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Research article

Standing waves for quasilinear Schrödinger equations involving double exponential growth

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Abstract: We will focus on the existence of nontrivial, nonnegative solutions to the following quasilinear Schrödinger equation

$$\begin{cases} -\mathrm{div}\Big(\log\dfrac{e}{|x|}\nabla u\Big) - \mathrm{div}\Big(\log\dfrac{e}{|x|}\nabla(u^2)\Big)u &= g(x,u), \quad x\in B_1,\\ u &= 0, \qquad\quad x\in\partial B_1,\end{cases}$$

where B_1 denotes the unit ball centered at the origin in $\mathbb{R}^2$ and g behaves like $\exp(e^{s^4})$ as s tends to infinity, the growth of the nonlinearity is motivated by a Trudinder-Moser inequality version, which admits double exponential growth. The proof involves a change of variable (a dual approach) combined with the mountain pass theorem.

Keywords: quasilinear Schrödinger equation; double exponential growth; mountain pass theorem; Trudinger-Moser inequality; dual approach
Mathematics Subject Classification: 35J62, 35A15, 35J20

1. Introduction

This paper deals with the existence of a nontrivial, nonnegative solution for the following quasilinear stationary Schrödinger equation

$$\begin{cases} -\mathrm{div}(w(x)\nabla u) - \mathrm{div}(w(x)\nabla(u^2))u &= g(x,u), \quad x\in B_1,\\ u &= 0, \qquad\quad x\in\partial B_1,\end{cases} \tag{1.1}$$

where B_1 denotes the unit ball centered at the origin, $w(x) = \log(e/|x|)$ and the nonlinearity g possesses maximal growth range. For the case $w \equiv 1$ the problem (1.1) is reduced to

$$-\Delta u - \Delta(|u|^2)u = g(x,u), \quad x\in B_1, \tag{1.2}$$

whose solutions are related to the existence of solitary wave solutions $\psi(t,x) = e^{i\omega t}u(x)$ for the time-dependent nonlinear Schrödinger equation:

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + W(x)\psi - \Delta(|\psi|^2)\psi - h(|\psi|^2)\psi, \qquad (t,x) \in \mathbb{R}^+ \times \Omega, \tag{1.3}$$

where $\phi : \mathbb{R} \times \Omega \to \mathbb{C}$, $\lambda > 0$, $W : \Omega \to \mathbb{R}$ is a continuous potential and for suitable function $h : \mathbb{R}^+ \to \mathbb{R}$. it was shown that the class of equations of the type (1.3) arise many applications. Kurihara in [15] showed that the persistence of the solitonic behavior of superfluid films is modeled by (1.3). Also, The authors in [6, 24] used (1.3) to modeled the propagation of a short intense laser pulse through an underdense or cold plasma. For examples of other applications see for instance [1,2,24,32]. Equation of the type (1.2) was treated by many authors using several strategies such as via constrained minimization techniques [10,23], using a change of variable (a dual approach) [7,12,19], a perturbation method [8, 9], and a Nehari method [13] among others.

The strategy of variational methods considers an Euler-Lagrange functional $J : X \to \mathbb{R}$ for a suitable space X, where its critical points result in weak solutions of a nonlinear equation. We say that u is a weak solution of the problem (1.1) if

$$\int_{B_1} w(x)\nabla u\nabla\phi\, dx + \int_{B_1} w(x)\nabla(u^2)\nabla(u\phi)\, dx = \int_{B_1} g(x,u)\phi\, dx,$$

for all $\phi \in X$ (for some suitable space X to be determinate), which motivates to consider the following Euler-Lagrange functional $J : X \to \mathbb{R}$

$$J(u) = \frac{1}{2}\int_{B_1} w(x)|\nabla u|^2\, dx + \frac{1}{4}\int_{B_1} w(x)|\nabla(u^2)|^2\, dx - \int_{B_1} G(x,u)\, dx.$$

Observe that

$$\int_{B_1} w(x)|\nabla(u^2)|^2\, dx = 4\int_{B_1} w(x)|u|^2|\nabla u|^2\, dx.$$

Thus, we can rewrite J as

$$J(u) = \frac{1}{2}\int_{B_1} w(x)(1+2|u|^2)|\nabla u|^2\, dx - \int_{B_1} G(x,u)\, dx. \tag{1.4}$$

Observe that the energy functional J requires that $w(x)|u|^2|\nabla u|^2$ belongs to $L^1(B_1)$. However, there is no a suitable Sobolev space where the above energy functional is well-defined and belongs to the C^1-class. Following the ideas developed by Liu, Wang and Wang [12], we make a change of variable to reformulate the quasilinear equation (1.1) to obtain a semilinear problem whose associated functional is well-defined.

