac{\partial^2}{\partial\vartheta^2}|f_4(\vartheta, S(\vartheta), F)|((n+1)^{\rho} - \rho(n)^{\rho})\frac{n(n+4+2\rho)}{2}, \tag{6.29}$$

$$|R_n^{\rho_5}| \leq \frac{\rho(h)^{\rho+2}}{2ABC(\rho)\Gamma(\rho+2)} \max_{[0,t_{n+1}]} |\frac{\partial^2}{\partial\vartheta^2}|f_5(\vartheta, S(\vartheta), F)|((n+1)^{\rho} - \rho(n)^{\rho})\frac{n(n+4+2\rho)}{2}, \tag{6.30}$$

$$|R_n^{\rho_6}| \leq \frac{\rho(h)^{\rho+2}}{2ABC(\rho)\Gamma(\rho+2)} \max_{[0,t_{n+1}]} |\frac{\partial^2}{\partial\vartheta^2}|f_6(\vartheta, S(\vartheta), F)|((n+1)^{\rho} - \rho(n)^{\rho})\frac{n(n+4+2\rho)}{2}, \tag{6.31}$$

$$|R_n^{\rho_7}| \leq \frac{\rho(h)^{\rho+2}}{2ABC(\rho)\Gamma(\rho+2)} \max_{[0,t_{n+1}]} |\frac{\partial^2}{\partial\vartheta^2}|f_7(\vartheta, S(\vartheta), F)|((n+1)^{\rho} - \rho(n)^{\rho})\frac{n(n+4+2\rho)}{2}, \tag{6.32}$$

$$|R_n^{\rho_8}| \leq \frac{\rho(h)^{\rho+2}}{2ABC(\rho)\Gamma(\rho+2)} \max_{[0,t_{n+1}]} |\frac{\partial^2}{\partial\vartheta^2}|f_8(\vartheta, S(\vartheta), F)|((n+1)^{\rho} - \rho(n)^{\rho})\frac{n(n+4+2\rho)}{2}. \tag{6.33}$$

This completes the proof. □

7. Numerical results and discussions

This section presents the proposed fractional-order model for simulation-based analysis aimed at identifying potential COVID-19 transmission among various age groups in the community. The effectiveness of the theoretical outcomes is established by using advanced techniques. Intriguing findings are achieved by implementing the non-integer parametric choices of the COVID-19 system, as given in [27]. These parameters were set to $\alpha = 0.304, \beta = 0.3, \rho_1 = 0.575, \rho_2 = 0.377, \rho_3 = 0.048, \mu = 0.0192, d_1 = 0.00010, d_2 = 0.0005, d_3 = 0.20, \gamma_1 = 0.025, \gamma_2 = 0.0125$, and $\gamma_3 = 0.00625$. The susceptible population decreases due to an increase in exposed individuals at a rate of β, as shown in Figures 1 and 2, respectively, as a result of decreasing the fractional values. The number of young infected individuals increases sharply but stabilizes over time. In contrast, the number of adult infected individuals increases at a slower rate, taking a longer time to reach a stable situation. These trends can be observed in Figures 3 and 4, respectively. Figure 5 illustrates the decline of the older and comorbid individuals to zero after a certain time period due to their weaker immune system, most of them die and some may become recovered. Figure 6 shows that the number of recovered individuals gradually increases due to a decrease in fractional values, while the number of dead individuals increases due to the infected individuals in all age groups; however, it becomes stable after a certain time due to an increase in the number of recovered individuals as can be seen in Figure 7. In Figures 1–7, the solutions for all compartments are varied according to the desired values by decreasing the fractional values. MATLAB coding was employed to find the numerical solution for a fractional order COVID-19 model by using different fractional values. It is observed that researchers may predict what should happen in the future by referencing this research. The Atangana Toufik technique provides reliable findings for all compartments, particularly under the condition of non-integer fractional values, compared to integer values.

