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Research article

On fractional state-dependent delay integro-differential systems under the Mittag-Leffler kernel in Banach space

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Abstract: The existence of Atangana-Baleanu fractional-order semilinear integro-differential systems and semilinear neutral integro-differential systems with state-dependent delay in Banach spaces is investigated in this paper. We establish the existence findings by using Monch's fixed point theorem and the concept of measures of non-compactness. A functioning example is provided at the end in order to illustrate the findings reached from the theoretical study.

Keywords: AB fractional-order derivative; mild solution; state-dependent delay; fixed point theorem (FPT)
Mathematics Subject Classification: 26A33, 34A08

1. Introduction

In this paper, we first study the existence of mild solutions of Atangana-Baleanu fractional-order semilinear integro-differential systems (ABFSIDSs) involving state-dependent delay (SDD) in the model

$$\mathscr{D}^{\omega}_{ABC} w(\tau) = Aw(\tau) + \mathcal{F}\left(\tau, w_{\sigma(\tau,w_\tau)}, \int_0^\tau p(\tau, s, w_{\sigma(s,w_s)})ds\right), \quad \tau \in \mathscr{J} = [0, +\infty), \tag{1.1}$$

$$w(\tau) = \varphi(\tau) \in \mathfrak{C}_h, \tag{1.2}$$

where $0 < \omega < 1$ and A is the infinitesimal generator of the ω-resolvent family $\widehat{\mathscr{H}_\omega}(\tau)_{\tau\geq 0}$. $\mathscr{H}_\omega(\tau)_{\tau\geq 0}$ is the solution operator on a real Banach space $(E, |\cdot|)$. The Atangana-Baleanu-Caputo (ABC) derivative is denoted by $\mathscr{D}^{\omega}_{ABC}$, $\mathcal{F} : \mathscr{J} \times \mathfrak{C}_h \times E \to E$, $p : \mathcal{D} \times \mathfrak{C}_h \to E$, where $\mathcal{D} = \{(\tau, s) \in \mathscr{J} \times \mathscr{J} : 0 \leq s \leq \tau < +\infty\}$, and $\mathfrak{C}_h$ is the phase space to be described in preliminaries and $\sigma : \mathscr{J} \times \mathfrak{C}_h \to (-\infty, +\infty)$ is a given function which satisfies certain assumptions to be specified later on.

We make the assumption that the function $w_\tau : (-\infty, 0] \to E, w_\tau(\delta) = w(\tau + \delta),\ \delta \leq 0$, belongs to an abstract phase space $\mathfrak{C}_h$.

In the second part of this paper, we also establish the existence results of Atangana-Baleanu fractional-order semilinear neutral integro-differential systems (ABFSNIDSs) with SDD of the form

$$\mathscr{D}^{\omega}_{ABC}[w(\tau) - \mathscr{P}(\tau, w_{\sigma(\tau,w_\tau)})] = Aw(\tau) + \mathcal{F}\left(\tau, w_{\sigma(\tau,w_\tau)}, \int_0^\tau p(\tau, s, w_{\sigma(s,w_s)})ds\right), \quad \tau \in \mathscr{J} = [0, +\infty), \tag{1.3}$$

$$w(\tau) = \varphi(\tau) \in \mathfrak{C}_h, \tag{1.4}$$

where $\mathscr{P} : \mathscr{J} \times \mathfrak{C}_h \to E$ is a given function and the other functions specified in (1.3)–(1.4) are the same as defined in (1.1)–(1.2).

The usefulness of fractional differential and integral equations has increased significantly as a result of their extensive use in the modelling of physical processes and events, particularly anomalous systems with memory (processes with long-range interactions and long-term memory). There are many distinct kinds of fractional derivatives, including but not limited to Riemann–Liouville fractional derivatives, Caputo fractional derivatives, and Hadamard fractional derivatives; each of these types of fractional derivatives has its own specific constraints. For more information on how these fractional operators have been applied in many mathematical situations, see for example [1, 8, 12, 15–17, 26, 30–32, 36–40].

Over the years, researchers have come a long way in their pursuit for better fractional differential operators. An alternate definition for fractional derivatives has just been put out by Caputo and Fabrizio [18], and it is as follows:

$${}^{CF}\mathscr{D}^{\omega}_{d^+}h(\tau) = \frac{B(\omega)}{1-\omega}\frac{d}{d\tau}\int_d^\tau \exp\left[-\frac{\omega}{1-\omega}(\tau - \nu)\right]h(\nu)d\nu, \quad 0 < \omega < 1,$$

where $B(0) = B(1) = 1$ and $B(\omega)$ is called a normalization function. The authors of this work decided to use an exponential kernel in place of the more common power law kernel $(\tau - \nu)^{\omega-1}$. This may be written as $\exp\left[-\frac{\omega}{1-\omega}(\tau - \nu)\right]$. In addition to this, $\frac{1}{\Gamma(1-\omega)}$ was changed to $\frac{1}{\sqrt{2\pi(1-\omega^2)}}$. This current concept not only provides a better explanation of the dynamics of a non-local phenomenon, but it also effectively replies to the question of whether it is possible to have a fractional operator that possesses a non-singular kernel (see [2, 4, 6, 7, 14, 35], and their references for more information). Interestingly, this fractional differential operator has been useful in modelling some physical phenomena, such as in HIV/AIDS with the treatment compartment model [42] and in RC-electrical circuits [5].

The novel Atangana-Baleanu (AB) fractional derivative was recently proposed by Atangana and Baleanu [9] in both the Riemann-Liouville and Caputo meanings. The generalized Mittag-Leffler function is utilized in the form of a kernel in this specific derivation. The extended Mittag-Leffler function's non-local behavior makes it possible to more accurately describe the macroscopic behavior and memory effects of systems with non-local interactions. Additionally, the fractional derivative developed by Atangana and Baleanu retains all the properties that were previously recognized to be connected to other fractional derivatives. This makes it crucial to study fractional systems utilizing AB fractional derivatives; for example, see [33, 36, 37, 47, 50].

Functional differential equations with SDD typically emerge in applications as models of equations. As a result, research into this class of equations has garnered a lot of attention in recent years. This can be attributed to the fact that these types of equations appear frequently in applications. For more details on the applications of this theory, we suggest the reader to refer to [15, 23, 39, 43, 44, 49]. Systems with memory have the characteristic that the mathematical physics description of their state at a certain moment includes those states that the systems had been in at earlier times. As a result, a partial integro-differential equation is produced by adding an integral component to the fundamental partial differential equation. The concept of an "aftereffect", which was first proposed in physics, is widely acknowledged to be crucial. It is not sufficient to use conventional or partial differential equations to represent processes having an "aftereffect". Using integro-differential equations is one method for solving this issue. Conceptually, modelling climatic systems has been done effectively in the past using delay differential equations (DDEs). The existence of feedback loops with a delay period, often related to the amount of time needed to move energy across the world via oceans and/or the atmosphere, is a crucial component of these models. As of now, it is generally believed that these delays are constant. Recent research has shown that even simple DDEs with non-constant delay times—that is, delay times that change depending on the state of the system—can create dynamic behaviour that is surprisingly complex. The notion of heat conduction in fading memory materials gives rise to partial neutral differential equations with non-constant delay (or SDD) like (1.3)–(1.4). The internal energy and the heat flux are thought to depend linearly on the temperature $u(\cdot)$ and its gradient $\Delta u(\cdot)$ in the conventional theory of heat conduction. In these circumstances, the classical heat equation provides an adequate description of the temperature development in many types of materials. This explanation, however, falls short when it comes to fading memory materials. This phenomenon has commonly been described by using the following equation

$$\frac{d^{\omega}}{d\tau^{\omega}}\left[u(\tau,p)+\int_{-\infty}^{\tau}k_1(\tau-s)u(s,p)ds\right]=c\Delta u(\tau,p)+\int_{-\infty}^{\tau}k_2(\tau-s)\Delta u(s,p)ds,\quad 0<\omega<1,$$
$$u(\tau,p)=0,\quad p\in\partial\Omega.$$

In this system, $\Omega\subset\mathbb{R}^n$ is open, bounded and has a smooth boundary, $(\tau,p)\in[0,\infty)\times\Omega$, $u(\tau,p)$ represents the temperature in p at the time τ, c is a physical constant and $k_i:\mathbb{R}\to\mathbb{R}$, $i=1,2$, are the internal energy and the heat flux relaxation respectively. If we assume that a solution $u(\cdot)$ of the above system is known on $(-\infty,0]$ and that $u(\cdot)$ is smooth enough, we can re-write the above system in the form of

$$\begin{aligned}\frac{d^{\omega}}{d\tau^{\omega}}&\left[u(\tau,p)+\int_{-\infty}^{\tau}k_1(\tau-s)u\left(s,p(s-\theta(s,p(s)))\right)ds\right]=c\Delta u(\tau,p)\\&+\int_{-\infty}^{\tau}k_2(\tau-s)\Delta u\left(s,p(s-\theta(s,p(s))),\int_0^s e(s,\eta,p(\eta-\theta(\eta,p(\eta))))d\eta\right)ds,\\u(\tau,p)&=0,\quad p\in\partial\Omega,\end{aligned}$$

where $\theta(\tau,p(\tau))\geq 0$ is a given function. It is easy to see that this system can be represented in the abstract form given by (1.3)–(1.4).

Due to the extensive usage highlighted in many domains (for example see [11, 25, 33, 34, 36–39, 47, 50]) the study on qualitative behaviors of AB fractional differential equations has recently been published in the literature. In [34, 47], the authors studied the existence and uniqueness results for fractional neutral integro-differential equations with an AB derivative under the Banach contraction principle. Further, they developed the Ulam-Hyer stability of the addressing systems. In [33], the authors analyzed the existence results for AB fractional differential equations with non-instantaneous impulses. Results are obtained via non-compactness of the semigroup and fixed point theory. Then, in [38], the authors extended the works of [33] to integro-differential systems in Banach spaces through the use of Kuratowski's measure of non-compactness (KNMC) of the semigroup and fixed point theory. Later, in [11], the author analyzed the existence results for AB fractional neutral systems with non-instantaneous impulses through measures of non-compactness and the K-set contraction principle. As further research, in [37], the authors studied the existence of solutions of AB fractional Volterra-Fredholm integro-differential inclusions through Martelli's fixed point theorem. They generalized the results of [33, 38]. In [36], the authors initiated a study of the existence results of AB fractional neutral differential systems with infinite delay via Banach contraction principle, nonlinear alternative of Leray-Schauder type and Krasnoselskii-Schaefer fixed point theorem joined with ρ-resolvent operators in Banach spaces. In their work, the authors improved and generalized the existing results of [33,34,38,47]. Later, in [50], the authors extended the existence results into the controllability of AB semilinear fractional integro-differential equations with non-instantaneous impulses. The outcomes are proved through Darbo's fixed point theorem. Then, in [25], the authors studied the approximate controllability results of AB neutral fractional stochastic hemivariational inequality under a suitable fixed point theorem on multivalued maps. Recently, Mallika Arjunan et al. [39] examined the existence results of AB fractional fractional differential inclusions with SDD and a Mittag-Leffler kernel in Banach space. The results were obtained by using contractive and condensing maps. A study of recent work shows that the problem of fractional integro-differential equations with the form of (1.1)–(1.4) with AB derivatives and Monch's fixed point theorem has not yet been addressed. This is the main motivation for this work.

