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Research article

Common fixed point theorems on complex valued extended b-metric spaces for rational contractions with application

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Abstract: The purpose of this article is to establish common fixed point results on complex valued extended b-metric spaces for the mappings satisfying rational expressions on a closed ball. Our investigations generalize some well-known results of literature. Furthermore, we supply a significant example to show the authenticity of established results. As application, we solve Urysohn integral equations by our main results.

Keywords: common fixed point; complex valued extended b-metric space; rational expressions; closed ball
Mathematics Subject Classification: 37C25, 47H10, 54H25

1. Introduction

The notion of complex valued metric space is introduced by Azam et al. [1] in 2011 and established common fixed points of self mappings satisfying rational contractions. Later on, Rouzkard et al. [2] gave generalized contraction and extended the leading theorem of Azam et al. [1]. Subsequently, Sintunavarat et al. [3] replaced the constants involved in the contraction with control functions of one variable and generalized the results of Azam et al. [1] and Rouzkard et al. [2]. Sitthikul et al. [4] used control functions of two vaiables in the contraction and established common fixed point theorems in context of complex valued metric space. Although many researchers [5–10] worked in this space and proved different generalized results. Mukheimer [11] gave the notion of complex valued b-metric space (CVbMS) by involving a constant $\pi \geq 1$ in the triangle inequality and generalized the concept of complex valued metric space (CVMS). Kumar [12] and Rao et al. [13] proved common fixed point results in CVbMS for generalized contractions. Naimatullah et al. [14] replaced contant with a control function and extended the concept of complex valued b-metric space (CVbMS) to complex valued extended b-metric space (CVEbMS). They proved fixed points of multivalued mappings for contractions involving rational expressions in CVEbMS. For more details in this direction, we refer the readers to [15–22].

In this paper, we obtain common fixed point theorems in complex valued extended b-metric spaces (CVEbMS) for rational contractions with contractiveness on a closed ball. We also provide a significant example to show the originality of obtained results.

2. Preliminaries

Azam et al. [1] gave the notion of complex valued metric space (CVMS) in this way.

Definition 1. *(See [1])Let $\omega_1, \omega_2 \in \mathbb{C}$ (set of complex numbers). A partial order $\precsim$ on $\mathbb{C}$ is defined as follows*

$$\omega_1 \precsim \omega_2 \;\Leftrightarrow\; Re(\omega_1) \leqslant Re(\omega_2),\; Im(\omega_1) \leqslant Im(\omega_2).$$

It follows that

$$\omega_1 \precsim \omega_2,$$

if one of these assertions is satisfied:

$$\begin{aligned}
&\text{(a) } \mathrm{Re}(\omega_1) = \mathrm{Re}(\omega_2),\; \mathrm{Im}(\omega_1) < \mathrm{Im}(\omega_2),\\
&\text{(b) } \mathrm{Re}(\omega_1) < \mathrm{Re}(\omega_2),\; \mathrm{Im}(\omega_1) = \mathrm{Im}(\omega_2),\\
&\text{(c) } \mathrm{Re}(\omega_1) < \mathrm{Re}(\omega_2),\; \mathrm{Im}(\omega_1) < \mathrm{Im}(\omega_2),\\
&\text{(d) } \mathrm{Re}(\omega_1) = \mathrm{Re}(\omega_2),\; \mathrm{Im}(\omega_1) = \mathrm{Im}(\omega_2).
\end{aligned}$$

Definition 2. *(See [1]) Let $\mathcal{W} \neq \emptyset$ and $d : \mathcal{W} \times \mathcal{W} \to \mathbb{C}$ satisfy*

(i) $0 \precsim d(\omega, \varrho)$ and $d(\omega, \varrho) = 0$ if and only if $\omega = \varrho$;
(ii) $d(\omega, \varrho) = d(\varrho, \omega)$;
(iii) $d(\omega, \varrho) \precsim d(\omega, \nu) + d(\nu, \varrho)$;

for all $\omega, \varrho, \nu \in \mathcal{W}$, then $(\mathcal{W}, d)$ is said to be complex valued metric space (CVMS).

Example 3. *(See [1]) Let $\mathcal{W} = [0, 1]$. Define $d : \mathcal{W} \times \mathcal{W} \to \mathbb{C}$ by*

$$d(\omega, \varrho) = \begin{cases} 0, & \text{if } \omega = \varrho, \\ \frac{i}{2}, & \text{if } \omega \neq \varrho, \end{cases}$$

for all $\omega, \varrho \in \mathcal{W}$, then $(\mathcal{W}, d)$ is CVMS.

Mukheimer [11] gave the conception of complex valued b-metric space (CVbMS) in this way.

Definition 4. *(See [11]) Let $\mathcal{W} \neq \emptyset$ and $\pi \geq 1$ be a real number. If a mapping $d : \mathcal{W} \times \mathcal{W} \to \mathbb{C}$ satisfy*

(i) $0 \precsim d(\omega, \varrho)$ and $d(\omega, \varrho) = 0$ if and only if $\omega = \varrho$;
(ii) $d(\omega, \varrho) = d(\varrho, \omega)$;
(iii) $d(\omega, \varrho) \precsim \pi\left[d(\omega, \nu) + d(\nu, \varrho)\right]$;

for all $\omega, \varrho, \nu \in \mathcal{W}$, then $(\mathcal{W}, d)$ is called a complex valued b- metric space (CVbMS).

Example 5. *(See [11]) Let* $\mathcal{W} = [0,1]$. *Define* $d : \mathcal{W} \times \mathcal{W} \to \mathbb{C}$ *by*

$$d(\omega, \varrho) = |\omega - \varrho|^2 + i|\omega - \varrho|^2,$$

for all $\omega, \varrho \in \mathcal{W}$, *then* $(\mathcal{W}, d)$ *is CVbMS with* $\pi = 2$.

Recently, Naimatullah et al. [14] defined the notion of complex valued extended *b*-metric space (CVE*b*MS) in this way.

Definition 6. *(See [14]) Let* $\mathcal{W} \neq \emptyset$ *and* $\varphi : \mathcal{W} \times \mathcal{W} \to [1, \infty)$. *If a mapping* $d : \mathcal{W} \times \mathcal{W} \to \mathbb{C}$ *satisfy*

(i) $0 \precsim d(\omega, \varrho)$ and $d(\omega, \varrho) = 0$ if and only if $\omega = \varrho$;
(ii) $d(\omega, \varrho) = d(\varrho, \omega)$;
(iii) $d(\omega, \varrho) \precsim \varphi(\omega, \varrho)\left[d(\omega, \nu) + d(\nu, \varrho)\right]$;

for all $\omega, \varrho, \nu \in \mathcal{W}$, then $(\mathcal{W}, d)$ is called CVE*b*MS.

Example 7. *(See [14]) Let* $\mathcal{W} \neq \emptyset$ *and* $\varphi : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ *be defined by*

$$\varphi(\omega, \varrho) = \frac{1 + \omega + \varrho}{\omega + \varrho},$$

and $d : \mathcal{W} \times \mathcal{W} \to \mathbb{C}$ *by*

(i) $d(\omega, \varrho) = \frac{i}{\omega\varrho}$, for all $0 < \omega, \varrho \leq 1$;

(ii) $d(\omega, \varrho) = 0$ if and only if $\omega = \varrho$, for all $0 \leq \omega, \varrho \leq 1$;

(iii) $d(\omega, 0) = d(0, \omega) = \frac{i}{\omega}$, for all $0 < \omega \leq 1$.

