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Research article

New measure of circular intuitionistic fuzzy sets and its application in decision making

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Abstract: Circular intuitionistic fuzzy sets are further extensions of intuitionistic fuzzy sets with a stronger ability to express uncertain information than intuitionistic fuzzy sets. This paper firstly defines a new distance measure for circular intuitionistic fuzzy sets based on the theory of circular intuitionistic fuzzy sets, considering the information of four aspects: membership degree, non-membership degree, radius and the assignment of hesitation degree, and proves that the new distance satisfies the distance measure conditions. Secondly, by constructing a manual testing framework, the new distance is analyzed in comparison with the existing distance metric to show the rationality of the new method. Finally, the method is applied to fuzzy multi-criteria decision making to further demonstrate the effectiveness and practicality of the method.

Keywords: circular intuitionistic fuzzy set; distance; hesitation degree; fuzzy multi-criteria decision making
Mathematics Subject Classification: 28E10, 90B50,91B06

1. Introduction

Fuzziness exists in all aspects of people's lives and is an indispensable part of daily production practices and decision-making management. The fuzzy set theory was first proposed in 1965 by Zadeh [1], an American computer and cybernetics expert, to provide an effective method for dealing with fuzziness. Zadeh used a single scale, membership degree, in fuzzy sets to represent the fuzzy state of support or opposition to something. Later, Bulgarian scholar Atanassov [2] proposed the concept of intuitionistic fuzzy sets based on fuzzy sets. In intuitionistic fuzzy sets, the membership degree, non-membership degree and hesitation degree are considered simultaneously, and can express three states:

support, opposition and neutrality, which is more flexible in dealing with uncertain information. Based on this good property, intuitionistic fuzzy sets have become a research hotspot for many scholars, not only to enrich the theoretical results of their properties, operations and metrics [3–7], but also to apply the set widely in practical life, such as pattern recognition [8–11], decision analysis [12–15], etc.

However, with the advancement of technology and the continuous development of society, it is difficult for people to express the affiliation and non-affiliation with precise real values in practical decision making. For this reason, in 1989, Gargov and Atanassov [16] generalized the intuitionistic fuzzy set and proposed the concept of an interval intuitionistic fuzzy set. Interval intuitionistic fuzzy sets have a stronger ability to express uncertainty when describing uncertain, imprecise problems, but expressing the degree of membership through intervals lacks certain representativeness. Therefore, another generalization of intuitionistic fuzzy sets circular intuitionistic fuzzy set (CIFS) [17] has been proposed, which not only has explicit membership degree and non-membership degree but also reflects the degree of uncertainty in the information by adding a parameter radius. Circular intuitionistic fuzzy sets (CIFSs) are used to simulate the fuzziness and uncertainty in real problems by means of circles in geometry. This special geometric shape is more inclusive. In the same case, circular intuitionistic fuzzy sets can reduce information loss and translate the fuzzy nature of objective problems more accurately into mathematical language. Additionally, circular intuitionistic fuzzy sets are also very suitable for dealing with complex multi-attribute group decision problems, due to the addition of a radius parameter, it can effectively fuse multiple fuzzy information into a circular intuitionistic fuzzy number, so that complex decision problems can be simplified. Therefore, under the combined effect of four parameters, affiliation, non-affiliation, hesitation and radius, the circular intuitionistic fuzzy set can express the information more comprehensively, and is more flexible and realistic in dealing with conceptual fuzzy uncertainty. Currently, circular intuitionistic fuzzy sets have been successfully applied to multiple attribute decision making [18–21], medical diagnosis[18], telework evaluation [22] and many other fields.

In the practical decision-making process, score function and distance measurements are important tools for comparing and ranking fuzzy concepts. First, the score function was firstly proposed by Chen [23] and other scholars in the intuitionistic fuzzy set, and then improved by Hong [24] and others in 2000 by analyzing the problems of the original function, and later Çakır [25] extended the score function to the circular intuitionistic fuzzy set, and with the development of fuzzy multi-attribute decision making, more and more score functions were proposed and applied to practical decision making. Second, distance measure is an important concept in intuitionistic fuzzy sets, and distance measure also has a very important role in circular intuitionistic fuzzy sets. The traditional distance measure of intuitionistic fuzzy sets cannot be used in the new set, so we focus on the problem of distance measure among circular intuitionistic fuzzy sets. Therefore, this paper focuses on the problem of distance measurement between circular intuitionistic fuzzy sets. At present, there are few studies related to distance measures of circular intuitionistic fuzzy sets in the existing literature. Atanassov and Marinovz [26] proposed four distance measures of circular intuitionistic fuzzy sets on the basis of intuitionistic fuzzy set distances. In the same year, Oaty and Kahraman [20] proposed the combined AHP and VIKOR integrated evaluation method on circular intuitionistic fuzzy sets and gave a separation distance metric to calculate the best and worst case. Although the above methods are useful, there are still some shortcomings in the differentiation ability between distances for some circular intuitionistic fuzzy sets.

Our motivation and objectives are as follows:

(1) Due to the special geometric structure of circular intuitionistic fuzzy sets, it is necessary to study the problems related to circular intuitionistic fuzzy sets because of their richer expression for fuzzy information compared with other fuzzy sets. In addition, in the existing literature, there are few relevant studies about the study of circular intuitionistic fuzzy distance metrics, and the methods in the literature [20, 26] can play a certain metric role for circular intuitionistic fuzzy sets, but there are still some shortcomings in the ability to distinguish the distances between some special circular intuitionistic fuzzy sets.

(2) Although four distance measures for circular intuitionistic fuzzy sets have been proposed in [11], these four distance measures only consider the case of the same radius between different circular intuitionistic fuzzy sets, so the applicability of the distance measures in [26] is limited to some specific circular intuitionistic fuzzy sets, and the scope of use is relatively narrow.

(3) Although ALkan et al. [27] introduced circular intuitionistic fuzzy sets into the TOPSIS decision process, its decision principle is processed by transforming circular intuitionistic fuzzy sets into intuitionistic fuzzy sets, and does not really give the solution of circular intuitionistic fuzzy sets. In addition, their decision process is very cumbersome and increases the computational complexity in practical applications.

Therefore, to address the above mentioned research gaps, this paper will give a new circular intuitionistic fuzzy set distance metric to overcome the shortcomings of existing methods. Since hesitation is a very important element in reflecting fuzzy information, the response is an unknown and uncertain. This unknown can be interpreted as a premise that can become either positive or negative. Thus, if an object is further explained, the hesitation may be transformed in the direction of affiliation and disaffiliation. Hence, there may be a certain degree of distribution of the hesitation degree to the affiliation degree and the non-affiliation degree. So, in this paper, we will start from the hesitation degree and explore the potential relationship between the affiliation degree, the non-affiliation degree, the hesitation degree and the radius in the circular intuitionistic fuzzy set.

This paper is divided into six parts. First, the background knowledge is given in Section 1. In Section 2, the concept of circular intuitionistic fuzzy sets and related research are introduced, and the definition of the distance axiom is given. In Section 3, the existing circular intuitionistic fuzzy set distance metric is summarized, and a new circular intuitionistic fuzzy set distance metric is proposed by considering the distribution of hesitancy for affiliation and non-affiliation and the effect of hesitancy on radius. Then, it is proved that the new distance metric satisfies the conditions defined by the distance axiom. In Section 4, a manual testing framework for circular intuitionistic fuzzy sets is established, and the existing distance metric is compared and analyzed with the new distance metric using manual data experiments to prove the rationality of the new distance metric. In Section 5, the new distance metric is applied to multi-criteria decision-making to demonstrate its feasibility and validity. Finally, Section 6 concludes the paper and gives suggestions for future research directions.

2. Pre-requisite knowledge

First, we briefly review some basic concepts of circular intuitionistic fuzzy sets and distance metrics and related theories.