We will now proceed to a description of the work. Section 2 presents the concept of phase space axioms, measures of non-compactness, and the AB fractional derivative, as well as several notations and a review of some concepts and previous findings. The results are based on Monch's fixed point theorem, which we give in Section 3. In Section 4, we provide an example to illustrate the validity of our primary findings.

2. Preliminaries

In this section, we will briefly discuss a few lemmas, definitions of fractional differential equations [3, 9, 36, 37, 39], Hale and Kato axioms [29] and KMNC, which will be applied throughout the remainder of the paper.

Let $\mathcal{BUC} : (-\infty, 0] \to E$ be the space of bounded uniformly continuous functions.

Let $\mathcal{BC} : (-\infty, +\infty) \to E$ be the Banach space of all bounded and continuous functions (BCFs) equipped with the general norm

$$\|w\|_{\mathcal{BC}} = \sup_{\tau\in(-\infty,+\infty)} |w(\tau)|.$$

Lastly, let $\mathcal{BC}' : [0, +\infty) \to E$ be the Banach space of all BCFs equipped with the general norm

$$\|w\|_{\mathcal{BC}'} = \sup_{\tau\in[0,+\infty)} |w(\tau)|.$$

Denote Y_τ and Y'_τ as the space of all BCFs of $\mathcal{BC}$ and $\mathcal{BC}'$ respectively.

It is important to note that when the delay reaches an infinite value, we should constructively discuss the theoretical phase space $\mathfrak{C}_h$. We discuss phase spaces $\mathfrak{C}_h$ in this paper, which are identical to those defined in our previous work [19]. In light of this, we will not go into further detail.

As a direct result of the work done by Hale and Kato [28, 29], we assume that the phase space $(\mathfrak{C}_h, \|\cdot\|_{\mathfrak{C}_h})$ will be a semi-normed linear space of functions mapping $(-\infty, 0]$ into E and meeting the subsequent elementary axioms.

If $w : (-\infty, b) \to E, b > 0$ in a way that $w_0 \in \mathfrak{C}_h$, for every $\tau \in \mathscr{J}$, the following presumptions should be valid:

(C1) $w_\tau \in \mathfrak{C}_h$.
(C2) $\|w_\tau\|_{\mathfrak{C}_h} \leq Q_1(\tau) \sup\limits_{0\leq y\leq\tau} \|w(y)\| + Q_2(\tau) \|w_0\|_{\mathfrak{C}_h}$.
(C3) $\|w(\tau)\| \leq \overline{Z} \|w_\tau\|_{\mathfrak{C}_h}$, where $\overline{Z} > 0$ is a constant and $Q_1 : [0, \infty) \to [0, \infty)$ is continuous, $Q_2 : [0, \infty) \to [0, \infty)$ is locally bounded and Q_1 and Q_2 are independent of $w(\cdot)$. In addition to this, $\|\varphi(0)\| \leq \overline{Z}\|\varphi\|_{\mathfrak{C}_h}$ can be deduced for all $\varphi \in \mathfrak{C}_h$. For further information, refer to [22].
(C4) w_τ is a $\mathfrak{C}_h$-valued continuous function on $\mathscr{J}$ and $\mathfrak{C}_h$ is complete.
(C5) The function $\tau \to \varphi_\tau$ is well described and continuous from the set

$$\mathcal{R}(\sigma^-) = \{\sigma(s, \varphi) : (s, \varphi) \in \mathscr{J} \times \mathfrak{C}_h, \sigma(s, \varphi) \leq 0\}$$

into $\mathfrak{C}_h$ and there is a BCF $Z^\varphi : \mathcal{R}(\sigma^-) \to (0, \infty)$ to confirm that $\|\varphi_\tau\|_{\mathfrak{C}_h} \leq Z^\varphi(\tau) \|\varphi\|_{\mathfrak{C}_h}$ for each $\tau \in \mathcal{R}(\sigma^-)$.

For simplicity, we denote

$$Q_1^* = \sup\{Q_1(\tau) : \tau \in \mathscr{J}\} \quad \text{and} \quad Q_2^* = \sup\{Q_2(\tau) : \tau \in \mathscr{J}\}.$$

Definition 2.1. A function $\mathcal{F} : \mathscr{J} \times \mathfrak{C}_h \to E$ is said to be of the Caratheodory if

(i) $\tau \to \mathcal{F}(\tau, \nu)$ is measurable for all $\nu \in \mathfrak{C}_h$;
(ii) $\nu \to \mathcal{F}(\tau, \nu)$ is continuous for almost each $\tau \in \mathscr{J}$.

Lemma 2.2. *[43] Let $w : (-\infty, +\infty) \to E$ be a function in such a way that $w_0 = \varphi$ and if (C5) holds, then*

$$\|w_\nu\|_{\mathfrak{C}_h} \leq \left(Q_2^* + Z^\varphi\right) \|\varphi\|_{\mathfrak{C}_h} + Q_1^* \sup\{\|w(\Theta)\|_E : \Theta \in [0, \max\{0, \nu\}]\} \qquad \nu \in \mathcal{R}(\sigma^-) \cup \mathscr{J},$$

where $Z^\varphi = \sup\limits_{\tau\in\mathcal{R}(\sigma^-)} Z^\varphi(\tau)$.

Now, let us review some of the fundamental aspects of the concept of the AB fractional derivative.

Definition 2.3. [9] The AB fractional integral of order $\omega \in (0, 1)$ of a function $w : (d, \xi) \to \mathbb{R}$ is defined by

$${}^{AB}I^{\omega}_{d^+} w(\tau) = \frac{1-\omega}{B(\omega)} w(\tau) + \frac{\omega}{B(\omega)\Gamma(\omega)} \int_d^\tau (\tau - y)^{\omega-1} w(y) dy,$$

where the normalizing function $B(\omega) = (1 - \omega) + \frac{\omega}{\Gamma(\omega)}$ is the one that satisfies the criterion $B(0) = B(1) = 1$.

Definition 2.4. [9] Let $w \in H^1(0, +\infty)$, $0 < +\infty$. The AB fractional derivative of a function w of order $\omega \in (0, 1)$, is described in the Caputo sense with the base 0 at $\tau \in (0, +\infty)$ by

$$\mathscr{D}^{\omega}_{ABC} w(\tau) = \frac{B(\omega)}{1-\omega} \int_0^{\tau} w'(s) E_{\omega} \Big(-\frac{\omega}{1-\omega} (\tau - y)^{\omega} \Big) dy.$$

The Mittag-Leffler function is denoted by $E_{\omega}(\cdot)$ in this expression.

Definition 2.5. [45] Consider $\rho(A) = \{\lambda \in \mathbb{C} : (\lambda I - A) : D(A) \to E$ is bijective $\}$. The spectrum of A is described as the complement $\mathbb{C} \backslash \rho(A)$ denoted by $\sigma(A)$. By the closed graph theorem, for a bounded linear operator A on E, the resolvent of the operator A is defined by $R(\lambda, A) = (\lambda I - A)^{-1}$, where $\lambda \in \rho(A)$.

Definition 2.6. [45] Closed and linear operator A is said to be sectorial if there exist constants $M > 0, \overline{\omega} \in \mathbb{R}$ and $\alpha \in \left[\frac{\pi}{2}, \pi\right]$, such that the following conditions are satisfied:

(i) $\Sigma_{(\alpha,\overline{\omega})} = \{\lambda \in \mathbb{C} : \lambda \neq \overline{\omega}, |\arg(\eta - \overline{\omega})| < \alpha\} \subset \rho(A)$,
(ii) $\|R(\lambda, A)\| \leq \frac{M}{|\lambda - \omega|}, \lambda \in \Sigma_{(\alpha,\overline{\omega})}$.

To avoid repetitions of various terminology used in this work, we recommend that the readers refer to [27, 36, 37]: sectorial operator [27] and solution operator (see Definitions 2.6 and 2.7 in [36]). We recommend that the reader read [3,9,33,46,48] for further information on this topic and its applications.

The idea that there exist measures of non-compactness is the foundation for more than one of our conclusions. Keeping this in mind, let us review some of the characteristics that are associated with this idea. When looking for essential knowledge, the reader should refer to [13, 24, 49]. We employ KMNCs exclusively throughout this paper.

Definition 2.7 ([13, 24] (KMNC)). Let a family of bounded subsets of E be denoted by the symbol $V(E)$. If this is the case, then $\chi : V(E) \to R_+$ can be characterized as

$$\chi(C) := \inf\{\delta > 0 : C = \cup_{\ell=1}^{n} C_{\ell} \text{ with } \operatorname{diam}(C_{\ell}) \leq \delta \text{ for } \ell = 1, 2, \ldots, n\}, \quad C \in V(E).$$

Lemma 2.8 ([13,24]). *We establish the following results for any bounded sets C, C_1, and C_2 containing the element E:*

(i) $\chi(C) = 0$ if and only if C is a compact set in E;
(ii) $\chi(C) = \chi(\overline{C})$, where $\overline{C}$ means the closure of C;
(iii) Each $C_1 \subset C_2$ implies $\chi(C_1) \leq \chi(C_2)$;
(iv) $\chi(C_1 + C_2) \leq \chi(C_1) + \chi(C_2)$;
(v) $\chi(C_1 \cup C_2) = \max\{\chi(C_1), \chi(C_2)\}$;
(vi) $\chi(\mu C) = |\mu| \chi(C)$ for any $\mu \in \mathbb{R}$.

Lemma 2.9. *[20] If $C \subset E$ is bounded for a Banach space E, then a countable subset $C_0 \subset C$ exists, for which $\chi(C) \leq 2\chi(C_0)$ exists.*

Lemma 2.10 ([20,21]). *Let E be a Banach space, and let $C = \{w_n\} \subset \mathscr{C}([\tau_1, \tau_2], E)$ be a bounded and countable set for the constants $-\infty < \tau_1 < \tau_2 < +\infty$. Then $\chi(C(\tau))$ is a Lebesgue integral on $[\tau_1, \tau_2]$ and*

$$\chi\Big(\Big\{ \int_{\tau_1}^{\tau_2} w_n(\tau) d\tau : n \in \mathbb{N} \Big\}\Big) \leq 2 \int_{\tau_1}^{\tau_2} \chi(C(\tau)) d\tau.$$

Given [36, Lemma 1], we will now describe the mild solution of the systems (1.1)–(1.2) and (1.3)–(1.4).

Definition 2.11. A function $w : (-\infty, +\infty) \to E$ is said to be a mild solution of the system (1.1)–(1.2) when the following conditions are satisfied: $w_0 = \varphi \in \mathfrak{C}_h$ on $(-\infty, 0]$ and the limitation of $w(\cdot)$ to $[0, +\infty)$ is continuous and satisfies the subsequent equation:

$$w(\tau) = \begin{cases} \varphi(\tau), \quad \tau \in (-\infty, 0] \\ \mathbb{E}\mathscr{H}_\vartheta(\tau)\varphi(0) + \dfrac{\mathbb{E}\mathbb{F}(1-\omega)}{B(\omega)\Gamma(\omega)} \displaystyle\int_0^\tau (\tau - s)^{\omega-1} \mathcal{F}\left(s, w_{\sigma(s,w_s)}, \int_0^s p(s, \nu, w_{\sigma(\nu,w_\nu)})d\nu\right) ds \\ + \dfrac{\omega\mathbb{E}^2}{B(\omega)} \displaystyle\int_0^\tau \widehat{\mathscr{H}_\omega}(\tau - s)\mathcal{F}\left(s, w_{\sigma(s,w_s)}, \int_0^s p(s, \nu, w_{\sigma(\nu,w_\nu)})d\nu\right) ds, \quad \tau \in \mathscr{J}, \end{cases} \tag{2.1}$$

where $\mathbb{E} = \eta(\eta I - A)^{-1}$ and $\mathbb{F} = -\widetilde{\eta}A(\eta I - A)^{-1}$ with $\eta = \frac{B(\omega)}{1-\omega}$, $\widetilde{\eta} = \frac{\omega}{1-\omega}$ and

$$\mathscr{H}_\omega(\tau) = E_\omega(-\mathbb{F}\tau^\omega) = \frac{1}{2\pi i}\int_\Gamma e^{y\tau} y^{\omega-1}(y^\omega I - \mathbb{F})^{-1} dy, \tag{2.2}$$

$$\widehat{\mathscr{H}_\omega}(\tau) = \tau^{\omega-1}E_{\omega,\omega}(-\mathbb{F}\tau^\omega) = \frac{1}{2\pi i}\int_\Gamma e^{y\tau}(y^\omega I - \mathbb{F})^{-1} dy; \tag{2.3}$$

Γ denotes the Bromwich path [10].