Then $(\mathcal{W}, d)$ is a CVE*b*MS.

Example 8. *Let* $\mathcal{W} = [0, \infty)$ *and* $\varphi : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ *be a function defined by* $\varphi(\omega, \varrho) = 1 + \omega + \varrho$ *and* $d : \mathcal{W} \times \mathcal{W} \to \mathbb{C}$ *by*

$$d(\omega, \varrho) = \begin{cases} 0, & \text{if } \omega = \varrho, \\ i, & \text{if } \omega \neq \varrho. \end{cases}$$

Then $(\mathcal{W}, d)$ *is a CVEbMS.*

Lemma 9. *(See [14]) Let* $(\mathcal{W}, d)$ *be a CVEbMS and* $\{\omega_n\} \subseteq \mathcal{W}$, *then* $\{\omega_n\}$ *converges to* ω *if and only if* $|d(\omega_n, \omega)| \to 0$, *as* $n \to \infty$.

Lemma 10. *(See [14]) Let* $(\mathcal{W}, d)$ *be a CVEbMS and* $\{\omega_n\} \subseteq \mathcal{W}$, *then* $\{\omega_n\}$ *is a Cauchy sequence if and only if* $|d(\omega_n, \omega_m)| \to 0$ *as* $n, m \to \infty$.

3. Results

Now we state our main result in this way.

Theorem 11. *Let* $(\mathcal{W}, d)$ *be a complete CVEbMS with* $\varphi : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ *and* $\beth_1, \beth_2 : \mathcal{W} \to \mathcal{W}$. *Suppose that there exist* $\aleph_1, \aleph_2, \aleph_3, \aleph_4, \aleph_5 \in [0, 1)$ *with* $\aleph_1 + \aleph_2 + \aleph_3 + 2\aleph_4 + 2\aleph_5 < 1$ *such that*

$$\begin{aligned} d(\beth_1\omega, \beth_2\varrho) \precsim\ & \aleph_1 d(\omega, \varrho) + \aleph_2 \frac{d(\omega, \beth_1\omega)\, d(\varrho, \beth_2\varrho)}{1 + d(\omega, \varrho)} + \aleph_3 \frac{d(\varrho, \beth_1\omega)\, d(\omega, \beth_2\varrho)}{1 + d(\omega, \varrho)} \\ & + \aleph_4 \frac{d(\omega, \beth_1\omega)\, d(\omega, \beth_2\varrho)}{1 + d(\omega, \varrho)} + \aleph_5 \frac{d(\varrho, \beth_1\omega)\, d(\varrho, \beth_2\varrho)}{1 + d(\omega, \varrho)}, \end{aligned} \tag{3.1}$$

for all $\omega_0, \omega, \varrho \in \overline{B(\omega_0, r)}$, $0 \prec r \in \mathbb{C}$ *and*

$$|d(\omega_0, \beth_1\omega_0)| \leq (1 - \lambda)|r|, \tag{3.2}$$

where $\lambda = \max\{(\frac{\aleph_1+\aleph_4}{1-\aleph_2-\aleph_4}), (\frac{\aleph_1+\aleph_5}{1-\aleph_2-\aleph_5})\}$. *And for each* $\omega_0 \in \overline{B(\omega_0, r)}$ *and* $\lim_{n,m\to+\infty} \varphi(\omega_n, \omega_m)\lambda < 1$, *then there exists a unique point* $\omega^* \in \overline{B(\omega_0, r)}$ *such that* $\omega^* = \beth_1\omega^* = \beth_2\omega^*$.

Proof. Let $\omega_0 \in \mathcal{W}$ and define

$$\omega_{2n+1} = \beth_1\omega_{2n} \text{ and } \omega_{2n+2} = \beth_2\omega_{2n+1},$$

for all $n = 0, 1, 2, \ldots$. Now we show that $\omega_n \in \overline{B(\omega_0, r)}$, for all $n \in \mathbb{N}$. By the fact that $\lambda = \max\{(\frac{\aleph_1+\aleph_4}{1-\aleph_2-\aleph_4}), (\frac{\aleph_1+\aleph_5}{1-\aleph_2-\aleph_5})\} < 1$ and inequality (3.2), we have

$$|d(\omega_0, \beth_1\omega_0)| \leq |r|.$$

It implies that $\omega_1 \in \overline{B(\omega_0, r)}$. Let $\omega_2, ..., \omega_j \in \overline{B(\omega_0, r)}$ for some $j \in N$. If $j = 2n + 1$, where $n = 0, 1, 2, \ldots \frac{j-1}{2}$ or $j = 2n + 2$, where $n = 0, 1, 2, \ldots, \frac{j-2}{2}$. By (3.1), we have

$$\begin{aligned} d(\omega_{2n+1}, \omega_{2n+2}) &= d(\beth_1\omega_{2n}, \beth_2\omega_{2n+1}) \\ &\precsim \aleph_1 d(\omega_{2n}, \omega_{2n+1}) + \aleph_2 \frac{d(\omega_{2n+1}, \beth_2\omega_{2n+1})\, d(\omega_{2n}, \beth_1\omega_{2n})}{1 + d(\omega_{2n}, \omega_{2n+1})} \\ &\quad + \aleph_3 \frac{d(\omega_{2n}, \beth_2\omega_{2n+1})\, d(\omega_{2n+1}, \beth_1\omega_{2n})}{1 + d(\omega_{2n}, \omega_{2n+1})} \\ &\quad + \aleph_4 \frac{d(\omega_{2n}, \beth_2\omega_{2n+1})\, d(\omega_{2n}, \beth_1\omega_{2n})}{1 + d(\omega_{2n}, \omega_{2n+1})} \\ &\quad + \aleph_5 \frac{d(\omega_{2n+1}, \beth_2\omega_{2n+1})\, d(\omega_{2n+1}, \beth_1\omega_{2n})}{1 + d(\omega_{2n}, \omega_{2n+1})}. \end{aligned}$$

Now $\omega_{2n+1} = \beth_1\omega_{2n}$ implies that $d(\omega_{2n+1}, \beth_1\omega_{2n}) = 0$, so we have

$$\begin{aligned} d(\omega_{2n+1}, \omega_{2n+2}) &\precsim \aleph_1 d(\omega_{2n}, \omega_{2n+1}) + \aleph_2 \frac{d(\omega_{2n+1}, \omega_{2n+2})\, d(\omega_{2n}, \omega_{2n+1})}{1 + d(\omega_{2n}, \omega_{2n+1})} \\ &\quad + \aleph_4 \frac{d(\omega_{2n}, \omega_{2n+2})\, d(\omega_{2n}, \omega_{2n+1})}{1 + d(\omega_{2n}, \omega_{2n+1})}. \end{aligned}$$

This implies that

$$\begin{aligned}|d(\omega_{2n+1},\omega_{2n+2})| &\leq \aleph_1|d(\omega_{2n},\omega_{2n+1})| + \aleph_2\frac{|d(\omega_{2n+1},\omega_{2n+2})||d(\omega_{2n},\omega_{2n+1})|}{|1+d(\omega_{2n},\omega_{2n+1})|}\\ &\quad +\aleph_4\frac{|d(\omega_{2n},\omega_{2n+2})||d(\omega_{2n},\omega_{2n+1})|}{|1+d(\omega_{2n},\omega_{2n+1})|}.\end{aligned}$$