Definition 1. *Let X be a given fixed universe, then $C = \{\langle x, \mu_C(x), \nu_C(x); r | x \in X\}$ is said to be a* *circular intuitionistic fuzzy set on X. Where the functions $\mu_A : X \to [0,1]$,$\nu_A : X \to [0,1]$, denote the membership degree and non-membership degree of C, respectively, and for any $x \in X$, there is $r : X \to \left[0, \sqrt{2}\right]$ a circle radius around each element and there is $0 \le \mu_C(x) + \nu_C(x) \le 1$ holds. Further, $\pi_C(x) = 1 - \mu_C(x) - \nu_C(x)$ is called the hesitancy degree of element x in circular intuitionistic fuzzy set C.*

Definition 2. *[27] With respect to a circular intuitionistic fuzzy set C_i , there exists a set of intuitionistic fuzzy set pairs $\{\langle m_{i,1}, n_{i,1}\rangle, \langle m_{i,2}, n_{i,2}\rangle, ...\}$ that make the circular intuitionistic fuzzy set is calculated as follows:*

$$\langle \mu(C_i), \nu(C_i)\rangle = \left\langle \frac{\sum_{j=1}^{k_i} m_{i,j}}{k_i}, \frac{\sum_{j=1}^{k_i} n_{i,j}}{k_i} \right\rangle, \tag{2.1}$$

where k_i denotes the number of decision makers.

$$r_i = 1 \le \overset{\max}{j} \le k_i \sqrt{\left(\mu(C_i) - m_{i,j}\right)^2 + \left(\nu(C_i) - n_{i,j}\right)^2}. \tag{2.2}$$

For the circular intuitionistic fuzzy set $W = \{C_1, C_2, ...\}$, it can be rewritten as

$$A_r = \{\langle C_i, \mu(C_i), \nu(C_i); r\rangle | C_i \in W\} = \{\langle C_i, O_r(\mu(C_i), \nu(C_i))\rangle | C_i \in W\}. \tag{2.3}$$

Definition 3. *[18] Let C_r be a circular intuitionistic fuzzy set, when $L^* = \{\langle a, b\rangle | a, b \in [0,1] \ \& \ a + b \le 1\}$, C_r can be represented as $C_r^* = \{\langle x, O_r(\mu_C(x), \nu_C(x))\rangle | x \in X\}$, where O_r is a function representing a circle with a radius of r and a center of $(\mu_C(x), \nu_C(x))$.*

$$O_r(\mu_C(x), \nu_C(x)) = \left\{ \begin{matrix} \langle a, b\rangle | a, b \in [0.1] \\ \& \ \sqrt{(\mu_C(x) - a)^2 + (\nu_C(x) - b)^2} \le r \end{matrix} \right\} \cap L^* = \left\{ \begin{matrix} \langle a, b\rangle | a, b \in [0.1] \\ \& \ \sqrt{(\mu_C(x) - a)^2 + (\nu_C(x) - b)^2} \le r \\ \& \ a + b \le 1 \end{matrix} \right\}.$$

Circular intuitionistic fuzzy sets are extensions of standard intuitionistic fuzzy sets, each of the form $C = C_0 = \{x, O_0(\mu_C(x), \nu_C(x)); 0 | x \in X\}$, so that the circular intuitionistic fuzzy set of $r > 0$ cannot be represented by a standard intuitionistic fuzzy set.

Geometrically, each element in the intuitionistic fuzzy set can be represented as a point in the range of the intuitionistic fuzzy set, but in the circular intuitionistic fuzzy set, each element is represented by a circle with center $\langle \mu_C(x), \nu_C(x)\rangle$ and radius$r$, as shown in Figure 1. In Figure 1, it can be seen that there are five possible forms of the circle in the circular intuitionistic fuzzy set.

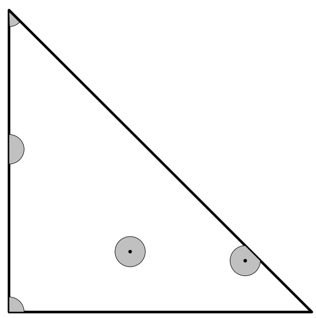

Figure 1. CIFS geometrical representation.

In order to better apply circular intuitionistic fuzzy sets to multi-attribute decision making, some basic circular intuitionistic fuzzy set algorithms are given below.

Definition 4. *[18] Let* $C_1 = \{\langle x, \mu_{C_1}(x), \nu_{C_1}(x); r | x \in X.\}$, $C_2 = \{\langle x, \mu_{C_2}(x), \nu_{C_2}(x); r | x \in X\}$ *be two circular intuitionistic fuzzy sets , the operations between them can be defined as follows.*

$$C_1 \oplus_{\min} C_2 = \left\{x, \mu_{C_1}(x) + \mu_{C_2}(x) - \mu_{C_1}(x) * \mu_{C_2}(x), \nu_{C_1}(x) * \nu_{C_2}(x); \min(r_1, r_2) | x \in X\right\}, \tag{2.4}$$

$$C_1 \oplus_{\max} C_2 = \left\{x, \mu_{C_1}(x) + \mu_{C_2}(x) - \mu_{C_1}(x) * \mu_{C_2}(x), \nu_{C_1}(x) * \nu_{C_2}(x); \max(r_1, r_2) | x \in X\right\}, \tag{2.5}$$

$$C_1 \otimes_{\min} C_2 = \left\{x, \mu_{C_1}(x) * \mu_{C_2}(x), \nu_{C_1}(x) + \nu_{C_2}(x) - \nu_{C_1}(x) * \nu_{C_2}(x); \min(r_1, r_2) | x \in X\right\}, \tag{2.6}$$

$$C_1 \otimes_{\max} C_2 = \left\{x, \mu_{C_1}(x) * \mu_{C_2}(x), \nu_{C_1}(x) + \nu_{C_2}(x) - \nu_{C_1}(x) * \nu_{C_2}(x); \max(r_1, r_2) | x \in X\right\}. \tag{2.7}$$

Inspired by the Eqs (2.6) and (2.7) in Definition 4, this paper adds an extension to the operations between circular intuitionistic fuzzy sets, and the extended operations for circular intuitionistic fuzzy sets is given below.

Definition 5. *Let* $C_1 = \{\langle x, \mu_{C_1}(x), \nu_{C_1}(x); r | x \in X\}$, $C_2 = \{\langle x, \mu_{C_2}(x), \nu_{C_2}(x); r | x \in X\}$ *be two circular intuitionistic fuzzy sets, then*

$$C_1 \odot C_2 = \left\{x, \mu_{C_1}(x) * \mu_{C_2}(x), \nu_{C_1}(x) + \nu_{C_2}(x) - \nu_{C_1}(x) * \nu_{C_2}(x); \frac{r_1 + r_2}{2}\right\}. \tag{2.8}$$

Definition 6. *[25] Let* $C = \{\langle x, \mu_C(x), \nu_C(x); r | x \in X.\}$ *be a circular intuitionistic fuzzy set, then the score function on the circular intuitionistic fuzzy set is*

$$S_C(C) = \left(\mu_C - \nu_C + \sqrt{2}r(2\lambda - 1)\right)/3. \tag{2.9}$$

When $\lambda = 0$ *indicates a completely pessimistic view,* $\lambda = 1$ *indicates an optimistic view, and* $\lambda = 0.5$ *indicates indifference on the part of the decision maker.*

To characterize the distance of a circular intuitionistic fuzzy set, a general definition of the distance metric is given next.

Definition 7. *Let X be a non-empty universe and say that the real-valued function* $D : X \times X \to [0, \infty)$ *is the distance between elements in X if for any* $x, y, z \in X$, *D satisfies the condition*

a) *Non-negativity,* $D(x, y) \geq 0, D(x, y) = 0$ *when and only when* $x = y$.
b) *Symmetry,* $D(x, y) = D(y, x)$.
c) *trigonometric inequalities,* $D(x, y) + D(y, z) \geq D(x, z)$.

3. Proposal of new circular intuitionistic fuzzy set distance measure

This part first summarizes and concludes the existing circular intuitionistic fuzzy set distance metric, and then proposes a new definition of circular intuitionistic fuzzy set metric by considering the distribution of hesitation degree on affiliation degree and non-affiliation degree and the influence of hesitation degree on radius. Finally, it is proved theoretically that the distance definition proposed in this part satisfies the conditions of the distance axiom.

3.1. Distance analysis of existing circular intuitionistic fuzzy sets

Before proposing a new distance measure, the existing circular intuitionistic fuzzy set distance measures are first briefly reviewed and analyzed.

Let A and B be two circular intuitionistic fuzzy sets on the universe $x \in X$, where

$$A = \left\{\left\langle x, \mu_A(x), \nu_A(x); \quad r_A | x \in X\right\rangle\right\},$$

$$B = \left\{\left\langle x, \mu_B(x), \nu_B(x); \quad r_B | x \in X\right\rangle\right\}.$$

Based on the intuitionistic fuzzy set distance, 4 metrics of circular intuitionistic fuzzy set distance were proposed by Atanassov [26] in 2021.

$$H_2(A,B) = \frac{1}{2}\left(\frac{|r_A - r_B|}{\sqrt{2}} + \frac{1}{2C_X} \cdot \sum_{x \in X}\left(|\mu_A(x) - \mu_B(x)| + |\nu_A(x) - \nu_B(x)|\right)\right), \tag{3.1}$$

$$E_2(A,B) = \frac{1}{2}\left(\frac{|r_A - r_B|}{\sqrt{2}} + \sqrt{\frac{1}{2C_X} \cdot \left(\sum_{x \in X}\left((\mu_A(x) - \mu_B(x))^2 + (\nu_A(x) - \nu_B(x))^2\right)\right)}\right), \tag{3.2}$$

$$H_3(A,B) = \frac{1}{2}\left(\frac{|r_A - r_B|}{\sqrt{2}} + \frac{1}{2C_X} \cdot \sum_{x \in X}\left(|\mu_A(x) - \mu_B(x)| + |\nu_A(x) - \nu_B(x)| + |\pi_A(x) - \pi_B(x)|\right)\right), \tag{3.3}$$

$$E_3(A,B) = \frac{1}{2}\left(\frac{|r_A - r_B|}{\sqrt{2}} + \sqrt{\frac{1}{2C_X} \cdot \left(\sum_{x \in X}\left((\mu_A(x) - \mu_B(x))^2 + (\nu_A(x) - \nu_B(x))^2 + (\pi_A(x) - \pi_B(x))^2\right)\right)}\right) \tag{3.4}$$

In Eqs (3.1) and (3.2) only the difference between membership degree and non-membership degree is considered and the effect of hesitation degree on distance is ignored, according to the analysis in the literature [28], the role of hesitation degree in distance metrics is very important, so Eqs (3.1) and (3.2) have some defects.