Nonlinearities of the form $g(x,u) \sim |u|^{p-2}u$ for $p \in (4, 2\cdot 2^*)$ where 2^* denotes the critical Sobolev exponent, this is, $2^* = 2N/(N-2)$ if $N \geq 3$ and $2^* = +\infty$ if $N = 2$, were considered by several authors, see [10,11,23]. It is important to mention that the presence of the term $u\Delta(u^2)$ allow us to consider p in a double range of values as in the standard case. In dimension $N = 2$, we notice that every polynomial growth is admitted on g. However, the embedding $H_0^1(\Omega) \hookrightarrow L^\infty(\Omega)$ does not hold. In this case, another kind of maximal growth was obtained independently by Yudovich [31], Pohozaev [22], and

Trudinger [25]. It was proved that $e^{\alpha|u|^2} \in L^1(\Omega)$ for all $u \in H_0^1(\Omega)$ and $\alpha > 0$. Furthermore, Moser [18] improved this result by finding the best exponent α. More precisely, it was showed that

$$\sup_{u\in H_0^1(\Omega),\|\nabla u\|_2=1} \int_\Omega e^{\alpha|u|^2}\,dx,$$

is finite for $\alpha \leq 4\pi$ and infinite if $\alpha > 4\pi$. Above statements are well known as Trudinger-Moser inequalities.

Motivated for the above results Eq (1.2) involving exponential growth was treated in [17, 20, 27, 28] among others. More precisely, the above papers considered the following condition on g: there exists a constant $\alpha_0 > 0$ such that

$$\lim_{s\to+\infty} \frac{g(x,s)}{e^{\alpha|s|^4}} = \begin{cases} +\infty & \text{for } \alpha < \alpha_0, \\ 0 & \text{for } \alpha > \alpha_0. \end{cases} \tag{1.5}$$

Similarly to the polynomial growth the expression $e^{\alpha|s|^2}$ is replace by $e^{\alpha|s|^4}$ due to the appearance of the term $\Delta(|u|^2)u$. Trudinger-Moser inequalities were extended in many directions [3–5, 16, 26] among others. Calanchi and Ruf in [4] established a Trudinger-Moser version which admits double exponential growth using a Sobolev weighted space.

Let $H^1_{0,rad}(B_1, w)$ be the Sobolev weighted space defined as the subspace of the radially symmetric functions in the closure of $C_0^\infty(B_1)$ with respect to the norm

$$\|u\| := \Big(\int_{B_1} |\nabla u|^2 w(x)\,dx\Big)^{\frac{1}{2}}.$$

where $w(x) = \log\Big(\dfrac{e}{|x|}\Big)$. The space $H^1_{0,rad}(B_1, w)$ is a separable Banach space (see [14]). Moreover, we note that $H^1_{0,rad}(B, w)$ is a Hilbert space endowed with inner product

$$\langle u, v\rangle := \int_{B_1(0)} w(x)\nabla u \nabla v\,dx, \quad \text{for all } u,\ v \in H^1_{0,rad}(B, w).$$

In what follows, we denote by E the space $H^1_{0,rad}(B, w)$ and by E^* the dual space of E with the usual norm. We start with the next embedding result.

Lemma 1.1. *The space E is continuously embedded in $L^p(B_1)$ for $1 \leq p < +\infty$, and compactly embedded in $L^p(B_1)$ for $1 < p < +\infty$.*

Proof. From the radial lemma version on the space E (see [4]), we have

$$|u(x)| \leq \frac{1}{\sqrt{2\pi}} \log^{1/2}\Big(\log\big(\frac{e}{|x|}\big)\Big)\|u\|.$$

Let consider $p > 2$. Then,

$$\int_{B_1} |u(x)|^p\,dx \leq \frac{\|u\|^p}{(2\pi)^{p/2-1}} \int_0^1 r \log^{p/2}\Big(\log\big(\frac{e}{r}\big)\Big)\,dx. \tag{1.6}$$

Observe that the function $\Phi(r) = r\log^{p/2}\Big(\log\big(\frac{e}{r}\big)\Big)$ is continuous in $(0,1]$ and

$$\lim_{r\to 0^+}\Phi(r) = e\lim_{r\to 0^+}\frac{\log^{p/2}\Big(\log\big(\frac{e}{r}\big)\Big)}{\log^{p/2}\big(\frac{e}{r}\big)}\cdot\frac{\log^{p/2}\big(\frac{e}{r}\big)}{\frac{e}{r}} = e\left[\lim_{t\to+\infty}\frac{\log t}{t}\right]^{p/2}\cdot\left[\lim_{s\to+\infty}\frac{\ln^{p/2}s}{s}\right] = 0.$$

Therefore, the integral of the right hand of (1.6) is finite. Then, there exists $C > 0$ such that $\|u\|_p \leq C\|u\|$ for every $p > 2$. Using the fact that the domain B_1 is bounded, we get $H^1_{0,rad}(B_1, w) \hookrightarrow L^p(B_1)$ for every $p \geq 1$. In order to prove compactness, we consider a sequence $(u_n) \subset E$ such that $u_n \rightharpoonup 0$ in E and $u_n \to 0$ almost everywhere in B_1. Setting $P(s) = |s|^p$ and $Q(s) = |s|^{p-\epsilon} + |s|^{p+\epsilon}$ where $p > 1$ and $0 < \epsilon < p - 1$.

