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*Research article*

# Inclusion results for the class of fuzzy $\alpha$-convex functions

**Shujaat Ali Shah[1,*], Ekram Elsayed Ali[2,3], Asghar Ali Maitlo[1], Thabet Abdeljawad[4,5,*] and Abeer M. Albalahi[2]**

[1] Department of Mathematics and Statistics, Quaid-e-Awam University of Engineering, Science and Technology, Nawabshah 67450, Pakistan
[2] Department of Mathematics, College of Science, University of Ha'il, Ha'il 81451, Saudi Arabia
[3] Department of Mathematics and Computer Science, Faculty of Science, Port Said University, Port Said 42521, Egypt
[4] Department of Mathematics and Sciences, Prince Sultan University, P.O. Box 66833, Riyadh, 11586, KSA
[5] Department of Medical Research, China Medical University, Taichung 40402, Taiwan

* **Correspondence:** Email: shahglike@yahoo.com, tabdeljawad@psu.edu.sa.

**Abstract:** The notion of a fuzzy subset is used to introduce certain subclasses of analytic functions. Mainly, this article presents several inclusion results and integral preserving properties. Also, certain applications of the analytic functions in terms of fuzzy structure will be discussed.

**Keywords:** univalent functions; fuzzy differential subordination; fuzzy starlike functions; fuzzy convex functions
**Mathematics Subject Classification:** 30C45, 30C50

## 1. Introduction

Let $\mho = \{\mathfrak{y} \in \mathbb{C} : |\mathfrak{y}| < 1\}$ denote the unit disk of the complex plane, with $A[a, n]$ consisting of the analytic functions of the form

$$f(\mathfrak{y}) = a + \sum_{k=n}^{\infty} a_k \mathfrak{y}^k, \quad \mathfrak{y} \in \mho,$$

with $A[0, 1] = A$, where $A$ is the class of normalized analytic functions in $\mho$.

In recent days, several authors contributed with great interest to the study of fuzzy differential subordinations. For examples, see [8–12]. Before discussing our main work, we recall some useful basic concepts and preliminaries, as follows.

**Definition 1.1.** *[4] Let $\mathcal{Y}$ be a nonempty set. If $\mathfrak{F}$ is a mapping from $\mathcal{Y}$ to $[0,1]$, then $\mathfrak{F}$ is called a fuzzy subset on $\mathcal{Y}$.*

Alternatively, the fuzzy subset is also defined as the following.

**Definition 1.2.** *[4] A pair $(M, \mathfrak{F}_M)$ is called a fuzzy subset on $\mathcal{Y}$, where $\mathfrak{F}_M : \mathcal{Y} \to [0,1]$ is the membership function of the fuzzy set $(M, \mathfrak{F}_M)$, and $M = \{x \in \mathcal{Y} : 0 < \mathfrak{F}_M(x) \leq 1\} = \sup(M, \mathfrak{F}_M)$ is the support of fuzzy set $(M, \mathfrak{F}_M)$.*

**Definition 1.3.** *[4] Two fuzzy subsets $(M, \mathfrak{F}_M)$ and $(N, \mathfrak{F}_N)$ of $\mathcal{Y}$ are equal if and only if $M = N$, whereas $(M, \mathfrak{F}_M) \subseteq (N, \mathfrak{F}_N)$ if and only if $\mathfrak{F}_M(x) \leq \mathfrak{F}_N(x)$, $x \in \mathcal{Y}$.*

**Definition 1.4.** *[12] Let $\mathcal{R} \subset \mathbb{C}$ and $\mathfrak{y}_0$ be a fixed point in $\mathcal{R}$. Then, analytic function $f$ is subordinate to the analytic function $g$ (written as $f \prec_{\mathfrak{F}} g$ (or $f(\mathfrak{y}) \prec_{\mathfrak{F}} g(\mathfrak{y})$)) if;*

$$f(\mathfrak{y}_0) = g(\mathfrak{y}_0) \quad \text{and} \quad \mathfrak{F}(f(\mathfrak{y})) \leq \mathfrak{F}(g(\mathfrak{y})), \ \mathfrak{y} \in \mathcal{R}.$$

