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Research article

Feedback stabilization for prey predator general model with diffusion via multiplicative controls

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Abstract: In this paper, we consider a predator–prey model given by a reaction–diffusion system. This model encompasses the classic Holling I, Holling II, Holling III, and Holling IV functional responses. We investigate the stabilization problem of the considered system using multiplicative controls. By linearizing the system and using the maximum principle, we construct a multiplicative control that exponentially stabilizes the system towards its steady-state solutions. The proposed feedback control allows us to reach a large class of steady-state solutions. The global well-posedness is obtained via Banach fixed point. Applications and numerical simulations to Holling responses I, II, III, and IV are presented.

Keywords: prey predator with diffusion; Holling responses; feedback stabilization; multiplicative controls; numerical simulations
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1. Introduction

The dynamics between prey and predator species in natural systems can be represented by a coupled system of nonlinear reaction-diffusion equations. This type of system is usually represented by a system of partial differential equations, rather than a system of ordinary differential equations, which are often used to represent interactions between prey and predator populations without diffusion (see for instance, [13] and [30] and references therein). We are concerned with the following mathematical model, which expresses the conservation of predator and prey densities:

$$\begin{cases} \dfrac{\partial y_1(t,x)}{\partial t} = \Delta y_1 + y_1 h(y_1) - y_1 y_2 k(y_1), & x \in \Omega, t > 0 \\ \dfrac{\partial y_2(t,x)}{\partial t} = \delta \Delta y_2 - a y_2 + b y_1 y_2 k(y_1), & x \in \Omega, t > 0 \\ \dfrac{\partial y_1}{\partial \nu} = \dfrac{\partial y_2}{\partial \nu} = 0, \quad (t,x) \in \Sigma = (0,\infty) \times \partial\Omega \\ y_1(0,x) = y_1^0(x), \quad y_2(0,x) = y_2^0(x), \quad x \in \bar{\Omega} \end{cases} \tag{1.1}$$

where y_1 and y_2 represent the prey and predator population densities at time t respectively. The function $y_1 h(y_1)$ is the intrinsic growth rate of the prey y_1 and signifies its growth rate in the absence of the predator. It can be linear if $h(y_1) = r_1$, logistic if $h(y_1) = r_1(1 - y_1/\kappa_1)$, Gompertz if $h(y_1) = h_0 \ln(\kappa_1/y_1)\,(h_0, r_1, \kappa_1 > 0)$ etc. See [1, 2] and the references therein. The predator's functional response to prey is $y_1 k(y_1)$, which represents the number of prey individuals consumed per unit area and unit time per predator. It includes as particular cases of various classical functional responses: $y_1 k(y_1) = \beta y_1$ (Holling type I), $y_1 k(y_1) = \beta y_1/(1 + \mu y_1)$ (Holling type II), $y_1 k(y_1) = \beta y_1^2/\left(1 + \mu y_1^2\right)$ (Holling type III; see [3]), $y_1 k(y_1) = \beta y_1/\left(\gamma + \mu y_1 + y_1^2\right)$ (Holling type IV; see [4]) etc. Here $\beta, \gamma > 0, \mu \geq 0$.

There is a large amount of literature related to the mathematical study of prey-predator systems of the diffusion type. In [24], Morita and Tachibana showed the existence of an entire solution (i.e., a solution that exists for all $(t,x) \in \mathbb{R}^2$) of the predator-prey reaction-diffusion systems with Holling type I. The proof is carried out by applying the comparison principle and an appropriate pair of a subsolution and a supersolution. For Holling II functional response, Garvie and Trenchea proved the existence of a solution by using semigroup theory and application of the invariant region method of Smoller (see [15]). In [1], Apreutesei and Dimitriu studied the well-posedness of a predator-prey reaction-diffusion system with Holling type III. They show the existence of the solutions provided that the initial data are positive and satisfy a specific regularity. For more results on the well-posedness of these systems, we refer, for instance, to [8–10] and the reference therein. Recently, Mi et al. [23] considered a nonlocal predator–prey model with double mutation; they defined pair of upper and lower solutions, and they designed a new comparison principle that ensures the existence of the solutions.

In our case, the situation is quite complicated due to the generalized nonlinearity considered, which encompasses all Holling functional responses nonlinearities. Consequently, we use the Banach fixed point and the stabilization of an associated system to obtain the global well-posedness. The existence and stability results concerning the steady states of these systems have been extensively studied see, for example, [2, 5, 7, 21] and the references therein. The results show that the system exhibits very unusual behavior for some parameter values, while some steady states are stable under system parameter constraints. It would be interesting to investigate a method that allows driving the systems to these equilibrium states without adding additional constraints on the system parameters that might contradict the measures taken during the modelling. To this end we use multiplicative controls to stabilize the system, this choice is determined by the real application. In fact, for the prey predator model, these controls can be interpreted as harvesting efforts. A huge amount of literature has been devoted to stabilizing uncoupled linear systems via multiplicative controls. For example, we mention the works of [4, 17, 26] and the references therein. However, the choice of such control in the nonlinear cases generates new difficulties. Indeed, the first difficulty is that the stabilization of nonlinear parabolic coupled systems by using multiplicative controls remains an open problem. Another difficulty of the problem lies in the fact that these controls are nonlinear, which doubles the nonlinearity of the system. Note also that the controllability of bilinear systems is an open problem see [3]. For hyperbolic coupled systems we refer to [19], where the authors characterize the stabilization of a class of coupled hyperbolic systems by using multiplicative controls. They showed the equivalence between stabilization and the observability of the uncontrolled system. In this work, we hope to achieve the stabilization result. More precisely, let

$$f(y_1, y_2) = y_1 h(y_1) - y_1 y_2 k(y_1) \quad \text{and} \quad g(y_1, y_2) = -a y_2 + b y_1 y_2 k(y_1). \tag{1.2}$$

We obtain from (1.1) the following system

$$\begin{cases} \dfrac{\partial y_1(t,x)}{\partial t} = \Delta y_1 + f(y_1, y_2), & x \in \Omega, t > 0 \\ \dfrac{\partial y_2(t,x)}{\partial t} = \delta \Delta y_2 + g(y_1, y_2), & x \in \Omega, t > 0 \\ \dfrac{\partial y_1}{\partial v} = \dfrac{\partial y_2}{\partial v} = 0, & (t,x) \in \Sigma = (0,\infty) \times \partial\Omega \\ y_1(0,x) = y_1^0(x), \quad y_2(0,x) = y_2^0(x), & x \in \bar{\Omega} \end{cases} \tag{1.3}$$

We say that (y_1^e, y_2^e) is an equilibrium state of (1.3) if and only if:
1- $(y_1^e, y_2^e) \in H^2(\Omega) \cap H_0^1(\Omega)$.
2- (y_1^e, y_2^e) solves the following elliptic system

$$\begin{cases} \Delta y_1^e + f(y_1^e, y_2^e) = 0, & x \in \Omega \\ \delta \Delta y_2^e + g(y_1^e, y_2^e) = 0, & x \in \Omega \\ \dfrac{\partial y_1^e}{\partial v} = \dfrac{\partial y_2^e}{\partial v} = 0, & x \in \partial\Omega \end{cases}$$

Translate (y_1^e, y_2^e) into zero via the following change of variable $z_1 = y_1 - y_1^e$ et $z_2 = y_2 - y_2^e$. Obviously (z_1, z_2) solves the following system

$$\begin{cases} \dfrac{\partial z_1(t,x)}{\partial t} = \Delta z_1 + f\left(z_1 + y_1^e, z_2 + y_2^e\right) - f(y_1^e, y_2^e), & x \in \Omega, t > 0 \\ \dfrac{\partial z_2(t,x)}{\partial t} = \delta \Delta z_2 + g\left(z_1 + y_1^e, z_2 + y_2^e\right) - g(y_1^e, y_2^e), & x \in \Omega, t > 0 \\ \dfrac{\partial z_1}{\partial v} = \dfrac{\partial z_2}{\partial v} = 0, & (t,x) \in \Sigma = (0,\infty) \times \partial\Omega \\ z_1(0,x) = z_1^0(x) := y_1^0(x) - y_1^e, \quad z_2(0,x) = z_2^0(x) := y_2^0(x) - y_2^e, & x \in \bar{\Omega} \end{cases} \tag{1.4}$$

then stabilizing (1.3) towards (y_1^e, y_1^e) is reduced to the stability of the null solution to system (1.4). By injecting a multiplicative control into the prey and predator equations, we obtain from (1.4) the following system

$$\begin{cases} \dfrac{\partial z_1(t,x)}{\partial t} = \Delta z_1 + f\left(z_1 + y_1^e, z_2 + y_2^e\right) - f(y_1^e, y_2^e) + v(t) B z_1, & x \in \Omega, t > 0 \\ \dfrac{\partial z_2(t,x)}{\partial t} = \delta \Delta z_2 + g\left(z_1 + y_1^e, z_2 + y_2^e\right) - g(y_1^e, y_2^e) + v(t) B z_2, & x \in \Omega, t > 0 \\ \dfrac{\partial z_1}{\partial v} = \dfrac{\partial z_2}{\partial v} = 0, & (t,x) \in \Sigma = (0,\infty) \times \partial\Omega \\ z_1(0,x) = z_1^0(x) := y_1^0(x) - y_1^e, \quad z_2(0,x) = z_2^0(x) := y_2^0(x) - y_2^e, & x \in \bar{\Omega} \end{cases} \tag{1.5}$$

where $v(t)$ is a feedback control to be determined. The control operator B is assumed to be bounded from $L^2(\Omega)$ to $L^2(\Omega)$. The terms $v(t)Bz_1$ and $v(t)Bz_2(t)$ can be regarded as the effort applied to harvest the prey and predator, respectively. System (1.5) represents the evolution of predator-prey densities under the action of harvesting. In the following, we construct the the control feedback $v(t)$, ensuring the exponential stabilization of (1.5).

The rest of the paper is organized as follows. In section 2, we present the assumptions and main results. We start with the stabilization of the linearized system and extend the result to the nonlinear system, where we show the well-posedness and exponential stabilization using Banach fixed points. In section 3, we illustrate the obtained results for different Holling responses.

2. Assumptions and main results

For a bounded open set $\Omega \subset \mathbb{R}^N$, we denote by H the Lebesgue space $L^2(\Omega)$ endowed with the inner product $\langle .,. \rangle$ and its corresponding norm $\|.\|$, $\mathcal{H}$ the Cartesian product $L^2(\Omega) \times L^2(\Omega)$ with the norm $\|.\|_{\mathcal{H}}$ and $\mathbb{H} = H^1(\Omega) \times H^1(\Omega)$ with the norm $\|.\|_{\mathbb{H}}$.
The following assumptions will be in effect everywhere in the following:

(H_1) $(y_1^e, y_2^e) \in C(\overline{\Omega})$
(H_2) $f, g \in C^1(\mathbb{R} \times \mathbb{R})$ satisfy the growth condition

$$|f(y,z)| + |g(y,z)| \leqslant C \sum_{i=1}^{m_0} (|y|^{r_i} + |z|^{r_i}) \qquad \text{for all } y, z \in \mathbb{R}, \tag{2.1}$$

where m_0 is a positive integer and r_i, $1 \leqslant i \leqslant m_0$, are such that

$$1 \leq r_1 < r_2 ... < r_{m_0} \leq m_0$$

(H_3) $y_1(t,x) \geq 0$ and $y_2(t,x) \geq 0$ provided that $y_1^0(x) \geq 0$ and $y_2^0(x) \geq 0$.