We observe that $P(s)/Q(s) \to 0$, as $|s| \to \infty$, and $P(s)/Q(s) \to 0$, as $|s| \to 0$. Using the continuous embeddings $E \hookrightarrow L^{q-\epsilon}(B_1)$ and $E \hookrightarrow L^{q+\epsilon}(B_1)$, we obtain $\sup_{n\geq 1}\int_{B_1}|Q(u_n)|\,dx < +\infty$. Moreover, $P(u_n) \to 0$ for almost everywhere in B_1. Applying the Strauss compactness lemma (see [29]), we get $\int_{B_1} P(u_n)\,dx \to 0$, as $n \to +\infty$. Hence, $u_n \to 0$ in $L^p(B_1)$ for $p > 1$. □

Next, we state the Trudinger-Moser version inequality which we will use throughout this paper.

Proposition 1.1. *(See [4]) It holds*

$$\int_{B_1}\exp(\alpha e^{u^2})\,dx < +\infty, \quad \textit{for all} \quad u \in E \quad \textit{and} \quad \alpha > 0. \tag{1.7}$$

Furthermore, if $\alpha \leq 2$, there exists a positive constant C such that

$$\sup_{u\in E,\ \|u\|\leq 1}\int_{B_1}\exp(\alpha e^{2\pi u^2})\,dx \leq C. \tag{1.8}$$

Concerning to the nonlinearity g, we assume the following conditions.

(g_1) $g \in C(B_1 \times \mathbb{R})$ and $g(x, s) = o(s)$ near the origin and $g(x, s) = 0$ for all $x \in B_1$ and $s \leq 0$.

(g_2) There exists a constant $\mu > 2$ such that

$$0 < 2\mu G(x, s) \leq sg(x, s), \quad \text{for all } \ x \in B_1 \ \text{ and } \ s > 0,$$

where $G(x, s) = \int_0^s g(x, t)\,dt$.

(g_3) There exist constants $s_0 > 0$ and $M > 0$ such that

$$0 < G(x, s) \leq Mg(x, s), \quad \text{for all} \quad s > s_0.$$

(g_4) There exists a constant $\alpha_0 > 0$ such that

$$\lim_{s\to+\infty}\frac{g(x, s)}{\exp(e^{\alpha s^4})} = \begin{cases} 0, & \alpha > \alpha_0, \\ +\infty, & \alpha < \alpha_0, \end{cases}$$

uniformly in $x \in B_1$.

(g_5) There exist constants $p > 4$ and $C_p > 0$ such that

$$g(x, s) \geq C_p s^{p-1}, \quad \text{for all } x \in B_1 \text{ and } s \geq 0,$$

where

$$C_p > \max\left\{30, p\left[\Big(\frac{\alpha_0(\mu-2)}{4\pi\mu}\Big)\Big(9 \cdot 18^{2/(p-2)} - 18^{p/(p-2)}\Big)\right]^{(p-2)/2}\right\}.$$

The following theorem contains our main result.

Theorem 1.1. *Suppose that* (g_1)–(g_5) *holds. Then, the quasilinear equation (1.1) possesses a nontrivial nonnegative weak solution.*

The main difference of our result with previous works is the assumption (g_3) which says that g has double exponential critical growth in the sense of Proposition 1.1, we consider the term $\exp(e^{\alpha s^4})$ instead of $\exp(e^{\alpha s^2})$ in analogy to (1.5). In order to prove the existence of solution of the Eq (1.1) we make a change a variable to reformulate the energy functional (1.4) and apply the mountain pass theorem in the new energy functional.

The paper is organized as follows: Section 2 contains some preliminaries results and we set up the reformulate energy functional. In Section 3, we show that the reformulate energy functional possesses the pass mountain geometry. In Section 4, we estimate the Palais-Smale sequences and minimax levels of the new functional. Finally, in Sections 5, we present the proof of our main theorem.