**Remark 1.1.** *We can assume such a function $J_i : \mathbb{C} \to [0,1]$, $(i = 1,2,3,4)$, as any of the following.*

$$J_1(\mathfrak{y}) = \frac{|\mathfrak{y}|}{1+|\mathfrak{y}|}, \ J_2(\mathfrak{y}) = \frac{1}{1+|\mathfrak{y}|}, \ J_3(\mathfrak{y}) = |\sin|\mathfrak{y}||, \ J_4(\mathfrak{y}) = |\cos|\mathfrak{y}||.$$

**Remark 1.2.** *If $\mathcal{R} = \mho$ in Definition 1.4, then the fuzzy subordination coincides with the classical subordination.*

Let $b \in \mathbb{C} \setminus \mathbb{Z}^-$, $s \in \mathbb{C}$ when $|\mathfrak{y}| < 1$, and $\Re(s) > 1$ when $|\mathfrak{y}| = 1$. The authors in [14], introduced an operator $J_b^s : A \to A$ by

$$\begin{aligned} J_b^s f(\mathfrak{y}) &= \psi(s,b) * f(\mathfrak{y}) \\ &= \mathfrak{y} + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b}\right)^s a_n \mathfrak{y}^n, \end{aligned} \tag{1.1}$$

where $*$ denotes the convolution or Hadamard product. This operator is known as the Srivastava-Attiya operator.

By using (1.1), one can easily verify the following identity.

$$\mathfrak{y}\left(J_b^{s+1} f(\mathfrak{y})\right)' = (1+b) J_b^s f(\mathfrak{y}) - b J_b^{s+1} f(\mathfrak{y}). \tag{1.2}$$

The operator $J_b^s$ generalizes some well-known operators studied and introduced in [2,3,5,7].

Motivated by the above literature, we are going to define and study certain subclasses by using the concept of fuzzy subsets. We denote by $\mathcal{T}$ the class of analytic, convex univalent functions $\varphi(\mathfrak{y})$ with $\varphi(0) = 1$ and $\Re(\varphi(\mathfrak{y})) > 0$ in $\mho$.

Now, we are defining the class $\mathfrak{F}M_\alpha(\varphi)$ of fuzzy generalized $\alpha$-convex functions as the following.

**Definition 1.5.** *Let $\varphi \in \mathcal{T}$, $\alpha \in [0,1]$. Then, $f \in A$ is said to be in $\mathfrak{F}M_\alpha(\varphi)$ if and only if*

$$(1-\alpha)\frac{\mathfrak{y} f'(\mathfrak{y})}{f(\mathfrak{y})} + \alpha \frac{(\mathfrak{y} f'(\mathfrak{y}))'}{f'(\mathfrak{y})} \prec_{\mathfrak{F}} \varphi(\mathfrak{y}).$$

*Moreover, let us define*

$$\mathfrak{F}M_0(\varphi) = \mathfrak{F}ST(\varphi) = \left\{ f \in A : \frac{\mathfrak{y} f'(\mathfrak{y})}{f(\mathfrak{y})} \prec_{\mathfrak{F}} \varphi(\mathfrak{y}) \right\}$$

*and*

$$\mathfrak{F}M_1(\varphi) = \mathfrak{F}C(\varphi) = \left\{ f \in A : \frac{(\mathfrak{y} f'(\mathfrak{y}))'}{f'(\mathfrak{y})} \prec_{\mathfrak{F}} \varphi(\mathfrak{y}) \right\}.$$

The following Alexander type relation can be proved easily by using Lemma 2.3.

$$f \in \mathfrak{F}C(\varphi) \text{ if and only if } \mathfrak{y} f' \in \mathfrak{F}ST(\varphi).$$

Now, we define the following classes by making use of the integral operator given by (1.1).