Equations (3.3) and (3.4), although taking into account the influence of hesitation degree on the distance generated and satisfying the definition of relevant distance, cannot make a correct distinction between the distance between some special circular intuitionistic fuzzy sets.

Example 1. *For the distances between the circular intuitionistic fuzzy sets* $\{\langle x, 0.3, 0.4; 0.3\rangle\}$ *and* $\{\langle x_2, 0.4, 0.3; 0.2\rangle\}$ *and the distances between the circular intuitionistic fuzzy sets* $\{\langle x, 0.4, 0.2; 0.35\rangle\}$ *and* $\{\langle x_2, 0.5, 0.2; 0.25\rangle\}$ *, the deviation between their membership degree and radius are both 0.1, but the deviation of non- membership degree for the former set is 0.1 (the deviation of hesitation degree is 0) and the deviation of non- membership degree for the latter set is 0 (the deviation of hesitation degree is 0.1). Since the degree of hesitation degree portraying the fuzzy object can neither be affirmed nor denied is different from the degree of membership and non-membership, it can be assumed that these two sets of distances should be different. However, the results obtained by both Eqs (3.3) and (3.4) are 0.085 do not reflect this difference, so Eqs (3.3) and (3.4) have some shortcomings.*

3.2. New circular intuitionistic fuzzy set distance metric

Circular intuitionistic fuzzy sets use membership degree, non-membership degree and radius to describe the degree of uncertainty of information and hesitation degree to describe its degree of unknown. Therefore, the status and role between hesitation degree and membership degree, non-membership degree and radius are different in the circular intuitionistic fuzzy set. And through the above analysis, it can be found that there are still some unreasonableness in the practical application of introducing the difference of hesitation degree directly into the distance metric. Based on this, this paper will give a new distance metric for circular intuitionistic fuzzy sets, which not only satisfies the definition of distance, but also solves the problem that the existing distance exists unreasonably. Next, the specific formulation of the new metric is given by the voting model.

Suppose the universe $E = \{e_1, e_2, e_3\}$ indicates that there are 3 experts voting on an object A, and the results of the experts' votes are represented by $\langle e_i, \mu(e_i), \nu(e_i), \pi(e_i)\rangle, i = 1, 2, 3$, where $\mu_A(e_i)$ indicates the degree of support of each expert e_i for A, $\nu_A(e_i)$ indicates the degree of opposition, and $\pi_A(e_i) = 1-\mu_A(e_i)-\nu_A(e_i)$ indicates the degree of abstention. In the above condition, if assume that the three experts' voting attitudes towards A are $\langle e_1, 0.6, 0.1, 0.3\rangle$,$\langle e_2, 0.4, 0.3, 0.3\rangle$ and $\langle e_3, 0.5, 0.1, 0.4\rangle$, the number of circular intuitionistic fuzzy of the three voting attitudes is $C_A = \langle 0.5, 0.17; 0.17\rangle$. If after some persuasion the experts may change all the abstentions to support, then the best result will be obtained , $\langle e_1, 0.9, 0.1, 0\rangle$,$\langle e_2, 0.7, 0.3, 0\rangle$,$\langle e_3, 0.9, 0.1, 0\rangle$, then get $C_{A^1} = \langle 0.87, 0.17; 0.15\rangle$. Conversely, if the expert changes all the abstentions to oppose, then the result of the circular intuition fuzzy number will again become $C_{A^2} = \langle 0.50, 0.48; 0.13\rangle$, and of course there are cases where the expert divides the abstentions equally between support and opposition. So after some persuasion the voting attitude produces some changes in terms of membership degree, non- membership degree and radius, and satisfies both $\mu_{A^i}(e) + \nu_{A^i}(e) = 1$, $r_c(e) \le \sqrt{2}$, and this paper calls this situation as the distribution of hesitation degree to membership degree and non-membership degree. Therefore, in order to better deal with imprecise information in practice, this paper considers introducing the assignment of hesitation degree to affiliation degree and non-affiliation degree into the circular intuitionistic fuzzy set distance, thus indirectly introducing hesitation degree into the distance metric. The definition of the assignment of hesitation degree to membership degree and non- membership degree is given below first.

Definition 8. *Let* $A=\{\langle x, \mu_A(x), \nu_A(x); r_A\rangle | x_i \in X\}$ *be a circular intuitionistic fuzzy set on X and the allocation of membership degree and non-membership degree by hesitancy degree* $\pi_A(x_i)$ *is defined as*

$$\begin{aligned} CD^A_{\pi\to\mu} &= \frac{1}{2}\left[\pi_A(x) + 2\mu_A(x)\right], \\ CD^A_{\pi\to\nu} &= \frac{1}{2}\left[\pi_A(x) + 2\nu_A(x)\right]. \end{aligned} \tag{3.5}$$

Since the degree of hesitation degree responds to the unknown degree of information, it can be interpreted as an ambiguous attitude of the decision maker to affirm or deny an uncertain object. Therefore, it can be considered that there may be a degree of partial affirmation and a degree of partial negation in hesitation degree, so in this paper, the hesitation degree is equally divided into membership and non-membership degrees in Eq (3.5).

In the definition of circular intuitionistic fuzzy set radius is obtained from the calculation of membership degree and membership degree, so the change of membership degree and non-membership degree will also cause the change of radius. Based on this, the definition of radius of circular intuitionistic fuzzy set under the new hesitation degree assignment is given.

Definition 9. *Let there exist a set of intuitionistic fuzzy set pairs* $\{\langle m_{i1}, n_{i1}\rangle, \langle m_{i1}, n_{i1}\rangle, ...\}$ *, then the radius of the circular intuitionistic fuzzy set* C_i *is*

$$r_{ij} = \max_{1\le j\le k_i} \left| \sqrt{\left(\left(\mu(C_i) + \frac{1}{2}\pi(C_i)\right) - \left(m_i + \frac{1}{2}l_i\right)\right)^2 + \left(\left(\nu(C_i) + \frac{1}{2}\pi(C_i)\right) - \left(n_i + \frac{1}{2}l_i\right)\right)^2} \right|, \tag{3.6}$$

where $r_{ij} \in [0, \sqrt{2}]$, $\pi(C_i) = 1 - \mu(C_i) - \nu(C_i)$, $l_{ij} = 1 - m_i - n_i$,$\langle \mu(C_i), \nu(C_i)\rangle = \left\langle \frac{\sum_{j=1}^{k_i} m_{i.j}}{k_i}, \frac{\sum_{j=1}^{k_i} n_{i.j}}{k_i} \right\rangle$.

The following distance metric formula for circular intuitionistic fuzzy sets under hesitation degree assignment is given based on the inspiration of the distance metric for intuitionistic fuzzy sets in the literature [29].

Definition 10. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x); r_A\rangle\}$, $B = \{\langle x, \mu_B(x), \nu_B(x); r_B\rangle\}$ *be two arbitrary circular intuitionistic fuzzy sets of on the universe* X $(x \in X)$*.Then the distance metric between A and B is defined as*

$$d_C(A, B) = \frac{|r_A - r_B|}{\sqrt{2}} + \sqrt{\frac{1}{4}\left[(\Delta_\mu^{AB})^2 + (\Delta_\nu^{AB})^2 + (\Delta_{\pi\to\mu}^{AB})^2 + (\Delta_{\pi\to\nu}^{AB})^2\right]}, \tag{3.7}$$

where $\Delta_\mu^{AB}=|\mu_A(x) - \mu_B(x)|$, $\Delta_\nu^{AB}=|\nu_A(x) - \nu_B(x)|$, $\Delta_{\pi\to\mu}^{AB} = |CD_{\pi\to\mu}^A - CD_{\pi\to\mu}^B|$, $\Delta_{\pi\to\nu}^{AB} = |CD_{\pi\to\nu}^A - CD_{\pi\to\nu}^B|$.

The construction of $d_C = (A, B)$ is divided into two main aspects, one is to calculate the difference of radius between two circular intuitionistic fuzzy sets, and the other is to use the idea of Euclidean distance in real space, and to construct the distance formula by considering $\left(\mu_A(x), \nu_A(x), CD_{\pi\to\mu}^A(x), CD_{\pi\to\nu}^A(x)\right)$ and $\left(\mu_B(x), \nu_B(x), CD_{\pi\to\mu}^B(x), CD_{\pi\to\nu}^B(x)\right)$ as two points in a four-dimensional real space. Therefore, the following theorem can be obtained.