2. Preliminaries

Following [7, 12], we make a change of variable $v = f^{-1}(u)$ where f is an invertible function satisfying

$$f'(t) = \frac{1}{\sqrt{1 + 2f^2(t)}} \quad \text{and} \quad f(-t) = -f(t).$$

Then, J is reformulate as follows

$$I(v) := J(f(v)) = \frac{1}{2}\int_{B_1} w(x)|\nabla v|^2\, dx - \int_{B_1} G(x, f(v))\, dx,$$

for all $v \in E$, and by standard arguments, one can verify that

$$I'(v)\phi = \frac{1}{2}\int_{B_1} w(x)\nabla v \cdot \nabla\phi\, dx - \int_{B_1} g(x, f(v)) f'(v)\phi\, dx,$$

for all $v,\ \phi \in E$. Moreover, $I(v)$ is the energy functional associated to the semilinear equation

$$\begin{cases} -\mathrm{div}\Big(\log\dfrac{e}{|x|}\nabla v\Big) = g(x, f(v)) f'(v), & x \in B_1(0), \\ v = 0, & x \in \partial B_1(0). \end{cases} \tag{2.1}$$

It was observed in [7], if v is a weak solution of the problem (2.1) then $v = f^{-1}(u)$ is a weak solution of the problem (1.1).

The function f satisfies the following properties (see [7, 21]).

Lemma 2.1. *The function f satisfies the following properties*

(i) $f \in C^1(\mathbb{R})$ *is strictly increasing, in particular, it is invertible.*
(ii) $|f(t)| \leq |t|$ *and* $0 < |f'(t)| \leq 1$ *for all* $t \in \mathbb{R}$.
(iii) $f(t)/t \to 0$ *as* $t \to 0$ *and* $0 < f'(t) \leq 1$ *for all* $t \in \mathbb{R}$.
(iv) $f(t)/\sqrt{t} \to 2^{1/4}$ *as* $t \to +\infty$.
(v) $f(t)/2 \leq tf'(t) \leq f(t)$ *for all* $t \geq 0$.
(vi) $|f(t)| \leq 2^{1/4}|t|^{1/2}$ *for all* $t \in \mathbb{R}$.

3. Geometry of the pass mountain theorem

This section is devoted to set the geometry of the linking theorem of the functional I.

Lemma 3.1. *Let* $v \in E$, $\alpha > 0$ *and* $q > 1$ *with* $2\alpha\|v\|^2 < \pi$, *then there exists* $C > 0$ *such that*

$$\int_{B_1} |f(v)|^q \exp(e^{\alpha|f(v)|^4})\, dx \leq C\|f(v)\|^q.$$

Proof. Using the Cauchy-Schwarz inequality and Lemma 2.1-(vi), we have

$$\begin{aligned}
\int_{B_1} |f(v)|^q \exp(e^{\alpha|f(v)|^4})\, dx &\leq \|f(v)\|_{2q}^q \left(\int_{B_1} \exp(2e^{2\alpha|v|^2})\, dx\right)^{1/2} \\
&\leq |f(v)\|_{2q}^q \left(\int_{B_1} \exp\Big(2e^{2\alpha\|v\|^2\left(\frac{|v|}{\|v\|}\right)^2}\Big)\, dx\right)^{1/2} \\
&\leq C|f(v)\|^q,
\end{aligned}$$

where we have used Proposition 1.1 and Lemma 1.1. □

Lemma 3.2. *Suppose that* (g_1) *and* (g_4) *are hold. Then, there exist* $\rho > 0$ *and* $\sigma > 0$ *satisfying*

$$I(v) \geq \sigma, \quad \textit{for all } u \in E \textit{ with } \|v\| = \rho.$$

Proof. From (g_1) and (g_4), $\alpha > 0$ and $q > 2$, we can find $c > 0$ such that

$$|G(x,s)| \leq \epsilon|s|^2 + c|s|^q \exp(e^{\alpha|s|^4}) \quad \text{for all} \quad s \in \mathbb{R}.$$

Then,

$$I(v) = \frac{1}{2}\|v\|^2 - \int_{B_1} G(x, f(v))\, dx \geq \frac{1}{2}\|v\|^2 - \int_{B_1} \Big(\epsilon|f(v)|^2 + c|f(v)|^q \exp(e^{\alpha|f(v)|^4}\Big)\, dx.$$

Using Lemma 3.1, Lemma 2.1-(ii) and Lemma 1.1, we obtain

$$I(v) \geq \Big(\frac{1}{2} - \epsilon C\Big)\|v\|^2 - C\|f(v)\|^q \geq \Big(\frac{1}{2} - \epsilon C\Big)\|v\|^2 - C_0\|v\|^q,$$

provided that $\|u\| \leq \rho_0$ for some $\rho_0 > 0$ such that $2\alpha\rho_0^2 < \pi$. Now, taking $0 < \epsilon < 1/4C$ and $\rho_1 > 0$ such that $(1/2 - \epsilon C)\rho_1^2 - C_0\rho_1^q > \rho_1^2/4$. The result follows considering $\rho = \min\{\rho_0, \rho_1\}$ and $\sigma = \rho_1^2/4$. □