Since $|1+d(\omega_{2n},\omega_{2n+1})| > |d(\omega_{2n},\omega_{2n+1})|$, so we have

$$|d(\omega_{2n+1},\omega_{2n+2})| \leq \aleph_1|d(\omega_{2n},\omega_{2n+1})| + \aleph_2|d(\omega_{2n+1},\omega_{2n+2})| + \aleph_4|d(\omega_{2n},\omega_{2n+2})|.$$

Which implies that by triangular inequality

$$|d(\omega_{2n+1},\omega_{2n+2})| \leq \frac{(\aleph_1+\aleph_4)}{(1-\aleph_2-\aleph_4)}|d(\omega_{2n},\omega_{2n+1})|. \tag{3.3}$$

Similarly, we get

$$\begin{aligned}d(\omega_{2n+2},\omega_{2n+3}) &= d(\beth_1\omega_{2n+2},\beth_2\omega_{2n+1})\\ &\precsim \aleph_1 d(\omega_{2n+2},\omega_{2n+1}) + \aleph_2\frac{d(\omega_{2n+1},\beth_2\omega_{2n+1})\,d(\omega_{2n+2},\beth_1\omega_{2n+2})}{1+d(\omega_{2n+2},\omega_{2n+1})}\\ &\quad +\aleph_3\frac{d(\omega_{2n+2},\beth_2\omega_{2n+1})\,d(\omega_{2n+1},\beth_1\omega_{2n+2})}{1+d(\omega_{2n+2},\omega_{2n+1})}\\ &\quad +\aleph_4\frac{d(\omega_{2n+2},\beth_2\omega_{2n+1})\,d(\omega_{2n+2},\beth_1\omega_{2n+2})}{1+d(\omega_{2n+2},\omega_{2n+1})}\\ &\quad +\aleph_5\frac{d(\omega_{2n+1},\beth_2\omega_{2n+1})\,d(\omega_{2n+1},\beth_1\omega_{2n+2})}{1+d(\omega_{2n+2},\omega_{2n+1})}.\end{aligned}$$

Now $\omega_{2n+2} = \beth_2\omega_{2n+1}$ implies that $d(\omega_{2n+2},\beth_2\omega_{2n+1}) = 0$, so we have

$$\begin{aligned}d(\omega_{2n+2},\omega_{2n+3}) &\precsim \aleph_1 d(\omega_{2n+2},\omega_{2n+1}) + \aleph_2\frac{d(\omega_{2n+1},\omega_{2n+2})\,d(\omega_{2n+2},\omega_{2n+3})}{1+d(\omega_{2n+2},\omega_{2n+1})}\\ &\quad +\aleph_5\frac{d(\omega_{2n+1},\omega_{2n+2})\,d(\omega_{2n+1},\omega_{2n+3})}{1+d(\omega_{2n+2},\omega_{2n+1})}.\end{aligned}$$

This implies that

$$\begin{aligned}|d(\omega_{2n+2},\omega_{2n+3})| &\leq \aleph_1|d(\omega_{2n+2},\omega_{2n+1})| + \aleph_2\frac{|d(\omega_{2n+1},\omega_{2n+2})||d(\omega_{2n+2},\omega_{2n+3})|}{|1+d(\omega_{2n+1},\omega_{2n+2})|}\\ &\quad +\aleph_5\frac{|d(\omega_{2n+1},\omega_{2n+2})||d(\omega_{2n+1},\omega_{2n+3})|}{|1+d(\omega_{2n+2},\omega_{2n+1})|}.\end{aligned}$$

Since $|1+d(\omega_{2n+2},\omega_{2n+1})| > |d(\omega_{2n+2},\omega_{2n+1})|$, so we have

$$|d(\omega_{2n+2},\omega_{2n+3})| \leq \aleph_1|d(\omega_{2n+2},\omega_{2n+1})| + \aleph_2|d(\omega_{2n+2},\omega_{2n+3})| + \aleph_5|d(\omega_{2n+1},\omega_{2n+3})|.$$

Which implies that by triangular inequality

$$|d(\omega_{2n+2},\omega_{2n+3})| \leq \frac{(\aleph_1+\aleph_5)}{(1-\aleph_2-\aleph_5)}|d(\omega_{2n+2},\omega_{2n+1})|. \tag{3.4}$$

Putting $\lambda = \max\{(\frac{\aleph_1+\aleph_4}{1-\aleph_2-\aleph_4}), (\frac{\aleph_1+\aleph_5}{1-\aleph_2-\aleph_5})\}$, we obtain that

$$|d(\omega_j, \omega_{j+1})| \leq \lambda^j |d(\omega_0, \omega_1)| \text{ for some } j \in N. \tag{3.5}$$

Now

$$\begin{aligned} |d(\omega_0, \omega_{j+1})| &\leq |d(\omega_0, \omega_1)| + ... + |d(\omega_j, \omega_{j+1})| \\ &\leq |d(\omega_0, \omega_1)| + ... + \lambda^j |d(\omega_0, \omega_1)| \\ &= |d(\omega_0, \omega_1)|[1 + ... + \lambda^{j-1} + \lambda^j] \\ &\leq (1-\lambda)(|r|) \frac{(1-\lambda^{j+1})}{1-\lambda} \\ &\leq |r|, \end{aligned}$$

gives $\omega_{j+1} \in \overline{B(\omega_0, r)}$. Hence $\omega_n \in \overline{B(\omega_0, r)}$ for all $n \in \mathbb{N}$. One can easily prove that

$$|d(\omega_n,\ \omega_{n+1})| \leq \lambda^n |d(\omega_0,\ \omega_1)|,$$

for all $n \in \mathbb{N}$. Now for $m > n$ and by triangular inequality, we have

$$\begin{aligned} |d(\omega_n, \omega_m)| &\leq \varphi(\omega_n, \omega_m)\lambda^n |d(\omega_0, \omega_1)| \\ &\quad + \varphi(\omega_n, \omega_m)\varphi(\omega_{n+1}, \omega_m)\lambda^{n+1} |d(\omega_0, \omega_1)| \\ &\quad + \cdots + \\ &\quad \varphi(\omega_n, \omega_m)\varphi(\omega_{n+1}, \omega_m)\cdots\varphi(\omega_{m-2}, \omega_m)\varphi(\omega_{m-1}, \omega_m)\lambda^{m-1} |d(\omega_0, \omega_1)| \\ &\leq |d(\omega_0, \omega_1)| \begin{bmatrix} \varphi(\omega_n, \omega_m)\lambda^n \\ +\varphi(\omega_n, \omega_m)\varphi(\omega_{n+1}, \omega_m)\lambda^{n+1} + \cdots + \\ \varphi(\omega_n, \omega_m)\varphi(\omega_{n+1}, \omega_m)\cdots\varphi(\omega_{m-2}, \omega_m)\varphi(\omega_{m-1}, \omega_m)\lambda^{m-1} \end{bmatrix}. \end{aligned}$$

Since $\lim_{n,m\to+\infty} \varphi(\omega_n, \omega_m)\lambda < 1$, so the series $\sum_{n=1}^{\infty} \lambda^n \prod_{i=1}^{p} \varphi(\omega_i, \omega_m)$ converges by ratio test for each $m \in \mathbb{N}$. Let

$$\wp = \sum_{n=1}^{\infty} \lambda^n \prod_{i=1}^{p} \varphi(\omega_i, \omega_m),\ \wp_n = \sum_{j=1}^{n} \lambda^j \prod_{i=1}^{p} \varphi(\omega_i, \omega_m).$$