Theorem 1. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x); r_A\rangle\}$, $B = \{\langle x, \mu_B(x), \nu_B(x); r_B\rangle\}$,$C = \{\langle x, \mu_C(x), \nu_C(x); r_C\rangle\}$ *be three arbitrary circular intuitionistic fuzzy sets on the theoretical universe* X $(x \in X)$ *and* $d_C(A, B)$ *be the distance metric on the circular intuitionistic fuzzy set. Therefore* $d_C(A, B)$ *should satisfy the following three properties:*

a) *Non-negativity,* $d_C(A, B) \ge 0, d_C(A, B) = 0$ *when and only when* $A = B$.

b)*symmetry,* $d_C(A, B) = d_C(B, A)$.

c) *trigonometric inequality,* $D(A, B) + D(B, C) \ge D(A, C)$.

Proof. To prove whether $d_C(A, B)$ is a distance metric, it is sufficient to show that $d_C(A, B)$ satisfies the three conditions in Definition 7. Let $A, B, C \in CIFS\,s(X)$.

a) Obviously $d_C(A, B) \ge 0$, if $A = B$, then $\Delta_\mu^{AB} = \Delta_\nu^{AB} = \Delta_{\pi\to\mu}^{AB} = \Delta_{\pi\to\nu}^{AB} = 0$, $r_A = r_B = 0$, and thus $d_C(A, B) = 0$. On the contrary, if $d_C(A, B) = 0$, $\Delta_\mu^{AB} = \Delta_\nu^{AB} = 0$, $\mu_A(x) = \mu_B(x)$,$\nu_A(x) = \nu_B(x)$,thus $A = B$. Form this $d_C(A, B)$ satisfies the condition of Definition 3.

b) For $d_C(A, B)$, $d_C(A, B) = d_C(B, A)$ is constant. So $d_C(A, B)$satisfies the condition$b$ in Definition 7.

c) By taking the second half of the absolute value inequality $|x| - |y| \le |x \pm y| \le |x| + |y|$ from $|x| + |y| \ge |x + y|$.

$$|(r_C - r_B) + (r_B - r_A)| \leq |r_B - r_A| + |r_C - r_B| \Rightarrow |r_C - r_A| \leq |r_B - r_A| + |r_C - r_B|.$$

Both sides of the inequality are divided by $\sqrt{2}$

$$\frac{|r_C - r_A|}{\sqrt{2}} \leq \frac{|r_B - r_A|}{\sqrt{2}} + \frac{|r_C - r_B|}{\sqrt{2}},$$

make

$$d(A,B) = \sqrt{\frac{1}{4}\left[(\Delta_{\mu}^{AB})^2 + (\Delta_{\nu}^{AB})^2 + (\Delta_{\pi\to\mu}^{AB})^2 + (\Delta_{\pi\to\nu}^{AB})^2\right]}.$$

Since $\sqrt{\frac{1}{4}\left[(\Delta_{\mu}^{AB})^2 + (\Delta_{\nu}^{AB})^2 + (\Delta_{\pi\to\mu}^{AB})^2 + (\Delta_{\pi\to\nu}^{AB})^2\right]}$ can be considered as the Euclidean distance between two points $(\mu_A(x), \nu_A(x), CD_{\pi\to\mu}^{A}(x), CD_{\pi\to\nu}^{A}(x))$ and $(\mu_B(x),\ \nu_B(x),\ CD_{\pi\to\mu}^{B}(x), CD_{\pi\to\nu}^{B}(x))$ in a four-dimensional space divided by 2, it follows from the triangular inequality of the Euclidean distance that $d(A,B)$ satisfies the triangular inequality. So $d(A,C) \leq d(A,B) + d(B,C)$. Finally by adding the two sides of the inequality to get

$$\frac{|r_C - r_A|}{\sqrt{2}} + \sqrt{\frac{1}{4}\left[(\Delta_{\mu}^{AC})^2 + (\Delta_{\nu}^{AC})^2 + (\Delta_{\pi\to\mu}^{AC})^2 + (\Delta_{\pi\to\nu}^{AC})^2\right]} \leq \frac{|r_B - r_A|}{\sqrt{2}} + \sqrt{\frac{1}{4}\left[\begin{array}{c}(\Delta_{\mu}^{AB})^2 + (\Delta_{\nu}^{AB})^2 \\ + (\Delta_{\pi\to\mu}^{AB})^2 + (\Delta_{\pi\to\nu}^{AB})^2\end{array}\right]}$$
$$+ \frac{|r_C - r_B|}{\sqrt{2}} + \sqrt{\frac{1}{4}\left[(\Delta_{\mu}^{BC})^2 + (\Delta_{\nu}^{BC})^2 + (\Delta_{\pi\to\mu}^{BC})^2 + (\Delta_{\pi\to\nu}^{BC})^2\right]}.$$

Thus the distance $d_C(A,B)$ satisfies the condition c in Definition 7.

In summary, it can be concluded that since the property $a \sim c$ in Definition 7 is satisfied, $d_C(A,B)$ is the distance metric of the circular intuitionistic fuzzy set. □

Definition 11. *Let A, B be two arbitrary circular intuitionistic fuzzy sets on the universe $X = \{x_1, x_2, \cdots, x_n\}$, then the normalized circular intuitionistic fuzzy set distance between A and B is defined as follows*

$$Nd_C(A,B) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\left|\Delta_r^{AB}(i)\right|}{\sqrt{2}} + \sqrt{\frac{1}{4}\left[\left(\Delta_{\mu}^{AB}(i)\right)^2 + \left(\Delta_{\nu}^{AB}(i)\right)^2 + \left(\Delta_{\pi\to\mu}^{AB}(i)\right)^2 + \left(\Delta_{\pi\to\nu}^{AB}(i)\right)^2\right]}\right), \tag{3.8}$$

where $\Delta_{\mu}^{AB}(i) = |\mu_A(x_i) - \mu_B(x_i)|$, $\Delta_{\nu}^{AB}(i) = |\nu_A(x_i) - \nu_B(x_i)|$, $\Delta_r^{AB}(i) = |r_A(x_i) - r_B(x_i)|$, $\Delta_{\pi\to\mu}^{AB}(i) = |CD_{\pi\to\mu}^{A}(x_i) - CD_{\pi\to\mu}^{B}(x_i)|$, $\Delta_{\pi\to\nu}^{AB}(i) = |CD_{\pi\to\nu}^{A}(x_i) - CD_{\pi\to\nu}^{B}(x_i)|$.

Similar to Theorem 1, the normalized distance metric $Nd_C(A,B)$ yields the following conclusion.

Theorem 2. *Let A and B be two circular intuitionistic fuzzy sets on the theoretical universe $X = \{x_1, x_2, \cdots, x_n\}$, then $Nd_C(A,B)$ is a distance metric on the circular intuitionistic fuzzy set.*

Proof. Based on the Euclidean distance and Theorem 1, it is clear that the conclusion holds. □

4. Numerical comparison analysis

To illustrate the effectiveness of the distance metric method in discriminating circular intuitionistic fuzzy sets in this paper, a six-group manual test framework was established to test the ability to discriminate the distance of circular intuitionistic fuzzy sets by referring to the distance metric test data in the literature [29–33]. Although these examples cannot cover all counterexamples, they are representative.

Since there are few distance measures for circular intuitionistic fuzzy sets, the divergence measure for circular intuitionistic fuzzy sets from the literature [34] will be added in this section for joint testing. The divergence measure and the distance measure have the same role in distinguishing the similarity between circular intuitionistic fuzzy sets, so they can be used together for comparison.

In 2022, Muhammad et al. proposed four formulas for the scattering measure of circular intuitionistic fuzzy sets as shown below.Let A and B be two circular intuitionistic fuzzy sets on set $x_i \in X = \{x_1, x_2, \cdots, x_n\}$, where $A = \{\langle x_i, \mu_A(x_i), \nu_A(x_i); r_A(x_i) | x_i \in X\rangle\}$,$B = \{\langle x_i, \mu_B(x_i), \nu_B(x_i); r_B(x_i) | x_i \in X\rangle\}$.

$$\overline{D}_1(A,B) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{(\mu_A(x_i)-\mu_B(x_i))^2}{1+\mu_A(x_i)+\mu_B(x_i)} + \frac{(\nu_A(x_i)-\nu_B(x_i))^2}{1+\nu_A(x_i)+\nu_B(x_i)} + \frac{(r_A(x_i)-r_B(x_i))^2}{1+r_A(x_i)+r_B(x_i)}\right]. \tag{4.1}$$

$$\overline{D}_2(A,B) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{(\mu_A(x_i)-\mu_B(x_i))^2}{1+\mu_A(x_i)+\mu_B(x_i)} + \frac{(\nu_A(x_i)-\nu_B(x_i))^2}{1+\nu_A(x_i)+\nu_B(x_i)}\right]\exp\left(\frac{(r_A(x_i)-r_B(x_i))^2}{1+r_A(x_i)+r_B(x_i)}\right). \tag{4.2}$$

$$\overline{D}_3(A,B) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{|\mu_A(x_i)-\mu_B(x_i)|}{1+\mu_A(x_i)+\mu_B(x_i)} + \frac{|\nu_A(x_i)-\nu_B(x_i)|}{1+\nu_A(x_i)+\nu_B(x_i)} + \left(\frac{|r_A(x_i)-r_B(x_i)|}{1+r_A(x_i)+r_B(x_i)}\right)\right]. \tag{4.3}$$

$$\overline{D}_4(A,B) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{|\mu_A(x_i)-\mu_B(x_i)|}{1+\mu_A(x_i)+\mu_B(x_i)} + \frac{|\nu_A(x_i)-\nu_B(x_i)|}{1+\nu_A(x_i)+\nu_B(x_i)}\right]\exp\left(\frac{|r_A(x_i)-r_B(x_i)|}{1+r_A(x_i)+r_B(x_i)}\right). \tag{4.4}$$

The distance between six sets of circular intuitionistic fuzzy sets is calculated using the existing circular intuitionistic fuzzy set distance metric, divergence metric and the metric proposed in this paper, respectively. The results are shown in Table 1.