Lemma 3.3. *Suppose that* (g_1), (g_3) *and* (g_4) *are hold. Then, there exists* $e \in E$ *with* $\|e\| > \rho$ *such that*

$$I(e) < 0 < \sigma \leq \inf_{\|v\|=\rho} I(v),$$

where ρ *is given by Lemma 3.2.*

Proof. Using (g_3), we can find constants positives C_1 and C_2 such that $G(x,s) \geq C_1|s|^{2\mu} - C_2$ for all $x \in B_1$ and $s \geq 0$. Let $0 \neq \phi \in C_0^\infty(B_1)$ such that $0 < \phi(x) \leq 1$. Then,

$$I(t\phi) = \frac{t^2}{2}\|\phi\|^2 - \int_{B_1} G(x, f(tv))\,dx \leq \frac{t^2}{2}\|\phi\|^2 - \int_{B_1}\Big(C_1|f(t\phi)|^{2\mu} - C_2\Big)\,dx.$$

By Lemma 2.1-(v) the function $f(t)/t$ is non-increasing for $t > 0$. Since, $0 \leq \phi(|x|)t \leq t$, we get $f(t\phi(|x|)) \geq f(t)\phi(|x|)$ for all $t > 0$ and $x \in B_1$. Then,

$$I(tv) \leq \frac{t^2}{2}\|\phi\|^2 + C_2|B_1| - C_1\int_{B_1}|f(t)|^{2\mu}\phi^{2\mu}\,dx = \frac{t^2}{2}\Big(\|\phi\|^2 - 2C_1\frac{|f(t)|^{2\mu}}{t^2}\|\phi\|_{2\mu}^{2\mu}\Big) + C_2|B_1|.$$

By Lemma 2.1-(iv) and the fact that $\mu > 2$, we obtain

$$\lim_{t\to+\infty}\frac{|f(t)|^{2\mu}}{t^2} = \lim_{t\to+\infty}\Big(\frac{f(t)}{\sqrt{t}}\Big)^{2\mu}t^{\mu-2} = +\infty.$$

Thus, there exists $T > 0$ such that $\|\phi\|^2 - 2C_1\frac{|f(t)|^{2\mu}}{2t^2}\|\phi\|_{2\mu}^{2\mu} \leq -1$ for $t \geq T$. Therefore, taking $t_0 > T$ sufficiently large we can get $e = t_0\phi$, satisfying $I(e) < 0$ and $\|e\| > \rho$. □

4. On Palais-Smale sequences

Lemma 4.1. *Suppose that* (g_1), (g_2) *and* (g_4) *are hold. Let* (v_n) *be a Palais-Smale sequence of I at level d, that is,* $I(v_n) \to d$ *and* $\|I'(v_n)\|_{E^*} \to 0$ *as* $n \to +\infty$*. Then,* $\|u_n\| \leq c$ *for all* $n \in \mathbb{N}$ *and some* $c > 0$*.*

Proof. Observe that

$$I(v_n) - \frac{1}{\mu}I'(v_n)v_n = \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|u\|^2 - \int_{B_1} G(x, f(v_n))\,dx + \frac{1}{\mu}\int_{B_1} g(x, f(v_n))f'(v_n)v_n\,dx.$$

By Lemma 2.1-(v), we have $f(t)/2 \leq tf'(t)$. Then,

$$I(v_n) - \frac{1}{\mu}I'(v_n)v_n \geq \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|u\|^2 - \frac{1}{2\mu}\Big(\int_{B_1}\Big(2\mu G(x, f(v_n)) - g(x, f(v_n))v_n\Big)\,dx.$$

Using (g_2), we get

$$I(v_n) - \frac{1}{\mu}I'(v_n)v_n \geq \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|v_n\|^2.$$

On the other hand, using the fact that $I(v_n) \to d$ and $\|I'(v_n)\|_{E^*} \to 0$ as $n \to +\infty$. We may assume that $|I(v_n)| \leq d + 1$ and $\|I(v_n)\|_{E^*} \leq \mu$ for all $n \in \mathbb{N}$. Hence,

$$\Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|v_n\|^2 \leq d + 1 + \|v_n\|, \quad \text{for all } n \in \mathbb{N},$$

which implies the assertion of the lemma. □

Lemma 4.2. *Supposed that* (g_1)*–*(g_4) *are hold. Let* v_n *be a Palais-Smale sequence for the functional* I*. Then, there exists a subsequence (not renamed) of* (v_n) *such that*

$$\int_{B_1} g(x, f(v_n))f'(v_n)\phi\, dx \to \int_{B_1} g(x, f(v))f'(v)\phi\, dx,$$

for all $\phi \in C_0^\infty(B_1)$*, as* $n \to +\infty$*, where* v *is the weak limit of* (v_n) *in* E*.*

