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Research article

Investigation of time fractional nonlinear KdV-Burgers equation under fractional operators with nonsingular kernels

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Abstract: In this manuscript, the Korteweg-de Vries-Burgers (KdV-Burgers) partial differential equation (PDE) is investigated under nonlocal operators with the Mittag-Leffler kernel and the exponential decay kernel. For both fractional operators, the existence of the solution of the KdV-Burgers PDE is demonstrated through fixed point theorems of α-type F contraction. The modified double Laplace transform is utilized to compute a series solution that leads to the exact values when fractional order equals unity. The effectiveness and reliability of the suggested approach are verified and confirmed by comparing the series outcomes to the exact values. Moreover, the series solution is demonstrated through graphs for a few fractional orders. Lastly, a comparison between the results of the two fractional operators is studied through numerical data and diagrams. The results show how consistently accurate the method is and how broadly applicable it is to fractional nonlinear evolution equations.

Keywords: double Laplace transform; KdV equation; Burgers equation; fractional operators
Mathematics Subject Classification: 35R11

1. Introduction

The Korteweg–de Vries–Burgers (KdV-Burgers) model, which comes up in several practical situations, such as the turbulence of undular bores in shallow water [1], the transport of liquids carrying gas bubbles [2], the waves that go through an elastic pipe that is filled with a viscous liquid [3], and weakly nonlinear plasma waves that have specific dissipative properties [4], has received a lot of interest over the last few decades. It may also be utilized as a nonlinear model in ferroelectricity theory, turbulence, circuit theory, and other fields [5, 6]. The typical version of the KdV-Burger's equation is

$$\frac{\partial \mathcal{U}}{\partial t} + \sigma \frac{\partial^3 \mathcal{U}}{\partial x^3} + \zeta \frac{\partial^2 \mathcal{U}}{\partial x^2} + \eta \mathcal{U} \frac{\partial \mathcal{U}}{\partial x} = 0, \quad t > 0. \tag{1.1}$$

The KdV [7] and Burgers models [8] are commonly believed to be combined in Eq (1.1). Johnson [9] discovered that a specific limit of the matter led to the proposed model, where $\mathcal{U}(x, t)$ is proportional to the radial disturbance of the pipe wall, and x and t are the characterizing and temporal variables, in a field of wave propagation in fluid-filled elastic pipes. The model (1.1) was correct in the far-field of a near-field solution that was originally linear (small amplitude). Nonlinearity ($\mathcal{U}\frac{\partial \mathcal{U}}{\partial x}$), dispersion ($\frac{\partial^3 \mathcal{U}}{\partial x^3}$), and dissipation ($\frac{\partial^2 \mathcal{U}}{\partial x^2}$) all exist in this equation, which is the basic version of a wave model. Fractional differential equations (FDEs) are extensions of differential equations (DEs) having integer order. FDEs have ample applications in different domains of sciences [10–14]. Due to the wide applications of FDEs, several operators have been defined in the literature [15,16]. The recent operators that are frequently used for studying DEs are the Caputo-Fabrizio (CF) [17] and Atangana-Baleanu (AB) [18] operators. These operators are dependent on the exponential and Mittag-Leffler kernels, respectively. The literature has several applications for the CF and the AB operators. For instance, HIV-1 infection has been investigated via the CF operator in [19]. Ahmad et al. studied the fractional-order Ambartsumian equation through the CF operator [20]. The Φ 4-model has been investigated using the CF and AB operators by Rahman et al. [21]. More applications can be found in the literature [22–24]. The fractional non-linear KdV-Burger's equation is taken into consideration in the form:

$$\frac{\partial^\alpha \mathcal{U}}{\partial t^\alpha} + \sigma \frac{\partial^3 \mathcal{U}}{\partial x^3} + \zeta \frac{\partial^2 \mathcal{U}}{\partial x^2} + \eta \mathcal{U} \frac{\partial \mathcal{U}}{\partial x} = 0, \quad 0 < \alpha \leq 1, \quad t > 0, \tag{1.2}$$

with

$$\mathcal{U}(x, 0) = R(x),$$

where $\mathcal{U}$ is a function of x and t, x represents the space variable, t represents the time variable and η is a positive constant. We analyze Eq (1.2) in two ways: writing it first in the Atangana-Baleanu-Caputo (ABC) sense and then in the CF sense.