Assumptions (H_1) and (H_2) for Holling types I, II, III and IV, imply, in particular that:
$f_y(y_1^e, y_2^e), f_z(y_1^e, y_2^e), g_y(y_1^e, y_2^e), g_z(y_1^e, y_2^e) \in L^\infty(\Omega)$. Assumption (H_3) is proved for Holling type I, II, III and IV; see, for instance, [1, 8].
Consider System (1.5),

$$\begin{cases} \dfrac{\partial z_1(t,x)}{\partial t} = \Delta z_1 + f\left(z_1 + y_1^e, z_2 + y_2^e\right) - f(y_1^e, y_2^e) + v(t)Bz_1, & x \in \Omega, t > 0 \\ \dfrac{\partial z_2(t,x)}{\partial t} = \delta \Delta z_2 + g\left(z_1 + y_1^e, z_2 + y_2^e\right) - g(y_1^e, y_2^e) + v(t)Bz_2, & x \in \Omega, t > 0 \\ \dfrac{\partial z_1}{\partial v} = \dfrac{\partial z_2}{\partial v} = 0, \quad (t,x) \in \Sigma = (0,\infty) \times \partial\Omega \\ z_1(0,x) = z_1^0(x) := y_1^0(x) - y_1^e, \quad z_2(0,x) = z_2^0(x) := y_2^0(x) - y_2^e, & x \in \bar{\Omega} \end{cases} \tag{2.2}$$

The linearized system associated with (2.2) is given by

$$\begin{cases} \dfrac{\partial z_1(t,x)}{\partial t} = \Delta z_1 + f_y\left(y_1^e, y_2^e\right) z_1 + f_z\left(y_1^e, y_2^e\right) z_2 + v(t)Bz_1, & x \in \Omega, t > 0 \\ \dfrac{\partial z_2(t,x)}{\partial t} = \delta \Delta z_2 + g_y\left(y_1^e, y_2^e\right) z_1 + g_z\left(y_1^e, y_2^e\right) z_2 + v(t)Bz_2, & x \in \Omega, t > 0 \\ \dfrac{\partial y_1}{\partial v} = \dfrac{\partial y_2}{\partial v} = 0, \quad (t,x) \in \Sigma = (0,\infty) \times \partial\Omega \\ z_1(0,x) = z_1^0(x) := y_1^0(x) - y_1^e, \quad z_2(0,x) = z_2^0(x) := y_2^0(x) - y_2^e, & x \in \bar{\Omega} \end{cases} \tag{2.3}$$

2.1. Exponential stabilization of the linearized system

In this section, we establish the exponential stabilization of the linearized system (2.3).

Spectral proprieties

Let $a = \sup_{x\in\Omega} |f_y(y_1^e(x), y_2^e(x))|$, $b = \sup_{x\in\Omega} |f_z(y_1^e(x), y_2^e(x))|$, $c = \sup_{x\in\Omega} |g_y(y_1^e(x), y_2^e(x))|$ and $d = \sup_{x\in\Omega} |g_z(y_1^e(x), y_2^e(x))|$.
Let $A := \Delta + aI$ where $D(A) = D(\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$. It is clear that A is a self-adjoint operator with a compact resolvent; hence, the spectrum of A reduces to its point spectrum. More precisely, the eignvalues $(\lambda_i)_{i\in\mathbb{N}^*}$ of A are reals. We suppose that there exists a finite positive integer N such that $\{\lambda_i \geq 0, \quad \forall i \in \{1, ..., N\}\}$ which is guaranteed thanks to the assumption $f_y(y_1^e, y_2^e) \in L^\infty(\Omega)$.
Let us consider the following auxiliary linear system:

$$\begin{cases} \dfrac{\partial Z_1(t,x)}{\partial t} = AZ_1 + bZ_2 + v(t)BZ_1, \quad x \in \Omega, t > 0 \\ \dfrac{\partial Z_2(t,x)}{\partial t} = \delta AZ_2 + cZ_1 + c_0 Z_2 + v(t)BZ_2, \quad x \in \Omega, t > 0 \\ \dfrac{\partial Z_1}{\partial \nu} = \dfrac{\partial Z_2}{\partial \nu} = 0, \quad (t,x) \in \Sigma = (0,\infty) \times \partial\Omega \\ Z_1(0,x) = Z_1^0 := \max_{x\in\Omega} z_1^0(x), \quad Z_2(0,x) = Z_2^0 := \max_{x\in\Omega} z_2^0(x), \quad x \in \bar{\Omega} \end{cases} \tag{2.4}$$

where $c_0 = \sup_{x\in\Omega} |d - \delta a|$. We mention that a simple application of the maximum principle (see [29]) gives that $0 \leq z_1(t) \leq Z_1(t)$ and $0 \leq z_2(t) \leq Z_2(t)$ for

$$0 \leq z_1^0(x) \leq Z_1^0 := \max_{x\in\Omega} z_1^0(x) \quad \text{and} \quad 0 \leq z_2^0(x) \leq Z_2^0 := \max_{x\in\Omega} z_2^0(x)$$

Theorem 2.1. *Let B be a bounded operator on H; suppose that assumptions (H_1), (H_2) and (H_3) hold; then, the feedback*

$$v(t) = \begin{cases} (-D-\eta)\dfrac{\left(\|Z_1(t)\|^2 + \|Z_2(t)\|^2\right)}{\langle BZ_1(t), Z_1(t)\rangle + \langle BZ_2(t), Z_2(t)\rangle} & \text{if } (Z_1, Z_2) \neq (0,0) \\ 0 \qquad \text{else} \end{cases} \tag{2.5}$$

where $D = (2\lambda + b + c + 2\delta\lambda + 2c_0)$, $\lambda = \max_{1\leq i\leq N}(\lambda_i)$ and $\eta > 0$, ensures the exponential stabilization of system (2.3).

Remark 2.1. *In the case where $B = I_d$, the feedback control $v(t)$ will be a constant, that is, $v(t) = -D-\eta$*

Proof. On the one hand, from the first equation of (2.4), we have

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\|Z_1(t)\|^2 &= \langle AZ_1(t), Z_1(t)\rangle + b\langle Z_2(t), Z_1(t)\rangle + v(t)\langle BZ_1(t), Z_1(t)\rangle \\ &\leq \lambda\|Z_1(t)\|^2 + b\langle Z_2(t), Z_1(t)\rangle + v(t)\langle BZ_1(t), Z_1(t)\rangle \qquad \lambda = \max_{1\leq i\leq N}(\lambda_i) \\ &\leq \lambda\|Z_1(t)\|^2 + b\|Z_2(t)\|\|Z_1(t)\| + v(t)\langle BZ_1(t), Z_1(t)\rangle \\ &\leq \lambda\|Z_1(t)\|^2 + \frac{b}{2}\|Z_1(t)\|^2 + \frac{b}{2}\|Z_2(t)\|^2 + v(t)\langle BZ_1(t), Z_1(t)\rangle \end{aligned}$$

then

$$\frac{\partial}{\partial t}||Z_1(t)||^2 \leq (2\lambda + b)\,||Z_1(t)||^2 + b||Z_2(t)||^2 + 2v(t)\langle BZ_1(t), Z_1(t)\rangle. \tag{2.6}$$

On the other hand, from the second equation of (2.4), we have

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}||Z_2(t)||^2 &\leq \delta\lambda||Z_2(t)||^2 + c\langle Z_1(t), Z_2(t)\rangle + c_0||Z_2(t)||^2 + v(t)\langle BZ_2(t), Z_2(t)\rangle \\ &\leq \left(\delta\lambda + c_0 + \frac{c}{2}\right)||Z_2(t)||^2 + \frac{c}{2}||Z_1(t)||^2 + v(t)\,\langle BZ_2(t), Z_2(t)\rangle\end{aligned}$$

then

$$\frac{\partial}{\partial t}||Z_2(t)||^2 \leq (2\delta\lambda + 2c_0 + c)\,||Z_2(t)||^2 + c||Z_1(t)||^2 + 2v(t)\,\langle BZ_2(t), Z_2(t)\rangle\,. \tag{2.7}$$

Combining (2.6) and (2.7), we obtain

$$\frac{\partial}{\partial t}\,||(Z_1(t), Z_2(t))||^2_{\mathcal{H}} \leq D\,||(Z_1(t), Z_2(t))||^2_{\mathcal{H}} + 2v(t)\Big(\langle BZ_1(t), Z_1(t)\rangle + \langle BZ_2(t), Z_2(t)\rangle\Big)$$

where $D = (2\lambda + b + c + 2\delta\lambda + 2c_0)$.
Using the expression of $v(t)$, we obtain

$$\frac{\partial}{\partial t}\,||(Z_1(t), Z_2(t))||^2_{\mathcal{H}} \leq -2\eta\,||(Z_1(t), Z_2(t))||^2_{\mathcal{H}}\,. \tag{2.8}$$

Integrating over $[k, k+1]$ for $k \in \mathbb{N}^*$, we obtain

$$||(Z_1(k+1), Z_2(k+1))||^2_{\mathcal{H}} - ||(Z_1(k), Z_2(k))||^2_{\mathcal{H}} \leq -2\eta\,||(Z_1(k+1), Z_2(k+1))||^2_{\mathcal{H}}\,, \tag{2.9}$$

then

$$||(Z_1(k+1), Z_2(k+1))||^2_{\mathcal{H}} \leq \frac{1}{2\eta + 1}\,||(Z_1(k), Z_2(k))||^2_{\mathcal{H}}\,.$$

By the recurrence argument, one can obtain

$$||(Z_1(k), Z_2(k))||_{\mathcal{H}} \leq \frac{1}{(2\eta + 1)^{\frac{k}{2}}}\,\left\|(Z_1^0, Z_2^0)\right\|_{\mathcal{H}}\,. \tag{2.10}$$

Since $||(Z_1(k+1), Z_2(k+1))||$ decreases then for $t \geq k$, we have

$$||(Z_1(t), Z_2(t))||_{\mathcal{H}} \leq \mathrm{e}^{-mt}\,\left\|(Z_1^0, Z_2^0)\right\|_{\mathcal{H}}\,, \tag{2.11}$$

where $m = \ln(1 + 2\eta) > 0$. Recalling that $0 \leq z_1(t) \leq Z_1(t)$ and $0 \leq z_2(t) \leq Z_2(t)$ for

$$0 \leq z_1^0(x) \leq Z_1^0 := \max_{x\in\Omega} z_1^0(x) \quad \text{and} \quad 0 \leq z_2^0(x) \leq Z_2^0 := \max_{x\in\Omega} z_2^0(x)$$

hence,

$$||(z_1(t), z_2(t))||_{\mathcal{H}} \leq \mathrm{e}^{-mt}\,\left\|(Z_1^0, Z_2^0)\right\|_{\mathcal{H}}\,, \tag{2.12}$$

and therefore, there exists a positive constant M_3 such that

$$||(z_1(t), z_2(t))||_{\mathcal{H}} \leq M_3\,\mathrm{e}^{-mt}\,\left\|(z_1^0, z_2^0)\right\|_{\mathcal{H}}\,. \tag{2.13}$$

This completes the proof of Theorem 2.1. □

2.2. Nonlinear setting

In this section, we shall prove that if $\|(z_1^0, z_2^0)\|_{\mathcal{H}} \leq \epsilon$ for ϵ small enough, then the local solution of (2.2) is global by using a Banach fixed point. Moreover, we show that this solution is exponentially stabilizable.