Proof. From Lemma 4.1, we have that (v_n) is bounded. Thus, we can assume that v_n converge weakly to $v \in E$. According to Lemma 1.1, we can assume that $v_n \to v$ in $L^p(B_1)$ for all $p \geq 1$, using Lemma 2.1-(ii) and Dominated convergence theorem we have $f(v_n) \to f(v)$ in $L^p(B_1)$ for all $p \geq 1$. Combining Proposition 1.1 with the inequality $|f'(t)| \leq 1$ for all $t \in \mathbb{R}$ given by Lemma 2.1-(ii), we obtain $g(x, f(v))f'(v) \in L^1(B_1)$. On the other hand, using the boundedness of the sequence (v_n) in E, we have

$$I'(v_n)v_n = \frac{\|v_n\|^2}{2} - \int_{B_1} g(x, f(v_n))f'(v_n)v_n\, dx \to 0 \quad \text{as } n \to +\infty.$$

where we have used that (v_n) is a Palais-Smale sequence. Thus, there exists some positive constant C such that

$$\int_{B_1} g(x, f(v_n))f'(v_n)v_n\, dx \leq C \quad \text{for all } n \geq 1. \tag{4.1}$$

Given $\epsilon > 0$ there exists $\delta > 0$ such that for any measurable subset $A \subset B_1$

$$\int_A |f(v)|\, dx < \epsilon \quad \text{and} \quad \int_A g(x, f(v_))f'(v)v\, dx < \epsilon \quad \text{if } |A| < \delta.$$

Using the fact that $f(v) \in L^1(B_1)$ there exists $M_0 > 0$ satisfying

$$|\{x \in B_1 : |f(v(x))| \geq M_0\}| \leq \delta. \tag{4.2}$$

Setting $M = \max\{M_0, C/\epsilon\}$ where C and M_0 are given by (4.1) and (4.2) respectively. Note that, we can write

$$\left| \int_{B_1} |g(x, f(v_n))f'(v_n)|\, dx - \int_{B_1} |g(x, f(v))f'(v)|\, dx \right| \leq I_{1,n} + I_{2,n} + I_{3,n},$$

where

$$I_{1,n} = \int_{\{x \in B_1 : f(v(x)) \geq M\}} |g(x, f(v_n))f'(v_n)|\, dx,$$

$$I_{2,n} = \left| \int_{\{x \in B_1 : f(v_n(x)) < M\}} |g(x, f(v_n))f'(v_n)|\, dx - \int_{\{x \in B_1 : f(v(x)) < M\}} |g(x, f(v))f'(v)|\, dx \right|,$$

and

$$I_{3,n} = \int_{\{x \in B_1 : f(v(x)) \geq M\}} |g(x, f(v))f'(v)|\, dx.$$

As in [28] we can show that $I_{1,n}$, $I_{2,n}$ and $I_{3,n}$ go to 0, as $n \to +\infty$. Therefore,

$$g(x, f(v_n))f'(v_n) \to g(x, f(v))f'(v), \quad \text{in } L^1(B_1).$$

Now, taking $\phi \in C_0^\infty(B_1)$. Then,

$$\int_{B_1} \left|g(x, f(v_n))f'(v_n)\phi - g(x, f(v))f'(v)\phi\right| dx \le \|\phi\|_\infty \|g(x, f(v_n))f'(v_n) - g(x, f(v))f'(v)\|_1 \to 0,$$

as $n \to +\infty$, and the assertion of the lemma follows. □

Lemma 4.3. *Supposed that (g_1)–(g_4) are hold. Let (v_n) be a Palais-Smale sequence for the functional I. Then, there exists a subsequence (not renamed) of (v_n) such that*

$$\int_{B_1} G(x, f(v_n))\, dx \to \int_{B_1} G(x, f(v))\, dx, \quad \textit{as } n \to +\infty,$$

where v is the weak limit of (v_n) in E.

Proof. From Lemma 4.1, we have that (v_n) is bounded. Thus, we can assume that v_n converge weakly to $v \in E$. According to Lemma 1.1, we can assume that $v_n \to v$ in $L^p(B_1)$ for all $p \ge 1$ and $v_n \to v$ almost everywhere in B_1. Using (g_2)–(g_3) there exists $M > 0$ such that

$$|G(x, f(v_n))| \le Mg(x, f(v_n)) \quad \text{for all } \ x \in B_1. \tag{4.3}$$

As in Lemma 4.2, we can get

$$\int_{B_1} g(x, f(v_n))\, dx \to \int_{B_1} g(x, f(v))\, dx.$$

Since $g(x, f(v_n)) \to g(x, f(v))$ almost everywhere in B_1. By (4.3) and the generalized dominated convergence theorem, we obtain

$$\int_{B_1} G(x, f(v_n))\, dx \to \int_{B_1} G(x, f(v))\, dx.$$

□

Let ϕ be a function in $C_{0,rad}^\infty(B_1)$ satisfying

(i) $\phi \equiv 1$ in $B_{1/2}(0)$ and $\phi \equiv 0$ in $B^c_{1/\sqrt{2}}(0)$.
(ii) $0 \le \phi(x) \le 1$ for all $x \in B_1$.
(iii) $0 \le |\nabla\phi(x)| \le 1$ for all $x \in B_1$.