Thus, for $m > n$, the above inequality can be written as

$$|d(\omega_n, \omega_m)| \leq |d(\omega_0, \omega_1)| [\wp_{m-1} - \wp_n].$$

Now, by taking the limit as $n, m \to +\infty$, we get

$$\lim_{n,m\to+\infty} |d(\omega_n, \omega_m)| \to 0.$$

By lemma (10), we conclude that the sequence $\{\omega_n\}$ is a Cauchy sequence in $\overline{B(\omega_0, r)}$. Consequently there exists $\omega^* \in \overline{B(\omega_0, r)}$ such that $\lim_{n\to+\infty} \omega_n = \omega^*$. It follows that $\omega^* = \beth_1\omega^*$, otherwise $d(\omega^*, \beth_1\omega^*) = \upsilon > 0$ and we would then have

$$
\begin{aligned}
\upsilon &\precsim \varphi(\omega^*, \beth_1\omega^*)\left(d(\omega^*, \omega_{2n+2}) + d(\omega_{2n+2}, \beth_1\omega^*)\right) \\
&= \varphi(\omega^*, \beth_1\omega^*)\left(d(\omega^*, \omega_{2n+2}) + d(\beth_2\omega_{2n+1}, \beth_1\omega^*)\right) \\
&\precsim \varphi(\omega^*, \beth_1\omega^*)\begin{pmatrix} d(\omega^*, \omega_{2n+2}) + \aleph_1 d(\omega_{2n+1}, \omega^*) + \aleph_2 \frac{d(\omega_{2n+1}, \beth_2\omega_{2n+1}) d(\omega^*, \beth_1\omega^*)}{1+d(\omega^*, \omega_{2n+1})} \\ +\aleph_3 \frac{d(\omega_{2n+1}, \beth_1\omega^*) d(\omega^*, \beth_2\omega_{2n+1})}{1+d(\omega^*, \omega_{2n+1})} \\ +\aleph_4 \frac{d(\omega^*, \beth_2\omega_{2n+1}) d(\omega^*, \beth_1\omega^*)}{1+d(\omega^*, \omega_{2n+1})} \\ +\aleph_5 \frac{d(\omega_{2n+1}, \beth_2\omega_{2n+1}) d(\omega_{2n+1}, \beth_1\omega^*)}{1+d(\omega^*, \omega_{2n+1})} \end{pmatrix},
\end{aligned}
$$

which implies that

$$
|\upsilon| \leqslant \varphi(\omega^*, \beth_1\omega^*)\begin{pmatrix} |d(\omega^*, \omega_{2n+2})| + \aleph_1 |d(\omega_{2n+1}, \omega^*)| + \aleph_2 \frac{|d(\omega_{2n+1}, \omega_{2n+2})||\upsilon|}{|1+d(\omega^*, \omega_{2n+1})|} \\ +\aleph_3 \frac{|d(\omega_{2n+1}, \beth_1\omega^*)||d(\omega^*, \omega_{2n+2})|}{|1+d(\omega^*, \omega_{2n+1})|} \\ +\aleph_4 \frac{|d(\omega^*, \omega_{2n+2})||\upsilon|}{|1+d(\omega^*, \omega_{2n+1})|} \\ +\aleph_5 \frac{|d(\omega_{2n+1}, \omega_{2n+2})||d(\omega_{2n+1}, \beth_1\omega^*)|}{|1+d(\omega^*, \omega_{2n+1})|} \end{pmatrix}.
$$

That is $|\upsilon| = 0$, which is a contradiction. Thus $\omega^* = \beth_1\omega^*$.Similarly, we can prove that $\omega^* = \beth_2\omega^*$. □

Now we show uniqueness of common fixed point. We suppose $\omega^/$ in $\mathcal{W}$ is another common fixed point of $\beth_1$ and $\beth_2$ that is $\omega^/ = \beth_1\omega^/ = \beth_2\omega^/$ which is distinct from ω^* that is $\omega^* \neq \omega^/$. Now by (3.1), we have

$$
\begin{aligned}
d(\omega^*, \omega^/) &= d(\beth_1\omega^*, \beth_2\omega^/) \\
&\precsim \aleph_1 d(\omega^*, \omega^/) + \aleph_2 \frac{d(\omega^*, \beth_1\omega^*) d\left(\omega^/, \beth_2\omega^/\right)}{1 + d(\omega^*, \omega^/)} \\
&\quad +\aleph_3 \frac{d\left(\omega^/, \beth_1\omega^*\right) d\left(\omega^*, \beth_2\omega^/\right)}{1 + d(\omega^*, \omega^/)} \\
&\quad +\aleph_4 \frac{d(\omega^*, \beth_1\omega^*) d\left(\omega^*, \beth_2\omega^/\right)}{1 + d(\omega^*, \omega^/)} \\
&\quad +\aleph_5 \frac{d\left(\omega^/, \beth_1\omega^*\right) d\left(\omega^/, \beth_2\omega^/\right)}{1 + d(\omega^*, \omega^/)}
\end{aligned}
$$

so that

$$
\begin{aligned}
|d(\omega^*, \omega^/)| &\leq \aleph_1 |d(\omega^*, \omega^/)| + \aleph_2 \frac{|d(\omega^*, \beth_1\omega^*)||d\left(\omega^/, \beth_2\omega^/\right)|}{|1 + d(\omega^*, \omega^/)|} \\
&\quad +\aleph_3 \frac{|d\left(\omega^/, \beth_1\omega^*\right)||d\left(\omega^*, \beth_2\omega^/\right)|}{|1 + d(\omega^*, \omega^/)|} \\
&\quad +\aleph_4 \frac{|d(\omega^*, \beth_1\omega^*)||d\left(\omega^*, \beth_2\omega^/\right)|}{|1 + d(\omega^*, \omega^/)|} \\
&\quad +\aleph_5 \frac{|d\left(\omega^/, \beth_1\omega^*\right)||d\left(\omega^/, \beth_2\omega^/\right)|}{|1 + d(\omega^*, \omega^/)|}.
\end{aligned}
$$

Since $|1 + d(\omega^*, \omega^/)| > |d(\omega^*, \omega^/)|$, so we have

$$|d(\omega^*, \omega^/)| \leq (\aleph_1 + \aleph_3)|d(\omega^*, \omega^/)|.$$

This is contradiction to $\aleph_1 + \aleph_3 < 1$. Hence, $\omega^/ = \omega^*$. Therefore ω^* is a unique common fixed point of $\beth_1$ and $\beth_2$.

Corollary 12. *Let $(\mathcal{W}, d)$ be a complete CVEbMS with $\varphi : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ and $\beth : \mathcal{W} \to \mathcal{W}$. Suppose that there exist $\aleph_1, \aleph_2, \aleph_3, \aleph_4, \aleph_5 \in [0, 1)$ with $\aleph_1 + \aleph_2 + \aleph_3 + 2\aleph_4 + 2\aleph_5 < 1$ such that*

$$\begin{aligned} d(\beth\omega, \beth\varrho) \quad &\precsim \quad \aleph_1 d(\omega, \varrho) + \aleph_2 \frac{d(\omega, \beth\omega)\, d(\varrho, \beth\varrho)}{1 + d(\omega, \varrho)} + \aleph_3 \frac{d(\varrho, \beth\omega)\, d(\omega, \beth\varrho)}{1 + d(\omega, \varrho)} \\ &+\aleph_4 \frac{d(\omega, \beth\omega)\, d(\omega, \beth\varrho)}{1 + d(\omega, \varrho)} + \aleph_5 \frac{d(\varrho, \beth\omega)\, d(\varrho, \beth\varrho)}{1 + d(\omega, \varrho)}, \end{aligned}$$

for all $\omega_0, \omega, \varrho \in \overline{B(\omega_0, r)}$, $0 \prec r \in \mathbb{C}$ and

$$|d(\omega_0, \beth_1\omega_0)| \leq (1 - \lambda)|r|,$$

where $\lambda = \max\{(\frac{\aleph_1+\aleph_4}{1-\aleph_2-\aleph_4}), (\frac{\aleph_1+\aleph_5}{1-\aleph_2-\aleph_5})\}$. And for each $\omega_0 \in \overline{B(\omega_0, r)}$ and $\lim_{n,m\to+\infty} \varphi(\omega_n, \omega_m)\, \lambda < 1$, there exists a unique point $\omega^* \in \overline{B(\omega_0, r)}$ such that $\omega^* = \beth\omega^*$.