Theorem 2.2. ***Well-posedness***
Let B be a bounded operator on H, suppose that assumptions (H_1), (H_2) and (H_3); then, for $(z_1^0, z_2^0) \in \mathcal{H}$ such that $\|(z_1^0, z_2^0)\|_{\mathcal{H}} \leq \epsilon$, where ϵ is sufficiently small, system (2.2) *admits a global solution $(z_1, z_2) \in L^r(0, \infty; \mathbb{H})$ for some $r \geq 1$.*

Proof. System (2.2) can be written as

$$\begin{cases} \dfrac{\partial}{\partial t}(z_1(t), z_2(t)) = \mathcal{A}(z_1, z_2) + \mathcal{A}_0(z_1, z_2) + \Phi(z_1, z_2) + v(t)\mathcal{B}(z_1, z_2), \quad t > 0, \\ \dfrac{\partial z_1}{\partial v} = \dfrac{\partial z_2}{\partial v} = 0, \quad (t, x) \in \Sigma = (0, \infty) \times \partial\Omega \\ (z_1(0), z_2(0)) = \left(z_1^0, z_2^0\right) \equiv \left(y_1^0 - y_1^e, y_2^0 - y_2^e\right), \end{cases} \tag{2.14}$$

where

$$\mathcal{A} = \begin{pmatrix} \Delta & 0 \\ 0 & \alpha\Delta \end{pmatrix}, \quad \mathcal{A}_0 = \begin{pmatrix} f_y(y_1^e(x), y_2^e(x))I_d & f_z(y_1^e(x), y_2^e(x))I_d \\ g_y(y_1^e(x), y_2^e(x))I_d & g_z(y_1^e(x), y_2^e(x))I_d \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} B \\ B \end{pmatrix}$$

and

$$\begin{aligned} \Phi(z_1, z_2) &\equiv \left(\Phi^1(z_1, z_2), \Phi^2(z_1, z_2)\right) \\ &= \left(f\left(z_1 + y_1^e, z_2 + y_2^e\right) - f\left(y_1^e, y_2^e\right), g\left(z_1 + y_1^e, z_2 + y_2^e\right) - g\left(y_1^e, y_2^e\right)\right) \\ &\quad - \mathcal{A}_0(z_1, z_2). \end{aligned}$$

According to assumptions (H_1) and (H_2), we deduce that

$$\begin{aligned} \|\Phi(z_1, z_2)\|_{\mathcal{H}} &\leq C \sum_{i=1}^{m} (\|z_1\|^{r_i} + \|z_2\|^{r_i}), \\ &\leq 2C \sum_{i=1}^{m} \|(z_1, z_2)\|_{\mathcal{H}}^{r_i}, \end{aligned}$$

for some positive integer m, where r_i are such that $1 \leq r_1 < \cdots < r_m \leq m$.
Let us consider

$$\left\{X \in L^r(0, \infty; \mathcal{H}); \quad \|X\|_{L^r(0,\infty;\mathbb{H})} \leq \rho\right\} = S(0, \rho),$$

where $1 \leq r \leq m$. We note by $(\Gamma(t, s))_{0 \leq s \leq t}$ the evolution system generated by $(z_1, z_2) :\longrightarrow \mathcal{A}(z_1, z_2) + \mathcal{A}_0(z_1, z_2) + v(t)\mathcal{B}(z_1, z_2)$ (see Definition 5.3 p 126 [25]). Let the map

$$(\Lambda(z_1, z_2))(t) \equiv \Gamma(t, 0)(z_1^0, z_2^0) + \int_0^t \Gamma(t, s)\Phi(z_1(s), z_2(s))ds. \tag{2.15}$$

From (2.13), we conclude that $\Gamma(t, .)(z_1, z_2) \in L^r(0, \infty; \mathcal{H})$, that is

$$\int_0^\infty \|\Gamma(t, s)(z_1, z_2)\|_{\mathcal{H}}^r ds < \infty, \qquad \forall (z_1, z_2) \in \mathcal{H}, \tag{2.16}$$

(see p 299 [12]). Moreover, using the superposition property of the evolution system $(\Gamma(t,s))_{0\leq s\leq t}$, we deduce from (2.13) and (2.16) that there exists a positive constant M_4 such that

$$\int_0^\infty \|\Gamma(t,s)(z_1,z_2)\|_{\mathbb{H}}^r ds \leq M_4\|(z_1,z_2)\|_{\mathcal{H}}^r. \tag{2.17}$$

• We start by showing the invariance of Λ. We define

$$(\mathcal{N}(z_1,z_2))(t) := \int_0^t \Gamma(t,s)\Phi(z_1(s),z_2(s))ds \tag{2.18}$$

By duality arguments as in [31] page 197, we obtain

$$\|(\mathcal{N}(z_1,z_2)(t)\|_{L^r(0,\infty;\mathbb{H})} \leq M_4\sum_{r=1}^m \rho^r, \tag{2.19}$$

for all $(z_1,z_2)\in S(0,\rho)$.
Then

$$\|(\mathcal{N}(z_1,z_2)(t)\|_{L^r(0,\infty;\mathbb{H})} \leq M_4\sum_{r=1}^m \rho^r = M_4\frac{1-\rho^m}{1-\rho}. \tag{2.20}$$

Substituting in (2.15), we obtain

$$\begin{aligned}\|(\Lambda(z_1,z_2))\|_{L^r(0,\infty;\mathbb{H})}^r &\leq 2^{r-1}\|\Gamma(t,0)(z_1^0,z_2^0)\|_{L^r(0,\infty;\mathbb{H})}^r + 2^{r-1}\Big\|\int_0^t \Gamma(t,s)\Phi(z_1(s),z_2(s))ds\Big\|_{L^r(0,\infty;\mathbb{H})}^r\\ &\leq 2^{r-1}M_3^r\|(z_1^0,z_2^0)\|_{\mathcal{H}}^r + 2^{r-1}\left(M_4\frac{1-\rho^m}{1-\rho}\right)^r, \qquad \text{(using (2.20))}\end{aligned}$$

and hence

$$\|\Lambda(z_1,z_2)\|_{L^r(0,\infty;\mathbb{H})}^r \leq 2^{r-1}M_3^r\|(z_1^0,z_2^0)\|_{\mathcal{H}}^r + 2^{r-1}\left(M_4\frac{1-\rho^m}{1-\rho}\right)^r.$$

Let us consider $\rho>0$, which is chosen to satisfy the following constraints

$$2^{r-1}M_3^r\|(z_1^0,z_2^0)\|_{\mathcal{H}}^r \leq \frac{1}{2}\rho^r, \quad 2^{r-1}\left(M_4\frac{1-\rho^{m+1}}{1-\rho}\right)^r \leq \frac{1}{2}\rho^r, \quad \rho\neq 1 \tag{2.21}$$

and hence

$$\|\Lambda(z_1,z_2)\|_{L^r(0,\infty;\mathbb{H})}^r \leq \rho^r.$$

Therefore

$$\Lambda(z_1,z_2)\in S(0,\rho).$$

• Now we show that Λ is a contraction map on $S(0,\rho)$.
Let us consider $(y_1,z_1),(y_2,z_2)\in S(0,r)$, then there exists a constant C such that:

$$\Big\|\Phi(y_1,z_1)-\Phi(y_2,z_2)\Big\|_{\mathcal{H}} \leq C\Big(\sum_{i=1}^m \|y_1-y_2\|^{r_i} + \|z_1-z_2\|^{r_i}\Big),$$

$$\leq C\|y_1-y_2\|\sum_{i=1}^{m}\|y_1-y_2\|^{r_i-1}+C\|z_1-z_2\|\sum_{i=1}^{m}\|z_1-z_2\|^{r_i-1},$$

$$\leq C\Big\|(y_1-y_2),(z_1-z_2)\Big\|_{\mathcal{H}}\sum_{i=1}^{m}\|y_1-y_2\|^{r_i-1}$$

$$+C\Big\|(y_1-y_2),(z_1-z_2)\Big\|_{\mathcal{H}}\sum_{i=1}^{m}\|z_1-z_2\|^{r_i-1},$$

$$\leq C\Big\|(y_1,z_1)-(y_2,z_2)\Big\|_{\mathcal{H}}\Big(\sum_{i=1}^{m}\|y_1-y_2\|^{r_i-1}+\sum_{i=0}^{m}\|z_1-z_2\|^{r_i-1}\Big). \tag{2.22}$$

$$\leq 2C\Big\|(y_1,z_1)-(y_2,z_2)\Big\|_{\mathcal{H}}\Big(\sum_{i=1}^{m}(\|(y_1,z_1)\|+\|(y_1,z_1)\|)^{r_i-1}\Big), \tag{2.23}$$

where the following argument is used

$$\|y_1-y_2\|\leq\|y_1\|+\|y_2\|\leq\|(y_1,z_1)\|_{\mathcal{H}}+\|(y_2,z_2)\|_{\mathcal{H}},$$

$$\|z_1-z_2\|\leq\|z_1\|+\|z_2\|\leq\|(y_1,z_1)\|_{\mathcal{H}}+\|(y_2,z_2)\|_{\mathcal{H}},$$

In the other hand, similar to (2.19), one can show as well that

$$\Big\|\Lambda(y_1,z_1)-\Lambda(y_2,z_2)\Big\|_{L^r(0,\infty;\mathbb{H})}\leq 2C\Big\|(y_1,z_1)-(y_2,z_2)\|_{L^r(0,\infty;\mathbb{H})}\sum_{r=1}^{m}(2\rho)^{r-1},$$

$$\leq 2C\frac{1-(2\rho)^m}{1-2\rho}\Big\|(y_1,z_1)-(y_2,z_2)\|_{L^r(0,\infty;\mathbb{H})}.$$

Then Λ is a contraction on $S(0,\rho)$ for ρ chosen such that

$$2C\frac{1-(2\rho)^m}{1-2\rho}<1,\qquad \rho\neq\frac{1}{2} \tag{2.24}$$

then according to the Banach fixed point, system (2.2) has for (z_1^0,z_2^0) sufficiently small, a unique solution

$$(z_1,z_2)\in L^r(0,\infty;\mathbb{H}).$$

□

Now we characterize the exponential stabilization of (2.2). Theorem 2.3 below is the main result of this paper.

Theorem 2.3. *Let B be a bounded operator on H, suppose that assumptions (H_1), (H_2) and (H_3) hold, then the following feedback*

$$v(t)=\begin{cases}\dfrac{(-D-\eta)\Big(\|Z_1(t)\|^2+\|Z_2(t)\|^2\Big)}{\langle BZ_1(t),Z_1(t)\rangle+\langle BZ_2(t),Z_2(t)\rangle} & \textit{if}\,(Z_1,Z_2)\neq(0,0)\\ 0 & \textit{else}\end{cases} \tag{2.25}$$

where $D=2\lambda+b+c+2\delta\lambda+2c_0$, $\lambda=\max_{1\leq i\leq N}(\lambda_i)$, exponentially stabilizes (2.2).

Proof. We start by showing that the solution (z_1, z_2) of (2.2) obeys the following estimate:

$$\|(z_1, z_2)\|^r_{L^r(0,\infty;\mathbb{H})} \le \frac{2^{r-1}M_3^r}{1-C_{\rho,1}}\|(z_1^0, z_2^0)\|^r_{\mathcal{H}} := C_\rho\|(z_1^0, z_2^0)\|^r_{\mathcal{H}}, \tag{2.26}$$

where

$$C_{\rho,1} = 2^r C_3\Big(\frac{1-\rho^m}{1-\rho}\Big)^r.$$

.

In fact, according to the variation of constants formula, we have

$$(\Lambda(z_1, z_2))(t) \equiv \Gamma(t,0)(z_1^0, z_2^0) + \int_0^t \Gamma(t,s)\Phi(z_1(s), z_2(s))ds, \tag{2.27}$$

then

$$\begin{aligned}\left\|\Lambda(z_1, z_2)\right\|^r_{L^r(0,\infty;\mathbb{H})} &= \int_0^\infty \|\Lambda(z_1(t), z_2(t))\|^r_{\mathbb{H}}dt \\ &\le 2^{r-1}M_3^r\|(z_1^0, z_2^0)\|^r_{\mathcal{H}} + 2^{r-1}\int_0^\infty \Big\|\int_0^t \Gamma(t,s)\Phi(z_1(s), z_2(s))ds\Big\|^r_{\mathbb{H}}dt. \end{aligned} \tag{2.28}$$

Using (2.19) in (2.28), we deduce that there exists a positive constant $C_3 := CM_4$ such that

$$\begin{aligned}&\|(z_1, z_2)\|^r_{L^r(0,\infty;\mathbb{H})} := \left\|\Lambda(z_1, z_2)\right\|^r_{L^r(0,\infty;\mathbb{H})}, \\ &\le 2^{r-1}M_3^r\|(z_1^0, z_2^0)\|^r_{\mathcal{H}} + 2^r C_3\Big(\sum_{r=0}^m \|(z_1, z_2)\|^r_{L^r(0,\infty;\mathbb{H})}\Big)^r, \\ &\le 2^{r-1}M_3^r\|(z_1^0, z_2^0)\|^r_{\mathcal{H}} \\ &\quad + 2^r C_3\|(z_1, z_2)\|^r_{L^r(0,\infty;\mathbb{H})}\Big(\sum_{r=1}^m \|(z_1, z_2)\|^{r-1}_{L^r(0,\infty;\mathbb{H})}\Big)^r, \\ &\le 2^{r-1}M_3^r\|(z_1^0, z_2^0)\|^r_{\mathcal{H}} \\ &\quad + 2^r C_3\|(z_1, z_2)\|^r_{L^r(0,\infty;\mathbb{H})}\Big(\sum_{r=1}^m \rho^{r-1}\Big)^r, \\ &\le 2^{r-1}M_3^r\|(z_1^0, z_2^0)\|^r_{\mathcal{H}} \\ &\quad + 2^r C_3\Big(\frac{1-\rho^m}{1-\rho}\Big)^{m+1}\|(z_1, z_2)\|^r_{L^r(0,\infty;\mathbb{H})},\end{aligned}$$

then

$$\|(z_1, z_2)\|^r_{L^r(0,\infty;\mathbb{H})} \le \frac{2^{r-1}M_3^r}{1-C_{\rho,1}}\|(z_1^0, z_2^0)\|^r_{\mathcal{H}} := C_\rho\|(z_1^0, z_2^0)\|^r_{\mathcal{H}}, \tag{2.29}$$

where

$$C_{\rho,1} = 2^r C_3\Big(\frac{1-\rho^m}{1-\rho}\Big)^r.$$

Now, we prove the following lemma to achieve the proof of Theorem 2.3.