Definition 2.12. A function $w : (-\infty, +\infty) \to E$ is said to be a mild solution of the system (1.3)–(1.4) when the following conditions are satisfied: $w_0 = \varphi \in \mathfrak{C}_h$ on $(-\infty, 0]$ and the limitation of $w(\cdot)$ to $[0, +\infty)$ is continuous and satisfies the subsequent equation:

$$w(\tau) = \begin{cases} \varphi(\tau), \quad \tau \in (-\infty, 0] \\ \mathbb{E}\mathscr{H}_\vartheta(\tau)[\varphi(0) - \mathscr{P}(0, \varphi)] \\ + \dfrac{\mathbb{E}\mathbb{F}(1-\omega)}{B(\omega)\Gamma(\omega)} \displaystyle\int_0^\tau (\tau - s)^{\omega-1} \mathcal{F}\left(s, w_{\sigma(s,w_s)}, \int_0^s p(s, \nu, w_{\sigma(\nu,w_\nu)})d\nu\right) ds \\ + \dfrac{\mathbb{E}\mathbb{F}}{\Gamma(\omega)} \displaystyle\int_0^\tau (\tau - s)^{\omega-1} \mathscr{P}(s, w_{\sigma(s,w_s)}) ds \\ + \dfrac{\omega\mathbb{E}^2}{B(\omega)} \displaystyle\int_0^\tau \widehat{\mathscr{H}_\omega}(\tau - s)\mathcal{F}\left(s, w_{\sigma(s,w_s)}, \int_0^s p(s, \nu, w_{\sigma(\nu,w_\nu)})d\nu\right) ds \\ - \mathbb{E}\mathbb{F} \displaystyle\int_0^\tau \widehat{\mathscr{H}_\omega}(\tau - s)\mathscr{P}(s, w_{\sigma(s,w_s)}) ds, \quad \tau \in \mathscr{J}, \end{cases} \tag{2.4}$$

where $\mathbb{E} = \eta(\eta I - A)^{-1}$ and $\mathbb{F} = -\widetilde{\eta}A(\eta I - A)^{-1}$ with $\eta = \frac{B(\omega)}{1-\omega}$, $\widetilde{\eta} = \frac{\omega}{1-\omega}$ and

$$\mathscr{H}_\omega(\tau) = E_\omega(-\mathbb{F}\tau^\omega) = \frac{1}{2\pi i}\int_\Gamma e^{y\tau} y^{\omega-1}(y^\omega I - \mathbb{F})^{-1} dy,$$

$$\widehat{\mathscr{H}_\omega}(\tau) = \tau^{\omega-1}E_{\omega,\omega}(-\mathbb{F}\tau^\omega) = \frac{1}{2\pi i}\int_\Gamma e^{y\tau}(y^\omega I - \mathbb{F})^{-1} dy;$$

Γ denotes the Bromwich path [10].

Remark 2.13. We must first define the operator estimates that are addressed in (2.2) and (2.3), respectively, in order to examine and demonstrate the main conclusions of this paper.

As a direct result of our previous work [36], we are able to write the operator estimates as

$$\|\mathscr{H}_\omega(\tau)\| \leq \widehat{C}_{\mathscr{H}} \quad \text{and} \quad \|\widehat{\mathscr{H}_\omega}(\tau)\| \leq \tau^{\omega-1}\widehat{C}_{\widehat{\mathscr{H}}}.$$

Please refer to [27,33,48] for further information.

At the conclusion of this section, we present the significant fixed-point theorem, which is a particularly helpful tool for presenting our findings [15,41].

Theorem 2.14. *(Monch fixed point)*

Let us assume that B is a bounded, closed, and convex subset of a Banach space such that $0 \in B$, and let us further assume that Υ is a continuous mapping of B into itself. Then Υ has a fixed point, if the assumption

$$K = \overline{conv}\,\Upsilon(K) \quad or \quad K = \Upsilon(K) \cup 0 \implies \chi(K) = 0$$

holds for every subset K of B.

3. Existence results for ABFSDSs and ABFSNDSs

This section provides and demonstrates the existence results for the models (1.1)–(1.2) and (1.3)–(1.4) in accordance with Monch's fixed-point theorem.

We will start by imposing some essential restrictions on $p, \mathcal{F}, \mathbb{E}$, and $\mathbb{F}$.

(A0) The function $p : \mathcal{D} \times \mathfrak{C}_h \to E$ fulfills the following:

(i) For every $(\tau, s) \in \mathcal{D}$, the function $p(\tau, s, \cdot) : \mathfrak{C}_h \to E$ is continuous and for each $u \in \mathfrak{C}_h$, the function $p(\cdot, \cdot, u) : \mathcal{D} \to E$ is strongly measurable.

(ii) There exists an integrable function $\Omega_1 : \mathscr{J} \to [0, \infty)$ to ensure that

$$|p(\tau, s, u)| \leq \Omega_1(\tau)\|u\|_{\mathfrak{C}_h} \quad \text{for a.e. } \tau, s \in \mathscr{J},\ u \in \mathfrak{C}_h.$$

Assume that the finite bound of $\int_0^\tau \Omega_1(s)ds$ is $\widetilde{P}_0$.

(iii) There exists an integrable function $\overline{\mu} : \mathscr{J} \times \mathscr{J} \to (0, +\infty)$ to ensure that

$$\chi(p(\tau, s, C_1)) \leq \overline{\mu}(\tau, s)\left[\sup_{-\infty < x \leq 0} \chi(C_1(x))\right] \quad \text{for a.e. } \tau, s \in \mathscr{J},$$

where $C_1(x) = \{v(x) : v \in C_1\}$; χ is the KMNC and denote $\widetilde{\mu}^* = \int_0^s \overline{\mu}(s, v)dv < \infty$.

(A1) The Caratheodory function $\mathcal{F} : \mathscr{J} \times \mathfrak{C}_h \times E \to E$ fulfills the subsequent assumptions: There exists an integrable function $\Omega : (-\infty, +\infty) \to [0, +\infty)$ in a way that:

$$|\mathcal{F}(\tau, u, v)| \leq \Omega(\tau)(\|u\|_{\mathfrak{C}_h} + \|v\|), \quad \tau \in \mathscr{J},\ u \in \mathfrak{C}_h,\ v \in E$$

and

$$\Omega^* := \sup_{\tau \in \mathscr{J}} \int_0^\tau (\tau - s)^{\omega-1}\Omega(s)ds < \infty.$$

(A2) Let $C_2 \subset \mathfrak{C}_h, \overline{C} \subset E$ and each $\tau \in \mathscr{J}$; we have

$$\chi(\mathcal{F}(\tau, C_2, \overline{C})) \leq \Omega(\tau)\left[\sup_{-\infty<x\leq 0} \chi(C_2(x)) + \chi(\overline{C})\right],$$

where $C_2(x) = \{u(x) : u \in C_2\}$.

(A3) $\mathbb{E}$ and $\mathbb{F}$ are the bounded linear operators and there exist positive constants $\mathbb{E}$ and $\mathbb{F}$ such that $\|\mathbb{E}\| \leq \rho$ and $\|\mathbb{F}\| \leq \overline{\rho}$.

Theorem 3.1. *If the assumptions (A0)–(A3) hold and*

$$\widehat{M} = \left[2(1+\widetilde{\mu}^*)\left(\frac{\rho\overline{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)} + \frac{\omega\rho^2\widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)}\right)\Omega^*\right] < 1, \tag{3.1}$$

then the system (1.1)–(1.2) *has at least one mild solution on* Y_τ.

Proof. The operator $\Upsilon : Y_\tau \to Y_\tau$ is defined as follows:

$$(\Upsilon w)(\tau) = \begin{cases} \varphi(\tau), \quad \tau \leq 0, \\ \mathbb{E}\mathscr{H}_\vartheta(\tau)\varphi(0) + \dfrac{\mathbb{E}\mathbb{F}(1-\omega)}{B(\omega)\Gamma(\omega)} \displaystyle\int_0^\tau (\tau-s)^{\omega-1}\mathcal{F}\left(s, w_{\sigma(s,w_s)}, \int_0^s p(s,\nu,w_{\sigma(\nu,w_\nu)})d\nu\right)ds \\ +\dfrac{\omega\mathbb{E}^2}{B(\omega)} \displaystyle\int_0^\tau \widehat{\mathscr{H}_\omega}(\tau-s)\mathcal{F}\left(s, w_{\sigma(s,w_s)}, \int_0^s p(s,\nu,w_{\sigma(\nu,w_\nu)})d\nu\right)ds, \qquad \tau \in \mathscr{J}. \end{cases}$$

Let $u(\cdot) : (-\infty, +\infty) \to E$ be the function described by

$$u(\tau) = \begin{cases} \varphi(\tau), & \tau \leq 0 \\ \mathbb{E}\mathscr{H}_\omega(\tau)\varphi(0), & \tau \in \mathscr{J}. \end{cases}$$

Then $u_0 = \varphi$. Let $v \in \mathscr{C}(\mathscr{J}, E)$ with $v_0 = 0$; we denote by $\overline{v}$ the function given by

$$\overline{v}(\tau) = \begin{cases} 0, & \tau \leq 0 \\ v(\tau), & \tau \in \mathscr{J}. \end{cases}$$

If $w(\cdot)$ satisfies (2.1), we can decompose $w(\cdot)$ as $w(\tau) = v(\tau) + u(\tau)$, $\tau \geq 0$ which implies $w_\tau = v_\tau + u_\tau$ and the function $v(\cdot)$ satisfies

$$v(\tau) = \begin{cases} \dfrac{\mathbb{E}\mathbb{F}(1-\omega)}{B(\omega)\Gamma(\omega)} \displaystyle\int_0^\tau (\tau-s)^{\omega-1}\mathcal{F}\left(s, v_{\sigma(s,v_s+u_s)} + u_{\sigma(s,v_s+u_s)}, \int_0^s p(s,\nu,v_{\sigma(\nu,v_\nu+u_\nu)} + u_{\sigma(\nu,v_\nu+u_\nu)})d\nu\right)ds \\ +\dfrac{\omega\mathbb{E}^2}{B(\omega)} \displaystyle\int_0^\tau \widehat{\mathscr{H}_\omega}(\tau-s)\mathcal{F}\left(s, v_{\sigma(s,v_s+u_s)} + u_{\sigma(s,v_s+u_s)}, \int_0^s p(s,\nu,v_{\sigma(\nu,v_\nu+u_\nu)} + u_{\sigma(\nu,v_\nu+u_\nu)})d\nu\right)ds, \\ \tau \in \mathscr{J}. \end{cases}$$