Proof. Taking $\beth_1 = \beth_2 = \beth$ in Theorem 11. □

Example 13. *Suppose*

$$\mathcal{W}_1 = \{\upsilon \in \mathbb{C} : Re(\upsilon) \geq 0,\ Im(\upsilon) = 0\},$$
$$\mathcal{W}_2 = \{\upsilon \in \mathbb{C} : Im(\upsilon) \geq 0,\ Re(\upsilon) = 0\},$$

and $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$. Consider complex valued extended b-metric $d : \mathcal{W} \times \mathcal{W} \longrightarrow \mathbb{C}$ as follows:

$$d(\upsilon_1, \upsilon_2) = \begin{cases} \frac{2}{3}|\omega_1 - \omega_2|^2 + \frac{i}{2}|\omega_1 - \omega_2|^2, & \text{if } \upsilon_1, \upsilon_2 \in \mathcal{W}_1, \\ \frac{1}{2}|\varrho_1 - \varrho_2|^2 + \frac{i}{3}|\varrho_1 - \varrho_2|^2, & \text{if } \upsilon_1, \upsilon_2 \in \mathcal{W}_2, \\ \frac{2}{9}(\omega_1 + \varrho_2)^2 + \frac{i}{6}(\omega_1 + \varrho_2)^2, & \text{if } \upsilon_1 \in \mathcal{W}_1,\ \upsilon_2 \in \mathcal{W}_2, \\ \frac{i}{3}(\omega_2 + \varrho_1)^2 + \frac{2i}{9}(\omega_2 + \varrho_1)^2, & \text{if } \upsilon_1 \in \mathcal{W}_2,\ \upsilon_2 \in \mathcal{W}_1, \end{cases}$$

and $\varphi : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ by $\varphi(\omega, \varrho) = 2$. Then $(\mathcal{W}, d)$ is complete CVEbMS. Take $\upsilon_0 = \frac{1}{2} + 0i$ and $r = \frac{1}{3} + \frac{1}{4}i$. Then,

$$\overline{B(\upsilon_0, r)} = \begin{cases} \upsilon \in \mathbb{C} : 0 \leqslant \mathrm{Re}(\upsilon) \leqslant 1,\ \mathrm{Im}(\upsilon) = 0 & \text{if } \upsilon \in \mathcal{W}_1 \\ \upsilon \in \mathbb{C} : 0 \leqslant \mathrm{Im}(\upsilon) \leqslant 1,\ \mathrm{Re}(\upsilon) = 0 & \text{if } \upsilon \in \mathcal{W}_2. \end{cases}$$

Define $\beth_1, \beth_2 : \mathcal{W} \to \mathcal{W}$ as

$$\beth_1\upsilon = \begin{cases} 0 + \dfrac{\omega}{4}i \quad \text{if } \upsilon \in \mathcal{W}_1 \text{ with } 0 \leqslant \mathrm{Re}(\upsilon) \leqslant 1,\ \mathrm{Im}(\upsilon) = 0, \\ \dfrac{5\omega}{6} + 0i \quad \text{if } \upsilon \in \mathcal{W}_1 \text{ with } \mathrm{Re}(\upsilon) > 1,\ \mathrm{Im}(\upsilon) = 0, \\ \dfrac{\varrho}{5} + 0i \quad \text{if } \upsilon \in \mathcal{W}_2 \text{ with } 0 \leqslant \mathrm{Im}(\upsilon) \leqslant 1,\ \mathrm{Re}(\upsilon) = 0, \\ 0 + \dfrac{4\varrho}{5}i \quad \text{if } \upsilon \in \mathcal{W}_2 \text{ with } \mathrm{Im}(\upsilon) > 1,\ \mathrm{Re}(\upsilon) = 0. \end{cases}$$

$$\beth_2 \upsilon = \begin{cases} 0 + \dfrac{\omega}{6} i & \text{if } \upsilon \in \mathcal{W}_1 \text{ with } 0 \leq \text{Re}(\upsilon) \leq 1,\ \text{Im}(\upsilon) = 0, \\ \dfrac{4\omega}{5} + 0i & \text{if } \upsilon \in \mathcal{W}_1 \text{ withRe}(\upsilon) > 1,\ \text{Im}(\upsilon) = 0, \\ \dfrac{\varrho}{7} + 0i & \text{if } \upsilon \in \mathcal{W}_2 \text{ with } 0 \leqslant \text{Im}(\upsilon) \leqslant 1,\ \text{Re}(\upsilon) = 0, \\ 0 + \dfrac{5\varrho}{6} i & \text{if } \upsilon \in \mathcal{W}_2 \text{ with } \text{Im}(\upsilon) > 1,\ \text{Re}(\upsilon) = 0. \end{cases}$$

Then with $\aleph_1 = \frac{1}{6}, \aleph_2 = \frac{1}{24}, \aleph_3 = \frac{1}{2}, \aleph_4 = \frac{1}{25}$ and $\aleph_5 = \frac{1}{26}$, all the assumptions of Theorem 11 are satisfied and hence $0 + 0i \in \overline{B(\upsilon_0, r)}$ is a unique common fixed point $\beth_1$ and $\beth_2$.

Corollary 14. *Let $(\mathcal{W}, d)$ be a complete CVEbMS with $\varphi : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ and $\beth_1, \beth_2 : \mathcal{W} \to \mathcal{W}$. Suppose that there exist $\aleph_1, \aleph_2, \aleph_3, \aleph_4 \in [0, 1)$ with $\aleph_1 + \aleph_2 + \aleph_3 + 2\aleph_4 < 1$ such that*

$$\begin{aligned} d(\beth_1\omega, \beth_2\varrho) \precsim\ & \aleph_1 d(\omega, \varrho) + \aleph_2 \frac{d(\omega, \beth_1\omega)\, d(\varrho, \beth_2\varrho)}{1 + d(\omega, \varrho)} + \aleph_3 \frac{d(\varrho, \beth_1\omega)\, d(\omega, \beth_2\varrho)}{1 + d(\omega, \varrho)} \\ & + \aleph_4 \frac{d(\omega, \beth_1\omega)\, d(\omega, \beth_2\varrho)}{1 + d(\omega, \varrho)}, \end{aligned}$$

for all $\omega_0, \omega, \varrho \in \overline{B(\omega_0, r)}$, $0 \prec r \in \mathbb{C}$ and

$$|d(\omega_0, \beth_1\omega_0)| \leq (1 - \lambda)|r|,$$

where $\lambda = \max\{\frac{\aleph_1+\aleph_4}{1-\aleph_2-\aleph_4}, \frac{\aleph_1}{1-\aleph_2}\}$. And for each $\omega_0 \in \overline{B(\omega_0, r)}$, $\lim_{n,m\to+\infty} \varphi(\omega_n, \omega_m)\lambda < 1$, then there exists a unique point $\omega^* \in \overline{B(\omega_0, r)}$ such that $\omega^* = \beth_1\omega^* = \beth_2\omega^*$.