Lemma 2.1. *There exist a time $T > 0$ and a constant $0 < \gamma < 1$ such that*

$$\|(z_1(T), z_2(T))\|_{\mathcal{H}} \leq \gamma \|(z_1^0, z_2^0)\|_{\mathcal{H}}. \tag{2.30}$$

Proof. According to the variation of constants formula (2.15), there exists a positive constant $K := \sup_{s\in[0,T]} \|\Gamma(t,s)\|_{\mathcal{L}(\mathcal{H})}$ such that we have

$$\begin{aligned}
\|(z_1(T), z_2(T))\|_{\mathcal{H}} &\leq M_3\, e^{-\lambda T} \|(z_1^0, z_2^0)\|_{\mathcal{H}} + K \int_0^T \|\Phi(z_1(s), z_2(s))\|_{\mathcal{H}} ds,\\
&\leq M_3\, e^{-\lambda T} \|(z_1^0, z_2^0)\|_{\mathcal{H}} + K \int_0^\infty \|\Phi(z_1(s), z_2(s))\|_{\mathcal{H}} ds,\\
&\leq M_3\, e^{-\lambda T} \|(z_1^0, z_2^0)\|_{\mathcal{H}} + 2KC \sum_{r=1}^{m} \|(z_1(s), z_2(s))\|^r_{L^r(0,\infty;\mathbb{H})},\\
&\leq M_3\, e^{-\lambda T} \|(z_1^0, z_2^0)\|_{\mathcal{H}} + M_5 \sum_{r=1}^{m} \|(z_1^0, z_2^0)\|^r_{\mathcal{H}}, \qquad \text{(using (2.29))}\\
&\leq M_3\, e^{-\lambda T} \|(z_1^0, z_2^0)\|_{\mathcal{H}} + M_5 \|(z_1^0, z_2^0)\|_{\mathcal{H}} \sum_{r=1}^{m} \|(z_1^0, z_2^0)\|^{r-1}_{\mathcal{H}},\\
&\leq M_3\, e^{-\lambda T} \|(z_1^0, z_2^0)\|_{\mathcal{H}} + M_5 \|(z_1^0, z_2^0)\|_{\mathcal{H}} \sum_{r=1}^{m} \left(\frac{\rho}{2M_3}\right)^{r-1}, \qquad \text{(using (2.21))}\\
&\leq M_3\, e^{-\lambda T} \|(z_1^0, z_2^0)\|_{\mathcal{H}} + M_5 \frac{1-(\frac{\rho}{2M_3})^m}{1-\frac{\rho}{2M_3}} \|(z_1^0, z_2^0)\|_{\mathcal{H}},\\
&\leq (M_3\, e^{-\lambda T} + M_5 \frac{1-(\frac{\rho}{2M_3})^m}{1-\frac{\rho}{2M_3}} \|)(z_1^0, z_2^0)\|_{\mathcal{H}}
\end{aligned}$$

then

$$\begin{aligned}
\|z_1(T), z_2(T)\|_{\mathcal{H}} &\leq (M_3\, e^{-\lambda T} + M_5 \frac{1-(\frac{\rho}{2M_3})^m}{1-\frac{\rho}{2M_3}} \|)(z_1^0, z_2^0)\|_{\mathcal{H}},\\
&\leq \Big(M_3\, e^{-\lambda T} + \alpha \rho^r\Big) \|(z_1^0, z_2^0)\|_{\mathcal{H}}, \qquad \text{(for some } \alpha > 0)\\
&\leq \Big(M_3\, e^{-\lambda T} + h(\rho)\Big) \|(z_1^0, z_2^0)\|_{\mathcal{H}},
\end{aligned}$$

where $h(\rho) = \alpha \rho^m$; we have

$$h(0) = 0, \quad h'(\rho) > 0.$$

We can take $\rho = R > 0$ sufficiently small and T sufficiently large so that

$$\gamma := C M_3\, e^{-\lambda T} + h(R) < 1;$$

then, (2.30) yields. ◻

We reiterate the argument obtaining

$$\|(z_1(nT), z_2(nT))\|_{\mathcal{H}} \leq \gamma \|(z_1((n-1)T), z_2((n-1)T))\|_{\mathcal{H}} \leq \gamma^n \|(z_1^0, z_2^0)\|_{\mathcal{H}}, \quad n = 1, 2, \ldots$$

Let us consider $t_1 = nT$, then for all $t \geq t_1$ we conclude that

$$\|(z_1(t), z_2(t))\|_{\mathcal{H}} \leq \gamma^n \|(z_1^0, z_2^0)\|_{\mathcal{H}};$$

hence, there exist two positives constants N and α_1 such that

$$\|(z_1(t), z_2(t))\|_{\mathcal{H}} \leq N \, \mathrm{e}^{-\alpha_1 t} \, \|(z_1^0, z_2^0)\|_{\mathcal{H}}.$$

□

3. Applications and numerical simulations

In this section, we apply the exponential stabilization result for the different Holling response functions I, II, III and IV. We present numerical simulations for each example in two-dimensional space.

In the following, we consider the two-dimensional closed rectangular habitat $\Omega := \{(x, y)/\ 0 \leq x \leq a,\ 0 \leq y \leq b\}$. The eigenvalues and eigenfunctions respectively of the Laplacian operator Δ with the Neumann boundary in Ω are given by (see [16])

$$\lambda_{M,N} = -\pi^2 \left[\left(M^2/a^2 \right) + \left(N^2/b^2 \right) \right]; \quad M, N = 0, 1, 2, \ldots \tag{3.1}$$

and

$$\psi_{M,N}(x, y) = \cos(M\pi x/a) \cos(N\pi y/b). \tag{3.2}$$

3.1. Holling type I

Let consider the following prey-predator-diffusion with a Holling type I functional response.

$$\begin{cases} \dfrac{\partial y_1(t, x, y)}{\partial t} = \Delta y_1 + f(y_1, y_2), \quad (x, y) \in \Omega, t > 0 \\ \dfrac{\partial y_2(t, x, y)}{\partial t} = \delta \Delta y_2 + g(y_1, y_2), \quad (x, y) \in \Omega, t > 0 \\ \dfrac{\partial y_1}{\partial v} = \dfrac{\partial y_2}{\partial v} = 0, \quad (t, x, y) \in \Sigma = (0, \infty) \times \partial\Omega \\ y_1(0, x, y) = y_1^0(x, y), \quad y_2(0, x) = y_2^0(x, y), \quad (x, y) \in \Omega \end{cases} \tag{3.3}$$

where $f(y_1, y_2) = r_1 y_1 (1 - y_1/\kappa_1) - \beta y_1 y_2$ and $g(y_1, y_2) = -r_2 y_2 + b\beta y_1 y_2$.
Steady state solutions analysis:
System (3.3) has the following constant steady states

$$(0, 0),\ (\kappa_1, 0),\ (y^*, z^*) \tag{3.4}$$

where (y^*, z^*) is the solution of the following system

$$\begin{cases} r_1 y^* (1 - y^*/\kappa_1) - \beta y^* z^* = 0, \\ -r_2 z^* + b y^* z^* = 0. \end{cases} \tag{3.5}$$

and $(y(x), z(x))$, where $(y(x), z(x))$ is a non-constant positive function when it exists. The asymptotic behavior of these steady states has been studied extensively; see, e.g., [6, 11]. Furthermore, Kishimoto and Weinberger [18] showed that (3.3) has no stable positive steady-state solution when the domain Ω is convex, while, according to Theorem 2.3, these equilibrium states can be reached; more precisely, let $z_1 = y_1 - y_1^e$ and $z_2 = y_2 - y_2^e$; then, we have the following system:

$$\begin{cases} \dfrac{\partial z_1(t,x,y)}{\partial t} = \Delta z_1 + f(z_1 + y_1^e, z_2 + y_2^e) - f(y_1^e, y_2^e) + v(t)Bz_1, \quad (x,y) \in \Omega, t > 0 \\ \dfrac{\partial z_2(t,x,y)}{\partial t} = \delta \Delta z_2 + g(z_1 + y_1^e, z_2 + y_2^e) - g(y_1^e, y_2^e) + v(t)Bz_2, \quad (x,y) \in \Omega, t > 0 \\ \dfrac{\partial z_1}{\partial v} = \dfrac{\partial z_2}{\partial v} = 0, \quad (t,x,y) \in \Sigma = (0,\infty) \times \partial\Omega \\ z_1(0,x,y) = z_1^0(x,y) := z_1^0(x,y) - y_1^e, \quad z_2(0,x) = z_2^0(x,y) := z_2^0(x,y) - y_2^e, \quad (x,y) \in \Omega \end{cases} \tag{3.6}$$

where $BY = \dfrac{2}{1+\mu(x,y)} Y$, $\forall Y \in L^2(\Omega)$ and $\mu \in L^\infty(\Omega)$ such that $\mu(x,y) \geq 1$, $\forall (x,y) \in \Omega$, is exponentially stabilizable for all equilibrium states (y_1^e, y_2^e).

Let us verify the conditions of Theorem 2.3:

-Following [20], system (3.3) has a unique smooth non-negative solution for $y_1^0(x) \geq 0$ and $y_2^0(x) \geq 0$.

- By simple calculus, there exist positive constants C_1, C_2, C_3, C_4, C_5 and C_6 such that

$$|f(y_1(t,x), y_2(t,x))| \leq C_1|y_1(t,x)| + C_2|y_1(t,x)|^2 + C_3|y_2(t,x)|^2$$

and

$$|g(y_1(t,x), y_2(t,x))| \leq C_4|y_2(t,x)| + C_5|y_1(t,x)|^2 + C_6|y_2(t,x)|^2;$$

then (H_2) holds for $r_1 = 1$, $r_2 = 2$, $m_0 = 2$ and $C = \max\{C_1, C_2, C_3, C_4, C_5, C_6\}$.

Numerical simulations:

Let $\Omega = [0,300] \times [0,300]$, $B = I_d$, $r_1 = 1$, $\kappa_1 = 1$, $e = 2.5$, $\beta = 0.4$, $\delta = 1$, $r_2 = 0.6$ and $b = 5$. The choice of parameters for numerical simulation was inspired from [22]. We illustrate numerically the exponential stabilization of (3.6) towards $(y_1^e, y_2^e) = (0,0)$. To this end we start by the explicitation of the feedback control $v(t)$. On the one hand, by simple calculus we have:

$$a = 1, b = 0, c = 0, d = 0.6 \text{ and } c_0 = 0.4. \tag{3.7}$$

On the other hand, the eigenvalues of $A := \Delta + aI$ are:

$$\lambda_{M,N} = 1 - \pi^2[M^2/300^2 + N^2/300^2]; \qquad M, N = 0, 1, 2...$$

It is clear that A has a finite number of positive eigenvalues, the largest one being $\lambda_{0,0} := 1$. Then following Theorem 2.3, we obtain $v(t) = -5$. Let

$$y_1^0(x,y) = 0.1 + \sin(\frac{\pi}{400}x)\sin(\frac{\pi}{500}y), \quad \forall (x,y) \in, \Omega$$

and

$$y_2^0(x,y) = 0.1 + \sin(\frac{\pi}{400}x) + \sin(\frac{\pi}{400}y), \quad \forall (x,y) \in \Omega.$$

Using the 2D finite difference (see [14]), we obtain Figure 1, which shows the densities of the uncontrolled system, and Figure 2, which shows the densities of the controlled system.