One of the most significant areas of research for FDEs is the quest for accurate and numerical solutions to FDEs. To date, many strategies for obtaining numerical and precise solutions of FDEs have been developed. A number of FDEs have been examined using these approaches. For example, Ahmad et al. [25] used the Laplace transform to find series of third order dispersive fractional PDEs. A generalized differential transform approach has been developed to solve fractional order PDEs by Odibat and Momani [26]. The homotopy perturbation technique has been proposed to solve the KdV-Burger's fractional PDE by Wang [27]. The Laplace transform was also observed to have a number of advantages, including its convergence to an exact solution of a problem after a certain iteration and that it does not allow any perturbation or discretization. Here, we utilize the double Laplace (DL) transform to compute a series solution of the considered equation.

The rest of article is organized as follows: Section 2 contains some basic definitions and a remark. The existence and uniqueness of the IVPs are presented in Section 3. In Section 4, the proposed techniques are presented. Section 5 consists of the application and comparison between results and diagrams of the proposed method. Finally, the conclusion is presented in Section 6.

2. Preliminaries

In this part, we provide some definitions, remarks, and lemmas about fractional calculus. Additionally, we provide a definition of the DL transformation and decomposition technique.

Definition 1. *[18] Let $\mathcal{U} \in H^1(c,d)$, $c > d$ and $\beta \in (0,1]$. Then the ABC operator is expressed as*

$$^{ABC}D^{\beta}\mathcal{U}(t) = \frac{B(\beta)}{(1-\beta)} \int_c^t \mathcal{U}'(s) E_{\beta}\left(\frac{-\beta(t-s)^{\beta}}{(1-\beta)}\right) ds,$$

where $B(\beta)$ is a normalizing factor with the conditions $B(0) = B(1) = 1$, and E_{β} is the Mittag-Leffler function

$$E_{\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\beta k + 1)}, \quad 0 < \beta < 1.$$

Definition 2. *[17] Let $\mathcal{U} \in H^1(c,d)$, $d > c$ and $\beta \in (0,1]$. Then the CF operator is written as*

$$^{CF}D^{\beta}\mathcal{U}(t) = \frac{E(\beta)}{1-\beta} \int_c^t \mathcal{U}'(t) \exp\left(\frac{-\beta(t-s)}{1-\beta}\right) ds,$$

where $E(\beta)$ is the normalizing factor such that $E(0) = E(1) = 1$. When $\mathcal{U}(t) \notin H^1(c,d)$ then the above equation can be written for $u \,\epsilon\, L^{-1}(-\infty, d)$ and any $\beta \,\epsilon\, (0,1]$ as

$$^{CF}D^{\beta}\mathcal{U}(t) = \frac{\beta E(\beta)}{1-\beta} \int_{-\infty}^t (\mathcal{U}'(t) - \mathcal{U}(s)) - \exp\left(\frac{-\beta(t-s)}{1-\beta}\right) ds.$$

Remark 1. *For the above definitions, $n = [\beta] + 1$, $[\beta]$ is the greatest integer not greater than β, and "Γ" is the well-known gamma function that can be calculated as*

$$\Gamma(\beta) = \int_0^{\infty} e^{-s} s^{\beta-1} ds.$$

Definition 3. *Suppose that $\mathcal{U}$ is a function for $x, t > 0$. The DL transformation of $\mathcal{U}$ is expressed as [28]*

$$\mathrm{L}_x \mathrm{L}_t\,[\mathcal{U}] = \int_0^{\infty} e^{-px} \int_0^{\infty} e^{-st}\, \mathcal{U}\, dt\, dx,$$

where p and s are complex numbers.