Proof. Taking $\aleph_5 = 0$ in Theorem 11. □

Corollary 15. *Let $(\mathcal{W}, d)$ be a complete CVEbMS with $\varphi : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ and $\beth : \mathcal{W} \to \mathcal{W}$. Suppose that there exist $\aleph_1, \aleph_2, \aleph_3, \aleph_4 \in [0, 1)$ with $\aleph_1 + \aleph_2 + \aleph_3 + 2\aleph_4 < 1$ such that*

$$\begin{aligned} d(\beth\omega, \beth\varrho) \precsim\ & \aleph_1 d(\omega, \varrho) + \aleph_2 \frac{d(\omega, \beth\omega)\, d(\varrho, \beth\varrho)}{1 + d(\omega, \varrho)} + \aleph_3 \frac{d(\varrho, \beth\omega)\, d(\omega, \beth\varrho)}{1 + d(\omega, \varrho)} \\ & + \aleph_4 \frac{d(\omega, \beth\omega)\, d(\omega, \beth\varrho)}{1 + d(\omega, \varrho)}, \end{aligned}$$

for all $\omega_0, \omega, \varrho \in \overline{B(\omega_0, r)}$, $0 \prec r \in \mathbb{C}$ and

$$|d(\omega_0, \beth\omega_0)| \leq (1 - \lambda)|r|,$$

where $\lambda = \max\{\frac{\aleph_1+\aleph_4}{1-\aleph_2-\aleph_4}, \frac{\aleph_1}{1-\aleph_2}\}$. And for each $\omega_0 \in \overline{B(\omega_0, r)}$ and $\lim_{n,m\to+\infty} \varphi(\omega_n, \omega_m)\lambda < 1$,then there exists a unique point $\omega^* \in \overline{B(\omega_0, r)}$ such that $\omega^* = \beth\omega^*$.

Proof. Taking $\beth_1 = \beth_2 = \beth$ in Corollary 14. □

Corollary 16. *Let $(\mathcal{W}, d)$ be a complete CVEbMS with $\varphi : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ and $\beth_1, \beth_2 : \mathcal{W} \to \mathcal{W}$. Suppose that there exist $\aleph_1, \aleph_2, \aleph_3, \aleph_5 \in [0, 1)$ with $\aleph_1 + \aleph_2 + \aleph_3 + 2\aleph_5 < 1$ such that*

$$d(\beth_1\omega, \beth_2\varrho) \precsim \aleph_1 d(\omega, \varrho) + \aleph_2 \frac{d(\omega, \beth_1\omega)\, d(\varrho, \beth_2\varrho)}{1 + d(\omega, \varrho)} + \aleph_3 \frac{d(\varrho, \beth_1\omega)\, d(\omega, \beth_2\varrho)}{1 + d(\omega, \varrho)} + \aleph_5 \frac{d(\varrho, \beth_1\omega)\, d(\varrho, \beth_2\varrho)}{1 + d(\omega, \varrho)},$$

for all $\omega_0, \omega, \varrho \in \overline{B(\omega_0, r)}$, $0 \prec r \in \mathbb{C}$ *and*

$$|d(\omega_0, \beth_1\omega_0)| \leq (1-\lambda)|r|,$$

where $\lambda = \max\{(\frac{\aleph_1}{1-\aleph_2}), (\frac{\aleph_1+\aleph_5}{1-\aleph_2-\aleph_5})\}$. *And for each* $\omega_0 \in \overline{B(\omega_0, r)}$ *and* $\lim_{n,m\to+\infty} \varphi(\omega_n, \omega_m)\lambda < 1$, *then there exists a unique point* $\omega^* \in \overline{B(\omega_0, r)}$ *such that* $\omega^* = \beth_1\omega^* = \beth_2\omega^*$.

Proof. Taking $\aleph_4 = 0$ in Theorem 11. □

Corollary 17. *Let* $(\mathcal{W}, d)$ *be a complete CVEbMS with* $\varphi : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ *and* $\beth : \mathcal{W} \to \mathcal{W}$. *Suppose that there exist* $\aleph_1, \aleph_2, \aleph_3, \aleph_5 \in [0, 1)$ *with* $\aleph_1 + \aleph_2 + \aleph_3 + 2\aleph_5 < 1$ *such that*

$$d(\beth\omega, \beth\varrho) \precsim \aleph_1 d(\omega, \varrho) + \aleph_2 \frac{d(\omega, \beth\omega)\, d(\varrho, \beth\varrho)}{1 + d(\omega, \varrho)} + \aleph_3 \frac{d(\varrho, \beth\omega)\, d(\omega, \beth\varrho)}{1 + d(\omega, \varrho)} + \aleph_5 \frac{d(\varrho, \beth\omega)\, d(\varrho, \beth\varrho)}{1 + d(\omega, \varrho)},$$

for all $\omega_0, \omega, \varrho \in \overline{B(\omega_0, r)}$, $0 \prec r \in \mathbb{C}$ *and*

$$|d(\omega_0, \beth\omega_0)| \leq (1-\lambda)|r|,$$

where $\lambda = \max\{(\frac{\aleph_1}{1-\aleph_2}), (\frac{\aleph_1+\aleph_5}{1-\aleph_2-\aleph_5})\}$. *And for each* $\omega_0 \in \overline{B(\omega_0, r)}$ *and* $\lim_{n,m\to+\infty} \varphi(\omega_n, \omega_m)\lambda < 1$, *then there exists a unique point* $\omega^* \in \overline{B(\omega_0, r)}$ *such that* $\omega^* = \beth\omega^*$.

Proof. By setting $\beth_1 = \beth_2 = \beth$ in Corollary 16. □

Corollary 18. *Let* $(\mathcal{W}, d)$ *be a complete CVEbMS with* $\varphi : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ *and* $\beth_1, \beth_2 : \mathcal{W} \to \mathcal{W}$. *Suppose that there exist* $\aleph_1, \aleph_2, \aleph_3 \in [0, 1)$ *with* $\aleph_1 + \aleph_2 + \aleph_3 < 1$ *such that*

$$d(\beth_1\omega, \beth_2\varrho) \precsim \aleph_1 d(\omega, \varrho) + \aleph_2 \frac{d(\omega, \beth_1\omega)\, d(\varrho, \beth_2\varrho)}{1 + d(\omega, \varrho)} + \aleph_3 \frac{d(\varrho, \beth_1\omega)\, d(\omega, \beth_2\varrho)}{1 + d(\omega, \varrho)},$$

for all $\omega_0, \omega, \varrho \in \overline{B(\omega_0, r)}$, $0 \prec r \in \mathbb{C}$ *and*

$$|d(\omega_0, \beth_1\omega_0)| \leq (1-\lambda)|r|,$$

where $\lambda = \frac{\aleph_1}{1-\aleph_2}$. *And for each* $\omega_0 \in \overline{B(\omega_0, r)}$ *and* $\lim_{n,m\to+\infty} \varphi(\omega_n, \omega_m)\lambda < 1$, *then there exists a unique point* $\omega^* \in \overline{B(\omega_0, r)}$ *such that* $\omega^* = \beth_1\omega^* = \beth_2\omega^*$.