Table 1. Comparison of distance measures between CIFSs.

	1	**2**	**3**	**4**	**5**	**6**
A	$<x, 0.3, 0.3; 0.4>$	$<x, 0.3, 0.4; 0.4>$	$<x, 1.0, 0.0; 0.0>$	$<x, 0.5, 0.5; 0.0>$	$<x, 0.4, 0.2; 0.3>$	$<x, 0.4, 0.2; 0.35>$
B	$<x, 0.4, 0.4; 0.3>$	$<x, 0.4, 0.3; 0.3>$	$<x, 0.0, 0.0; 0.0>$	$<x, 0.0, 0.0; 0.0>$	$<x, 0.5, 0.3; 0.4>$	$<x, 0.5, 0.2; 0.25>$
$H_2(A,B)$	**0.0854**	**0.0854**	**0.2500**	**0.2500**	0.0854	0.0604
$E_2(A,B)$	**0.0854**	**0.0854**	0.3536	0.2500	0.0854	0.0707
$H_3(A,B)$	0.1354	**0.0854**	**0.5000**	**0.5000**	0.1354	**0.0854**
$E_3(A,B)$	0.1220	**0.0854**	0.5000	0.4330	0.1220	**0.0854**
$\overline{D}_1(A,B)$	**0.0176**	**0.0176**	0.5000	0.3333	0.0178	0.0115
$\overline{D}_2(A,B)$	**0.9742**	**0.9742**	**1.0000**	**1.0000**	0.9743	0.9677
$\overline{D}_3(A,B)$	**0.1765**	**0.1765**	0.5000	0.6667	0.1781	0.1151
$\overline{D}_4(A,B)$	**0.8817**	**0.8817**	**1.0000**	**1.0000**	0.8824	0.8319
$d_C(A,B)$	0.0854	0.1061	0.4330	0.2500	0.0854	0.0787

(1) For the first and second sets of circular intuitionistic fuzzy sets, the non-membership degree in $\{< x, 0.3, 0.3; 0.4 >\}$ is smaller than that in $\{< x, 0.3,0.4; 0.4 >\}$ with the same membership degree and radius, and the non-membership degree in $\{< x, 0.4, 0.4; 0.3 >\}$ is larger than that in $\{< x, 0.4,0.3; 0.3 >\}$, so it can be considered that the positive degree in $\{< x, 0.3, 0.3; 0.4 >\}$ is larger than that in $\{< x, 0.3, 0.4; 0.4 >\}$ and the positive degree in $\{< x, 0.4, 0.4; 0.3 >\}$ is smaller than that in $\{< x, 0.4, 0.3; 0.3 >\}$. Based on this, it can be known that the positive and negative degrees of the first and second circular intuitionistic fuzzy sets should be different when performing distance operations. However, in Table 1, it can be seen that the distance measures and calculate the distance between the first and second circular intuitionistic fuzzy sets are both 0.0854. Neither divergence measure $\overline{D}_1, \overline{D}_2, \overline{D}_3$ and $\overline{D}_4$ effectively distinguishes between the first and second circular intuitionistic fuzzy sets, so this result suggests that they cannot effectively distinguish between positive and negative differences.

(2) For the distance between the circular intuitionistic fuzzy sets $\{\langle x, 1, 0; 0\rangle\}$ and $\{\langle x, 0, 0; 0\rangle\}$ and the distance between the circular intuitionistic fuzzy sets $\{\langle x, 0.5, 0.5; 0\rangle\}$ and $\{\langle x, 0, 0; 0\rangle\}$,$\{\langle x, 1, 0; 0\rangle\}$ can be considered as a vote of 10 people with all voting in favor, $\{\langle x, 0.5, 0.5; 0\rangle\}$ as 5 people voting in favor and 5 people voting against, and $\{\langle x, 0, 0; 0\rangle\}$ as all abstaining from voting. It is therefore reasonable to assume that there should be some difference between the distances between $\{\langle x, 1, 0; 0\rangle\}$ and $\{\langle x, 0, 0; 0\rangle\}$ and the distances between $\{\langle x, 0.5, 0.5; 0\rangle\}$ and $\{\langle x, 0, 0; 0\rangle\}$. However, the results obtained for $H_3, \overline{D}_2$ and $\overline{D}_4$ are equal, which indicates that metric $H_3, \overline{D}_2, \overline{D}_4$ considers that the two sets of test set distances express the same meaning, which is obviously unreasonable.

(3) From the results shown in the table, the distance measures H_3 and E_3 are the same results in the second and sixth circular intuitionistic fuzzy sets. According to the analysis of Example 1 in this paper, the distances between these two sets of circular intuitionistic fuzzy sets should be different, so the distance measures H_3 and E_3 still have some shortcomings in distinguishing the differences between circular intuitionistic fuzzy sets.

Therefore, it can be found that the existing metrics still have some unreasonableness in distinguishing the distance of some special circular intuitionistic fuzzy sets. On the contrary, the circular intuitionistic fuzzy set metric proposed in this paper fully accounts for the potential differences caused by distance hesitation and introduces them into the distance metric in a distributive way. This new distance measure not only overcomes the problems arising from the existing method, but also the results obtained in Table 1 are more realistic. Therefore, the analysis of the new circular intuitionistic fuzzy set distance measure and the numerical comparison results in Table 1 show that the circular intuitionistic fuzzy set distance measure given in this paper is reasonable and feasible.

Additionally, it can be seen from the above experimental results that the distance metric proposed in this paper achieves the change of metric results by assigning the hesitation variable. When the hesitation variable is not assigned, there is a problem that some differences between circular intuitionistic fuzzy sets cannot be distinguished. However, when the hesitation variable is equally assigned to the membership and non-membership before the distance calculation, those originally indistinguishable circular intuitionistic fuzzy sets are successfully distinguished, so the assignment of the hesitation variable makes the fuzzy information expression more reasonable.

5. Application of the new distance measure in multi-criteria decision making

A concise version of the circular intuitionistic fuzzy set TOPSIS multi-criteria decision making method is first given in this section, and the flowchart in Figure 2 gives the framework of this decision making method. Then, the validity and feasibility of the new distance metric proposed in this paper are verified by using an example of pandemic hospital site selection. Finally, the multi-criteria decision model proposed in this paper is compared and analyzed with other multi-criteria decision models to reflect the advantages of this paper's multi-criteria decision model.

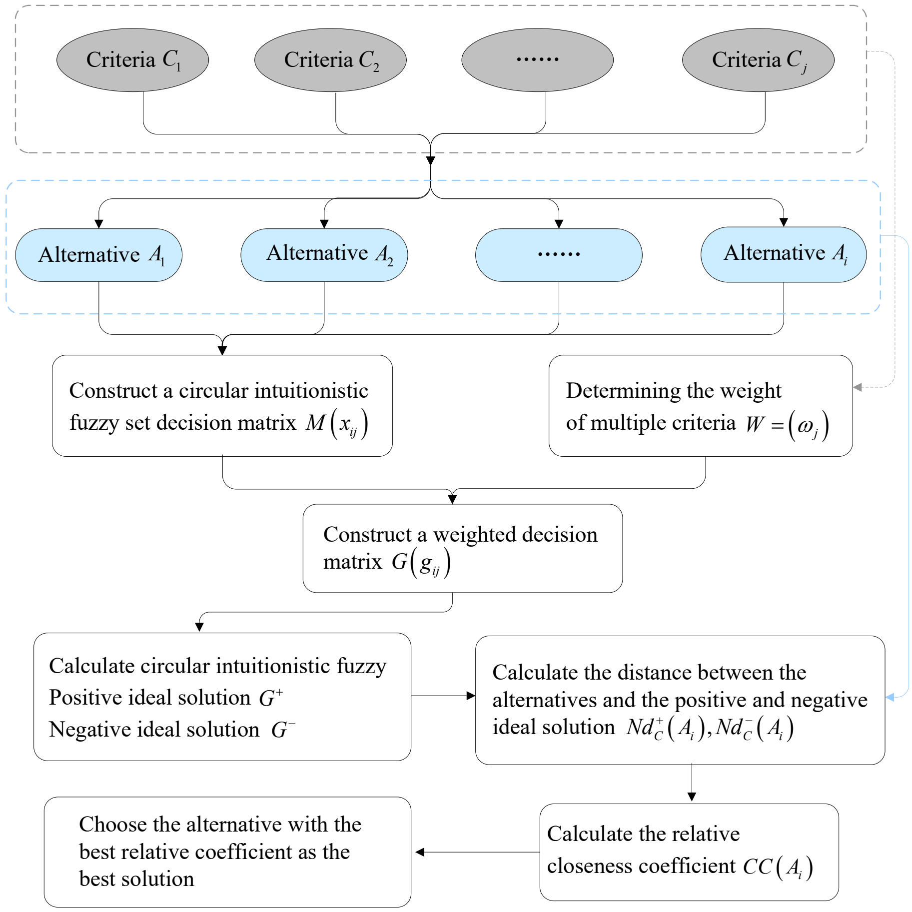

Figure 2. Flow chart of the new TOPSIS method for CIFS.