Let $Y_\tau^0 = \{v \in Y'_\tau : v_0 = 0 \in \mathfrak{C}_h\}$. Let $v \in Y_\tau^0$; then

$$\|v\|_{Y_\tau^0} = \|v_0\|_{\mathfrak{C}_h} + \sup\{|v(x)| : 0 \leq x < +\infty\} = \sup\{|v(x)| : 0 \leq x < +\infty\}.$$

Thus $(Y^0_\tau, \|\cdot\|_{Y^0_\tau})$ is a Banach space. Next, the operator $\overline{\Upsilon}: Y^0_\tau \to Y^0_\tau$ is defined by:

$$(\overline{\Upsilon}v)(\tau) = \begin{cases} \dfrac{\mathbb{E}\mathbb{F}(1-\omega)}{B(\omega)\Gamma(\omega)} \displaystyle\int_0^\tau (\tau - s)^{\omega-1} \\ (\times)\mathcal{F}\left(s, v_{\sigma(s,v_s+u_s)} + u_{\sigma(s,v_s+u_s)}, \displaystyle\int_0^s p(s,\nu, v_{\sigma(\nu,v_\nu+u_\nu)} + u_{\sigma(\nu,v_\nu+u_\nu)})d\nu\right)ds \\ +\dfrac{\omega\mathbb{E}^2}{B(\omega)} \displaystyle\int_0^\tau \widehat{\mathscr{H}_\omega}(\tau - s) \\ (\times)\mathcal{F}\left(s, v_{\sigma(s,v_s+u_s)} + u_{\sigma(s,v_s+u_s)}, \displaystyle\int_0^s p(s,\nu, v_{\sigma(\nu,v_\nu+u_\nu)} + u_{\sigma(\nu,v_\nu+u_\nu)})d\nu\right)ds, \quad \tau \in \mathscr{J}. \end{cases}$$

The operator Υ has a fixed point equal to one $\overline{\Upsilon}$, so it is time to prove that $\overline{\Upsilon}$ has a fixed point as well.

In order to demonstrate the result, we must initially get an approximation of the phase space axioms. As a result of the phase space axioms and Lemma 2.2, we have for every $t \in \mathscr{J}$,

$$\begin{aligned} \|v_{\sigma(\tau,v_\tau+u_\tau)} + u_{\sigma(\tau,v_\tau+u_\tau)}\|_{\mathfrak{C}_h} &\le \|v_{\sigma(\tau,v_\tau+u_\tau)}\|_{\mathfrak{C}_h} + \|u_{\sigma(\tau,v_\tau+u_\tau)}\|_{\mathfrak{C}_h} \\ &\le Q_1(\tau)|v(\tau)| + Q_2(\tau)\|v_0\|_{\mathfrak{C}_h} + Q_1(\tau)\|u(\tau)\|_E + Q_2(\tau)\|u_0\|_{\mathfrak{C}_h} \\ &\le Q_1(\tau)|v(\tau)| + Q_1(\tau)[\|\mathbb{E}\mathscr{H}_\omega(\tau)\varphi(0)\|_E] + (Q_2(\tau) + J^\varphi)\|\varphi\|_{\mathfrak{C}_h} \\ &\le Q_1^*|v(\tau)| + Q_1^*\rho\widehat{C}_{\mathscr{H}}\overline{Z}\|\varphi\|_{\mathfrak{C}_h} + (Q_2^* + Z^\varphi)\|\varphi\|_{\mathfrak{C}_h} \\ &= Q_1^*|v(\tau)| + (Q_1^*\rho\widehat{C}_{\mathscr{H}}\overline{Z} + Q_2^* + Z^\varphi)\|\varphi\|_{\mathfrak{C}_h}. \end{aligned}$$

Then, we have

$$\|v_{\sigma(\tau,v_\tau+u_\tau)} + u_{\sigma(\tau,v_\tau+u_\tau)}\|_{\mathfrak{C}_h} \le e + Q_1^*|v(\tau)| = e + Q_1^* q = Q', \tag{3.2}$$

where $e = (Q_1^*\rho\widehat{C}_{\mathscr{H}}\overline{Z} + Q_2^* + Z^\varphi)\|\varphi\|_{\mathfrak{C}_h}$ and $|v(\tau)| = \|v\|_{Y^0_\tau} \le q$.

We shall show that the operator $\overline{\Upsilon}$ satisfies all conditions of Monch's theorem. For better readability, we break the proof into several steps.

Step 1: $\overline{\Upsilon}$ maps Y^0_τ into Y^0_τ.

Evidently the map $\overline{\Upsilon}(v)$ is continuous on $[0, +\infty)$ for any $v \in Y^0_\tau$ and for every $\tau \in \mathscr{J}$, we have

$$\begin{aligned} |(\overline{\Upsilon}v)(\tau)| \le\ & \frac{\|\mathbb{E}\|\|\mathbb{F}\|(1-\omega)}{B(\omega)\Gamma(\omega)} \int_0^\tau (\tau - s)^{\omega-1} \\ & (\times)\left|\mathcal{F}\left(s, v_{\sigma(s,v_s+u_s)} + u_{\sigma(s,v_s+u_s)}, \int_0^s p(s,\nu, v_{\sigma(\nu,v_\nu+u_\nu)} + u_{\sigma(\nu,v_\nu+u_\nu)})d\nu\right)\right| ds \\ & + \frac{\omega\|\mathbb{E}\|^2}{B(\omega)} \int_0^\tau \widehat{\mathscr{H}_\omega}(\tau - s) \\ & (\times)\left|\mathcal{F}\left(s, v_{\sigma(s,v_s+u_s)} + u_{\sigma(s,v_s+u_s)}, \int_0^s p(s,\nu, v_{\sigma(\nu,v_\nu+u_\nu)} + u_{\sigma(\nu,v_\nu+u_\nu)})d\nu\right)\right| ds \\ \le\ & \frac{\|\mathbb{E}\|\|\mathbb{F}\|(1-\omega)}{B(\omega)\Gamma(\omega)} \int_0^\tau (\tau - s)^{\omega-1} \end{aligned}$$

$$
\begin{aligned}
&(\times)\Omega(s)\left(\|v_{\sigma(s,v_s+u_s)}+u_{\sigma(s,v_s+u_s)}\|_{\mathfrak{C}_h}+\int_0^s \Omega_1(\nu)(\|v_{\sigma(\nu,v_\nu+u_\nu)}+u_{\sigma(\nu,v_\nu+u_\nu)}\|_{\mathfrak{C}_h})d\nu\right)ds\\
&+\frac{\omega\|\mathbb{E}\|^2}{B(\omega)}\int_0^\tau \widehat{\mathscr{H}_\omega}(\tau-s)\\
&(\times)\Omega(s)\left(\|v_{\sigma(s,v_s+u_s)}+u_{\sigma(s,v_s+u_s)}\|_{\mathfrak{C}_h}+\int_0^s \Omega_1(\nu)(\|v_{\sigma(\nu,v_\nu+u_\nu)}+u_{\sigma(\nu,v_\nu+u_\nu)}\|_{\mathfrak{C}_h})d\nu\right)ds\\
\le{}& \frac{\rho\overline{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)}\int_0^\tau (\tau-s)^{\omega-1}\Omega(s)\left[e+Q_1^*|v(s)|+\widetilde{P}_0(e+Q_1^*|v(s)|)\right]ds\\
&+\frac{\omega\rho^2\widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)}\int_0^\tau (\tau-s)^{\omega-1}\Omega(s)\left[e+Q_1^*|v(s)|+\widetilde{P}_0(e+Q_1^*|v(s)|)\right]ds\\
\le{}& \left(\frac{\rho\overline{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)}+\frac{\omega\rho^2\widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)}\right)\int_0^\tau (\tau-s)^{\omega-1}\Omega(s)\left[e+Q_1^*|v(s)|+\widetilde{P}_0(e+Q_1^*|v(s)|)\right]ds\\
\le{}& \left(\frac{\rho\overline{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)}+\frac{\omega\rho^2\widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)}\right)\Omega^*(1+\widetilde{P}_0)e+\left(\frac{\rho\overline{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)}+\frac{\omega\rho^2\widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)}\right)\Omega^*(1+\widetilde{P}_0)Q_1^*\|v\|_{Y_\tau^0}.
\end{aligned}
$$

Therefore $\overline{\Upsilon}(v)\in Y_\tau^0$.

Furthermore, let $q>0$ be such that

$$
q\ge \frac{\left(\frac{\rho\overline{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)}+\frac{\omega\rho^2\widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)}\right)\Omega^*(1+\widetilde{P}_0)e}{1-\left(\frac{\rho\overline{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)}+\frac{\omega\rho^2\widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)}\right)\Omega^*(1+\widetilde{P}_0)Q_1^*},
$$

and $\mathscr{B}_q$ be the closed ball in Y_τ^0 centered at the origin and of radius q. Now, take $v\in\mathscr{B}_q$ and $\tau\in[0,+\infty)$; then

$$
|(\overline{\Upsilon}v)(\tau)|\le \left(\frac{\rho\overline{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)}+\frac{\omega\rho^2\widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)}\right)\Omega^*(1+\widetilde{P}_0)e+\left(\frac{\rho\overline{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)}+\frac{\omega\rho^2\widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)}\right)\Omega^*(1+\widetilde{P}_0)Q_1^*q.
$$

Hence

$$
\|\overline{\Upsilon}v\|_{Y_\tau^0}\le q.
$$

Step 2: $\overline{\Upsilon}$ is continuous in $\mathscr{B}_q$.

Let $\{v^n\}$ be a sequence such that $v^n\to v$ occurs in $\mathscr{B}_q$. Initially, we analyze the convergence of the sequences $\left(v^n_{\sigma(s,v_s^n)}\right)_{n\in\mathbb{N}}$, $s\in\mathscr{J}$.

If $s\in\mathscr{J}$ is such that $\sigma(s,v_s)>0$, then we have,

$$
\begin{aligned}
\left\|v^n_{\sigma(s,v_s^n)}-v_{\sigma(s,v_s)}\right\|_{\mathfrak{C}_h}&\le \left\|v^n_{\sigma(s,v_s^n)}-v_{\sigma(s,v_s^n)}\right\|_{\mathfrak{C}_h}+\left\|v_{\sigma(s,v_s^n)}-v_{\sigma(s,v_s)}\right\|_{\mathfrak{C}_h}\\
&\le Q_1^*\|v_n-v\|_{\mathfrak{C}_h}+\left\|v_{\sigma(s,v_s^n)}-v_{\sigma(s,v_s)}\right\|_{\mathfrak{C}_h}.
\end{aligned}
$$

From this, we notice that $v^n_{\sigma(s,v_s^n)}\to v_{\sigma(s,v_s)}$ in $\mathfrak{C}_h$ as $n\to\infty$ for every $s\in\mathscr{J}$ such that $\sigma(s,v_s)>0$.

In a similar manner, if $\sigma(s, v_s) < 0$, we obtain

$$\left\| v^n_{\sigma(s,v^n_s)} - v_{\sigma(s,v_s)} \right\|_{\mathfrak{C}_h} = \left\| \varphi^n_{\sigma(s,v^n_s)} - \varphi_{\sigma(s,v_s)} \right\|_{\mathfrak{C}_h} = 0.$$

From the above discussion, we realize that $v^n_{\sigma(s,v^n_s)} \to v_{\sigma(s,v_s)}$ in $\mathfrak{C}_h$ as $n \to \infty$ for every $s \in \mathscr{J}$ such that $\sigma(s, v_s) < 0$.