**Definition 1.6.** *Let* $\varphi \in \mathcal{T}$, $\alpha \in [0,1]$. *Then,*

$$f \in \mathfrak{F}M_\alpha^{s,b}(\varphi) \text{ if and only if } J_b^s f \in \mathfrak{F}M_\alpha(\varphi).$$

In addition, let us define

$$\mathfrak{F}M_0^{s,b}(\varphi) = \mathfrak{F}ST_b^s(\varphi) = \{ f \in A : J_b^s f \in \mathfrak{F}ST(\varphi) \}$$

and

$$\mathfrak{F}M_1^{s,b}(\varphi) = \mathfrak{F}C_b^s(\varphi) = \{ f \in A : J_b^s f \in \mathfrak{F}C(\varphi) \}.$$

Special cases:
(i) $\mathfrak{F}M_\alpha^{0,b}(\varphi) = \mathfrak{F}M_\alpha(\varphi)$
(ii) $\mathfrak{F}ST_b^0(\varphi) = \mathfrak{F}ST(\varphi)$
(iii) $\mathfrak{F}C_b^0(\varphi) = \mathfrak{F}C(\varphi)$
(iv) If we choose $\varphi(\mathfrak{y}) = \frac{1+\mathfrak{y}}{1-\mathfrak{y}}$, then we deduce the classes $\mathfrak{F}M_\alpha\left(\frac{1+\mathfrak{y}}{1-\mathfrak{y}}\right) = \mathfrak{F}M_\alpha$, $\mathfrak{F}ST\left(\frac{1+\mathfrak{y}}{1-\mathfrak{y}}\right) = \mathfrak{F}S^*$ and $\mathfrak{F}C\left(\frac{1+\mathfrak{y}}{1-\mathfrak{y}}\right) = \mathfrak{F}C$ of fuzzy $\alpha-$convex, fuzzy starlike and fuzzy convex functions, respectively.
It is noted that

$$f \in \mathfrak{F}C_b^s(\varphi) \text{ if and only if } \mathfrak{y} f' \in \mathfrak{F}ST_b^s(\varphi). \tag{1.3}$$

## 2. Main results

To discuss the main investigations, some required important results are stated as the following.

**Lemma 2.1.** *[9] Let* $P : \mho \to \mathbb{C}$, *with* $\Re(P(\mathfrak{y})) > 0$, *and* $\phi$ *be convex in* $\mho$. *If* $p$ *is analytic in* $\mho$, *and* $\Psi : \mathbb{C}^2 \times \mho \to \mathbb{C}$, *where*

$$\Psi(p(\mathfrak{y}), \mathfrak{y} p'(\mathfrak{y}); \mathfrak{y}) = p(\mathfrak{y}) + P(\mathfrak{y}).\mathfrak{y} p'(\mathfrak{y}),$$

*is analytic in* $\mho$, *then*

$$\mathfrak{F}_{\Psi(\mathbb{C}^2 \times \mho)}\left[p(\mathfrak{y}) + P(\mathfrak{y}).\mathfrak{y} p'(\mathfrak{y})\right] \leq \mathfrak{F}_{\phi(\mho)}(\phi(\mathfrak{y}))$$

*implies*

$$\mathfrak{F}_{p(\mho)}(p(\mathfrak{y})) \leq \mathfrak{F}_{\phi(\mho)}(\phi(\mathfrak{y})), \ \ \mathfrak{y} \in \mho.$$

**Lemma 2.2.** *[11] Let $\varepsilon, \zeta \in \mathbb{C}$, $\varepsilon \neq 0$, and a convex function $\phi$ satisfies*

$$\Re\left(\varepsilon\phi(\mathfrak{y}) + \zeta\right) > 0,\ \mathfrak{y} \in \mho.$$