Lemmas 3.2 and 3.3 with ϕ satisfying (i)–(iii), it follows that I satisfies the mountain pass theorem (see [30]). Therefore, there exists a Palais-Smale sequence $(v_n) \subset E$ such that

$$I(v_n) \to 0 \quad \text{and} \quad \|I'(v_n)\|_{E^*} \to 0, \quad \text{as } n \to +\infty, \tag{4.4}$$

where d can be characterized as

$$d = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)), \tag{4.5}$$

and

$$\Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0, \gamma(1) = e\}.$$

Moreover, by Lemma 3.2, we have thar $d \ge \sigma > 0$.

Lemma 4.4. *Supposed that* (g_1)–(g_5) *are hold. Then, the minimax level d given by (4.5) satisfies*

$$d < \frac{\pi}{2\alpha_0} \cdot \frac{2\mu}{\mu - 2}.$$

Proof. Let ϕ be a function in $C^\infty_{0,rad}(B_1)$ satisfying (i)–(iii). Then,

$$\begin{aligned}
\max_{t\geq 0} I(t\phi) &\leq \frac{t^2}{2}\int_{B_1} w(x)|\nabla\phi|^2\,dx + t^4\int_{B_1} w(x)|\phi|^2|\nabla\phi|^2\,dx - \int_{B_1} G(x,t\phi)\,dx \\
&\leq \frac{t^2}{2}\int_{B_1} w(x)|\nabla\phi|^2\,dx + t^4\int_{B_1} w(x)|\nabla\phi|^2\,dx - \frac{C_p t^p}{p}\int_{B_1}|\phi|^p\,dx \\
&\leq \frac{t^2}{2}\int_{B_1} w(x)\,dx + t^4\int_{B_1} w(x)\,dx - \frac{C_p t^p}{p}\int_{B_{1/2}} 1\,dx \\
&\leq \frac{3\pi t^2}{4} + \frac{3\pi t^4}{2} - \frac{\pi C_p t^p}{4p} = \frac{\pi}{4}\Big(3t^2 + 6t^4 - \frac{C_p}{p}t^p\Big).
\end{aligned}$$

From the intermediate value theorem, the function

$$\Phi(t) = 3t^2 + 6t^4 - \frac{C_p}{p}t^p \quad \text{for } t \geq 0,$$

attained its maximum in $t_0 \in (0,1)$ provided $C_p > 30$. Therefore,

$$\max_{t\geq 0}\Phi(t) \leq \max_{t\in[0,1]}\Phi(t) \leq \max_{t\in[0,1]}\Big(9t^2 - \frac{C_p}{p}t^p\Big) = \Big(9t^2 - \frac{C_p}{p}t^p\Big)\Big|_{(18p/C_p)^{1/(p-2)}}.$$

Hence, by assumption on C_p, we have

$$\max_{t\geq 0} I(t\phi) \leq \frac{\pi}{4}\Big(9\cdot 18^{2/(p-2)} - 18^{p/(p-2)}\Big)\Big(\frac{p}{C_p}\Big)^{2/(p-2)} < \frac{\pi}{2\alpha_0}\cdot\frac{2\mu}{\mu-2}.$$

Setting $\gamma_0 \in \Gamma$, defined by $\gamma_0(t) = tt_0\phi = te$, where t_0 is given by Lemma 3.3. Thus,

$$d \leq \max_{t\in[0,1]} I(\gamma_0(t)) \leq \max_{t\geq 0} I(t\phi) < \frac{\pi}{2\alpha_0}\cdot\frac{2\mu}{\mu-2}.$$

□

5. Proof of the main theorem

First, we show the existence of a nontrivial, nonnegative critical point of the functional I. Let (v_n) be the Palais-Smale sequence given by (4.4). From Lemma 4.1 the sequence (v_n) is bounded in E. Then, we can find a subsequence (not renamed) and $v \in E$ such that v_n converges weakly to v in E. In particular,

$$\lim_{n\to+\infty}\int_{B_1} w(x)\nabla v_n\nabla\phi\,dx = \int_{B_1} w(v)\nabla v\nabla\phi\,dx, \quad \text{for all } \phi \in C^\infty_{0,rad}(B_1).$$

From Lemma 4.2, we have

$$\lim_{n\to+\infty}\int_{B_1} g(x, f(v_n))f'(v_n)\phi\, dx = \int_{B_1} g(x, f(v))f'(v)\phi\, dx, \quad \text{for all } \phi \in C^{\infty}_{0,rad}(B_1).$$