Based on the above estimations, we can easily demonstrate that $v^n_{\sigma(s,v_s)} \to \varphi$ for every $s \in \mathscr{J}$ such that $\sigma(s, v_s) = 0$.

Conclusively

$$\begin{aligned}
|(\overline{\Upsilon}v^n)(\tau) - (\overline{\Upsilon}v)(\tau)| \leq & \left(\frac{\mu\overline{\mu}(1-\omega)}{B(\omega)\Gamma(\omega)} + \frac{\widehat{M}_{\widehat{\mathscr{B}}}\omega\mu^2}{B(\omega)} \right) \int_0^\tau (\tau - s)^{\omega-1} \\
(\times) \bigg| \mathcal{F}\bigg(& s, v^n_{\sigma(s,v^n_s+u_s)} + u_{\sigma(s,v^n_s+u_s)}, \int_0^s p(s, v, v^n_{\sigma(v,v^n_v+u_v)} + u_{\sigma(v,v^n_v+u_v)})dv \bigg) \\
& - \mathcal{F}\bigg(s, v_{\sigma(s,v_s+u_s)} + u_{\sigma(s,v_s+u_s)}, \int_0^s p(s, v, v_{\sigma(v,v_v+u_v)} + u_{\sigma(v,v_v+u_v)})dv \bigg) \bigg| ds.
\end{aligned}$$

Since $\mathcal{F}$ satisfies the Caratheodory conditions, we have

$$\begin{aligned}
& \mathcal{F}\bigg(s, v^n_{\sigma(s,v^n_s+u_s)} + u_{\sigma(s,v^n_s+u_s)}, \int_0^s p(s, v, v^n_{\sigma(v,v^n_v+u_v)} + u_{\sigma(v,v^n_v+u_v)})dv \bigg) \\
\to & \mathcal{F}\bigg(s, v_{\sigma(s,v_s+u_s)} + u_{\sigma(s,v_s+u_s)}, \int_0^s p(s, v, v_{\sigma(v,v_v+u_v)} + u_{\sigma(v,v_v+u_v)})dv \bigg) \quad \text{as} \quad n \to \infty.
\end{aligned}$$

From the Lebesgue dominated convergence theorem, we obtain

$$\|(\overline{\Upsilon}v^n) - (\overline{\Upsilon}v)\|_{\mathfrak{C}_h} \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore the operator $\overline{\Upsilon}$ is continuous in $\mathcal{B}_q$.

Further, in view of Step 1, we notice that $\overline{\Upsilon}(\mathcal{B}_q) \subset \mathcal{B}_q$.

Next, we demonstrate that the operator $\overline{\Upsilon}$ is equi-continuous on every compact interval $[0, \xi]$ of $[0, +\infty)$, for $\xi > 0$; and is equi-convergent in $\mathcal{B}_q$.

Step 3: $\overline{\Upsilon}$ maps bounded sets into equi-continuous sets in $\mathcal{B}_q$.

Take $0 \leq v_1 < v_2 \leq \xi$ and for each $v \in \mathcal{B}_q$, we sustain

$$\begin{aligned}
& |(\overline{\Upsilon}v)(v_2) - (\overline{\Upsilon}v)(v_1)|_E \\
& \leq \frac{\|\mathbb{E}\|\|\mathbb{F}\|(1-\omega)}{B(\omega)\Gamma(\omega)} \int_0^{v_1} [(v_2 - s)^{\omega-1} - (v_1 - s)^{\omega-1}] \\
& \quad (\times) \bigg| \mathcal{F}\bigg(s, v_{\sigma(s,v_s+u_s)} + u_{\sigma(s,v_s+u_s)}, \int_0^s p(s, v, v_{\sigma(v,v_v+u_v)} + u_{\sigma(v,v_v+u_v)})dv \bigg) \bigg| ds \\
& \quad + \frac{\|\mathbb{E}\|\|\mathbb{F}\|(1-\omega)}{B(\omega)\Gamma(\omega)} \int_{v_1}^{v_2} (v_2 - s)^{\omega-1} \\
& \quad (\times) \bigg| \mathcal{F}\bigg(s, v_{\sigma(s,v_s+u_s)} + u_{\sigma(s,v_s+u_s)}, \int_0^s p(s, v, v_{\sigma(v,v_v+u_v)} + u_{\sigma(v,v_v+u_v)})dv \bigg) \bigg| ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\omega\|\mathbb{E}\|^2}{B(\omega)}\int_0^{\nu_1}[\widehat{\mathscr{H}_\omega}(\nu_2-s)-\widehat{\mathscr{H}_\omega}(\nu_1-s)]\\
&(\times)\left|\mathcal{F}\left(s, v_{\sigma(s,v_s+u_s)}+u_{\sigma(s,v_s+u_s)}, \int_0^s p(s,\nu,v_{\sigma(\nu,v_\nu+u_\nu)}+u_{\sigma(\nu,v_\nu+u_\nu)})d\nu\right)\right|ds\\
&+\frac{\omega\|\mathbb{E}\|^2}{B(\omega)}\int_{\nu_1}^{\nu_2}\widehat{\mathscr{H}_\omega}(\nu_2-s)\left|\mathcal{F}\left(s, v_{\sigma(s,v_s+u_s)}+u_{\sigma(s,v_s+u_s)}, \int_0^s p(s,\nu,v_{\sigma(\nu,v_\nu+u_\nu)}+u_{\sigma(\nu,v_\nu+u_\nu)})d\nu\right)\right|ds.
\end{aligned}
$$

By (A1) and (3.2), we have

$$
\begin{aligned}
&\left|\mathcal{F}\left(s, v_{\sigma(s,v_s+u_s)}+u_{\sigma(s,v_s+u_s)}, \int_0^s p(s,\nu,v_{\sigma(\nu,v_\nu+u_\nu)}+u_{\sigma(\nu,v_\nu+u_\nu)})d\nu\right)\right|\\
&\leq \Omega(s)\left(\|v_{\sigma(s,v_s+u_s)}+u_{\sigma(s,v_s+u_s)}\|_{\mathfrak{C}_h}+\int_0^s\Omega_1(\nu)(\|v_{\sigma(\nu,v_\nu+u_\nu)}+u_{\sigma(\nu,v_\nu+u_\nu)}\|_{\mathfrak{C}_h})d\nu\right)\\
&\leq \Omega(s)(Q'+\widetilde{P}_0Q')\\
&\leq \Omega(s)(1+\widetilde{P}_0)Q'.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&|(\overline{\Upsilon}v)(\nu_2)-(\overline{\Upsilon}v)(\nu_1)|_E\\
&\leq \frac{\rho\overline{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)}(1+\widetilde{P}_0)Q'\int_0^{\nu_1}[(\nu_2-s)^{\omega-1}-(\nu_1-s)^{\omega-1}]\Omega(s)ds\\
&\quad+\frac{\rho\overline{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)}(1+\widetilde{P}_0)Q'\int_{\nu_1}^{\nu_2}(\nu_2-s)^{\omega-1}\Omega(s)ds\\
&\quad+\frac{\omega\rho^2}{B(\omega)}(1+\widetilde{P}_0)Q'\int_0^{\nu_1}[\widehat{\mathscr{H}_\omega}(\nu_2-s)-\widehat{\mathscr{H}_\omega}(\nu_1-s)]\Omega(s)ds\\
&\quad+\frac{\omega\rho^2}{B(\omega)}(1+\widetilde{P}_0)Q'\int_{\nu_1}^{\nu_2}\widehat{\mathscr{H}_\omega}(\nu_2-s)\Omega(s)ds.
\end{aligned}
$$

When $\nu_2 \to \nu_1$, the right-hand side of the above inequality tends to zero, since $\mathscr{H}_\omega(\tau)$ and $\widehat{\mathscr{H}_\omega}(\tau)$ are strongly continuous operators and the compactness of $\mathscr{H}_\omega(\tau)$ and $\widehat{\mathscr{H}_\omega}(\tau)$ for $\tau > 0$ implies the continuity in the uniform operators topology. So $\overline{\Upsilon}\left(\mathcal{B}_q\right)$ is equi-continuous.

Step 4: $\overline{\Upsilon}(\mathcal{B}_q)$ is equi-convergent.

Let $\tau \in [0,+\infty)$ and $v \in \mathcal{B}_q$, we obtain

$$
|(\overline{\Upsilon}v)(\tau)| \leq \left(\frac{\rho\overline{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)}+\frac{\omega\rho^2\widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)}\right)(1+\widetilde{P}_0)Q'\int_0^\tau(\tau-s)^{\omega-1}\Omega(s)ds.
$$

Then, we sustain

$$
\lim_{\tau\to+\infty}|(\overline{\Upsilon}v)(\tau)| \leq \left(\frac{\rho\overline{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)}+\frac{\omega\rho^2\widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)}\right)(1+\widetilde{P}_0)Q'\Omega^*.
$$

Hence,

$$
|(\overline{\Upsilon}v)(\tau)-(\overline{\Upsilon}v)(+\infty)| \to 0 \quad \text{as} \quad t \to +\infty.
$$

Now, let K be a subset of B_q such that $K \subset \overline{conv}\,(\Upsilon(K) \cup \{0\})$. In addition, by Lemma 2.9, we know that there is a countable set $C_0 = \{w_n\} \subset C$ such that $\chi(\overline{\Upsilon}(C)) \leq 2\chi(\overline{\Upsilon}(C_0))$ for any bounded set C. Thus for $\{w_n\} \subset C$, for the appropriate choice of K. For every $\tau \in [0, \xi]$, by utilizing Lemma 2.10 and conditions (A0)–(A2) and the properties of the measure χ, we obtain