*If $p$ is analytic in $\mho$ with $p(0) = \phi(0)$, and $\Psi\left(p(\mathfrak{y}), \mathfrak{y}p'(\mathfrak{y}); \mathfrak{y}\right) = p(\mathfrak{y}) + \frac{\mathfrak{y}p'(\mathfrak{y})}{\varepsilon p(\mathfrak{y})+\zeta}$ is analytic in $\mho$ with $\Psi\left(\phi(0), 0; 0\right) = \phi(0)$, then*

$$\mathfrak{F}_{\Psi\left(\mathbb{C}^2\times\mho\right)}\left[p(\mathfrak{y}) + \frac{\mathfrak{y}p'(\mathfrak{y})}{\varepsilon p(\mathfrak{y}) + \zeta}\right] \leq \mathfrak{F}_{\phi(\mho)}\left(\phi(\mathfrak{y})\right)$$

*implies*

$$\mathfrak{F}_{p(\mho)}\left(p(\mathfrak{y})\right) \leq \mathfrak{F}_{\phi(\mho)}\left(\phi(\mathfrak{y})\right),\ \ \mathfrak{y} \in \mho.$$

**Lemma 2.3.** *Let $P$ and $Q$ be analytic in $\mho$ with $P(0) = Q(0)$, and let $\phi(\mathfrak{y})$ be any convex function in $\mho$. If $Q(\mathfrak{y})$ maps $\mho$ onto a (possibly many-sheeted) domain which is starlike with respect to the origin, then*

$$\frac{P'(\mathfrak{y})}{Q'(\mathfrak{y})} \prec_{\mathfrak{F}} \phi(\mathfrak{y}) \text{ implies } \frac{P(\mathfrak{y})}{Q(\mathfrak{y})} \prec_{\mathfrak{F}} \phi(\mathfrak{y}).$$

*Proof.* Assume that

$$\frac{P(\mathfrak{y})}{Q(\mathfrak{y})} = R(\mathfrak{y}), \tag{2.1}$$

where $R(\mathfrak{y})$ is analytic in $\mho$. On logarithmic differentiation of (2.1), we obtain

$$\frac{P'}{P} - \frac{Q'}{Q} = \frac{R'}{R},$$

or equivalently,

$$\frac{\mathfrak{y}P'}{P} = \frac{\mathfrak{y}Q'}{Q} + \frac{\mathfrak{y}R'}{R}.$$

This implies

$$\frac{P'}{Q'} = R(\mathfrak{y}) + \frac{Q}{\mathfrak{y}Q'}\mathfrak{y}R'. \tag{2.2}$$

If we set $\frac{\mathfrak{y}Q'}{Q} = Y(\mathfrak{y})$, then $\Re(Y(\mathfrak{y})) > 0$. Since $\frac{P'}{Q'} \prec_{\mathfrak{F}} \varphi(\mathfrak{y})$, from (2.2),

$$R(\mathfrak{y}) + \frac{1}{Y}\mathfrak{y}R' \prec_{\mathfrak{F}} \varphi(\mathfrak{y}).$$

Lemma 2.1 and (2.2) imply that $\frac{P(\mathfrak{y})}{Q(\mathfrak{y})} \prec_{\mathfrak{F}} \varphi(\mathfrak{y})$. □

**Theorem 2.1.** *Let $\varphi \in \mathcal{T}$ and $0 \leq \alpha \leq 1$. Then,*

$$\mathfrak{F}M_\alpha\left(\varphi\right) \subset \mathfrak{F}ST\left(\varphi\right).$$

*Proof.* Let $f \in \mathfrak{F}M_\alpha\left(\varphi\right)$, and let

$$\frac{\mathfrak{y}f'(\mathfrak{y})}{f(\mathfrak{y})} = P(\mathfrak{y}). \tag{2.3}$$

We note that $P(\mathfrak{y})$ is analytic in $\mho$ with $P(0) = 1$.