Definition 4. *Application of the DL transform on the ABC operator is as follows:*

$$\mathrm{L}_x \mathrm{L}_t \left\{^{ABC}D_x^{\theta}\mathcal{U}\right\} = \frac{B(\theta)}{(1-\theta)(p^{\theta} + \frac{\theta}{(1-\theta)})} \left[p^{\theta}\,\bar{\mathcal{U}}(p,s) - \sum_{k=0}^{n-1} p^{\theta-1-k} \mathrm{L}_t \left\{\frac{\partial^k \mathcal{U}(0,t)}{\partial x^k}\right\}\right],$$

and

$$\mathrm{L}_x \mathrm{L}_t \left\{^{ABC}D_t^{\vartheta}\mathcal{U}\right\} = \frac{B(\vartheta)}{(1-\vartheta)(s^{\vartheta} + \frac{\vartheta}{(1-\vartheta)})} \left[s^{\vartheta}\,\bar{\mathcal{U}}(p,s) - \sum_{k=0}^{m-1} s^{\vartheta-1-k} \mathrm{L}_x \left\{\frac{\partial^k \mathcal{U}(x,0)}{\partial t^k}\right\}\right],$$

where, $\quad n = [\theta] + 1, m = [\vartheta] + 1.$

Definition 5. *Application of the DL transformation on the CF operator is as follows:*

$$\mathrm{L}_x\mathrm{L}_t\left\{{}^{CF}D_x^{\theta+n}\,\mathcal{U}\right\} = \frac{E(\theta)}{p+(1-\theta)p}\left[p^{n+1}\,\bar{\mathcal{U}}(p,s) - \sum_{i=0}^{n} p^{n-i}\mathrm{L}_t\left\{\frac{\partial^i\mathcal{U}(0,t)}{\partial x^i}\right\}\right],$$

and

$$\mathrm{L}_x\mathrm{L}_t\left\{{}^{CF}D_t^{\vartheta+m}\,\mathcal{U}\right\} = \frac{E(\vartheta)}{s+(1-\vartheta)s}\left[s^{m+1}\,\bar{\mathcal{U}}(p,s) - \sum_{i=0}^{m} s^{m-i}\mathrm{L}_x\left\{\frac{\partial^i\mathcal{U}(x,0)}{\partial t^i}\right\}\right],$$

where, $n = [\theta]+1, m = [\vartheta]+1$.

From the interpretation provided above, it is clear that

$$\mathrm{L}_x\mathrm{L}_t\mathcal{U}(x)\,v(t) = \bar{\mathcal{U}}(p)\bar{v}(s) = \mathrm{L}_x\mathcal{U}(x)\mathrm{L}_t v(t).$$

A complex double-integral formulation is used to represent the inverse DL transform $\mathrm{L}_x^{-1}\mathrm{L}_t^{-1}\{\bar{\mathcal{U}}\} = \mathcal{U}$:

$$\mathrm{L}_x^{-1}\mathrm{L}_t^{-1}\left\{\bar{\mathcal{U}}\right\} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}\int_{b-i\infty}^{b+i\infty} e^{px}\bar{\mathcal{U}}(p,s)\,dp\,ds,$$

where, $\bar{\mathcal{U}}\,(p,s)$, is an analytic function for all p and s that are described in the region $Re\ (p) \geq a$ and $Re\ (s) \geq b$, where $a, b \in \mathbb{R}$ to be chosen appropriately.

3. Existence of the initial value problems (IVPs)

The existence and uniqueness of the IVPs are studied in this part employing α-type F-contraction. For this purpose, assume that $(\mathcal{Z}, d)$ is a complete metric space, and $\daleth$ is the collection of strictly increasing functions $F : \mathcal{R}_+ \to \mathcal{R}$ having the following required characteristics:

- $\lim_{n\to\infty} F(c_n) = -\infty$ if and only if, for each $\{c_n\}$, $\lim_{n\to\infty}(c_n) = 0$;
- there exists $\upsilon \in (0,1)$ such that $\lim_{c\to 0^+} c^{\upsilon}F(c) = 0$.