Proof. By choosing $\aleph_4 = \aleph_5 = 0$ in Theorem 11. □

Corollary 19. *Let* $(\mathcal{W}, d)$ *be a complete CVEbMS with* $\varphi : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ *and* $\beth : \mathcal{W} \to \mathcal{W}$. *Suppose that there exist* $\aleph_1, \aleph_2, \aleph_3 \in [0, 1)$ *with* $\aleph_1 + \aleph_2 + \aleph_3 < 1$ *such that*

$$d(\beth\omega, \beth\varrho) \precsim \aleph_1 d(\omega, \varrho) + \aleph_2 \frac{d(\omega, \beth\omega)\, d(\varrho, \beth\varrho)}{1 + d(\omega, \varrho)} + \aleph_3 \frac{d(\varrho, \beth\omega)\, d(\omega, \beth\varrho)}{1 + d(\omega, \varrho)},$$

for all $\omega_0, \omega, \varrho \in \overline{B(\omega_0, r)}$, $0 \prec r \in \mathbb{C}$ *and*

$$|d(\omega_0, \beth\omega_0)| \leq (1-\lambda)|r|,$$

where $\lambda = \frac{\aleph_1}{1-\aleph_2}$. *And for each* $\omega_0 \in \overline{B(\omega_0, r)}$ *and* $\lim_{n,m\to+\infty} \varphi(\omega_n, \omega_m)\lambda < 1$, *then there exists a unique point* $\omega^* \in \overline{B(\omega_0, r)}$ *such that* $\omega^* = \beth\omega^*$.

Proof. Taking $\beth_1 = \beth_2 = \beth$ in Corollary 18. □

Corollary 20. *Let $(\mathcal{W}, d)$ be a complete CVEbMS with $\varphi : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ and $\beth_1, \beth_2 : \mathcal{W} \to \mathcal{W}$. Suppose that there exist $\aleph_1, \aleph_2 \in [0, 1)$ with $\aleph_1 + \aleph_2 < 1$ such that*

$$d(\beth_1\omega, \beth_2\varrho) \precsim \aleph_1 d(\omega, \varrho) + \aleph_2 \frac{d(\omega, \beth_1\omega)\, d(\varrho, \beth_2\varrho)}{1 + d(\omega, \varrho)},$$

for all $\omega_0, \omega, \varrho \in \overline{B(\omega_0, r)}$, $0 \prec r \in \mathbb{C}$ and

$$|d(\omega_0, \beth_1\omega_0)| \leq (1 - \lambda)|r|,$$

where $\lambda = \frac{\aleph_1}{1-\aleph_2}$. And for each $\omega_0 \in \overline{B(\omega_0, r)}$ and $\lim_{n,m\to+\infty} \varphi(\omega_n, \omega_m)\, \lambda < 1$, then there exists a unique point $\omega^* \in \overline{B(\omega_0, r)}$ such that $\omega^* = \beth_1\omega^* = \beth_2\omega^*$.

Proof. Taking $\aleph_3 = \aleph_4 = \aleph_5 = 0$ in Theorem 11. □

Corollary 21. *Let $(\mathcal{W}, d)$ be a complete CVEbMS with $\varphi : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ and $\beth : \mathcal{W} \to \mathcal{W}$. Suppose that there exist $\aleph_1, \aleph_2 \in [0, 1)$ with $\aleph_1 + \aleph_2 < 1$ such that*

$$d(\beth\omega, \beth\varrho) \precsim \aleph_1 d(\omega, \varrho) + \aleph_2 \frac{d(\omega, \beth\omega)\, d(\varrho, \beth\varrho)}{1 + d(\omega, \varrho)},$$

for all $\omega_0, \omega, \varrho \in \overline{B(\omega_0, r)}$, $0 \prec r \in \mathbb{C}$ and

$$|d(\omega_0, \beth\omega_0)| \leq (1 - \lambda)|r|,$$

where $\lambda = \frac{\aleph_1}{1-\aleph_2}$. And for each $\omega_0 \in \overline{B(\omega_0, r)}$ and $\lim_{n,m\to+\infty} \varphi(\omega_n, \omega_m)\, \lambda < 1$, then there exists a unique point $\omega^* \in \overline{B(\omega_0, r)}$ such that $\omega^* = \beth\omega^*$.

Proof. Taking $\beth_1 = \beth_2 = \beth$ in Corollary 20. □

Now we we establish the following result for two finite families of mappings as an application of Theorem 11.

Theorem 22. *If $\{\aleph_i\}_1^m$ and $\{\mathfrak{R}_i\}_1^n$ are two finite pairwise commuting finite families of self-mapping defined on a complex valued extended b-metric space with $\varphi : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ such that the mappings $\mathfrak{R}$ and $\mathfrak{I}$ (with $\mathfrak{I} = \aleph_1\aleph_2 \cdots \aleph_m$ and $\mathfrak{R} = \mathfrak{R}_1\mathfrak{R}_2 \cdots \mathfrak{R}_n$) satisfy (3.1) and (3.2) then the component mappings of these $\{\aleph_i\}_1^m$ and $\{\mathfrak{R}_i\}_1^n$ have a unique common fixed point.*

Proof. By Theorem 11, one can get $\mathfrak{I}\omega^* = \mathfrak{R}\omega^* = \omega^*$, which is unique. Now by pairwise commutativity of $\{\aleph_i\}_1^m$ and $\{\mathfrak{R}_i\}_1^n$,(for every $1 \leq k \leq m$) one can write $\aleph_k\omega^* = \aleph_k\aleph\omega^* = \aleph\aleph_k\omega^*$ and $\aleph_k\omega^* = \aleph_k\mathfrak{R}\omega^* = \mathfrak{R}\aleph_k\omega^*$ which manifest that $\aleph_k\omega^*$, for all k, is also a common fixed point of $\mathfrak{I}$ and $\mathfrak{R}$. Now utilizing the uniqueness, one can write $\mathfrak{I}_k\omega^* = \omega^*$ (for every k) which shows that ω^* is a common fixed point of $\{\mathfrak{I}_i\}_1^m$. By doing the same strategy, we can prove that $\mathfrak{R}_k\omega^* = \omega^*$ $(1 \leq k \leq n)$. Hence $\{\aleph_i\}_1^m$ and $\{\mathfrak{R}_i\}_1^n$ have a unique common fixed point. □

Corollary 23. *Let $(\mathcal{W}, d)$ be a complete CVEbMS with $\varphi : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ and $F, G : \mathcal{W} \to \mathcal{W}$. Suppose that there exist $\aleph_1, \aleph_2, \aleph_3, \aleph_4, \aleph_5 \in [0, 1)$ with $\aleph_1 + \aleph_2 + \aleph_3 + 2\aleph_4 + 2\aleph_5 < 1$ such that*

$$\begin{aligned} d(F^m\omega, G^n\varrho) \precsim\ & \aleph_1 d(\omega, \varrho) + \aleph_2 \frac{d(\omega, F^m\omega)\, d(\varrho, G^n\varrho)}{1 + d(\omega, \varrho)} + \aleph_3 \frac{d(\varrho, F^m\omega)\, d(\omega, G^n\varrho)}{1 + d(\omega, \varrho)} \\ & + \aleph_4 \frac{d(\omega, F^m\omega)\, d(\omega, G^n\varrho)}{1 + d(\omega, \varrho)} + \aleph_5 \frac{d(\varrho, F^m\omega)\, d(\varrho, G^n\varrho)}{1 + d(\omega, \varrho)}, \end{aligned}$$

for all $\omega_0, \omega, \varrho \in \overline{B(\omega_0, r)}$, $0 \prec r \in \mathbb{C}$ and

$$|d(\omega_0, G^n\omega_0)| \leq (1 - \lambda)|r|,$$

where $\lambda = \max\{(\frac{\aleph_1+\aleph_4}{1-\aleph_2-\aleph_4}), (\frac{\aleph_1+\aleph_5}{1-\aleph_2-\aleph_5})\}$. And for each $\omega_0 \in \overline{B(\omega_0, r)}$ and $\lim_{n,m\to+\infty} \varphi(\omega_n, \omega_m)\lambda < 1$, then there exists a unique point $\omega^* \in \overline{B(\omega_0, r)}$ such that $\omega^* = F\omega^* = G\omega^*$.