The logarithmic differentiation of (2.3) yields

$$\frac{(\mathfrak{y}f'(\mathfrak{y}))'}{\mathfrak{y}f'(\mathfrak{y})} - \frac{f'(\mathfrak{y})}{f(\mathfrak{y})} = \frac{P'(\mathfrak{y})}{P(\mathfrak{y})}.$$

Equivalently,

$$\frac{(\mathfrak{y}f'(\mathfrak{y}))'}{f'(\mathfrak{y})} = P(\mathfrak{y}) + \frac{\mathfrak{y}P'(\mathfrak{y})}{P(\mathfrak{y})}. \tag{2.4}$$

Since $f \in \mathfrak{F}M_\alpha(\varphi)$, from (2.3) and (2.4), we get

$$(1-\alpha)\frac{\mathfrak{y}f'(\mathfrak{y})}{f(\mathfrak{y})} + \alpha\frac{(\mathfrak{y}f'(\mathfrak{y}))'}{f'(\mathfrak{y})} = P(\mathfrak{y}) + \alpha\frac{\mathfrak{y}P'(\mathfrak{y})}{P(\mathfrak{y})} \prec_{\mathfrak{F}} \varphi(\mathfrak{y}). \tag{2.5}$$

We use (2.5) along with Lemma 2.2 to get $P(\mathfrak{y}) \prec_{\mathfrak{F}} \varphi(\mathfrak{y})$. Hence, $f \in \mathfrak{F}ST(\varphi)$. □

**Corollary 2.1.** *When $\alpha = 1$, we get $\mathfrak{F}C(\varphi) \subset \mathfrak{F}ST(\varphi)$. Furthermore, for $\varphi(\mathfrak{y}) = \frac{1+\mathfrak{y}}{1-\mathfrak{y}}$, we obtain $\mathfrak{F}C \subset \mathfrak{F}S^*$.*

**Theorem 2.2.** *Let $\varphi \in \mathcal{T}$ and $\alpha > 1$. Then,*

$$\mathfrak{F}M_\alpha(\varphi) \subset \mathfrak{F}C(\varphi).$$

*Proof.* Let $f \in \mathfrak{F}M_\alpha(\varphi)$. Then, by Definition 1.5,

$$(1-\alpha)\frac{\mathfrak{y}f'(\mathfrak{y})}{f(\mathfrak{y})} + \alpha\frac{(\mathfrak{y}f'(\mathfrak{y}))'}{f'(\mathfrak{y})} = p_1(\mathfrak{y}) \prec_{\mathfrak{F}} \varphi(\mathfrak{y}).$$

Now,

$$\begin{aligned} \alpha\frac{(\mathfrak{y}f'(\mathfrak{y}))'}{f'(\mathfrak{y})} &= (1-\alpha)\frac{\mathfrak{y}f'(\mathfrak{y})}{f(\mathfrak{y})} + \alpha\frac{(\mathfrak{y}f'(\mathfrak{y}))'}{f'(\mathfrak{y})} + (\alpha-1)\frac{\mathfrak{y}f'(\mathfrak{y})}{f(\mathfrak{y})} \\ &= (\alpha-1)\frac{\mathfrak{y}f'(\mathfrak{y})}{f(\mathfrak{y})} + p_1(\mathfrak{y}). \end{aligned}$$

This implies

$$\begin{aligned} \frac{(\mathfrak{y}f'(\mathfrak{y}))'}{f'(\mathfrak{y})} &= \left(\frac{1}{\alpha}-1\right)\frac{\mathfrak{y}f'(\mathfrak{y})}{f(\mathfrak{y})} + \frac{1}{\alpha}p_1(\mathfrak{y}) \\ &= \left(\frac{1}{\alpha}-1\right)p_2(\mathfrak{y}) + \frac{1}{\alpha}p_1(\mathfrak{y}). \end{aligned}$$

Since $p_1, p_2 \prec_{\mathfrak{F}} \varphi(\mathfrak{y})$, $\frac{(\mathfrak{y}f'(\mathfrak{y}))'}{f'(\mathfrak{y})} \prec_{\mathfrak{F}} \varphi(\mathfrak{y})$. This is our required result. □

**Theorem 2.3.** *For $\varphi \in \mathcal{T}$ and $0 \le \alpha_1 < \alpha_2 < 1$,*

$$\mathfrak{F}M_{\alpha_2}(\varphi) \subset \mathfrak{F}M_{\alpha_1}(\varphi).$$

*Proof.* From the previous result, this is straight forward for $\alpha_1 = 0$.