Definition 6. *[29] Let $Q : \mathcal{Z} \to \mathcal{Z}$ be self mapping with $\alpha : \mathcal{Z}\times\mathcal{Z} \to [0,\infty)$. If*

$$\alpha(\mathcal{X},\mathcal{W}) \geq 1 \quad\Rightarrow\quad \alpha(Q\mathcal{X},Q\mathcal{W}) \geq 1,$$

for all $\mathcal{X},\mathcal{W} \in \mathcal{Z}$, then Q is referred to as α-admissible.

Definition 7. *[30] Suppose that $Q : \mathcal{Z} \to \mathcal{Z}$, $\alpha : \mathcal{Z}\times\mathcal{Z} \to \{-\infty\}\cup[0,\infty)$, and there exists $\omega > 0$ such that*

$$\omega + \alpha(\mathcal{X},\mathcal{W})F(d(Q\mathcal{X},Q\mathcal{W})) \quad\leq\quad F(d(\mathcal{X},\mathcal{W}))$$

for each $\mathcal{X},\mathcal{W} \in \mathcal{Z}$ with $d(Q\mathcal{X},Q\mathcal{W}) > 0$. Then, Q is called an α-type F-contraction.

Theorem 1. *[30] Let $(\mathcal{Z}, d)$ be a complete metric space and $Q : \mathcal{Z} \to \mathcal{Z}$ be an α-type F-contraction such that*

1) *there exists* $\mathcal{X}_\circ \in \mathcal{Z}$ *such that* $\alpha(\mathcal{X}_\circ, Q\mathcal{X}_\circ) \geq 1$;
2) *if there exist* $\{\mathcal{X}_n\} \subseteq \mathcal{Z}$ *with* $\alpha(\mathcal{X}_n, \mathcal{X}_{n+1}) \geq 1$ *and* $\mathcal{X}_n \to \mathcal{X}$, *then* $\alpha(\mathcal{X}_n, \mathcal{X}) \geq 1$ *for all* $n \in \mathcal{N}$;
3) *F is continuous.*

Then, Q *has a fixed point* $\mathcal{X}^* \in \mathcal{Z}$. *Also for* $\mathcal{X}_\circ \in \mathcal{Z}$, *the sequence* $\{Q^n\mathcal{X}_\circ\}_{n\in\mathcal{N}}$ *is convergent to* $\mathcal{X}_\circ$

Let $\mathcal{Z} = C([0,1]^2, \mathcal{R})$, where C is the space of all continuous functions $\mathcal{X} : [0,1] \times [0,1] \to \mathcal{R}$, and $d(\mathcal{X}(x,t), \mathcal{W}(x,t)) = \sup_{x,t\in[0,1]}\{|\mathcal{X}(x,t) - \mathcal{W}(x,t)|\}$. Then we can write the IVP (1.1) in the CF fractional derivative sense as

$$ {}^{CF}D_t^{\alpha}\mathcal{X}(x,t) = \mathcal{F}(x,t,\mathcal{X}(x,t)), \ 0 < \alpha \leq 1, \tag{3.1} $$

with initial condition

$$ \mathcal{X}(x,0) = g(x), $$

where $\mathcal{F}(x,t,\mathcal{X}(x,t)) = -\sigma\mathcal{U}_{xxx} - \zeta\mathcal{U}_{xx} - \eta\mathcal{U}\mathcal{U}_x$.
The following theorem demonstrates the existence of a solution of the problem (3.1).