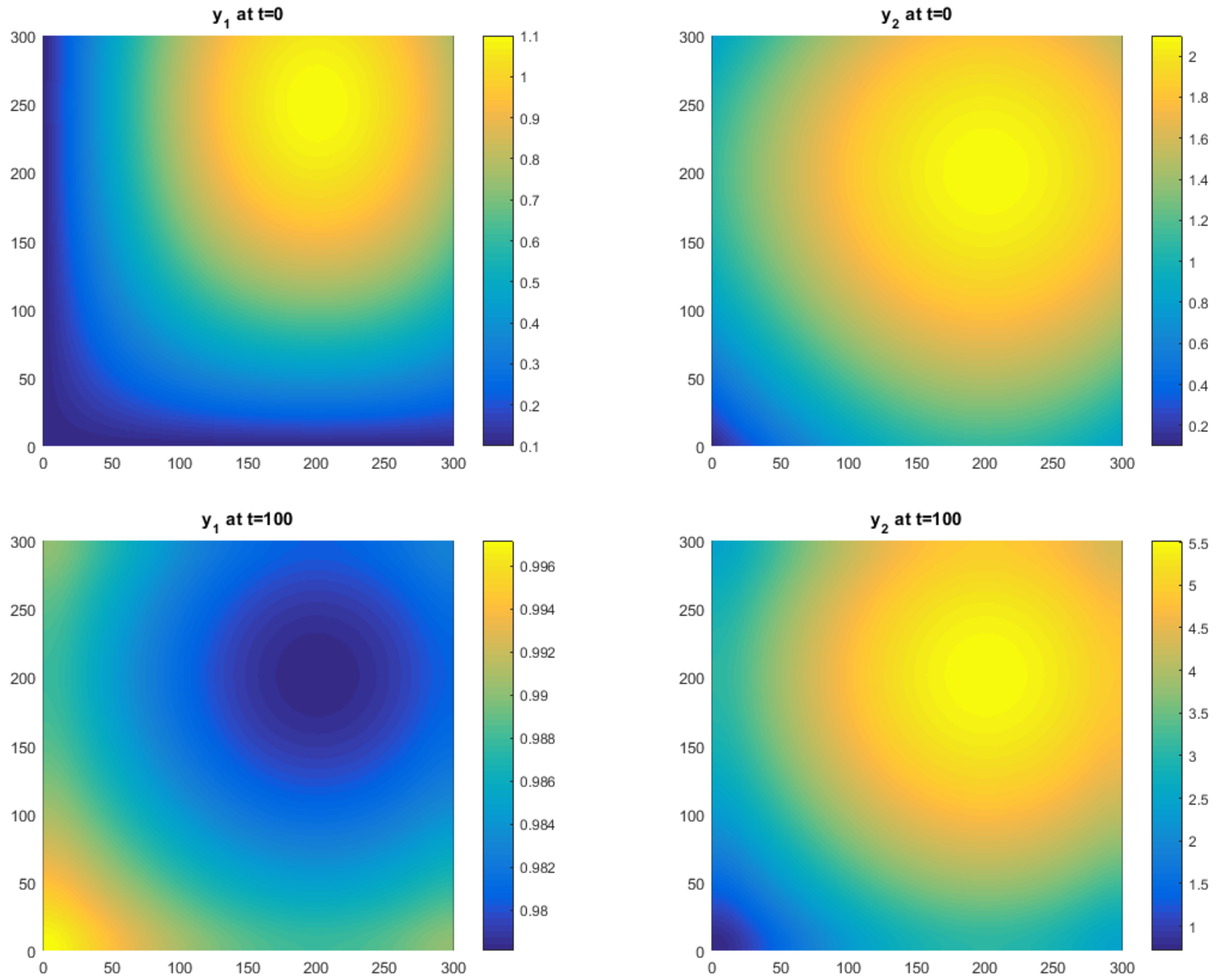

Figure 1. Uncontrolled prey and predator densities at $T = 0$ and $T = 100$ with Holling type I functional response.

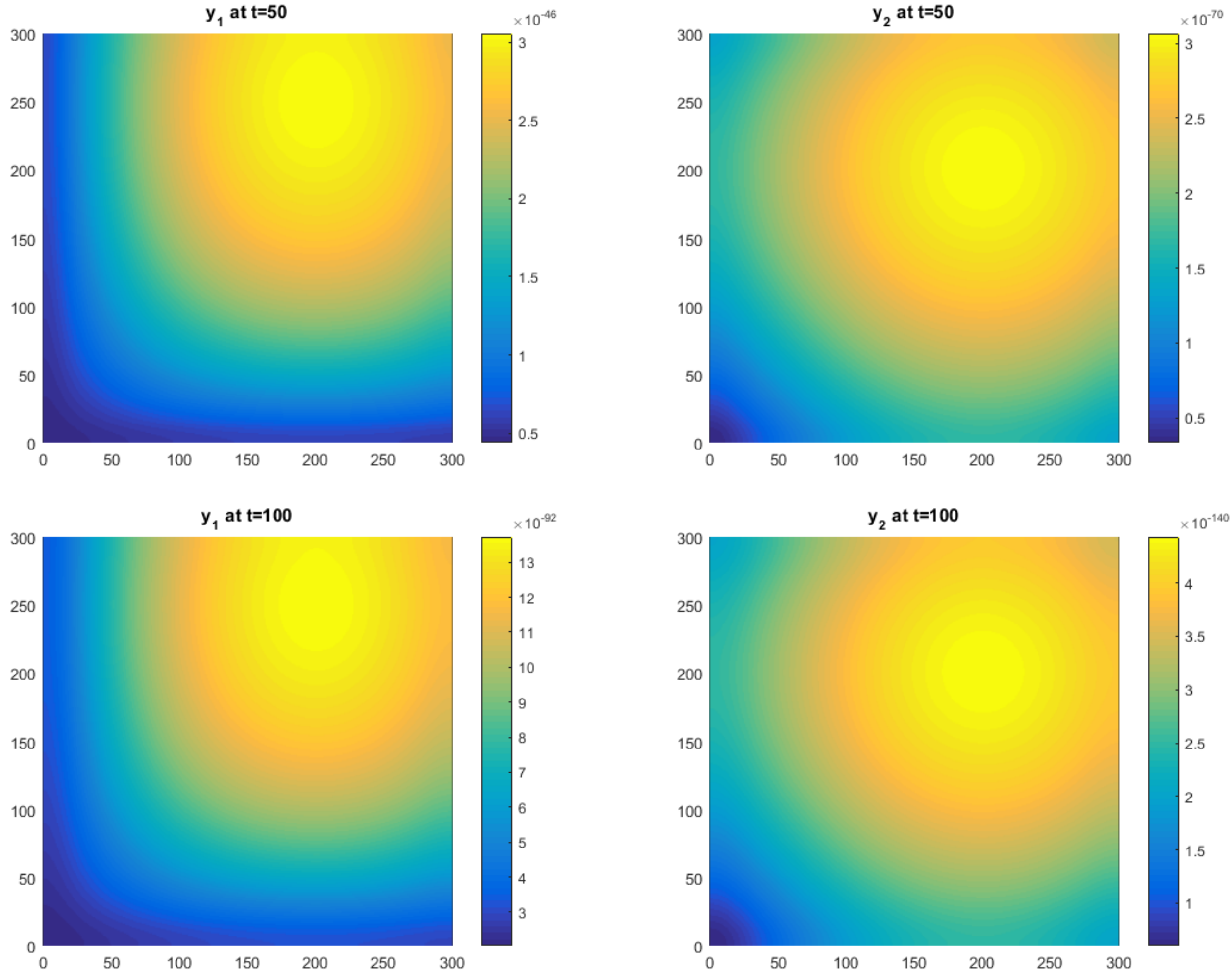

Figure 2. Controlled prey and predator densities at $T = 50$ and $T = 100$ with Holling type I functional response.

3.2. *Holling type II*

Let us consider the following prey-predator- diffusion with a Holling type II functional response.

$$\begin{cases} \dfrac{\partial y_1(t,x,y)}{\partial t} = \Delta y_1 + f(y_1, y_2), \quad (x,y) \in \Omega, t > 0 \\ \dfrac{\partial y_2(t,x,y)}{\partial t} = \delta \Delta y_2 + g(y_1, y_2), \quad (x,y) \in \Omega, t > 0 \\ \dfrac{\partial y_1}{\partial \nu} = \dfrac{\partial y_2}{\partial \nu} = 0, \quad (t,x,y) \in \Sigma = (0,\infty) \times \partial\Omega \\ y_1(0,x,y) = y_1^0(x,y), \quad y_2(0,x,y) = y_2^0(x,y), \quad (x,y) \in \Omega \end{cases} \tag{3.8}$$

where $f(y_1, y_2) = r_1 y_1(1 - y_1/\kappa_1) - \dfrac{\beta y_1 y_2}{1 + e y_1}$ and $g(y_1, y_2) = -r_2 y_2 + \dfrac{b\beta y_1 y_2}{1 + e y_1}$.

Steady state solutions analysis:

System (3.8) has the following constant steady states

$$(0,0), \quad (\kappa_1, 0), \quad (y^*, z^*) \tag{3.9}$$

where (y^*, z^*) is the positive solution of the system

$$\begin{cases} y^* + r_1 y^*(1 - y^*/\kappa_1) - \dfrac{\beta y^* z^*}{1 + e y^*} = 0, \\ -r_2 z^* + \dfrac{b\beta y^* z^*}{1 + e y^*} = 0; \end{cases} \tag{3.10}$$

and $(y(x), z(x))$ where $(y(x), z(x))$ is a non-constant positive function when it exists. Camara and Aziz-Alaoui (see [8] and [7]) show that for suitable conditions on r_1, β and e, system (3.8) has at least one positive solution. They have shown that $(0, 0)$ and $(\kappa_1, 0)$ are unstable, while, according to Theorem 2.3, these equilibrium states can be reached; more precisely, let $z_1 = y_1 - y_1^e$, $z_2 = y_2 - y_2^e$; then, we have the following system

$$\begin{cases} \dfrac{\partial z_1(t,x,y)}{\partial t} = \Delta z_1 + f(z_1 + y_1^e, z_2 + y_2^e) - f(y_1^e, y_2^e) + v(t)Bz_1, \quad (x,y) \in \Omega, t > 0 \\ \dfrac{\partial z_2(t,x,y)}{\partial t} = \delta\Delta z_2 + g(z_1 + y_1^e, z_2 + y_2^e) - g(y_1^e, y_2^e) + v(t)Bz_2, \quad (x,y) \in \Omega, t > 0 \\ \dfrac{\partial z_1}{\partial \nu} = \dfrac{\partial z_2}{\partial \nu} = 0, \quad (t,x,y) \in \Sigma = (0,\infty) \times \partial\Omega \\ z_1(0,x,y) = z_1^0(x,y) := z_1^0(x,y) - y_1^e, \quad z_2(0,x) = z_2^0(x,y) := z_2^0(x,y) - y_2^e, \quad (x,y) \in \Omega \end{cases} \tag{3.11}$$

where $BY = \mu(x,y)Y$, $\forall Y \in L^2(\Omega)$ and $\mu \in L^\infty(\Omega)$ such that $\mu(x,y) \geq 1$, $\forall (x,y) \in \Omega$, is exponentially stabilizable for all equilibrium states (y_1^e, y_2^e). Let us verify the conditions of Theorem 2.3:
-Following Lemma 14.20 [27], system (3.8) has a non-negative solution for $y_1^0(x) \geq 0$ and $y_2^0(x) \geq 0$.
-By simple calculus, there exist positive constants C_1, C_2, C_3 and C_4 such that

$$|f(y_1(t,x), y_2(t,x))| \leq C_1|y_1(t,x)| + C_2|y_1(t,x)|^2 + C_3|y_2(t,x)|$$

and

$$|g(y_1(t,x), y_2(t,x))| \leq C_4|y_2(t,x)|,$$

then (H_2) holds for $r_1 = 1$, $r_2 = 2$, $m_0 = 2$ and $C = \max\{C_1, C_2, C_3, C_4\}$.
Numerical simulations:
Let $\Omega = [0, 500] \times [0, 500]$, $B = I_d$, $r_1 = 1$, $\kappa_1 = 1$, $e = 2.5$, $\beta = 0.4$, $\delta = 1$, $r_2 = 0.6$, $b = 5$. Let explicit the feedback control that stabilizes the solution of (3.11) towards $(y_1^e, y_2^e) = (0, 0)$. The eigenvalues of $A := \Delta + aI$ are:

$$\lambda_{M,N} = 1 - \pi^2[M^2/500^2 + N^2/500^2]; \qquad M, N = 0, 1, 2...$$

It is clear that A has a finite number of positive eigenvalues, the largest one being $\lambda_{0,0} := 1$. By simple calculations we obtain

$$v(t) = -5. \tag{3.12}$$

Let

$$y_1^0(x,y) = 6/35 - 2 \times 10^{-7}\,(x - 0.1y - 225)\,(x - 0.1y - 675)\,, \quad \forall (x,y) \in \Omega,$$

and

$$y_2^0(x,y) = 116/245 - 3 \times 10^{-5}\,(x - 450) - 1.2 \times 10^{-4}\,(y - 150)\,, \quad \forall (x,y) \in \Omega.$$

Using the 2D finite difference (see [14]), we obtain Figure 3, which shows the densities of the uncontrolled system, and Figure 4, which shows the densities of the controlled system.