Let $f \in \mathfrak{F}M_{\alpha_2}(\varphi)$. Then, by Definition 1.5,

$$(1-\alpha_2)\frac{\mathfrak{y}f'(\mathfrak{y})}{f(\mathfrak{y})}+\alpha_2\frac{(\mathfrak{y}f'(\mathfrak{y}))'}{f'(\mathfrak{y})}=q_1(\mathfrak{y})\prec_F\varphi(\mathfrak{y}). \tag{2.6}$$

Now, we can easily write

$$(1-\alpha_1)\frac{\mathfrak{y}f'(\mathfrak{y})}{f(\mathfrak{y})}+\alpha_1\frac{(\mathfrak{y}f'(\mathfrak{y}))'}{f'(\mathfrak{y})}=\frac{\alpha_1}{\alpha_2}q_1(\mathfrak{y})+\left(1-\frac{\alpha_1}{\alpha_2}\right)q_2(\mathfrak{y}), \tag{2.7}$$

where we have used (2.6) and $\frac{\mathfrak{y}f'(\mathfrak{y})}{f(\mathfrak{y})}=q_2(\mathfrak{y})\prec_{\mathfrak{F}}\varphi(\mathfrak{y})$. Since $q_1,q_2\prec_{\mathfrak{F}}\varphi(\mathfrak{y})$, (2.7) implies our required result. □

**Remark 2.1.** *If $\alpha_2 = 1$ and $f\in\mathfrak{F}M_1(\varphi)=\mathfrak{F}C(\varphi)$, then the previous result gives us*

$$f\in\mathfrak{F}M_{\alpha_1}(\varphi),\ \textit{for}\ \ 0\le\alpha_1<1.$$

*Thus, we can deduce $\mathfrak{F}C(\varphi)\subset\mathfrak{F}ST(\varphi)$ by employing Theorem 2.1.*

Now, we examine certain properties of subclasses associated with the integral operator given by (1.1).

**Theorem 2.4.** *Let $\varphi\in\mathcal{T}$, $0\le\alpha\le1$, $s>0$ and $b\in\mathbb{N}$. Then,*

$$\mathfrak{F}M_{\alpha}^{s,b}(\varphi)\subset\mathfrak{F}ST_b^s(\varphi).$$

*Proof.* To prove this inclusion result, we apply a similar technique as used in Theorem 2.1 by setting

$$\frac{\mathfrak{y}\left(J_b^sf\right)'(\mathfrak{y})}{J_b^sf(\mathfrak{y})}=P(\mathfrak{y}),$$

for analytic $P(\mathfrak{y})$ in $\mho$ with $P(0) = 1$. □

**Theorem 2.5.** *Let $\varphi\in\mathcal{T}$, $s>0$ and $b\in\mathbb{N}$. Then,*

$$\mathfrak{F}ST_b^s(\varphi)\subset\mathfrak{F}ST_b^{s+1}(\varphi).$$

*Proof.* Let $f\in\mathfrak{F}ST_b^s(\varphi)$. Then,

$$\frac{\mathfrak{y}\left(J_b^sf\right)'(\mathfrak{y})}{J_b^sf(\mathfrak{y})}\prec_{\mathfrak{F}}\varphi(\mathfrak{y}).$$