Proof. Taking $\aleph_1 = \aleph_2 = \cdots = \aleph_m = F$ and $\Re_1 = \Re_2 = \cdots = \Re_n = G$, in Theorem 18. □

Corollary 24. *Let $(\mathcal{W}, d)$ be a complete CVEbMS with $\varphi : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ and $\beth : \mathcal{W} \to \mathcal{W}$. Suppose that there exist $\aleph_1, \aleph_2, \aleph_3, \aleph_4, \aleph_5 \in [0, 1)$ with $\aleph_1 + \aleph_2 + \aleph_3 + 2\aleph_4 + 2\aleph_5 < 1$ such that*

$$\begin{aligned} d(\beth^n\omega, \beth^n\varrho) \precsim\ & \aleph_1 d(\omega, \varrho) + \aleph_2 \frac{d(\omega\beth^n\omega)\, d(\varrho, \beth^n\varrho)}{1 + d(\omega, \varrho)} + \aleph_3 \frac{d(\varrho, \beth^n\omega)\, d(\omega, \beth^n\varrho)}{1 + d(\omega, \varrho)} \\ & + \aleph_4 \frac{d(\omega, \beth^n\omega)\, d(\omega, \beth^n\varrho)}{1 + d(\omega, \varrho)} + \aleph_5 \frac{d(\varrho, \beth^n\omega)\, d(\varrho, \beth^n\varrho)}{1 + d(\omega, \varrho)}, \end{aligned}$$

for all $\omega_0, \omega, \varrho \in \overline{B(\omega_0, r)}$, $0 \prec r \in \mathbb{C}$ and

$$|d(\omega_0, \beth^n\omega_0)| \leq (1 - \lambda)|r|,$$

where $\lambda = \max\{(\frac{\aleph_1+\aleph_4}{1-\aleph_2-\aleph_4}), (\frac{\aleph_1+\aleph_5}{1-\aleph_2-\aleph_5})\}$. And for each $\omega_0 \in \overline{B(\omega_0, r)}$ and $\lim_{n,m\to+\infty} \varphi(\omega_n, \omega_m)\lambda < 1$, then there exists a unique point $\omega^* \in \overline{B(\omega_0, r)}$ such that $\omega^* = \beth\omega^*$.

Taking $m = n$ and $F = G = \beth$ in Corollary 23.

4. Applications

Theorem 25. *Let $\mathcal{W} = C([a, b], \mathbb{R}^n)$, $a > 0$ and $d : \mathcal{W} \times \mathcal{W} \to \mathbb{C}$ be defined in this way*

$$d(\omega, \varrho) = \max_{t\in[a,b]} \|\omega(t) - \varrho(t)\|^2 \sqrt{1 + a^2} e^{i \tan^{-1} a},$$

and $\varphi : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ be defined by $\varphi(\omega, \varrho) = 2$. Then $(\mathcal{W}, d)$ is complete CVEbMS. Consider the Urysohn integral equations

$$\omega(t) = \int_a^b K_1(t, s, \omega(s))ds + \phi(t), \tag{4.1}$$

$$\omega(t) = \int_a^b K_2(t, s, \omega(s))ds + \psi(t), \tag{4.2}$$

for all $t \in [a, b] \subset \mathbb{R}$, $\omega, \phi, \psi \in \mathcal{W}$.

Assume that $K_1, K_2 : [a,b] \times [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ are such that $F_\omega, G_\omega \in \mathcal{W}$ for each $\omega \in \mathcal{W}$, where,

$$F_\omega(t) = \int_a^b K_1(t,s,\omega(s))ds, \quad G_\omega(t) = \int_a^b K_2(t,s,\omega(s))ds.$$

for all $t \in [a,b]$.

If there exist $\aleph_1, \aleph_2 \in [0,1)$ with $\aleph_1 + \aleph_2 < 1$ such that for every $\omega, \varrho \in \mathcal{W}$

$$\left\|F_\omega(t) - G_\varrho(t) + \phi(t) - \psi(t)\right\|^2 \sqrt{1+a^2}e^{i\tan^{-1}a} \precsim \aleph_1 A(\omega,\varrho)(t) + \aleph_2 B(\omega,\varrho)(t),$$

where

$$\begin{aligned} A(\omega,\varrho)(t) &= \|\omega(t) - \varrho(t)\|^2 \sqrt{1+a^2}e^{i\tan^{-1}a}, \\ B(\omega,\varrho)(t) &= \frac{\|F_\omega(t) + \phi(t) - \omega(t)\|^2 \left\|G_\varrho(t) + \psi(t) - \varrho(t)\right\|^2}{1 + \max\limits_{t\in[a,b]} A(\omega,\varrho)(t)} \sqrt{1+a^2}e^{i\tan^{-1}a}, \end{aligned}$$

then Urysohn integral equations (4.1) and (4.2) have a unique common solution.

Proof. Define $\Im_1, \Im_2 : \mathcal{W} \to \mathcal{W}$ by

$$\Im_1\omega = F_\omega + \phi, \quad \Im_2\omega = G_\omega + \psi.$$

Then

$$d(\Im_1\omega, \Im_2\varrho) = \max_{t\in[a,b]} \left\|F_\omega(t) - G_\varrho(t) + \phi(t) - \psi(t)\right\|^2 \sqrt{1+a^2}e^{i\tan^{-1}a},$$

$$d(\omega,\varrho) = \max_{t\in[a,b]} A(\omega,\varrho)(t),$$

$$\frac{d(\omega,\Im_1\omega)d(\varrho,\Im_2\varrho)}{1+d(\omega,\varrho)} = \max_{t\in[a,b]} B(\omega,\varrho)(t).$$

It is easily seen that

$$d(\Im_1\omega, \Im_2\varrho) \precsim \aleph_1 d(\omega,\varrho) + \aleph_2 \frac{d(\omega,\Im_1\omega)d(\varrho,\Im_2\varrho)}{1+d(\omega,\varrho)},$$

for every $\omega, \varrho \in \mathcal{W}$. By Theorem 11 with $\aleph_3 = \aleph_4 = \aleph_5 = 0$, the Urysohn integral equations (4.1) and (4.2) have a unique common solution. □

5. Conclusions

In this article, we have utilized the notion of complex valued extended b -metric space (CVE*b*MS) and secured common fixed point results for rational contractions on a closed ball. We have derived common fixed points and fixed points of single valued mappings for contractions on a closed ball. We expect that the obtained consequences in this article will form up to date relations for researchers who are employing in CVE*b*MS.

The future work in this way will target on studying the common fixed points of single valued and multivalued mappings in the setting of CVE*b*MS. Differential and integral equations can be solved as applications of these results.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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