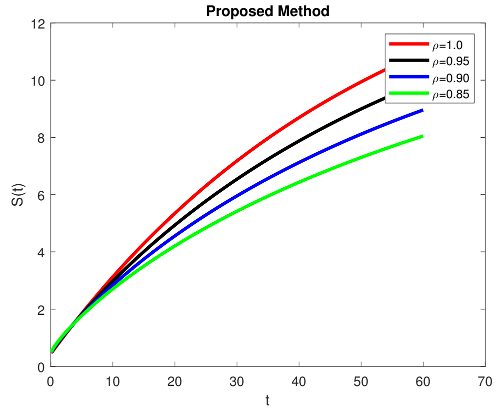

Figure 1. Simulation of $S(t)$ at different fractional values.

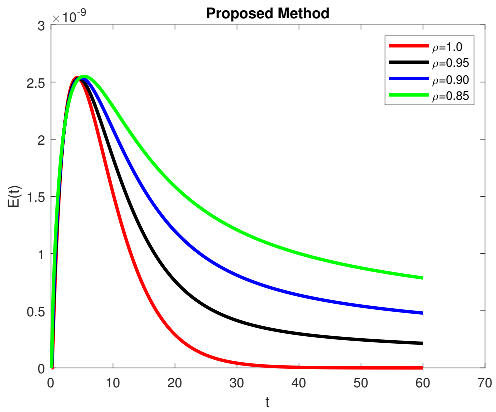

Figure 2. Simulation of $E(t)$ at different fractional values.

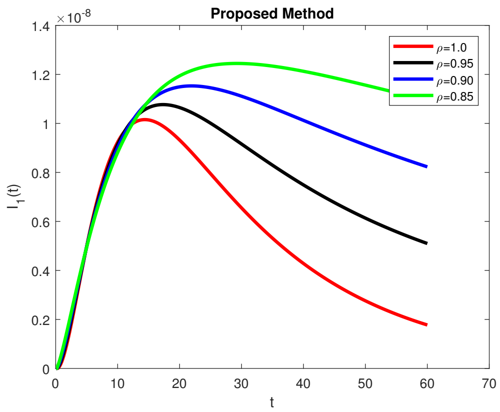

Figure 3. Simulation of $I_1(t)$ at different fractional values.

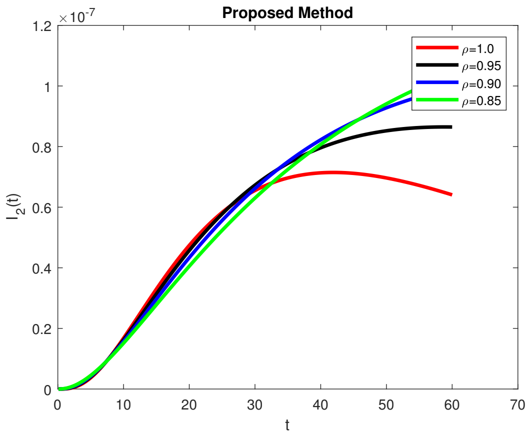

Figure 4. Simulation of $I_2(t)$ at different fractional values.

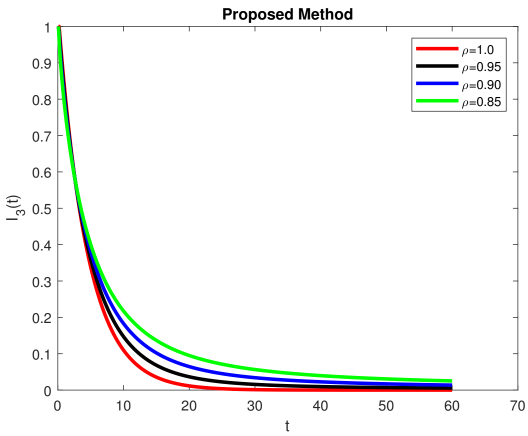

Figure 5. Simulation of $I_3(t)$ at different fractional values.

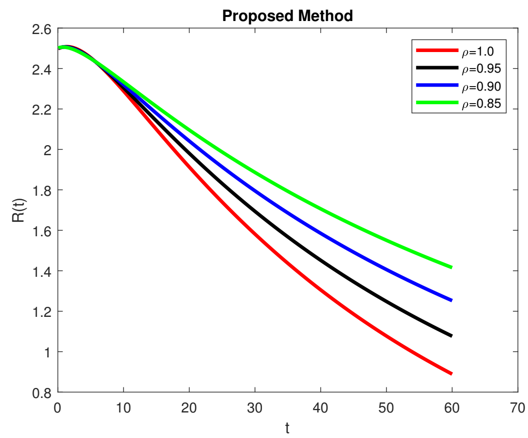

Figure 6. Simulation of $R(t)$ at different fractional values.