$$\begin{aligned}
\chi(\overline{\Upsilon}(w_n)) = {} & \chi\Bigg(\Bigg\{\frac{\mathbb{EF}(1-\omega)}{B(\omega)\Gamma(\omega)}\int_0^\tau (\tau-s)^{\omega-1} \\
& (\times)\mathcal{F}\left(s, w_{n\sigma(s,v_s+u_s)} + u_{\sigma(s,v_s+u_s)}, \int_0^s p(s,\nu,w_{n\sigma(\nu,v_\nu+u_\nu)} + u_{\sigma(\nu,v_\nu+u_\nu)})d\nu\right)ds\Bigg\}\Bigg) \\
& + \chi\Bigg(\Bigg\{\frac{\omega\mathbb{E}^2\overline{\Lambda}_1}{B(\omega)}\int_0^\tau (\tau-s)^{\omega-1} \\
& (\times)\mathcal{F}\left(s, w_{n\sigma(s,v_s+u_s)} + u_{\sigma(s,v_s+u_s)}, \int_0^s p(s,\nu,w_{n\sigma(\nu,v_\nu+u_\nu)} + u_{\sigma(\nu,v_\nu+u_\nu)})d\nu\right)ds\Bigg\}\Bigg) \\
= {} & \chi\Bigg(\Bigg\{\frac{\rho\bar{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)} + \frac{\omega\rho^2\widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)}\Bigg\}\int_0^\tau (\tau-s)^{\omega-1} \\
& (\times)\mathcal{F}\left(s, w_{n\sigma(s,v_s+u_s)} + u_{\sigma(s,v_s+u_s)}, \int_0^s p(s,\nu,w_{n\sigma(\nu,v_\nu+u_\nu)} + u_{\sigma(\nu,v_\nu+u_\nu)})d\nu\right)ds\Bigg) \\
\leq {} & 2\left(\frac{\rho\bar{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)} + \frac{\omega\rho^2\widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)}\right)\int_0^\tau (\tau-s)^{\omega-1}\Omega(s) \\
& (\times)\left[\sup_{-\infty<\theta\leq 0}\chi(w_n(\theta+s)+u(\theta+s)) + \int_0^s \overline{\mu}(s,\nu)\sup_{-\infty<\theta\leq 0}\chi(w_n(\theta+s)+u(\theta+s))d\nu\right]ds \\
\leq {} & 2\left(\frac{\rho\bar{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)} + \frac{\omega\rho^2\widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)}\right)\int_0^\tau (\tau-s)^{\omega-1}\Omega(s)\left[\sup_{0<\mu\leq s}\chi(w_n(\mu)) + \widetilde{\mu}^*\sup_{0<\mu\leq s}\chi(w_n(\mu))\right]ds \\
\leq {} & 2(1+\widetilde{\mu}^*)\left(\frac{\rho\bar{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)} + \frac{\omega\rho^2\widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)}\right)\int_0^\tau (\tau-s)^{\omega-1}\Omega(s)\sup_{0<s\leq\xi}\chi(w_n(s))ds \\
\leq {} & 2(1+\widetilde{\mu}^*)\left(\frac{\rho\bar{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)} + \frac{\omega\rho^2\widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)}\right)\chi(\{w_n\})\int_0^\tau (\tau-s)^{\omega-1}\Omega(s)ds \\
\leq {} & 2(1+\widetilde{\mu}^*)\left(\frac{\rho\bar{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)} + \frac{\omega\rho^2\widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)}\right)\Omega^*\chi(\{w_n\})
\end{aligned}$$

which ensures that

$$\chi(\overline{\Upsilon}(K)) \leq \left[2(1+\widetilde{\mu}^*)\left(\frac{\rho\bar{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)} + \frac{\omega\rho^2\widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)}\right)\Omega^*\right]\chi(K).$$

Then

$$\chi(K) \leq \chi(\overline{\Upsilon}(K)) \leq \widehat{M}\chi(K).$$

That is to say

$$\chi(K)(1-\widehat{M}) \leq 0.$$

From (3.1), we observe that $\chi(K)$=0 and for each $\tau \in \mathscr{J}$; then $K(\tau)$ is relatively compact in E. As a result of Steps 1–4 and Theorem 2.14, we conclude that $\overline{\Upsilon}$ has a fixed point v^*. Then $w^* = v^* + u$ is a fixed point of the operator Υ, which is a mild solution of the model (1.1)–(1.2). □

Next, we establish the existence result for the system (1.3)–(1.4). Now, we list the additional subsequent hypotheses:

(A4) Let $\mathscr{P} : \mathscr{J} \times \mathfrak{C}_h \to E$ be a Caratheodory function and we can find a continuous function $\Omega_{\mathscr{P}} : (-\infty, +\infty) \to [0, +\infty)$ in a way that:

$$|\mathscr{P}(\tau, u)| \leq \Omega_{\mathscr{P}}(\tau)\|u\|_{\mathfrak{C}_h}, \ \tau \in \mathscr{J}, u \in \mathfrak{C}_h$$

and

$$\Omega^*_{\mathscr{P}} := \sup_{\tau\in\mathscr{J}} \int_0^\tau (\tau - s)^{\omega-1}\Omega_{\mathscr{P}}(s)ds < \infty.$$

(A5) Let C be a bounded set, $C \subset \mathfrak{C}_h$ and every $\tau \in [0, +\infty)$; we have

$$\chi(\mathscr{P}(\tau, C)) \leq \Omega_{\mathscr{P}}(\tau)\chi(C).$$

(A6) Suppose $C \subset \mathfrak{C}_h$, where C is a bounded set; then, the function $\{\tau \to \mathscr{P}(\tau, u_\tau) : u \in C\}$ is equicontinuous on every compact interval $[0, \xi]$ of $[0, +\infty)$, for every $\xi > 0$.

Theorem 3.2. *If the assumptions (A0)–(A6) hold and*

$$\widehat{M}_1 = 2\left[(1 + \widetilde{\mu}^*)\left(\frac{\rho\bar{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)} + \frac{\omega\rho^2\widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)}\right)\Omega^* + \left(\frac{\rho\bar{\rho}}{\Gamma(\omega)} + \rho\bar{\rho}\widehat{C}_{\widehat{\mathscr{H}}}\right)\Omega^*_{\mathscr{P}}\right] < 1, \tag{3.3}$$

then the system (1.3)–(1.4) *has at least one mild solution on* Y_τ.

Proof. Consider the operator $\Upsilon_1 : Y_\tau \to Y_\tau$ defined by

$$(\Upsilon_1 w)(\tau) = \begin{cases} \varphi(\tau), \quad \tau \leq 0, \\ \mathbb{E}\mathscr{H}_\omega(\tau)[\varphi(0) - \mathscr{P}(0, \varphi)] \\ +\dfrac{\mathbb{E}\mathbb{F}(1-\omega)}{B(\omega)\Gamma(\omega)} \displaystyle\int_0^\tau (\tau - s)^{\omega-1}\mathcal{F}\left(s, w_{\sigma(s,w_s)}, \int_0^s p(s, v, w_{\sigma(v,w_v)})dv\right)ds \\ +\dfrac{\mathbb{E}\mathbb{F}}{\Gamma(\omega)} \displaystyle\int_0^\tau (\tau - s)^{\omega-1}\mathscr{P}(s, w_{\sigma(s,w_s)})ds \\ +\dfrac{\omega\mathbb{E}^2}{B(\omega)} \displaystyle\int_0^\tau \widehat{\mathscr{H}_\omega}(\tau - s)\mathcal{F}\left(s, w_{\sigma(s,w_s)}, \int_0^s p(s, v, w_{\sigma(v,w_v)})dv\right)ds \\ -\mathbb{E}\mathbb{F}\displaystyle\int_0^\tau \widehat{\mathscr{H}_\omega}(\tau - s)\mathscr{P}(s, w_{\sigma(s,w_s)})ds, \quad \tau \in \mathscr{J}. \end{cases}$$

In view of Theorem 3.1, define the operator $\overline{\Upsilon}_1 : Y_\tau^0 \to Y_\tau^0$ as

$$(\overline{\Upsilon}_1 v)(\tau) = \begin{cases} -\mathbb{E}\mathscr{H}_\omega(\tau)\mathscr{P}(0,\varphi) + \dfrac{\mathbb{E}\mathbb{F}(1-\omega)}{B(\omega)\Gamma(\omega)}\displaystyle\int_0^\tau (\tau-s)^{\omega-1} \\ (\times)\mathcal{F}\left(s, v_{\sigma(s,v_s+u_s)} + u_{\sigma(s,v_s+u_s)}, \displaystyle\int_0^s p(s,\nu, v_{\sigma(\nu,v_\nu+u_\nu)} + u_{\sigma(\nu,v_\nu+u_\nu)})d\nu\right)ds \\ +\dfrac{\mathbb{E}\mathbb{F}}{\Gamma(\omega)}\displaystyle\int_0^\tau (\tau-s)^{\omega-1}\mathscr{P}(s, v_{\sigma(s,v_s+u_s)} + u_{\sigma(s,v_s+u_s)})ds \\ +\dfrac{\omega\mathbb{E}^2}{B(\omega)}\displaystyle\int_0^\tau \widehat{\mathscr{H}_\omega}(\tau-s) \\ (\times)\mathcal{F}\left(s, v_{\sigma(s,v_s+u_s)} + u_{\sigma(s,v_s+u_s)}, \displaystyle\int_0^s p(s,\nu, v_{\sigma(\nu,v_\nu+u_\nu)} + u_{\sigma(\nu,v_\nu+u_\nu)})d\nu\right)ds \\ -\mathbb{E}\mathbb{F}\displaystyle\int_0^\tau \widehat{\mathscr{H}_\omega}(\tau-s)\mathscr{P}(s, v_{\sigma(s,v_s+u_s)} + u_{\sigma(s,v_s+u_s)})ds, \quad \tau\in\mathscr{J}. \end{cases}$$

Obviously Υ_1 has a fixed point equal to one $\overline{\Upsilon}_1$, so it stands to show that $\overline{\Upsilon}_1$ has a fixed point. We need to demonstrate that all assumptions of Theorem 2.14 are fulfilled by the operator $\overline{\Upsilon}_1$.

$$\begin{aligned} |(\overline{\Upsilon}_1 v)(\tau)| \le\ & \|\mathbb{E}\|\|\mathscr{H}_\omega(\tau)\|\|\mathscr{P}(0,\varphi)\| + \frac{\|\mathbb{E}\|\|\mathbb{F}\|(1-\omega)}{B(\omega)\Gamma(\omega)}\int_0^\tau (\tau-s)^{\omega-1} \\ & (\times)\left|\mathcal{F}\left(s, v_{\sigma(s,v_s+u_s)} + u_{\sigma(s,v_s+u_s)}, \int_0^s p(s,\nu, v_{\sigma(\nu,v_\nu+u_\nu)} + u_{\sigma(\nu,v_\nu+u_\nu)})d\nu\right)\right|ds \\ & + \frac{\omega\|\mathbb{E}\|^2}{B(\omega)}\int_0^\tau \|\widehat{\mathscr{H}_\omega}(\tau-s)\| \\ & (\times)\left|\mathcal{F}\left(s, v_{\sigma(s,v_s+u_s)} + u_{\sigma(s,v_s+u_s)}, \int_0^s p(s,\nu, v_{\sigma(\nu,v_\nu+u_\nu)} + u_{\sigma(\nu,v_\nu+u_\nu)})d\nu\right)\right|ds \\ & + \frac{\|\mathbb{E}\|\|\mathbb{F}\|}{\Gamma(\omega)}\int_0^\tau (\tau-s)^{\omega-1}|\mathscr{P}(s, v_{\sigma(s,v_s+u_s)} + u_{\sigma(s,v_s+u_s)})|ds \\ & + \|\mathbb{E}\|\|\mathbb{F}\|\int_0^\tau \|\widehat{\mathscr{H}_\omega}(\tau-s)\|\|\mathscr{P}(s, w_{\sigma(s,w_s)})|ds \\ \le\ & \rho\widehat{C}_{\mathscr{H}}\|\mathscr{P}(0,\varphi)\| \\ & + \left(\frac{\rho\overline{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)} + \frac{\omega\rho^2\widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)}\right)\int_0^\tau (\tau-s)^{\omega-1}\Omega(s)\left[e + Q_1^*|v(s)| + \widetilde{P}_0(e + Q_1^*|v(s)|)\right]ds \\ & + \left(\frac{\rho\overline{\rho}}{\Gamma(\omega)} + \rho\overline{\rho}\widehat{C}_{\widehat{\mathscr{H}}}\right)\int_0^\tau (\tau-s)^{\omega-1}\Omega_{\mathscr{P}}(s)[e + Q_1^*|v(s)|]ds \\ \le\ & \rho\widehat{C}_{\mathscr{H}}\|\mathscr{P}(0,\varphi)\| + \left(\frac{\rho\overline{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)} + \frac{\omega\rho^2\widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)}\right)\Omega^*(1+\widetilde{P}_0)e \\ & + \left(\frac{\rho\overline{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)} + \frac{\omega\rho^2\widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)}\right)\Omega^*(1+\widetilde{P}_0)Q_1^*\|v\|_{Y_\tau^0} \\ & + \left(\frac{\rho\overline{\rho}}{\Gamma(\omega)} + \rho\overline{\rho}\widehat{C}_{\widehat{\mathscr{H}}}\right)\Omega_{\mathscr{P}}^* e + \left(\frac{\rho\overline{\rho}}{\Gamma(\omega)} + \rho\overline{\rho}\widehat{C}_{\widehat{\mathscr{H}}}\right)\Omega_{\mathscr{P}}^* Q_1^*\|v\|_{Y_\tau^0} \end{aligned}$$

$$\leq \widetilde{A}_1 + Q_1^* \left[\left(\frac{\rho\bar{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)} + \frac{\omega\rho^2 \widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)} \right) \Omega^*(1+\widetilde{P}_0) + \left(\frac{\rho\bar{\rho}}{\Gamma(\omega)} + \rho\bar{\rho}\widehat{C}_{\widehat{\mathscr{H}}} \right) \Omega^*_{\mathscr{P}} \right] \|v\|_{Y^0_\tau},$$

where

$$\widetilde{A}_1 = \rho\widehat{C}_{\mathscr{H}} \|\mathscr{P}(0,\varphi)\| + \left(\frac{\rho\bar{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)} + \frac{\omega\rho^2 \widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)} \right) \Omega^*(1+\widetilde{P}_0)e + \left(\frac{\rho\bar{\rho}}{\Gamma(\omega)} + \rho\bar{\rho}\widehat{C}_{\widehat{\mathscr{H}}} \right) \Omega^*_{\mathscr{P}} e.$$

Therefore $\overline{\Upsilon}_1(v) \in Y^0_\tau$.