5.1. *Multi-criterion decision steps of CIFS based on new distance measure*

In order to test the practicality and effectiveness of the new circular intuitionistic fuzzy set distance proposed in this paper, the following will use the new distance measure to deal with the actual multi-criteria decision problem. TOPSIS is an effective method in solving multi-attribute decision problems, and this paper gives a circular intuitionistic fuzzy TOPSIS method based on a new distance measure, according to some improvements of the decision method in the literature [27]. The flow chart in

Figure 2 shows the framework of the new method. The specific process of the new decision method will be described in detail in text form.

Step 1 Let the solution set of the multi-criteria decision problem be $A_{\mathrm{i}} = \{A_1, A_2, ..., A_m\}, i = 1, 2, ..., m$; the decision criteria set be $C_j = \{C_1, C_2, ..., C_n\}, j = 1, 2, ..., n$; $w = (w_1, w_2, ..., w_n)$ be the weight vector of the criteria set and $w_j > 0$, $\sum_{j=1}^{n} w_j = 1$; DM be the decision maker, using experts in different fields to make judgments on the solutions and decision criteria.

Step 2 Collects the opinions of decision makers, constructs the expert linguistic decision matrix, and directly transforms the qualitative information into the information in the form of intuitionistic fuzzy numbers (IFNs) through Table 2.

Table 2. Intuitionistic fuzzy numbers semantic information quantization table.

Linguistic Value	IFNs
Certainly High Value-(CHV)	$\langle 0.9, 0.1\rangle$
Very High Value-(VHV)	$\langle 0.8, 0.15\rangle$
High Value-(HV)	$\langle 0.7, 0.25\rangle$
Above Average Value-(AAV)	$\langle 0.6, 0.35\rangle$
Average Value-(AV)	$\langle 0.5, 0.45\rangle$
Under Average Value-(UAV)	$\langle 0.4, 0.55\rangle$
Low Value-(LV)	$\langle 0.3, 0.65\rangle$
Very Low Value-(VLV)	$\langle 0.2, 0.75\rangle$
Certainly Low Value-(CLV)	$\langle 0.1, 0.9\rangle$

Step 3 Convert the intuitionistic fuzzy pairs of opinions of multiple decision makers of the same decision criteria and option in the decision matrix into aggregated intuitionistic fuzzy numbers $\langle \mu(C_i), \nu(C_i)\rangle$ using Eq (2.1). Then use Eq (3.6) to calculate the corresponding radius length to construct a circular intuitionistic fuzzy decision matrix $M = \left(x_{ij}\right)_{n\times m}$, where $x_{ij} = \left\{\left\langle \mu_{ij}, \nu_{ij}; r_{ij}\right\rangle\right\}$ denotes the circular intuitionistic fuzzy number of the alternative with respect to the criteria.

Step 4 Determine the weights of different criteria by quantifying the weight information Table 3 to obtain the weighted sum table of intuitionistic fuzzy criteria. Then the maximum radius$r$is calculated using Eq (3.6) to construct the circular intuitionistic fuzzy set criteria weight matrix W=$\left(w_j\right)_{1\times n}$, where $w_j = \left\{\left\langle \mu_j, \nu_j; r_j\right\rangle\right\}$.

Step 5 Construct the weighted decision matrix $G = \left(g_{ij}\right)_{m\times n}$ where $g_{ij} = \left\{\left\langle \mu_{ij}, \nu_{ij}; r_{ij}\right\rangle\right\}$ using Eq (2.8), the circular intuitionistic fuzzy set decision matrix M and the weight matrix W obtained in Step 4.

Step 6 Determine the positive ideal solution of the decision matrix G^+ and the negative ideal solution G^-

$$G^+ = \left\{\left\langle \left(\max_i g_{ij} | j \in \mathfrak{I}_1\right), \left(\min_i g_{ij} | j \in \mathfrak{I}_2\right)\right\rangle | j = 1, 2, ...n\right\}^T = \{g_1^+, g_2^+, ..., g_n^+\}^T, \tag{5.1}$$

$$G^- = \left\{\left\langle \left(\min_i g_{ij} | j \in \mathfrak{I}_1\right), \left(\max_i g_{ij} | j \in \mathfrak{I}_2\right)\right\rangle | j = 1, 2, ...n\right\}^T = \{g_1^-, g_2^-, ..., g_n^-\}^T, \tag{5.2}$$

where $g_j^+ = \left\{\left\langle \mu_j^+, \nu_j^+; r_j^+ \right\rangle\right\}$ represents the circular intuitionistic fuzzy set with the highest membership degree among j criteria, and $g_j^- = \left\{\left\langle \mu_j^-, \nu_j^-; r_j^- \right\rangle\right\}$ represents the circular intuitionistic fuzzy set with the lowest membership degree among j criteria. $\Im_1, \Im_2$ represents the beneficial criteria and the cost criteria.

Table 3. Quantitative table of weight information.

Linguistic terms	IFNs
Certainly High Importance-(CHI)	$\langle 0.9, 0.1 \rangle$
Very High Importance-(VHI)	$\langle 0.8, 0.15 \rangle$
High Importance-(HI)	$\langle 0.7, 0.25 \rangle$
Above Average Importance-(AAI)	$\langle 0.6, 0.35 \rangle$
Average Importance -(AI)	$\langle 0.5, 0.45 \rangle$
Under Average Importance -(UAI)	$\langle 0.4, 0.55 \rangle$
Low Importance -(LI)	$\langle 0.3, 0.65 \rangle$
Very Low Importance -(VLI)	$\langle 0.2, 0.75 \rangle$
Certainly Low Importance -(CLI)	$\langle 0.1, 0.9 \rangle$

Step 7 Calculate the distance between each alternative and the positive ideal solution $Nd_C^+(A_i)$ and the negative ideal solution $Nd_C^-(A_i)$ using the new distance Eq (3.8) proposed in this paper.

Step 8 Calculate the relative closeness coefficient $CC(A_i)$, use the relative closeness coefficient to rank the alternatives, and then select the best solution.

$$CC(A_i) = \frac{Nd_C^-(A_i)}{Nd_C^+(A_i) + Nd_C^-(A_i)}. \tag{5.3}$$

5.2. Application example

The validity of the new distance metric proposed in this paper is tested below using an example of hospital siting in the literature [27]. Mass infectious diseases have always been a very difficult problem for public health systems. In addition to posing a threat to people's lives, national as well as global public health events can even cause social disruption and have a significant negative impact on economic development. How to effectively respond to large-scale public health events is the direction of many scholars' research. In this paper, the problem of medical resource allocation will address from the location of hospitals in Istanbul.

Step 1 Firstly, seven options for hospital location were identified, A_1 - Bakırköy, A_2 - Sancaktepe, A_3 - Eyüp, A_4 - Esenyurt, A_5 - Çatalca, A_6 - Tuzla, A_7 - Ataşehir, which are scattered in different locations of Istanbul. Secondly seven criteria need to be considered in the selection process: C_1 (Cost), C_2 (Demographics), C_3 (Environmental Factors), C_4 (Transportation Opportunities), C_5 (Healthcare and Medical Practices), C_6 (Infrastructure), C_7 (Spread of the virus). This leads to the scenario set $A = \{A_1, A_2, \cdots, A_7\}$ and the criteria set $C = \{C_1, C_2, \cdots, C_7\}$. Additionally, three fuzzy multi-criteria decision experts were selected as decision makers who were abbreviated as $DM1, DM2, DM3$.

Step 2 Three decision makers (DM) were selected to evaluate each solution based on expertise and practical situation, and the expert decision matrix was obtained as shown in Table 4. Then the qualitative evaluation information is transformed into the form of intuitionistic fuzzy sets by semantic information quantification table (Table 2).

Table 4. Expert decision sheet.