Theorem 2. *There exists* $\mathcal{G} : \mathcal{R}^2 \to \mathcal{R}$ *such that*

1) $|\mathcal{F}(x,t,\mathcal{X}) - \mathcal{F}(x,t,\mathcal{W})| \leq \frac{2-\gamma M(\gamma)}{2}e^b|\mathcal{X}(x,t) - \mathcal{W}(x,t)|$ *for* $(x,t) \in [0,1]^2$ *and* $\mathcal{X}, \mathcal{W} \in \mathcal{R}$;
2) *there exists* $\mathcal{X}_1 \in \mathcal{Z}$ *such that* $\mathcal{G}(\mathcal{X}_\infty, Q\mathcal{X}_1) \geq 0$, *where* $Q : \mathcal{Z} \to \mathcal{Z}$ *is defined by*

$$ Q\mathcal{X} = \mathcal{X}_\circ + {}_0^{CF}I^{\gamma}\mathcal{F}(x,t,\mathcal{X}(x,t)); $$

3) *for* $\mathcal{X}, \mathcal{W} \in \mathcal{Z}$, $\mathcal{G}(\mathcal{X}, \mathcal{W}) \geq 0$ *implies that* $\mathcal{G}(Q\mathcal{X}, Q\mathcal{W}) \geq 0$;
4) $\{\mathcal{X}_n\} \subseteq \mathcal{Z}, \lim\limits_{n\to\infty} \mathcal{X}_n = \mathcal{X}$, *where* $\mathcal{X} \in \mathcal{Z}$ *and* $\mathcal{G}(\mathcal{X}_n, \mathcal{X}_{n+1}) \geq 0$ *implies that* $\mathcal{G}(\mathcal{X}_n, \mathcal{X}) \geq 0$, *for all* $n \in \mathcal{N}$.

Then, there exists at least one fixed point of Q *that is the solution of the given model (3.1).*

Proof. To prove that Q has a fixed point, we consider

$$
\begin{aligned}
|Q\mathcal{X} - Q\mathcal{W}||Q\mathcal{X} - Q\mathcal{W} + 1| &= |{}^{CF}I[\mathcal{F}(x,t\mathcal{X}) - \mathcal{F}(x,t\mathcal{W})]||{}^{CF}I[\mathcal{F}(x,t\mathcal{X}) - \mathcal{F}(x,t\mathcal{W})] + 1| \\
&\leq \Big(\frac{2(1-\gamma)}{(2-\gamma)\mathcal{M}(\gamma)}|\mathcal{F}(x,t\mathcal{X}) - \mathcal{F}(x,t\mathcal{W})| \\
&\quad + \frac{2\gamma}{(2-\gamma)\mathcal{M}(\gamma)}\int_0^{\tau}|\mathcal{F}(x,t\mathcal{X}) - \mathcal{F}(x,t\mathcal{W})|d\tau\Big) \\
&\quad \times\Big(\frac{2(1-\gamma)}{(2-\gamma)\mathcal{M}(\gamma)}|\mathcal{F}(x,t\mathcal{X}) - \mathcal{F}(x,t\mathcal{W})| \\
&\quad + \frac{2\gamma}{(2-\gamma)\mathcal{M}(\gamma)}\int_0^{\tau}|\mathcal{F}(x,t\mathcal{X}) - \mathcal{F}(x,t\mathcal{W})|d\tau + 1\Big) \\
&\leq \Big(\frac{2(1-\gamma)}{(2-\gamma)\mathcal{M}(\gamma)}.\frac{(2-\gamma\mathcal{M}(\gamma))}{2}e^{-b}|\mathcal{X} - \mathcal{W}| \\
&\quad + \frac{2\gamma}{(2-\gamma)\mathcal{M}(\gamma)}.\frac{2-\gamma\mathcal{M}(\gamma)}{2}\int_0^{\tau}e^{-b}|\mathcal{X} - \mathcal{W}|d\tau\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\times\Big(\frac{2(1-\gamma)}{(2-\gamma)\mathcal{M}(\gamma)}.\frac{(2-\gamma\mathcal{M}(\gamma))}{2}e^{-b}|\mathcal{X}-\mathcal{W}| \\
&+\frac{2\gamma}{(2-\gamma)\mathcal{M}(\gamma)}.\frac{2-\gamma\mathcal{M}(\gamma)}{2}\int_0^\tau e^{-b}|\mathcal{X}-\mathcal{W}|d\tau+1\Big) \\
\leq\;& \big(e^{-b}\sup_{x,t\in[0,1]}|\mathcal{X}(x,t)-\mathcal{W}(x,t)|\big)\big(e^{-b}\sup_{x,t\in[0,1]}|\mathcal{X}(x,t)-\mathcal{W}(x,t)|+1\big) \\
=\;& (e^{-b}d(\mathcal{X},\mathcal{W}))(e^{-b}d(\mathcal{X},\mathcal{W})+1) \\
=\;& e^{-b}[e^{-b}(d(\mathcal{X},\mathcal{W}))^2+d(\mathcal{X},\mathcal{W})] \\
\leq\;& e^{-b}[(d(\mathcal{X},\mathcal{W}))^2+d(\mathcal{X},\mathcal{W})].
\end{aligned}
$$