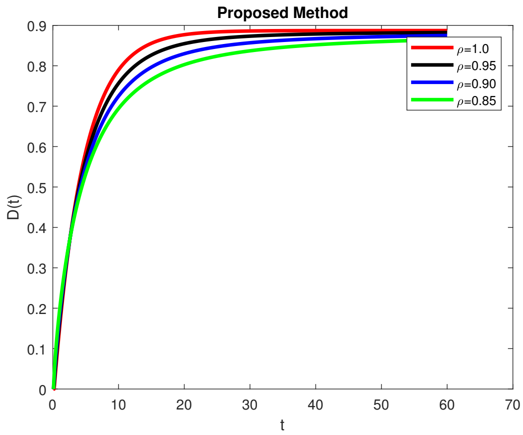

Figure 7. Simulation of $D(t)$ at different fractional values.

8. Explanation of results

In this section, extensive explanation of our developed results is presented to elucidate the pandemic effects for different age groups with the help of simulations. The susceptible population continuously rises and remains the same for the first 10 days. After 10 days, its values begin to diverge according to the fractional values; yet, there is a continued gradual increase until stability is achieved. It shows that the infection must have to be removed after a certain period of time due to a decrease in I_3 for the elderly individual with a comorbidity, because the individuals either become disease free or die due to a weaker immune system. The major treatment and screening of the positive COVID-19 diagnosed individuals causes a reduction in the old age group which can be seen in Figures 1–5 and it will also be implemented for undiagnosed individuals. Exposed individuals, who have been mildly infected by COVID-19 but exhibit no symptoms, experience a sharp increase over the initial 5 days, followed by a rapid decline over the subsequent 5 days. They survive better than the old age group individuals because these groups have strong immune systems; that is the reason for the sharp increase; but, just after 5 days, the number also starts sharply decreasing due to a fast recovery rate which can be seen in Figures 2–4. The number of young infected individuals and adult infected individuals increases but either start decreasing or approaching stability after 15 days and 20 days as can be seen in Figures 3 and 4 respectively. The number of recovered individuals gradually increases due to a decrease in fractional values due to the infected individuals in the young and adults age groups, but it approached stability after a certain period of time due to an increase in the number of recovered individuals, that can be seen in Figure 6. The number of deaths among the older age group increases significantly during the first 10 days, as shown in Figure 7, before approaching stability due to an increase in the number of recovered individuals. Deaths of individuals in the older age group are observed more than the individuals in the young and adult age groups. It is deduced that the number of recovered individuals increases gradually by decreasing the fractional value and it was due to the recovery of young and adult individuals; however, the deaths of individuals in the older age group affects its rate, while the number of dead individuals sharply increases due to the older-age individuals, because most of them died. It is also deduced from our justified outcomes that we need to give more protection to elderly individuals in the future to control the number of deaths in the DRC.

9. Conclusions

In this study, the direct or indirect dissemination of COVID-19 across various age groups in the DRC was examined. To assess the effectiveness of the suggested system and the accuracy of the results, the existence, positivity and boundedness of the system were validated, along with its singular solution. In order to determine the stability of COVID-19 globally, its global stability was explored by using the Lyapunov first derivation function. The solutions were generated by using the sophisticated Atangana-Toufik method. These produced solutions show trustworthy results for using cutting-edge methods for various age groups to regulate COVID-19's dreadful effects and to get rid of the community's death-causing component. All compartments are represented to show how COVID-19 effects respond to changes in parameter values. Through simulations, it is simpler and more comprehensible to observe how COVID-19 affects individuals of various ages over time. The authors contend that complicated dynamical behaviors that were previously unattainable have been disclosed by the synchronization of COVID-19 system. Such research provides more effective strategies to mitigate or manage the effects of COVID-19 and it is also valuable for decision-making. Future simulations involving various fractional value arrangements could be employed to explore potential behaviors within the context of dynamical systems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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