Furthermore, let $q > 0$ be such that

$$q \geq \frac{\widetilde{A}_1}{1 - Q_1^* \left\{ \left(\frac{\rho\bar{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)} + \frac{\omega\rho^2 \widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)} \right) \Omega^*(1+\widetilde{P}_0) + \left(\frac{\rho\bar{\rho}}{\Gamma(\omega)} + \rho\bar{\rho}\widehat{C}_{\widehat{\mathscr{H}}} \right) \Omega^*_{\mathscr{P}} \right\}},$$

and $\mathscr{B}_q$ be the same as defined in Theorem 3.1. Now, take $v \in \mathscr{B}_q$ and $\tau \in [0,+\infty)$; then

$$|(\overline{\Upsilon}_1 v)(\tau)| \leq \widetilde{A}_1 + Q_1^* \left[\left(\frac{\rho\bar{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)} + \frac{\omega\rho^2 \widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)} \right) \Omega^*(1+\widetilde{P}_0) + \left(\frac{\rho\bar{\rho}}{\Gamma(\omega)} + \rho\bar{\rho}\widehat{C}_{\widehat{\mathscr{H}}} \right) \Omega^*_{\mathscr{P}} \right] q.$$

Hence

$$\|\overline{\Upsilon}_1 v\|_{Y^0_\tau} \leq q.$$

Step 2: $\overline{\Upsilon}_1$ is continuous and equi-continuous in $\mathscr{B}_q$.

By thinking of Steps 2 and 3 of Theorem 3.1 and the conditions (A4) and (A6), we have come to the conclusion that the operator $\overline{\Upsilon}_1$ is both continuous and equi-continuous in the space $\mathscr{B}_q$.

Further, in view of Step 1, we notice that $\overline{\Upsilon}_1(\mathscr{B}_q) \subset \mathscr{B}_q$.

Next, we demonstrate that the operator $\overline{\Upsilon}_1$ is equi-convergent in $\mathscr{B}_q$.

Step 4: $\overline{\Upsilon}_1(\mathscr{B}_q)$ is equi-convergent.

Let $\tau \in [0,+\infty)$ and $v \in \mathscr{B}_q$; we get

$$\begin{aligned}|(\overline{\Upsilon}_1 v)(\tau)| \leq \widetilde{A}_1 + Q_1^*(1+\widetilde{P}_0) \left(\frac{\rho\bar{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)} + \frac{\omega\rho^2 \widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)} \right) \int_0^\tau (\tau-s)^{\omega-1}\Omega(s)|v(s)|ds \\ + \left(\frac{\rho\bar{\rho}}{\Gamma(\omega)} + \rho\bar{\rho}\widehat{C}_{\widehat{\mathscr{H}}} \right) Q_1^* \int_0^\tau (\tau-s)^{\omega-1}\Omega_{\mathscr{P}}(s)|v(s)|ds\end{aligned}$$

Then, we have

$$\lim_{\tau\to+\infty} |\overline{\Upsilon}_1(v)(\tau)| \leq \widetilde{A}_1 + Q_1^* \left[\left(\frac{\rho\bar{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)} + \frac{\omega\rho^2 \widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)} \right) \Omega^*(1+\widetilde{P}_0) + \left(\frac{\rho\bar{\rho}}{\Gamma(\omega)} + \rho\bar{\rho}\widehat{C}_{\widehat{\mathscr{H}}} \right) \Omega^*_{\mathscr{P}} \right] q.$$

Hence,

$$|(\overline{\Upsilon}_1 v)(\tau) - (\overline{\Upsilon}_1 v)(+\infty)| \to 0 \quad \text{as} \quad t \to +\infty.$$

As a result of Step 4 of Theorem 3.1 and the conditions (A4)–(A5), we have

$$\chi(\overline{\Upsilon}_1(w_n)) \leq 2(1+\widetilde{\mu}^*) \left(\frac{\rho\bar{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)} + \frac{\omega\rho^2 \widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)} \right) \chi(\{w_n\}) \int_0^\tau (\tau-s)^{\omega-1}\Omega(s)ds$$

$$+ 2\left(\frac{\rho\overline{\rho}}{\Gamma(\omega)} + \rho\overline{\rho}\widehat{C}_{\widehat{\mathscr{H}}}\right)\chi(\{w_n\})\int_0^\tau (\tau - s)^{\omega-1}\Omega_{\mathscr{P}}(s)ds$$
$$\leq 2\left[(1+\widetilde{\mu}^*)\left(\frac{\rho\overline{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)} + \frac{\omega\rho^2\widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)}\right)\chi(\{w_n\})\Omega^* + \left(\frac{\rho\overline{\rho}}{\Gamma(\omega)} + \rho\overline{\rho}\widehat{C}_{\widehat{\mathscr{H}}}\right)\chi(\{w_n\})\Omega^*_{\mathscr{P}}\right]$$

which ensures that

$$\chi(\overline{\Upsilon}_1(K)) \leq 2\left[(1+\widetilde{\mu}^*)\left(\frac{\rho\overline{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)} + \frac{\omega\rho^2\widehat{C}_{\widehat{\mathscr{H}}}}{B(\omega)}\right)\Omega^* + \left(\frac{\rho\overline{\rho}}{\Gamma(\omega)} + \rho\overline{\rho}\widehat{C}_{\widehat{\mathscr{H}}}\right)\Omega^*_{\mathscr{P}}\right]\chi(K).$$

Then

$$\chi(K) \leq \chi(\overline{\Upsilon}_1(K)) \leq \widehat{M}_1\chi(K).$$

That is to say

$$\chi(K)(1-\widehat{M}_1) \leq 0.$$

From (3.3), we observe that $\chi(K)$=0 for each $\tau \in \mathscr{J}$; then, $K(\tau)$ is relatively compact in E. As a result of Steps 1–4 and Theorem 2.14, we conclude that $\overline{\Upsilon}_1$ has a fixed point v^*. Then $w^* = v^* + u$ is a fixed point of the operator Υ_1, which is a mild solution of the model (1.3)–(1.4). □

4. Example

Consider the following partial integro-differential system, which includes an ABC derivative of the model

$$\mathscr{D}^{\frac{1}{2}}_{ABC}\left[u(\tau,y) + e^{-\tau}\int_{-\infty}^{\tau} e^{2(s-\tau)}\frac{u(s-\sigma_1(s)\sigma_2(\|u(s)\|),y)}{49}ds\right] = \frac{\partial^2}{\partial y^2}u(\tau,y)$$
$$+ e^{-\tau}\int_{-\infty}^{\tau} e^{2(s-\tau)}\frac{u(s-\sigma_1(s)\sigma_2(\|u(s)\|),y)}{64}ds$$
$$+ e^{-\tau}\int_0^\tau \sin(\tau-s)\int_{-\infty}^{s} e^{2(\nu-s)}\frac{u(\nu-\sigma_1(\nu)\sigma_2(\|u(\nu)\|),y)}{16}d\nu ds, \tag{4.1}$$
$$u(\tau,0) = 0 = u(\tau,\pi), \quad \tau\in[0,+\infty), \tag{4.2}$$
$$u(\tau,y) = \varphi(\tau,y), \quad -\infty < \tau \leq 0, 0 \leq y \leq \pi, \tag{4.3}$$

where $\mathscr{D}^{\vartheta}_{ABC}$ is the ABC derivative of order $0<\omega<1$; $\sigma_i : \mathbb{R}\to\mathbb{R}^+$, $i=1,2$ and $\varphi\in\mathfrak{C}_h$. We consider $E = L^2[0,\pi]$ having the norm $|\cdot|_{L^2}$ and determine the operator $A : D(A)\subset E\to E$ by $Aw = w''$ with the domain

$$D(A) = \{w\in E : w, w' \quad \text{are absolutely continuous}, \quad w''\in E,\ w(0) = w(\pi) = 0\}.$$

Then

$$Aw = \sum_{n=1}^{\infty} n^2\langle w, w_n\rangle w_n, \quad w\in D(A),$$

in which $w_n(s) = \sqrt{\frac{2}{\pi}}\sin(ns),\ n = 1, 2, \ldots,$. is the orthogonal set of eigenvectors of A. It is well-known that A is the infinitesimal generator of an analytic semigroup $(T(\tau))_{\tau\geq 0}$ in E and is provided by

$$T(\tau)w = \sum_{n=1}^{\infty} e^{-n^2\tau}\langle w, w_n\rangle w_n, \quad \text{for all} \quad w \in E, \quad \text{and every} \quad \tau > 0.$$

As $R(\lambda, A) = (\lambda I - A)^{-1}$ is a compact operator for all $\lambda \in \rho(A)$, $\{T(\tau)\}_{\tau\geq 0}$ is a uniformly bounded compact semigroup, which means that $A \in \mathscr{A}^{\omega}(\overline{\omega}_0, w_0)$. Also, subordination principle of the solution operator $\mathscr{H}_{\omega}(\tau)_{\tau\geq 0}$ is such that $\|\mathscr{H}_{\omega}(\tau)_{\tau\geq 0}\| \leq M$ for $\tau \in [0, 1] \subset [0, +\infty)$.