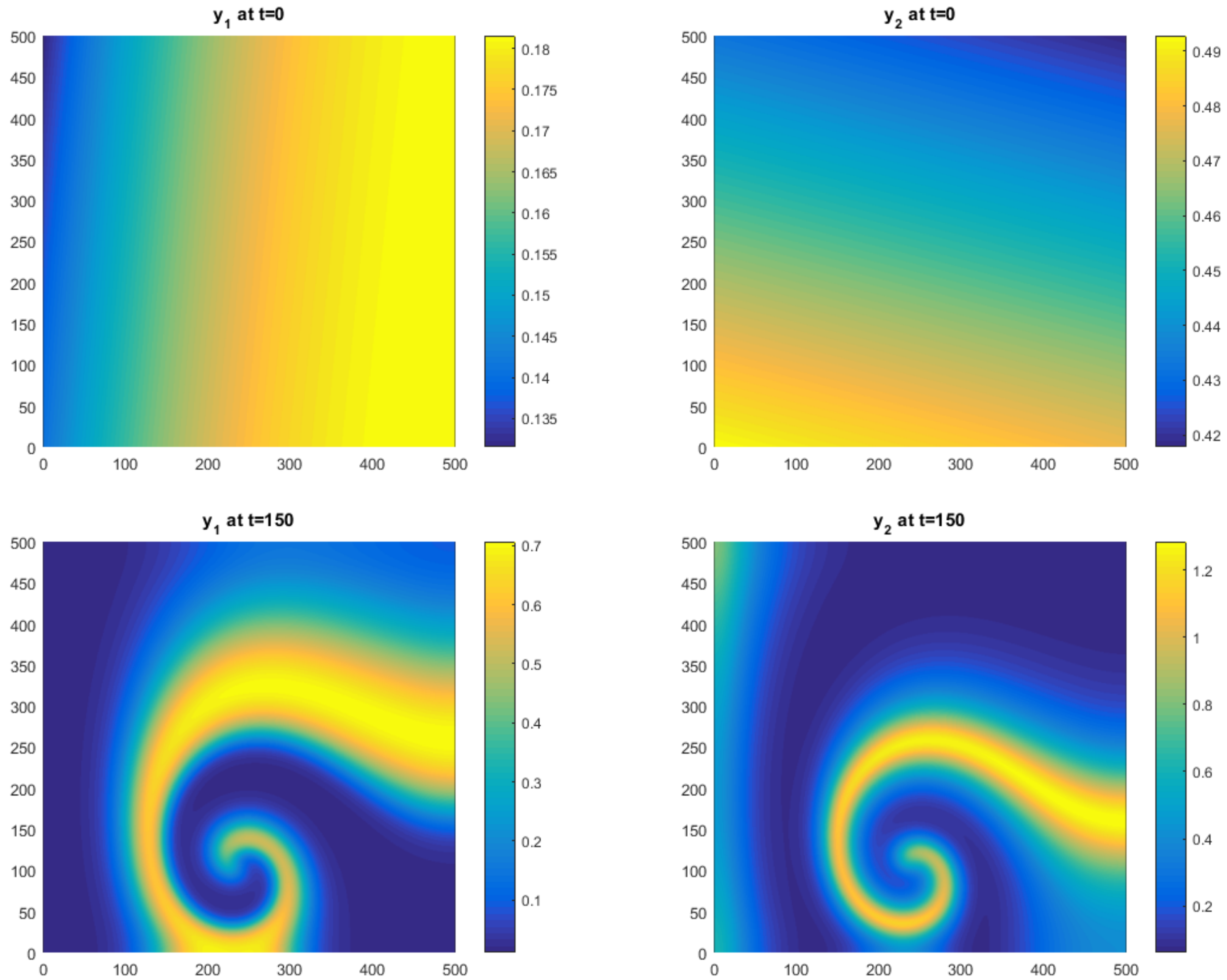

Figure 3. Uncontrolled prey and predator densities at $T = 0$ and $T = 150$ with Holling type II functional response.

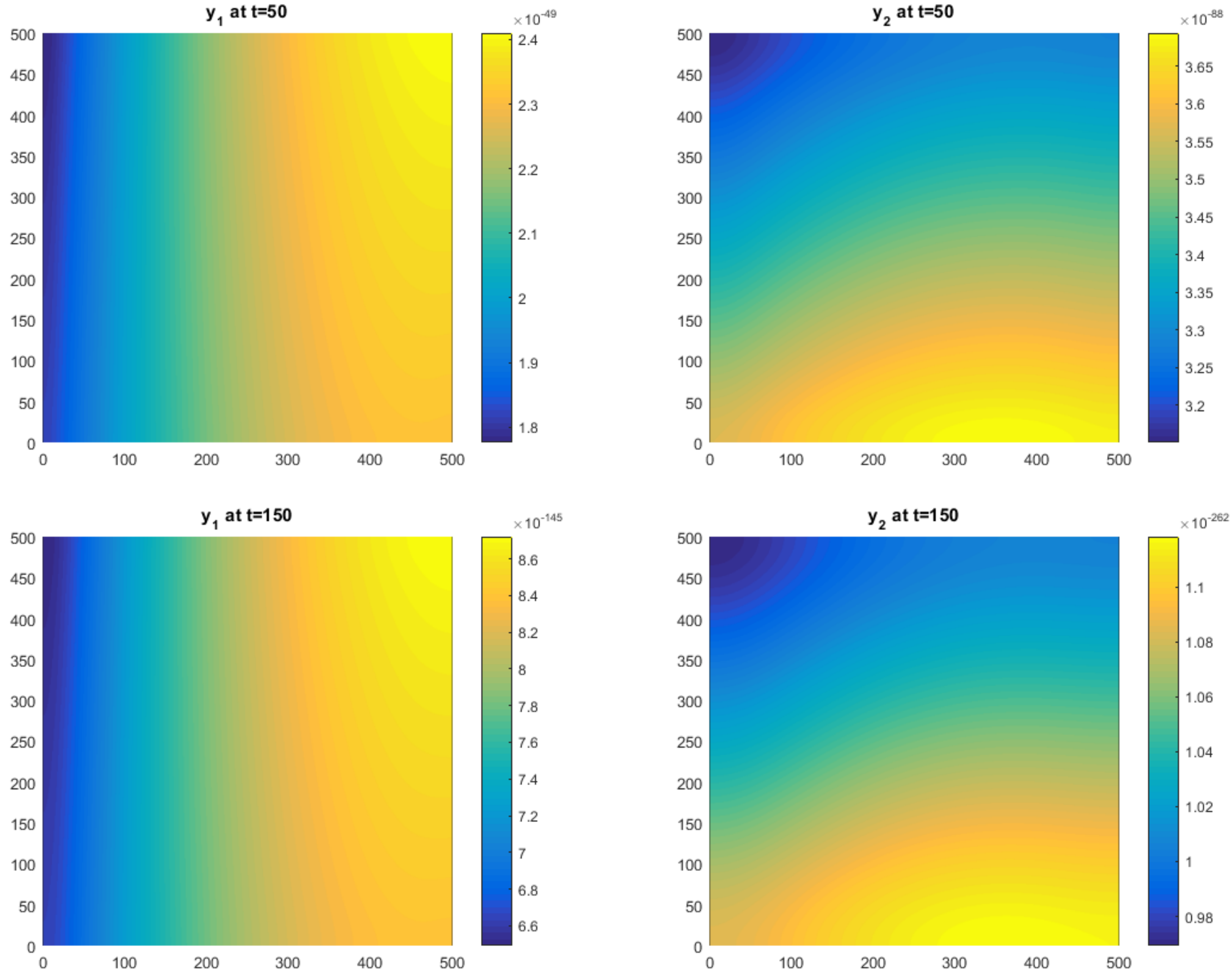

Figure 4. Controlled prey and predator densities at $T = 50$ and $T = 150$ with Holling type II functional response.

3.3. Holling type III

Let us consider the following prey-predator- diffusion with a Holling type III functional response:

$$\begin{cases} \dfrac{\partial y_1(t,x,y)}{\partial t} = \Delta y_1 + f(y_1, z_1), \quad (x,y) \in \Omega, t > 0 \\ \dfrac{\partial y_2(t,x,y)}{\partial t} = \delta \Delta y_2 + g(y_1, z_1), \quad (x,y) \in \Omega, t > 0 \\ \dfrac{\partial y_1}{\partial v} = \dfrac{\partial y_2}{\partial v} = 0, \quad (t,x,y) \in \Sigma = (0,\infty) \times \partial\Omega \\ y_1(0,x,y) = y_1^0(x,y), \quad y_2(0,x,y) = y_2^0(x,y), \quad (x,y) \in \Omega \end{cases} \tag{3.13}$$

where $f(y_1, y_2) = r_1 y_1(1 - y_1/\kappa_1) - \dfrac{\beta y_1^2 y_2}{1 + e y_1^2}$ and $g(y_1, y_2) = -r_2 y_2 + \dfrac{b\beta y_1^2 y_2}{1 + e y_1^2}$.

Steady state solutions analysis:

System (3.13) has the following constant steady states

$$(0,0), \quad (\kappa_1, 0), \quad (y^*, z^*) \tag{3.14}$$

where (y^*, z^*) is the solution of the following system

$$\begin{cases} ry^*(1 - y^*/\kappa_1) - \dfrac{\beta(y^*)^2 z^*}{1 + e(y^*)^2} = 0, \\ -r_2 z^* + \dfrac{b\beta(y^*)^2 z^*}{1 + e(y^*)^2} = 0. \end{cases} \tag{3.15}$$

Tian and Weng in [28] showed that (y^*, z^*) exists and is positive for appropriate assumptions on β, e, b and r_2, and they discussed the stability of this stationary solution. However, we have seen that, by using Theorem 2.3, these equilibrium states can be reached; more precisely, let $z_1 = y_1 - y_1^e$, $z_2 = y_2 - y_2^e$; then, we have then the following system

$$\begin{cases} \dfrac{\partial z_1(t,x,y)}{\partial t} = \Delta z_1 + f(z_1 + y_1^e, z_2 + y_2^e) - f(y_1^e, y_2^e) + v(t)Bz_1, \quad (x,y) \in \Omega, t > 0 \\ \dfrac{\partial z_2(t,x,y)}{\partial t} = \delta \Delta z_2 + g(z_1 + y_1^e, z_2 + y_2^e) - g(y_1^e, y_2^e) + v(t)Bz_2, \quad (x,y) \in \Omega, t > 0 \\ \dfrac{\partial z_1}{\partial \nu} = \dfrac{\partial z_2}{\partial \nu} = 0, \quad (t,x,y) \in \Sigma = (0,\infty) \times \partial\Omega \\ z_1(0,x,y) = z_1^0(x,y) := z_1^0(x,y) - y_1^e, \quad z_2(0,x) = z_2^0(x,y) := z_2^0(x,y) - y_2^e, \quad (x,y) \in \Omega \end{cases} \tag{3.16}$$

where $BY = (\mu(x,y) - 1)(\mu(x,y) + 1)Y$, $\forall Y \in L^2(\Omega)$ and $\mu \in L^\infty(\Omega)$ such that $\mu(x,y) \geq 1$, $\forall (x,y) \in \Omega$, is exponentially stabilizable for all equilibrium states (y_e^1, y_e^2). Now, let us verify the conditions of Theorem 2.3:

-Following Lemma 14.20 [27], system (3.13) has a non-negative solution for $y_1^0(x) \geq 0$ and $y_2^0(x) \geq 0$.