Thus, for $\mathcal{X},\mathcal{W}\in\mathcal{Z}$ with $\mathcal{G}(\mathcal{X},\mathcal{W})\geq 0$, we obtain

$$
\begin{aligned}
&(d(\mathcal{Q}\mathcal{X},\mathcal{Q}\mathcal{W}))^2+d(\mathcal{Q}\mathcal{X},\mathcal{Q}\mathcal{W}) \\
&\leq e^{-b}[(d(\mathcal{X},\mathcal{W}))^2+d(\mathcal{X},\mathcal{W})].
\end{aligned}
$$

Taking the ln on both sides, we have

$$
b+\ln[d(\mathcal{Q}\mathcal{X},\mathcal{Q}\mathcal{W}))^2+d(\mathcal{Q}\mathcal{X},\mathcal{Q}\mathcal{W}] \;\leq\; \ln[d(\mathcal{X},\mathcal{W}))^2+d(\mathcal{X},\mathcal{W}].
$$

If $\mathcal{F}:[0,\infty)\to\mathcal{R}$ is defined by $\mathcal{F}(\mathfrak{u})=\ln[\mathfrak{u}^2+\mathfrak{u}]$, $\mathfrak{u}>0$, then $\mathcal{F}\in\delta$.
Now, define $\alpha:\mathcal{Z}\times\mathcal{Z}\to\{-\infty\}\cup[0,\infty)$ as

$$
\alpha(\mathcal{X},\mathcal{W}) \;=\; \begin{cases} 1, & \text{if } \mathcal{G}(\mathcal{X}(x,t),\mathcal{W}(x,t))\geq 0 \;\; for\;\; all\;\; x,t\in[0,1], \\ -\infty, & \text{otherwise.}\end{cases}
$$

Then,

$$
b+\alpha(\mathcal{X},\mathcal{W})\mathcal{F}(d(\mathcal{Q}\mathcal{X},\mathcal{Q}\mathcal{W})) \;\leq\; \mathcal{F}(d(\mathcal{X},\mathcal{W})),
$$

for $\mathcal{X},\mathcal{W}\in\mathcal{Z}$ with $d(\mathcal{Q}\mathcal{X},\mathcal{Q}\mathcal{W})>0$. Now, by $\mathcal{G}3$,

$$
\begin{aligned}
\alpha(\mathcal{X},\mathcal{W})\geq 1 \;&\Rightarrow\; \mathcal{G}(\mathcal{X},\mathcal{W})\geq 0 \;\Rightarrow\; \mathcal{G}(\mathcal{Q}\mathcal{X},\mathcal{Q}\mathcal{W})\geq 0 \\
&\Rightarrow\; \alpha(\mathcal{Q}\mathcal{X},\mathcal{Q}\mathcal{W})\geq 1,
\end{aligned}
$$

for all $\mathcal{X},\mathcal{W}\in\mathcal{Z}$. From $\mathcal{G}2$, there exists $\mathcal{X}_\circ\in\mathcal{Z}$ such that $\alpha(\mathcal{X}_\circ,\mathcal{Q}\mathcal{X}_\circ)\geq 1$. Therefore by $\mathcal{G}4$ and Theorem 1, there exists $\mathcal{X}^*\in\mathcal{Z}$ such that $\mathcal{X}^*=\mathcal{Q}\mathcal{X}^*$. Hence, $\mathcal{X}^*$ is the solution of the problem (3.1) □