Criterion	DMs	Alternatives						
		A_1	A_2	A_3	A_4	A_5	A_6	A_7
	$DM1$	HV	LV	AV	VLV	AV	AV	AAV
C_1	$DM2$	HV	UAV	AV	LV	UAV	AV	HV
	$DM3$	AAV	LV	AAV	LV	UAV	AV	HV
	$DM1$	AAV	VHV	VHV	VHV	UAV	AAV	HV
C_2	$DM2$	AAV	HV	HV	VHV	LV	AV	HV
	$DM3$	AV	VHV	HV	CHV	LV	AAV	VHV
	$DM1$	LV	AV	UAV	VLV	HV	AV	UAV
C_3	$DM2$	LV	AV	UAV	LV	AAV	UAV	AV
	$DM3$	UAV	AAV	LV	LV	HV	AV	AV
	$DM1$	HV	AV	AAV	UAV	UAV	UAV	AAV
C_4	$DM2$	VHV	AV	HV	UAV	UAV	LV	AAV
	$DM3$	CHV	AV	AAV	AV	LV	UAV	HV
	$DM1$	AAV	AV	AAV	AV	UAV	LV	UAV
C_5	$DM2$	AAV	AV	AAV	AV	LV	UAV	UAV
	$DM3$	HV	UAV	HV	AAV	AV	AV	AV
	$DM1$	UAV	AV	AAV	CLV	VHV	AV	LV
C_6	$DM2$	UAV	AAV	AAV	CLV	CHV	AAV	LV
	$DM3$	LV	HV	HV	VLV	CHV	AAV	VLV
	$DM1$	HV	HV	HV	VHV	VLV	LV	HV
C_7	$DM2$	VHV	AAV	AAV	CHV	CLV	LV	HV
	$DM3$	VHV	HV	HV	CHV	VLV	UAV	VHV.

Step 3 The intuitionistic fuzzy evaluation information given by the decision maker is integrated according to Eq (2.1) and Eq (3.6) to obtain the circular intuitionistic fuzzy decision matrix, which is shown in Table 5.

Step 4 The criterion weight evaluation table given in the literature [19] (Table 6) was used to quantify the information using Table 3, and then the weight information was aggregated according to Eq (2.1) and Eq (3.6) to finally obtain the criterion weight matrix (Table 7).

Step 5 Determine the circular intuitionistic fuzzy decision matrix of each criterion C_j after weighting according to Eq (2.8), as shown in Table 8.

Step 6-7 After obtaining the weighted decision matrix, the positive and negative ideal solutions under different criteria are determined according to Eqs (5.1) and (5.2).

$$G^{+}=\left\{\begin{array}{l}\langle 0.142, 0.8150.094\rangle, \langle 0.639, 0.2920.082\rangle, \langle 0.445, 0.4860.094\rangle, \langle 0.586, 0.3480.112\rangle, \\ \langle 0.232, 0.7150.094\rangle, \langle 0.231, 0.7200.082\rangle, \langle 0.722, 0.2340.082\rangle\end{array}\right\},$$

$$G^{-}=\left\{\begin{array}{l}\langle 0.355, 0.5820.094\rangle, \langle 0.256, 0.6870.094\rangle, \langle 0.178, 0.7730.094\rangle, \langle 0.269, 0.6740.094\rangle, \\ \langle 0.147, 0.8120.118\rangle, \langle 0.036, 0.9520.106\rangle, \langle 0.139, 0.8270.106\rangle\end{array}\right\}.$$

Table 5. Circular intuitionistic fuzzy set decision matrix.

Criterion	Alternatives			
	A_1	A_2	A_3	A_4
C_1	$\langle 0.667, 0.283; 0.094\rangle$	$\langle 0.333, 0.617; 0.094\rangle$	$\langle 0.533, 0.417; 0.094\rangle$	$\langle 0.267, 0.683; 0.094\rangle$
C_2	$\langle 0.567, 0.383; 0.094\rangle$	$\langle 0.767, 0.183; 0.094\rangle$	$\langle 0.733, 0.217; 0.094\rangle$	$\langle 0.833, 0.133; 0.071\rangle$
C_3	$\langle 0.333, 0.617; 0.094\rangle$	$\langle 0.533, 0.417; 0.094\rangle$	$\langle 0.367, 0.583; 0.094\rangle$	$\langle 0.267, 0.683; 0.094\rangle$
C_4	$\langle 0.800, 0.167; 0.130\rangle$	$\langle 0.500, 0.450; 0.000\rangle$	$\langle 0.633, 0.317; 0.094\rangle$	$\langle 0.433, 0.517; 0.094\rangle$
C_5	$\langle 0.633, 0.317; 0.094\rangle$	$\langle 0.467, 0.483; 0.094\rangle$	$\langle 0.633, 0.317; 0.094\rangle$	$\langle 0.533, 0.417; 0.094\rangle$
C_6	$\langle 0.367, 0.583; 0.094\rangle$	$\langle 0.600, 0.350; 0.141\rangle$	$\langle 0.633, 0.317; 0.094\rangle$	$\langle 0.133, 0.850; 0.118\rangle$
C_7	$\langle 0.767, 0.183; 0.094\rangle$	$\langle 0.667, 0.283; 0.094\rangle$	$\langle 0.667, 0.283; 0.094\rangle$	$\langle 0.867, 0.117; 0.071\rangle$
	A_5	A_6	A_7	
C_1	$\langle 0.433, 0.517; 0.094\rangle$	$\langle 0.500, 0.450; 0.000\rangle$	$\langle 0.667, 0.283; 0.094\rangle$	
C_2	$\langle 0.333, 0.617; 0.094\rangle$	$\langle 0.567, 0.383; 0.094\rangle$	$\langle 0.733, 0.217; 0.094\rangle$	
C_3	$\langle 0.667, 0.283; 0.094\rangle$	$\langle 0.467, 0.483; 0.094\rangle$	$\langle 0.467, 0.483; 0.094\rangle$	
C_4	$\langle 0.367, 0.583; 0.094\rangle$	$\langle 0.367, 0.583; 0.094\rangle$	$\langle 0.633, 0.317; 0.094\rangle$	
C_5	$\langle 0.400, 0.550; 0.141\rangle$	$\langle 0.400, 0.550; 0.141\rangle$	$\langle 0.433, 0.517; 0.094\rangle$	
C_6	$\langle 0.867, 0.117; 0.071\rangle$	$\langle 0.567, 0.383; 0.094\rangle$	$\langle 0.267, 0.638; 0.094\rangle$	
C_7.	$\langle 0.167, 0.800; 0.118\rangle$	$\langle 0.333, 0.617; 0.094\rangle$	$\langle 0.733, 0.217; 0.094\rangle$	

Table 6. Criteria weighting evaluation table.

Criterion	$DM1$	$DM2$	$DM3$	Type	
				Cost	Benefit
C_1	AI	AI	AAI	✓	
C_2	VHI	VHI	HI		✓
C_3	AAI	HI	HI		✓
C_4	HI	HI	VHI		✓
C_5	LI	UAI	UAI		✓
C_6	LI	UAI	UAI		✓
C_7	VLI	LI	LI		✓

Table 7. Criteria weitht sheet.

Criterion	Criteria weight
C_1	$\langle 0.533, 0.417; 0.094\rangle$
C_2	$\langle 0.767, 0.183; 0.094\rangle$
C_3	$\langle 0.667, 0.283; 0.094\rangle$
C_4	$\langle 0.733, 0.217; 0.094\rangle$
C_5	$\langle 0.367, 0.583; 0.094\rangle$
C_6	$\langle 0.267, 0.683; 0.094\rangle$
C_7	$\langle 0.833, 0.133; 0.094\rangle$

Table 8. Weighted decision matrix.