-By simple calculus, there exist positive constants C_1, C_2, C_3 and C_4 such that

$$|f(y_1(t,x), y_2(t,x))| \leq C_1|y_1(t,x)| + C_2 y_1^2(t,x) + C_3|y_2(t,x)|$$

and

$$|g(y_1(t,x), y_2(t,x))| \leq C_4|y_2(t,x)|,$$

then (H_2) holds for $r_1 = 1$, $r_2 = 2$, $m_0 = 2$ and $C = \max\{C_1, C_2, C_3, C_4, C_5, C_6\}$.

Numerical simulations:

Let $\Omega = [0,350] \times [0,350]$, $B = I_d$, $r_1 = 1$, $\kappa_1 = 1$, $e = 2.5$, $\beta = 0.4$, $\delta = 1$, $r_2 = 0.6$, $b = 5$. Let explicit the feedback control that stabilizes the solution of (3.16) towards $(y_1^e, y_2^e) = (0,0)$. The eigenvalues of $A := \Delta + aI$ are:

$$\lambda_{M,N} = 1 - \pi^2[M^2/350^2 + N^2/350^2]; \qquad M, N = 0, 1, 2...$$

It is clear that A has a finite number of positive eigenvalues, the largest one being $\lambda_{0,0} := 1$. By simple calculations we obtain

$$v(t) = -3. \tag{3.17}$$

Let

$$y_1^0(x,y) = 1 - 0.1\sin(\frac{\pi}{400}x)\cos(\frac{\pi}{350}y), \quad \forall (x,y) \in \Omega$$

and

$$y_2^0(x,y) = 0.5 + 0.1\,sin(\frac{\pi}{200}x) + 0.01\cos(\frac{\pi}{300}y), \quad \forall (x,y) \in \Omega.$$

Using the 2D finite difference (see [14]), we obtain Figure 5, which shows the densities of the uncontrolled system, and Figure 6, which shows the densities of the controlled system.

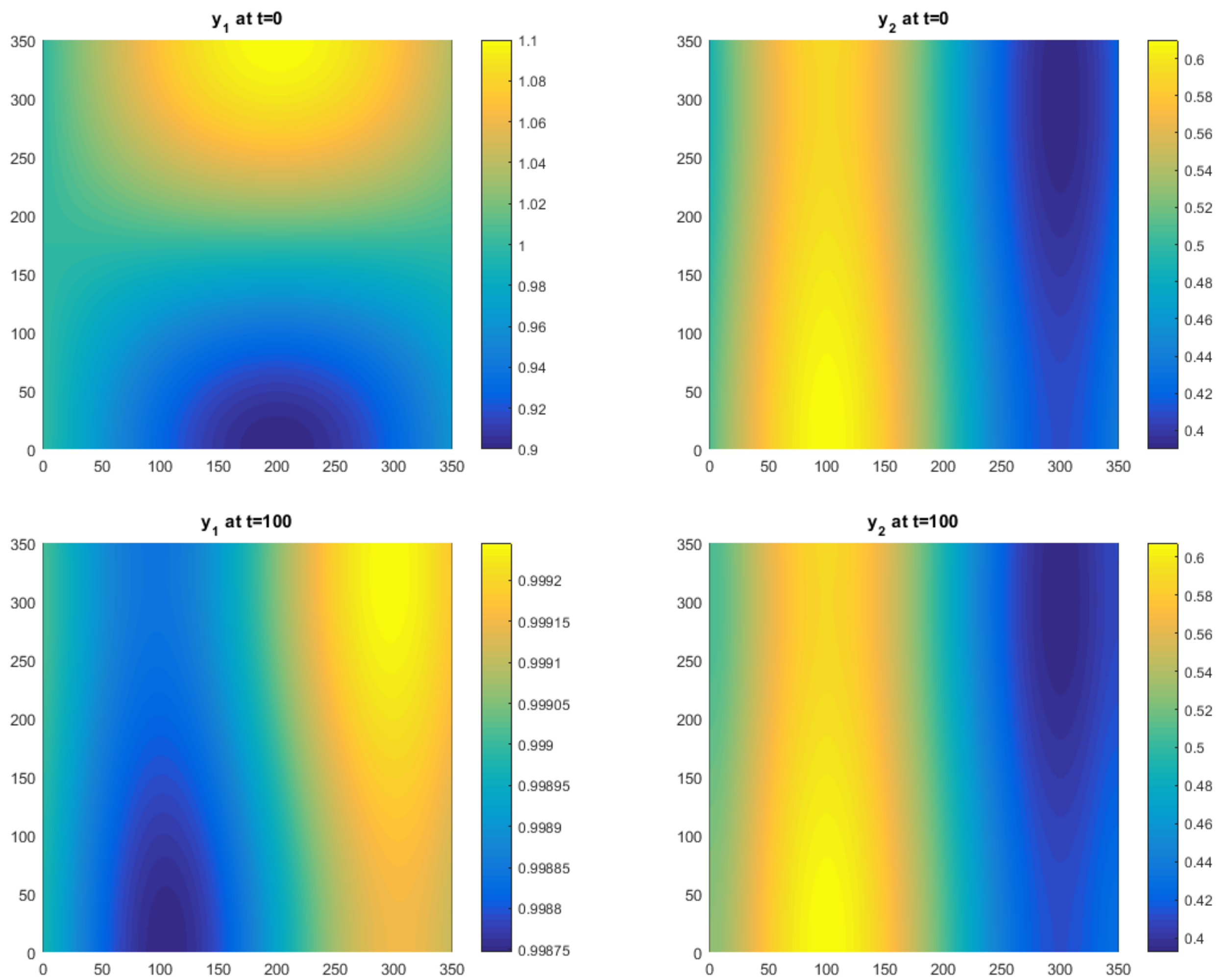

Figure 5. Uncontrolled prey and predator densities at $T = 0$ and $T = 100$ with Holling type III functional response.

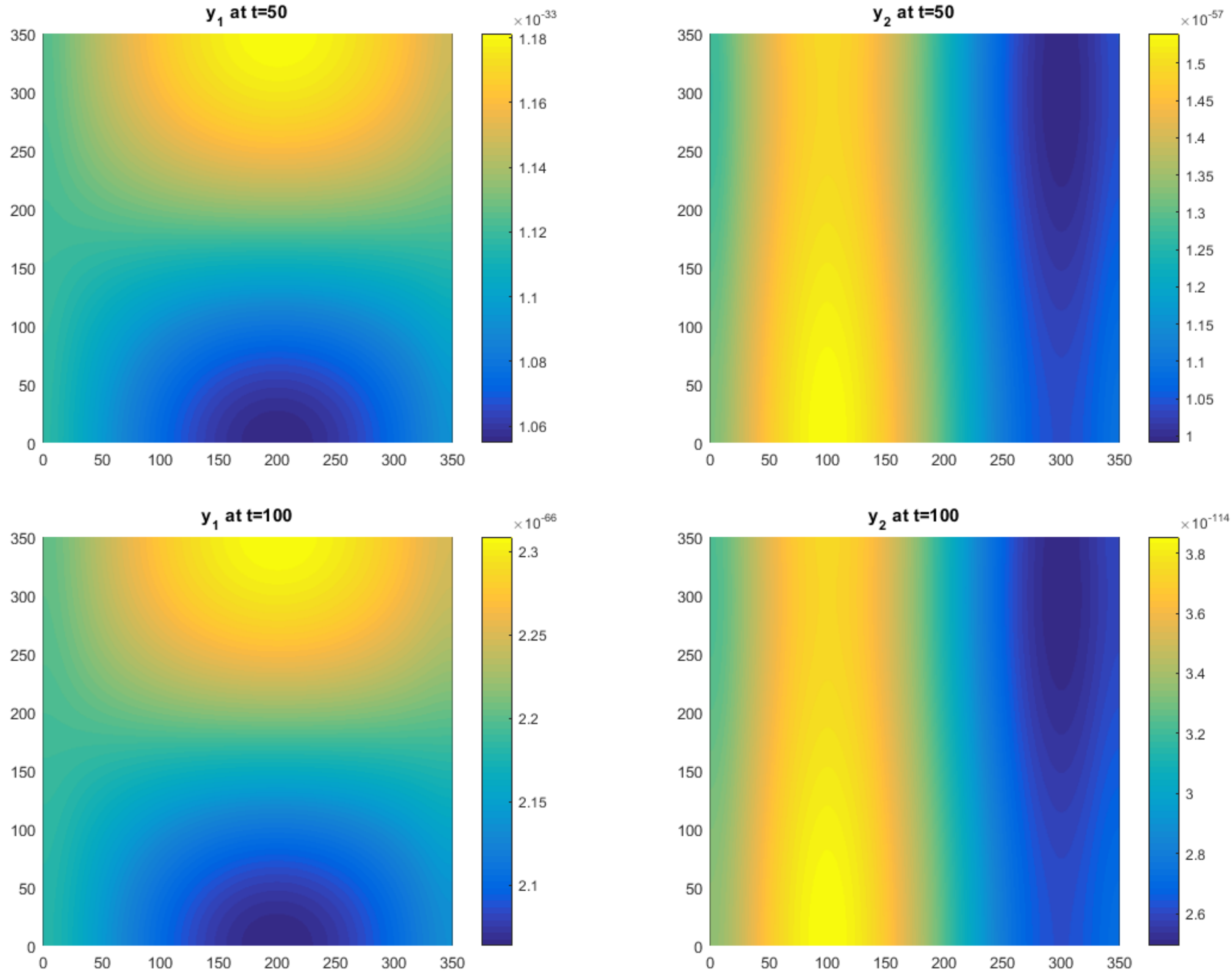

Figure 6. Controlled prey and predator densities at $T = 50$ and $T = 100$ with Holling type III functional response.

3.4. *Holling type IV*

Let us consider the following prey-predator- diffusion with a Holling type IV functional response:

$$\begin{cases} \dfrac{\partial y_1(t,x,y)}{\partial t} = \Delta y_1 + f(y_1, z_1), \quad (x,y) \in \Omega, t > 0 \\ \dfrac{\partial y_2(t,x,y)}{\partial t} = \delta \Delta y_2 + g(y_1, z_1), \quad (x,y) \in \Omega, t > 0 \\ \dfrac{\partial y_1}{\partial \nu} = \dfrac{\partial y_2}{\partial \nu} = 0, \quad (t,x,y) \in \Sigma = (0,\infty) \times \partial\Omega \\ y_1(0,x,y) = y_1^0(x,y), \quad y_2(0,x,y) = y_2^0(x,y), \quad (x,y) \in \Omega \end{cases} \tag{3.18}$$

where $f(y_1, y_2) = r_1 y_1 (1 - y_1/\kappa_1) - \dfrac{\beta y_1^2 y_2}{e_1 + e y_1 + e_2 y_1^2}$, $g(y_1, y_2) = -r_2 y_2 + \dfrac{b\beta y_1^2 y_2}{e_1 + e y_1 + e_2 y_1^2}$.