Similarly, we can write the IVP (1.1) in the ABC sense as

$$
{}^{ABC}D_t^\alpha\mathcal{X}(x,t) \;=\; \mathcal{F}(x,t,\mathcal{X}(x,t)),\; 0<\alpha\leq 1, \tag{3.2}
$$

with initial condition

$$
\mathcal{X}(x,0)=g(x),
$$

where $\mathcal{F}(x,t,\mathcal{X}(x,t))=-\sigma\mathcal{U}_{xxx}-\zeta\mathcal{U}_{xx}-\eta\mathcal{U}\mathcal{U}_x$.

The following theorem shows the existence of a solution of the problem (3.2).

Theorem 3. *There exists* $\mathcal{G} : \mathcal{R}^2 \to \mathcal{R}$ *such that*

1) $|\mathcal{F}(x,t,\mathcal{X}) - \mathcal{F}(x,t,\mathcal{W})| \leq \frac{\Gamma\gamma M(\gamma)}{(1-\gamma)\Gamma\gamma+1} e^{\frac{-b}{2}} |\mathcal{X}(x,t) - \mathcal{W}(x,t)|$ *for* $(x,t) \in [0,1]^2$ *and* $\mathcal{X}, \mathcal{W} \in \mathcal{R}$;
2) *there exists* $\mathcal{X}_1 \in \mathcal{Z}$ *such that* $\mathcal{G}(\mathcal{Y}_\infty, \mathcal{Q}\mathcal{X}_1) \geq 0$, *where* $\mathcal{Q} : \mathcal{Z} \to \mathcal{Z}$ *is defined by*

$$\mathcal{Q}\mathcal{X} = \mathcal{X}_\circ + {}_0^{ABC}I^\gamma \mathcal{F}(x,t,\mathcal{X}(x,t));$$

3) *for* $\mathcal{X}, \mathcal{W} \in \mathcal{Z}$, $\mathcal{G}(\mathcal{X}, \mathcal{W}) \geq 0$ *implies that* $\mathcal{G}(\mathcal{Q}\mathcal{X}, \mathcal{Q}\mathcal{W}) \geq 0$;
4) $\{\mathcal{X}_n\} \subseteq \mathcal{Z}, \lim_{n\to\infty} \mathcal{X}_n = \mathcal{X}$, *where* $\mathcal{X} \in \mathcal{Z}$ *and* $\mathcal{G}(\mathcal{X}_n, \mathcal{X}_{n+1}) \geq 0$ *implies that* $\mathcal{G}(\mathcal{X}_n, \mathcal{X}) \geq 0$, *for all* $n \in \mathcal{N}$.

Then, there exists at least one fixed point of $\mathcal{Q}$ *which is the solution of the problem (3.2).*

Proof.