Now, let

$$\frac{\mathfrak{y}\left(J_b^{s+1}f\right)'(\mathfrak{y})}{J_b^{s+1}f(\mathfrak{y})}=P(\mathfrak{y}), \tag{2.8}$$

for analytic $P(\mathfrak{y})$ in $\mho$ with $P(0) = 1$. We use (1.2) and (2.8) to obtain

$$\frac{\mathfrak{y}\left(J_b^{s+1}f\right)'(\mathfrak{y})}{J_b^{s+1}f(\mathfrak{y})}=(1+b)\frac{\mathfrak{y}\left(J_b^sf\right)'(\mathfrak{y})}{J_b^{s+1}f(\mathfrak{y})}-b;$$

equivalently,

$$(1+b)\frac{\mathfrak{y}\left(J_b^sf\right)'(\mathfrak{y})}{J_b^{s+1}f(\mathfrak{y})}=P(\mathfrak{y})+b.$$

The logarithmic differentiation yields,

$$\frac{\mathfrak{y}\left(J_b^s f\right)'(\mathfrak{y})}{J_b^s f(\mathfrak{y})} = P(\mathfrak{y}) + \frac{\mathfrak{y}P'(\mathfrak{y})}{P(\mathfrak{y})+b}. \tag{2.9}$$

Since $f \in \mathfrak{F}ST_b^s(\varphi)$, (2.9) implies

$$P(\mathfrak{y}) + \frac{\mathfrak{y}P'(\mathfrak{y})}{P(\mathfrak{y})+b} \prec_{\mathfrak{F}} \varphi(\mathfrak{y}). \tag{2.10}$$

We use (2.10) along with Lemma 2.2 to get $P(\mathfrak{y}) \prec_{\mathfrak{F}} \varphi(\mathfrak{y})$. Hence, $f \in \mathfrak{F}ST_b^{s+1}(\varphi)$. □

**Theorem 2.6.** *Let $\varphi \in \mathcal{T}$, $s > 0$ and $b \in \mathbb{N}$. Then,*

$$\mathfrak{F}C_b^s(\varphi) \subset \mathfrak{F}C_b^{s+1}(\varphi).$$

*Proof.* Let

$$\begin{aligned}
& f \in \mathfrak{F}C_b^s(\varphi) \\
\Leftrightarrow\ & \mathfrak{y}f' \in \mathfrak{F}ST_b^s(\varphi), && \text{(by (1.3))}, \\
\Rightarrow\ & \mathfrak{y}f' \in \mathfrak{F}ST_b^{s+1}(\varphi), && \text{(by Theorem 2.5)}, \\
\Leftrightarrow\ & f \in \mathfrak{F}C_b^{s+1}(\varphi), && \text{(by (1.3))}.
\end{aligned}$$

□

**Theorem 2.7.** *Let $f \in \mathbf{A}$. Then, $f \in \mathfrak{F}M_\alpha(\varphi)$, $\alpha \neq 0$, if and only if there exists $g \in \mathfrak{F}ST(\varphi)$ such that*

$$f(\mathfrak{y}) = \frac{1}{\alpha}\left[\int_0^t t^{\frac{1}{\alpha}-1}\left(\frac{g(t)}{t}\right)^{\frac{1}{\alpha}} d_q t\right]^{\alpha}. \tag{2.11}$$

*Proof.* Let $f \in \mathfrak{F}M_\alpha(\varphi)$. Then,

$$(1-\alpha)\frac{\mathfrak{y}f'(\mathfrak{y})}{f(\mathfrak{y})} + \alpha\frac{(\mathfrak{y}f'(\mathfrak{y}))'}{f'(\mathfrak{y})} \prec_{\mathfrak{F}} \varphi(\mathfrak{y}). \tag{2.12}$$

On some simple calculations of (2.11), we get

$$\mathfrak{y}(f)'.(f(\mathfrak{y}))^{\frac{1}{\alpha}} = (g(\mathfrak{y}))^{\frac{1}{\alpha}}. \tag{2.13}$$