Since, $\|I'(v_n)\|_{E^*} \to 0$ as $n \to +\infty$, we have $I'(v)\phi = 0$ for all $\phi \in C^{\infty}_{0,rad}(B_1)$. Using the density of $C^{\infty}_{0,rad}(B_1)$ in E, we obtain $I'(v)\phi = 0$, for all $\phi \in E$. Hence, $v \in E$ is a critical point of I. Now, we prove that v is nontrivial. Suppose that $v_n \rightharpoonup 0$ in E. By Lemma 1.1, we have that

$$v_n \to 0, \quad \text{in } L^p(B_1) \quad \text{for all } p \geq 1. \tag{5.1}$$

Since (v_n) is a Palais-Smale sequence as in Lemma 4.1, we get

$$\Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|v_n\|^2 \leq I(v_n) - \frac{1}{\mu}I'(v_n)v_n.$$

From (4.5), the boundedness of (v_n) in E and Lemma 4.4, we can assume that

$$\|v_n\|^2 \leq \frac{\pi}{2\alpha_0} - 2\delta \quad \text{for all } n \geq 1,$$

for some $\delta > 0$. Moreover, we can take $\alpha > \alpha_0$ such that

$$\alpha\|v_n\|^2 \leq \frac{\pi}{2} - \delta \quad \text{for all } n \geq 1. \tag{5.2}$$

Using (g_1)–(g_4) and the fact that $|f'(t)| \leq 1$, we can find $\epsilon > 0$ and $C > 0$ such that

$$|g(x, f(v_n))f'(v_n)v_n| \leq \epsilon|f(v_n)v_n| + C\exp(e^{\alpha|f(v_n)|^4})|v_n|, \quad \text{for all } n \geq 1.$$

From Lemma 2.1-(ii) and (vi), we obtain

$$|g(x, f(v_n))f'(v_n)v_n| \leq \epsilon|v_n|^2 + C\exp(e^{2\alpha|v_n|^2})|v_n|, \quad \text{for all } n \geq 1.$$

Applying the Cauchy-Schwarz inequality

$$\int_{B_1} |g(x, f(v_n))f'(v_n)v_n| \leq \epsilon\|v_n\|_2^2 + C\|v_n\|_2\Big(\int_{B_1} \exp(2e^{2\alpha|v_n|^2})\, dx\Big)^{1/2}, \quad \text{for all } n \geq 1.$$

Then,

$$\int_{B_1} |g(x, f(v_n))f'(v_n)v_n|\, dx \leq \epsilon\|v_n\|_2^2 + C\|v_n\|_2\Big(\int_{B_1} \exp\Big(2e^{2\alpha\|v_n\|^2\big(\frac{|v_n|}{\|v_n\|}\big)^2}\Big)\, dx\Big)^{1/2}.$$

Using Proposition 1.1, (5.1) and (5.2), we have

$$\int_{B_1} |g(x, f(v_n))f'(v_n)v_n|\, dx \to 0, \quad \text{as } n \to +\infty. \tag{5.3}$$

Note that

$$d + o_n(1) = I(v_n) = \frac{1}{2}\int_{B_1} w(x)|\nabla v_n|^2\, dx - \int_{B_1} G(x, f(v_n))\, dx$$

and

$$o_n(1) = I'(v_n)v_n = \int_{B_1} w(x)|\nabla v_n|^2\, dx - \int_{B_1} g(x, f(v_n))f'(v_n)v_n\, dx.$$

From Lemmas 4.3 and 5.3, we obtain $d = 0$ which represents a contradiction. Therefore, v is a nontrivial critical point. Taking $v^- = \max\{0, -v\}$, in particular, we have that $-v^- \in E$. Hence, $I'(v)(-v^-) = 0$, since g is nonnegative and $f'(s) > 0$ for all $s \in \mathbb{R}$, we obtain

$$\|\nabla v^-\|^2 = -\int_{B_1} g(x, f(v))f'(v)v^-\, dx \leq 0.$$

Then, $v^- = 0$, which implies that $v \geq 0$. Finally, $u = f(v)$ result a nontrivial nonnegative weak solution of the problem (1.1).

6. Conclusions

In this paper, a dual approach has been applied in order to reformulate the proposed quasilinear equation into a semilinear equation that is suitable for using variational methods. Next, we used a Trudinger-Moser inequality to prove that the geometry of the pass mountain theorem is satisfied. Thus, we obtain a nontrivial, nonnegative weak solutions for the new energy functional. Finally, we recover the solution to our problem through the change of the variable used. To the best of our knowledge, this is the first result to demonstrate the existence of solutions for a quasiliner Schrödinger equation involving double exponential growth in the literature. Observe that we required a nonnegative weight. According to our definition of logarithm weight, we restricted the domain to the unit ball. It is of interest to further our results to find standing waves for quasilinear Schrödinger equations involving double exponential growth on the whole space $\mathbb{R}^2$ and considering a potential function.

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Conflict of interest

The author declares no conflicts of interest.

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