Criterion	Alternatives			
	A_1	A_2	A_3	A_4
C_1	$\langle 0.355, 0.582; 0.094\rangle$	$\langle 0.178, 0.777; 0.094\rangle$	$\langle 0.284, 0.660; 0.094\rangle$	$\langle 0.142, 0.815; 0.094\rangle$
C_2	$\langle 0.435, 0.492; 0.094\rangle$	$\langle 0.588, 0.333; 0.094\rangle$	$\langle 0.562, 0.360; 0.094\rangle$	$\langle 0.639, 0.292; 0.082\rangle$
C_3	$\langle 0.222, 0.725; 0.094\rangle$	$\langle 0.356, 0.582; 0.094\rangle$	$\langle 0.245, 0.701; 0.094\rangle$	$\langle 0.178, 0.773; 0.094\rangle$
C_4	$\langle 0.586, 0.348; 0.112\rangle$	$\langle 0.367, 0.569; 0.047\rangle$	$\langle 0.464, 0.465; 0.094\rangle$	$\langle 0.318, 0.622; 0.094\rangle$
C_5	$\langle 0.232, 0.715; 0.094\rangle$	$\langle 0.171, 0.785; 0.094\rangle$	$\langle 0.232, 0.715; 0.094\rangle$	$\langle 0.196, 0.757; 0.094\rangle$
C_6	$\langle 0.098, 0.868; 0.094\rangle$	$\langle 0.160, 0.794; 0.118\rangle$	$\langle 0.169, 0.783; 0.094\rangle$	$\langle 0.036, 0.952; 0.106\rangle$
C_7	$\langle 0.639, 0.292; 0.094\rangle$	$\langle 0.555, 0.379; 0.094\rangle$	$\langle 0.555, 0.379; 0.094\rangle$	$\langle 0.722, 0.234; 0.082\rangle$
	A_5	A_6	A_7	
C_1	$\langle 0.231, 0.718; 0.094\rangle$	$\langle 0.267, 0.679; 0.047\rangle$	$\langle 0.355, 0.582; 0.094\rangle$	
C_2	$\langle 0.256, 0.687; 0.094\rangle$	$\langle 0.435, 0.496; 0.094\rangle$	$\langle 0.562, 0.360; 0.094\rangle$	
C_3	$\langle 0.445, 0.489; 0.094\rangle$	$\langle 0.311, 0.630; 0.094\rangle$	$\langle 0.311, 0.630; 0.094\rangle$	
C_4	$\langle 0.269, 0.674; 0.094\rangle$	$\langle 0.269, 0.674; 0.094\rangle$	$\langle 0.464, 0.465; 0.094\rangle$	
C_5	$\langle 0.147, 0.812; 0.118\rangle$	$\langle 0.147, 0.812; 0.118\rangle$	$\langle 0.159, 0.798; 0.094\rangle$	
C_6	$\langle 0.231, 0.720; 0.082\rangle$	$\langle 0.151, 0.805; 0.094\rangle$	$\langle 0.071, 0.900; 0.094\rangle$	
C_7	$\langle 0.139, 0.827; 0.106\rangle$	$\langle 0.278, 0.668; 0.094\rangle$	$\langle 0.611, 0.321; 0.094\rangle$	

Then the distance between each alternative and the positive and negative ideal solutions is calculated using the circular intuitionistic fuzzy set distance measure Eq (3.7) proposed in this paper, as shown in Table 9.

Table 9. The distance of each alternative to the positive and negative ideal solutions.

Distance	Alternatives						
	A_1	A_2	A_3	A_4	A_5	A_6	A_7
$Nd_C^+(A_i)$	0.131	0.113	0.117	0.121	0.222	0.216	0.138
$Nd_C^-(A_i)$	0.187	0.216	0.201	0.194	0.094	0.110	0.180

Step 8 From the distance data obtained for different alternatives in Table 9, the relative closeness coefficient of each alternative to the positive ideal solution is calculated using Eq (5.3) and ranked. The results of the relative closeness coefficient calculation are shown in Table 10.

The ranking of the different candidates according to the principle of maximum proximity $CC(A_i)$ is $A_2 \succ A_3 \succ A_4 \succ A_1 \succ A_7 \succ A_6 \succ A_5$, and the optimal hospital site is A_2.

However, as can be seen from the decision results in Table 10, the conclusion obtained by the method provided in literature [27] is $A_2 \succ A_3 \succ A_7 \succ A_4 \succ A_1 \succ A_6 \succ A_5$. Comparing the two results it is easy to see that the optimal and inferior solutions of both decision methods remain the same, the difference is in the ranking between the two alternatives of A_4 (Esenyurt) and A_7 (Ataşehir). So this study will use the score function Eq (2.9) to judge the ranking between A_4 and A_7 in Table 8, and since the decision maker is in an indifferent attitude in the application of this paper, $\lambda = 0.5$ is taken. According to Eq (2.8), $S_C(A_4) = -S_C(A_{41}) + \sum_{j=2}^{7} S_C\left(A_{4j}\right) = -0.290$, $S_C(A_7) = -S_C(A_{71}) + \sum_{j=2}^{7} S_C\left(A_{7j}\right) = -0.356$, $S_C(A_4) > S_C(A_7)$, so the alternative A_4 is ranked higher than A_7. In addition, Eq (2.9) is also used to calculate and rank the other alternatives , and the ranking results was $A_2 \succ A_3 \succ A_4 \succ A_1 \succ A_7 \succ A_6 \succ A_5$, which was consistent with the conclusion obtained in this paper. The specific results was shown in Table 10.

Table 10. Results of multi-criteria decision making by different methods.

Alternatives	CC_i^{CR}[27]	$CC(A_i)$	$S_C(A_i)$
A_1	0.613	1.612	-0.335
A_2	0.670	2.128	-0.215
A_3	0.643	1.908	-0.267
A_4	0.627	1.792	-0.290
A_5	0.320	0.517	-0.744
A_6	0.342	0.619	-0.694
A_7	0.635	1.489	-0.356

5.3. Comparison analysis

In this subsection, we compare and analyze the TOPSIS method based on hesitancy redistribution distance circular intuitionistic fuzzy set proposed in this paper with the multi-criteria decision making methods of intuitionistic fuzzy set [33], Pythagorean fuzzy set [33], circular intuitionistic fuzzy set [21, 27], and score function [25]. Different methods were used to conduct all these examples, and the final experimental results are shown in Table 11.

Table 11. Comparison with existing MCDM methods.

Reference	Method	Ranking
Fatih Emre Boran [33]	IF TOPSIS	$A_2 \succ A_3 \succ A_7 \succ A_4 \succ A_1 \succ A_6 \succ A_5$
Alkan Nurşah and Kahraman Cengiz [27]	C-IF TOPSIS	$A_2 \succ A_3 \succ A_7 \succ A_4 \succ A_1 \succ A_6 \succ A_5$
Muhammad Akram, Wieslaw A. Dudek and Farwa Ilyas [35]	Pythagorean fuzzy TOPSIS	$A_2 \succ A_3 \succ A_7 \succ A_4 \succ A_1 \succ A_6 \succ A_5$
Ting-Yu Chen [21]	C-IF TOPSIS Displaced anchoring mechanism	$A_2 \succ A_3 \succ A_4 \succ A_1 \succ A_7 \succ A_6 \succ A_5$
Esra Çakır, Mehmet Ali Taş and Ziya Ulukan [25]	Score function	$A_2 \succ A_3 \succ A_4 \succ A_1 \succ A_7 \succ A_6 \succ A_5$
Proposed method	$d_C(A, B)$based C-IF TOPSIS	$A_2 \succ A_3 \succ A_4 \succ A_1 \succ A_7 \succ A_6 \succ A_5$

By looking at the comparison results in Table 11, it can be observed that the ranking of the alternatives is slightly different. As analyzed in Section 4.2, the best and worst alternatives of the results obtained from different decision methods remain the same, the difference is in the ranking between the two alternatives A_4 (Esenyurt) and A_7 (Ataşehir). However, realistically, Esenyurt has one of the lowest construction costs in Istanbul, and the higher percentage of foreign residents also increases the probability of infectious diseases, so Esenyurt (A_4) is more suitable than Ataşehir (A_7) for establishing a hospital when a large-scale infectious disease outbreak occurs. Therefore, the results obtained from the multi-criteria decision making method proposed in this paper are more realistic.

In addition the decision making idea of literature [27] is to transform the circular intuitionistic fuzzy set problem into an intuitionistic fuzzy set for processing, and does not really give the solution of circular intuitionistic fuzzy set. Additionally, the decision process is very tedious, leading to an increase in computational complexity during practical applications. However, the proposed method in this paper reduces the computational complexity. By proposing a circular intuitionistic fuzzy set distance measure, linguistic information is transformed into circular intuitionistic fuzzy sets and then the ranking of the solutions is directly obtained by the distance measure, which makes the actual decision making process easier. The final result of "optimal hospital site selection" is consistent with that of the literature [27], which also indicates the effectiveness and practical significance of the proposed method.

6. Conclusions

As an extension of the intuitionistic fuzzy set, the circular intuitionistic fuzzy set has a stronger ability to express uncertain information and can better reflect the essential characteristics of the objective world. In this paper, a new distance measure is defined on the basis of circular intuitionistic fuzzy sets, and the corresponding theorems and proofs are given. This method not only considers the three factors of membership degree, non- membership degree and radius, but also considers the potential association between hesitation degree and membership degree and non- membership degree. In addition, we verify that the metric proposed in this paper overcomes the shortcomings of existing distance metrics through comparative data analysis. Finally, an example of epidemic hospital site selection is used to illustrate the effectiveness and rationality of the method in this paper. The limitation of this study is that the distribution ratio about the hesitation parameter in the distance metric can have more forms, so the distribution ratio of hesitation can be a direction for future research. Additionally, in the following research, we can further consider how to apply the distance metric proposed in this paper to different multi-criteria decision models, and then solve a wider range of multi-criteria decision problems. In addition, we can also consider how to use the distance metric proposed in this paper to solve problems related to pattern recognition and medical diagnosis.

Use of AI tools declaration

The authors declare that no Artificial Intelligence (AI) tools were used in the writing of this article.

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Conflict of interest

The authors declare no conflict of interests regarding the publication for the paper.

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