The logarithmic differentiation of (2.13), gives

$$(1-\alpha)\frac{\mathfrak{y}f'(\mathfrak{y})}{f(\mathfrak{y})} + \alpha\frac{(\mathfrak{y}f'(\mathfrak{y}))'}{f'(\mathfrak{y})} = \frac{\mathfrak{y}g'(\mathfrak{y})}{g(\mathfrak{y})}. \tag{2.14}$$

Our result follows by using (2.12) and (2.14). □

**Theorem 2.8.** *Let $f \in \mathfrak{F}M_\alpha^{s,b}(\varphi)$, and define*

$$f_\upsilon(\mathfrak{y}) = \frac{\upsilon+1}{\mathfrak{y}^\upsilon}\int_0^{\mathfrak{y}} t^{\upsilon-1}f(t)dt. \tag{2.15}$$

*Then, $f_\upsilon \in \mathfrak{F}ST_b^s(\varphi)$.*

*Proof.* Let $f \in \mathfrak{F}M_{\alpha}^{s,b}(\varphi)$ and $f_{\upsilon,b}^{s}(\mathfrak{y}) = J_{b}^{s}(f_{\upsilon}(\mathfrak{y}))$. We assume

$$\frac{\mathfrak{y}\left(f_{\upsilon,b}^{s}(\mathfrak{y})\right)'}{f_{\upsilon,b}^{s}(\mathfrak{y})} = Q(\mathfrak{y}), \tag{2.16}$$

for analytic $Q(\mathfrak{y})$ in $\mho$ with $Q(0) = 1$.

We use (2.15) to get

$$\frac{(\mathfrak{y}^{\upsilon} f_{\upsilon}(\mathfrak{y}))'}{\upsilon + 1} = \mathfrak{y}^{\upsilon-1} f(\mathfrak{y}).$$

This implies

$$\mathfrak{y}\,(f_{\upsilon}(\mathfrak{y}))' = (1+\upsilon)\, f(\mathfrak{y}) - \upsilon f_{\upsilon}(\mathfrak{y}). \tag{2.17}$$

We use (2.16), (2.17) and (1.1), to get

$$Q(\mathfrak{y}) = (1+\upsilon)\, \frac{\mathfrak{y}\left(f_{b}^{s}(\mathfrak{y})\right)}{f_{\upsilon,b}^{s}(\mathfrak{y})} - \upsilon,$$

where $f_{\upsilon,b}^{s}(\mathfrak{y}) = J_{b}^{s}(f_{\upsilon}(\mathfrak{y}))$, and $f_{b}^{s}(\mathfrak{y}) = J_{b}^{s}(f(\mathfrak{y}))$. From logarithmic differentiation, we have

$$\frac{\mathfrak{y}\left(f_{b}^{s}(\mathfrak{y})\right)'}{f_{b}^{s}(\mathfrak{y})} = Q(\mathfrak{y}) + \frac{\mathfrak{y}Q'(\mathfrak{y})}{Q(\mathfrak{y}) + \upsilon}. \tag{2.18}$$

Since $f \in \mathfrak{F}M_{\alpha}^{s,b}(\varphi) \subset \mathfrak{F}ST_{b}^{s}(\varphi)$, (2.18) implies

$$Q(\mathfrak{y}) + \frac{\mathfrak{y}Q'(\mathfrak{y})}{Q(\mathfrak{y}) + \upsilon} \prec_{\mathfrak{F}} \varphi(\mathfrak{y}).$$

This implies our required result by using Lemma 2.2. □

## 3. Conclusions

Nowadays, the link between two different fields, geometric function theory and fuzzy set theory, is well studied by various prominent mathematicians. In this connection, we have introduced certain subclasses associated with the linear operator by using the notion of a fuzzy subset and studied various interesting properties, such as inclusion results and integral preserving properties. This study is to be continued by using these two theories. Several newly defined subclasses of univalent or multivalent functions (see [1, 6, 13]) may be generalized in terms of fuzzy notions, and some original results for such subclasses will also be examined in the future.

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## Conflict of interest

The authors declare no conflict of interest.

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