Steady state solutions analysis:

System (3.18) has the following constant steady states

$$(0,0), \quad (\kappa_1, 0), \quad (y^*, z^*), \tag{3.19}$$

where (y^*, z^*) is the solution of the following system

$$\begin{cases} ry^*(1 - y^*/\kappa_1) - \dfrac{\beta(y^*)^2 z^*}{e_1 + ey^* + e_2(y^*)^2} = 0, \\ -r_2 z^* + \dfrac{b\beta(y^*)^2 z^*}{e_1 + ey^* + e_2(y^*)^2} = 0. \end{cases} \tag{3.20}$$

We refer to [9] for discussions on the existence and stability of these equilibrium states. However, we have seen that, by using Theorem 2.3, these equilibrium states can be reached; more precisely, let $z_1 = y_1 - y_1^e$, $z_2 = y_2 - y_2^e$, then the following system

$$\begin{cases} \dfrac{\partial z_1(t,x,y)}{\partial t} = \Delta z_1 + f(z_1 + y_1^e, z_2 + y_2^e) - f(y_1^e, y_2^e) + v(t)Bz_1, \quad (x,y) \in \Omega, t > 0 \\ \dfrac{\partial z_2(t,x,y)}{\partial t} = \delta\Delta z_2 + g(z_1 + y_1^e, z_2 + y_2^e) - g(y_1^e, y_2^e) + v(t)Bz_2, \quad (x,y) \in \Omega, t > 0 \\ \dfrac{\partial z_1}{\partial v} = \dfrac{\partial z_2}{\partial v} = 0, \quad (t,x,y) \in \Sigma = (0,\infty) \times \partial\Omega \\ z_1(0,x,y) = z_1^0(x,y) := z_1^0(x,y) - y_1^e, \quad z_2(0,x) = z_2^0(x,y) := z_2^0(x,y) - y_2^e, \quad (x,y) \in \Omega \end{cases} \tag{3.21}$$

where $BY = (1 - \mu(x))Y,\ \forall Y \in L^2(\Omega)$ and $\mu \in L^\infty(\Omega)$, is exponentially stabilizable for all equilibrium states (y_e^1, y_e^2).

Now, let us verify the conditions of Theorem 2.3:

-Following Lemma 14.20 [27], system (3.18) has a non-negative solution for $y_1^0(x) \geq 0$ and $y_2^0(x) \geq 0$.

-By simple calculus, there exist positive constants C_1, C_2, C_3 and C_4 such that

$$|f(y_1(t,x), y_1(t,x))| \leq C_1|y_1(t,x)| + C_2 y_1^2(t,x) + C_3|y_2(t,x)|$$

and

$$|g(y_1(t,x), y_1(t,x))| \leq C_4|y_2(t,x)|;$$

then (H_2) holds for $r_1 = 1$, $r_2 = 2$, $m_0 = 2$ and $C = \max\{C_1, C_2, C_3, C_4\}$.

Numerical simulation:

Let $\Omega = [0,400] \times [0,400]$, $B = I_d$, $r_1 = 1$, $\kappa_1 = 1$, $e = 0.7$; $e_1 = 0.5$, $e_2 = 2.5$, $\beta = 0.4$, $\delta = 1$, $r_2 = 0.6$ and $b = 5$. Let explicit the feedback control that stabilizes the solution of (3.21) towards $(y_1^e, y_2^e) = (0,0)$. The eigenvalues of $A := \Delta + aI$ are

$$\lambda_{M,N} = 1 - \pi^2[M^2/400^2 + N^2/400^2]; \qquad M, N = 0, 1, 2...$$

It is clear that A has a finite number of positive eigenvalues, the largest one being $\lambda_{0,0} := 1$. By simple calculations we obtain

$$v(t) = -2.85. \tag{3.22}$$

Let

$$y_1^0(x,y) = 1 - \sin(\frac{\pi}{400}x)\sin(\frac{\pi}{500}y), \quad \forall (x,y) \in \Omega,$$

and

$$y_2^0(x,y) = 1 + e^{-0.4x} + e^{-0.009y}, \quad \forall (x,y) \in \Omega.$$

Using the 2D finite difference (see [14]), we obtain Figure 7, which shows the densities of the uncontrolled system, and Figure 8, which shows the densities of the controlled system.

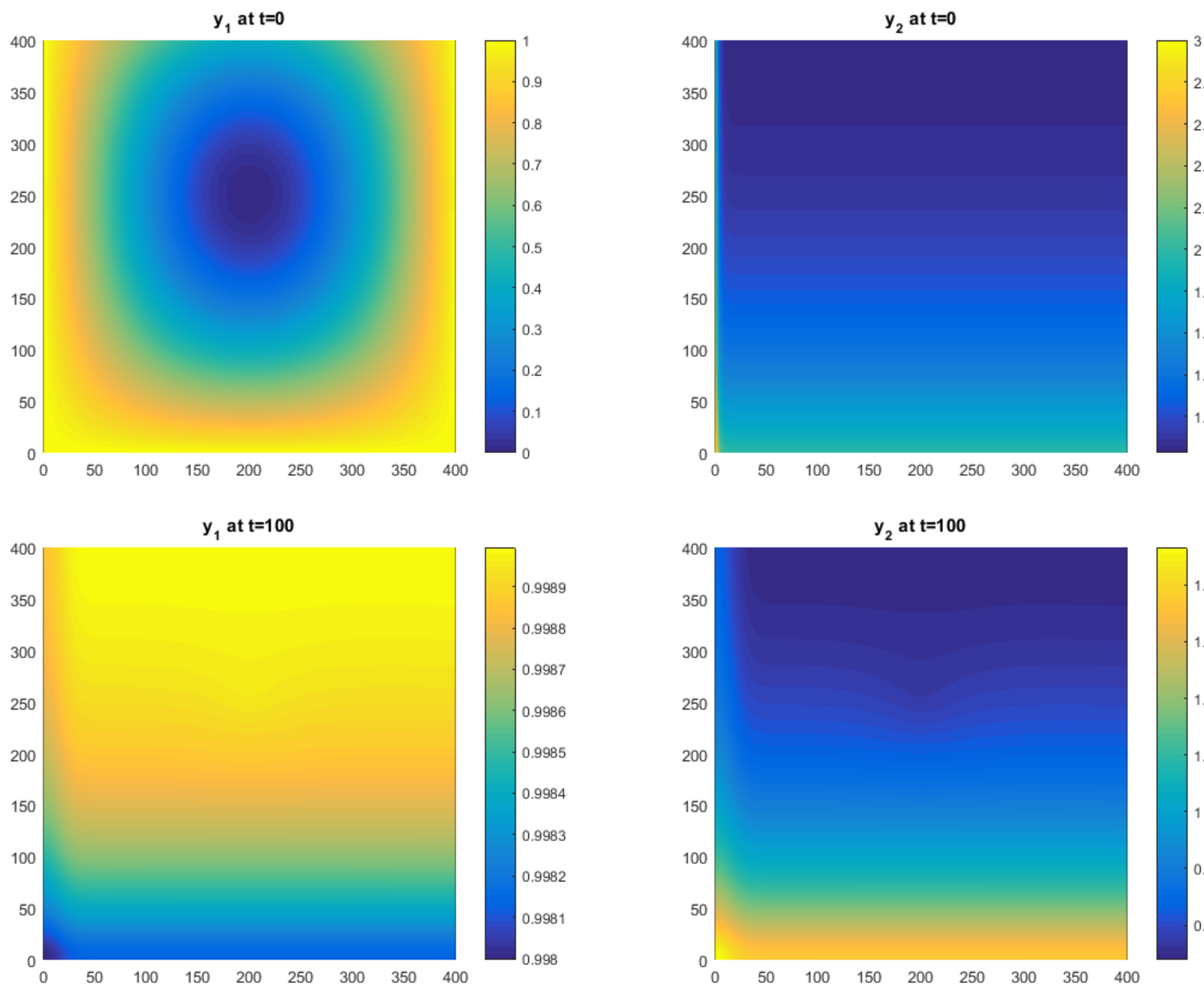

Figure 7. Uncontrolled prey and predator densities at $T = 0$ and $T = 100$ with Holling type IV functional response.

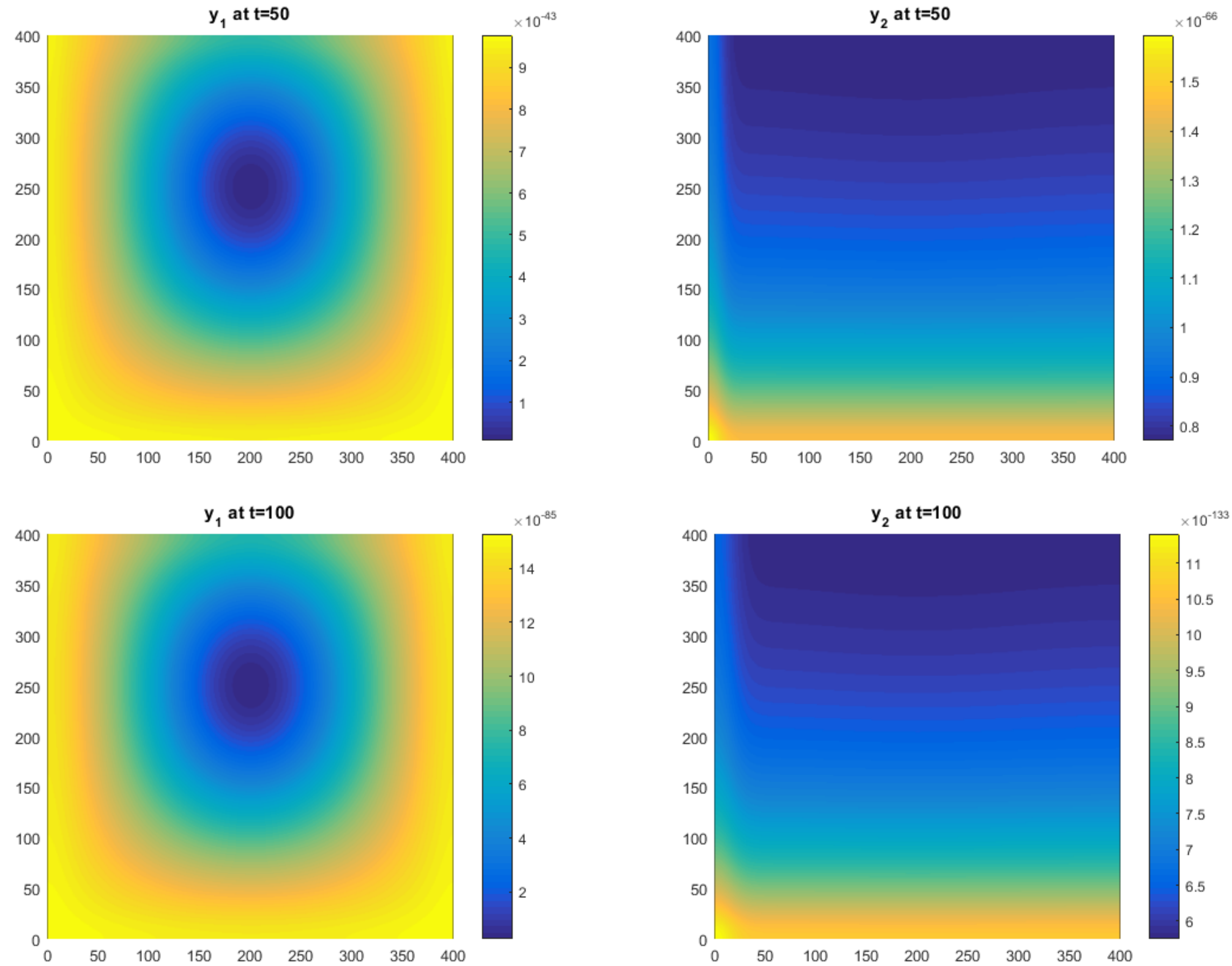

Figure 8. Controlled prey and predator densities at $T = 50$ and $T = 100$ with Holling type IV functional response.

4. Conclusions

The problem of exponential stabilization of reaction-diffusion systems simulating predatory prey systems has been investigated. We constructed a multiplicative control that exponentially stabilizes the solution of the system to its equilibrium state. The designed controller has the advantage of reaching all equilibrium states. Numerical simulations show the efficiency of the used control.

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Conflict of interest

The authors declare that there is no conflict of interests regarding this paper.

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