For the phase space, we choose $h = e^{2s},\ s < 0$; then, we set $l = \displaystyle\int_{-\infty}^{0} h(s)ds = \frac{1}{2} < \infty$, for $t \leq 0$ and determine

$$\|\psi\|_{\mathfrak{C}_h} = \int_{-\infty}^{0} h(s) \sup_{\theta\in[s,0]} \|\psi(\theta)\|_{L^2} ds.$$

We assume the following assumptions in order to put the system (4.1)–(4.3) into the abstract form given by (1.3)–(1.4):

Let $(\tau, \psi) \in [0, 1] \subset [0, +\infty) \times \mathfrak{C}_h$, where $\psi(\theta)(y) = \psi(\theta, y)$, $(\theta, y) \in (-\infty, 0] \times [0, \pi]$. Set

$$u(\tau)(y) = u(\tau, y), \quad \sigma(\tau, \psi) = \sigma_1(\tau)\sigma_2(\|\psi(0)\|);$$

we have

$$\mathscr{P}(\tau, \psi)(y) = e^{-\tau} \int_{-\infty}^{0} e^{2(s)} \frac{\psi}{49} ds,$$
$$\mathcal{F}(\tau, \psi, \widehat{\mathscr{H}\psi})(y) = e^{-\tau} \int_{-\infty}^{0} e^{2(s)} \frac{\psi}{64} ds + (\widehat{\mathscr{H}\psi})(x),$$

where

$$(\widehat{\mathscr{H}\psi})(y)(= p(\tau, s, u)(y)) = e^{-\tau} \int_{0}^{\tau} \sin(\tau - s) \int_{-\infty}^{0} e^{2(v)} \frac{\psi}{16} dv ds.$$

Then the problem (1.3)–(1.4) is an abstract formulation of the system (4.1)–(4.3).

Next, we verify the assumptions (A0)–(A6) for the above system (4.1)–(4.3) one by one.

Verification of A0:

The function $p(\tau, s, u)(y)$ is Caratheodory and for $\tau \in [0, 1] \subset [0, +\infty), \psi \in \mathfrak{C}_h$, we have

$$\begin{aligned}
|p(\tau, s, u)|_{L^2} &\leq \left(\int_0^{\pi} \left(e^{-\tau} \int_0^{\tau} \|\sin(\tau - s)\| \int_{-\infty}^{0} e^{2(v)} \left\| \frac{\psi}{16} \right\| dv ds \right)^2 dy \right)^{\frac{1}{2}} \\
&\leq \left(\int_0^{\pi} \left(\frac{1}{16} e^{-\tau} \int_{-\infty}^{0} e^{2(s)} \sup \|\psi\| ds \right)^2 dy \right)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{\pi}}{16} e^{-\tau} \|\psi\|_{\mathfrak{C}_h}
\end{aligned}$$

$$\leq \Omega_1(\tau)\|\psi\|_{\mathfrak{C}_h},$$

where $\Omega_1(\tau) = \frac{\sqrt{\pi}}{16}e^{-\tau}$ and $\frac{\sqrt{\pi}}{16} \sup_{\tau\in[0,1]\subset[0,+\infty)} \int_0^\tau e^{-s}ds = 0.070 = \widetilde{\mu}^*$. Also, we can see that each bounded set $C_1 \subset \mathfrak{C}_h$ and

$$\chi(p(\tau, s, C_1)) \leq \frac{\sqrt{\pi}}{16}e^{-\tau} \sup_{-\infty<x\leq 0} \chi(C_1(x)) \quad \text{for a.e. } \tau, s \in [0,1] \subset [0,+\infty).$$

Therefore p satisfies the condition (A0).

Verification of A1 and A2:

The function $\mathcal{F}(\tau, s, u)(y)$ is Caratheodory and for $\tau \in [0,1] \subset [0,+\infty), \psi \in \mathfrak{C}_h$, we have

$$\begin{aligned}
|\mathcal{F}(\tau, s, u)|_{L^2} &\leq \left(\int_0^\pi \left(e^{-\tau}\int_{-\infty}^0 e^{2(s)}\left\|\frac{\psi}{64}\right\|ds + e^{-\tau}\int_0^\tau \|\sin(\tau-s)\| \int_{-\infty}^0 e^{2(v)}\left\|\frac{\psi}{16}\right\|dvds\right)^2 dy\right)^{\frac{1}{2}}\\
&\leq \left(\int_0^\pi \left(\frac{1}{64}e^{-\tau}\int_{-\infty}^0 e^{2(s)}\sup\|\psi\|ds + \frac{1}{16}e^{-\tau}\int_{-\infty}^0 e^{2(s)}\sup\|\psi\|ds\right)^2 dy\right)^{\frac{1}{2}}\\
&\leq \frac{5\sqrt{\pi}}{64}e^{-\tau}\|\psi\|_{\mathfrak{C}_h}\\
&\leq \Omega(\tau)\|\psi\|_{\mathfrak{C}_h},
\end{aligned}$$

where $\Omega(\tau) = \frac{5\sqrt{\pi}}{64}e^{-\tau}$ and $\frac{5\sqrt{\pi}}{64} \sup_{\tau\in[0,1]\subset[0,+\infty)} \int_0^\tau (\tau-s)^{-\frac{1}{2}}e^{-s}ds = \frac{10\sqrt{\pi}}{64}F(\sqrt{\tau}) = 0.145 = \Omega^*$, where $F(z)$ is the Dawson integral. Also, we can see that each bounded set $C_2 \subset \mathfrak{C}_h, \overline{C} \subset E$, $\tau \in [0,1] \subset [0,+\infty)$ and

$$\chi(\mathcal{F}(\tau, C_2, \overline{C})) \leq \frac{5\sqrt{\pi}}{64}e^{-\tau}\left[\sup_{-\infty<x\leq 0}\chi(C_2(x)) + \chi(\overline{C})\right],$$

where $C_2(x) = \{u(x) : u \in C_2\}$.

From the above discussion, we notice that the assumptions (A1) and (A2) are verified.

Verification of A3:

In view of Definition 2.11, we have

$\mathbb{E} = \eta(\eta I - A)^{-1}$ and $\mathbb{F} = -\widetilde{\eta}A(\eta I - A)^{-1}$ with $\eta = \frac{B(\omega)}{1-\omega}$, $\widetilde{\eta} = \frac{\omega}{1-\omega}$ and $B(\omega) = 1-\omega+\frac{\omega}{\Gamma(\omega)}$. From (4.1), we know that $\omega = \frac{1}{2}$.

Therefore

$$B\left(\frac{1}{2}\right) = \left(1 - \frac{1}{2}\right) + \frac{\frac{1}{2}}{\Gamma\left(\frac{1}{2}\right)} = 0.7821.$$

From the above normalization function value, we have

$$\eta = \frac{B(\omega)}{1-\omega} = \frac{0.7821}{0.5} = 1.5642 \quad \text{and} \quad \widetilde{\eta} = \frac{\omega}{1-\omega} = \frac{0.5}{0.5} = 1.$$

From the boundedness on A and along with the above, we conclude that $\mathbb{E}$ and $\mathbb{F}$ are bounded linear operators and there are positive constants ρ and $\overline{\rho}$ such that $\|\mathbb{E}\| \leq \rho$ and $\|\mathbb{F}\| \leq \overline{\rho}$, respectively.

Verification of A4–A6:

The function $\mathscr{P}(\tau,\psi)(y)$ is Caratheodory and for $\tau \in [0,1] \subset [0,+\infty), \psi \in \mathfrak{C}_h$, we have

$$\begin{aligned}
|\mathscr{P}(\tau,\psi)|_{L^2} &\leq \left(\int_0^\pi \left(e^{-\tau}\int_{-\infty}^0 e^{2(s)}\left\|\frac{\psi}{49}\right\| ds\right)^2 dy\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^\pi \left(\frac{1}{49}e^{-\tau}\int_{-\infty}^0 e^{2(s)}\sup\|\psi\| ds\right)^2 dy\right)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{\pi}}{49}e^{-\tau}\|\psi\|_{\mathfrak{C}_h} \\
&\leq \Omega_{\mathscr{P}}(\tau)\|\psi\|_{\mathfrak{C}_h},
\end{aligned}$$

where $\Omega_{\mathscr{P}}(\tau) = \frac{\sqrt{\pi}}{49}e^{-\tau}$ and $\frac{\sqrt{\pi}}{49}\sup_{\tau\in[0,1]\subset[0,+\infty)}\int_0^\tau(\tau-s)^{-\frac{1}{2}}e^{-s}ds = \frac{2\sqrt{\pi}}{49}F(\sqrt{\tau}) = 0.039 = \Omega^*_{\mathscr{P}}$, where $F(z)$ is the Dawson integral. Also, we can see that each bounded set $C \subset \mathfrak{C}_h$ and

$$\chi(\mathscr{P}(\tau,C)) \leq \frac{\sqrt{\pi}}{49}e^{-\tau}\sup_{-\infty<x\leq 0}\chi(C(x)) \quad \text{for a.e. } \tau \in [0,1] \subset [0,+\infty).$$

For any $0 \leq \nu_1 < \nu_2 \leq \xi$ and for each $\nu \in \mathcal{B}_q$, we have

$$\begin{aligned}
|\mathscr{P}(\nu_2,\psi)(y) - \mathscr{P}(\nu_1,\psi)(y)|_{L^2} &\leq \frac{\sqrt{\pi}}{49}(e^{-\nu_2} - e^{-\nu_1})\|\psi\|_{\mathfrak{C}_h} \\
&\to 0 \quad \text{as} \quad \nu_2 \to \nu_1.
\end{aligned}$$

From this, we observe that the assumptions (A4)–(A6) are verified.

Since $\widetilde{\mu}^* = 0.070, \Omega^* = 0.145, \rho = \overline{\rho} = 0.5, \omega = \frac{1}{2}, B(\omega) = 0.7821, \Gamma\left(\frac{1}{2}\right) = 1.7724, \Omega^*_{\mathscr{P}} = 0.039, \widehat{C}_{\mathscr{H}} = 1$.

Furthermore, from Theorem 3.1, we obtain

$$\begin{aligned}
\widehat{M} &= \left[2(1+\widetilde{\mu}^*)\left(\frac{\rho\overline{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)} + \frac{\omega\rho^2\widehat{C}_{\mathscr{H}}}{B(\omega)}\right)\Omega^*\right] \\
&= 0.5329 < 1
\end{aligned}$$

and from Theorem 3.2, we obtain

$$\begin{aligned}
\widehat{M}_1 &= 2\left[(1+\widetilde{\mu}^*)\left(\frac{\rho\overline{\rho}(1-\omega)}{B(\omega)\Gamma(\omega)} + \frac{\omega\rho^2\widehat{C}_{\mathscr{H}}}{B(\omega)}\right)\Omega^* + \left(\frac{\rho\overline{\rho}}{\Gamma(\omega)} + \rho\overline{\rho}\widehat{C}_{\mathscr{H}}\right)\Omega^*_{\mathscr{P}}\right] \\
&= 0.5634 < 1.
\end{aligned}$$

Clearly, all assumptions of Theorems 3.1 and 3.2 are satisfied. Hence by the conclusion of Theorems 3.1 and 3.2, it follows that the system (4.1)–(4.3) has a mild solution.

5. Conclusions

The primary purpose of this investigation was to demonstrate that the ABC derivative, which is one of the most recent non-local derivatives with a non-singular kernel, is used extensively, and to encourage further development in the process of forming deep connections between this derivative and other scientific studies. This theory paved the way for new lines of inquiry in the scientific community, such as the analysis of qualitative and quantitative behavior in a variety of systems. The existence, stability and controllability outcomes for a variety of systems under a variety of assumptions are now being discussed by a large number of scholars.

In this paper, we applied the ABC derivative, which can be found in [9], to the differential structures (1.1)–(1.2) and (1.3)–(1.4) that were taken into consideration. By using the Monch fixed point theorem, it is possible to establish Theorems 3.1 and 3.2, which examine the existence of mild solutions to the systems (1.1)–(1.2) and (1.3)–(1.4). With an appropriate fixed point theorem, the effectiveness of such existing research may be developed for controllability with non-instantaneous impulses for suitable models.

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Conflict of interest

The authors declare that there is no conflict of interest.

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