$$\begin{aligned}
|\mathcal{Q}\mathcal{X} - \mathcal{Q}\mathcal{W}|^2 &= |{}_0^{AB}I^\gamma[\mathcal{F}(x,t,\mathcal{X}(x,t)) - \mathcal{F}(x,t,\mathcal{W}(x,t))]|^2 \\
&\leq |\{\frac{1-\gamma}{\mathcal{M}(\gamma)}[\mathcal{F}(x,t,\mathcal{X}) - \mathcal{F}(x,t,\mathcal{W})] + \frac{\gamma}{\mathcal{M}(\gamma)}{}_0I^\gamma[\mathcal{F}(x,t,\mathcal{X}(x,t)) - \mathcal{F}(x,t,\mathcal{W}(x,t))]\}|^2 \\
&\leq \{\frac{1-\gamma}{\mathcal{M}(\gamma)}|\mathcal{F}(x,t,\mathcal{X}) - \mathcal{F}(x,t,\mathcal{W})| + \frac{\gamma}{\mathcal{M}(\gamma)}{}_0I^\gamma|\mathcal{F}(x,t,\mathcal{X}(x,t)) - \mathcal{F}(x,t,\mathcal{W}(x,t))|\}^2 \\
&\leq \{\frac{1-\gamma}{\mathcal{M}(\gamma)}.\frac{\mathcal{M}(\gamma)\Gamma\gamma}{(1-\gamma)\Gamma\gamma+1}e^{\frac{-b}{2}}\sqrt{|\mathcal{X}-\mathcal{W}|^2} \\
&\quad + \frac{\gamma}{\mathcal{M}(\gamma)}\frac{\mathcal{M}(\gamma)\Gamma\gamma}{(1-\gamma)\Gamma\gamma+1}{}_0I^\gamma 1.e^{\frac{-b}{2}}\sqrt{|\mathcal{X}-\mathcal{W}|^2}\}^2 \\
&= \{\frac{\mathcal{M}(\gamma)\Gamma\gamma}{(1-\gamma)\Gamma\gamma+1}e^{\frac{-b}{2}}\sqrt{|\mathcal{X}-\mathcal{W}|^2}\}^2 \\
&\quad \{\frac{1-\gamma}{\mathcal{M}(\gamma)} + \frac{\gamma}{\mathcal{M}(\gamma)\gamma\Gamma\gamma}\}^2 \\
&\leq \{\frac{\mathcal{M}(\gamma)\Gamma\gamma}{(1-\gamma)\Gamma\gamma+1}e^{\frac{-b}{2}}\sqrt{\sup_{x,t\in[0,1]}|\mathcal{X}(x,t)-\mathcal{W}(x,t)|^2}\}^2 \\
&\quad \{\frac{1-\gamma}{\mathcal{M}(\gamma)} + \frac{\gamma}{\mathcal{M}(\gamma)\gamma\Gamma\gamma}\}^2 \\
&= \{\frac{\mathcal{M}(\gamma)\Gamma\gamma}{(1-\gamma)\Gamma\gamma+1}e^{\frac{-b}{2}}\sqrt{d(\mathcal{X},\mathcal{W})}\}^2 \\
&\quad \{\frac{1-\gamma}{\mathcal{M}(\gamma)} + \frac{\gamma}{\mathcal{M}(\gamma)\Gamma\gamma}\}^2 \\
&= e^{-b}d(\mathcal{X},\mathcal{W}).
\end{aligned}$$

Consequently,

$$d(\mathcal{Q}\mathcal{X}, \mathcal{Q}\mathcal{W}) \leq e^{-b}d(\mathcal{X},\mathcal{W}).$$

Applying “ln” on both sides, we have

$$\ln(d(\mathcal{Q}\mathcal{X}, \mathcal{Q}\mathcal{W})) \leq \ln(e^{-b}d(\mathcal{X},\mathcal{W})),$$

and

$$b + \ln(d(Q\mathcal{X}, Q\mathcal{W}) \leq \ln(d(\mathcal{X}, \mathcal{W})).$$

Let $F : [0, \infty) \to \mathcal{R}$ be defined by $F(\lambda) = \ln \lambda$, where $\lambda > 0$. Then it is easy to show that $F \in \daleth$.
Now, define $\alpha : \mathcal{Z} \times \mathcal{Z} \to \{-\infty\} \cup [0, \infty)$ by

$$\alpha(\mathcal{X}, \mathcal{W}) = \begin{cases} 1, & \text{if } \mathcal{G}(\mathcal{X}(x,t), \mathcal{W}(x,t)) \geq 0 \ \ for\ all\ x, t \in [0,1], \\ -\infty, & \text{otherwise.} \end{cases}$$

Thus, $b + \alpha(\mathcal{X}, \mathcal{W})F(d(Q\mathcal{X}, Q\mathcal{W})) \leq F(d(\mathcal{X}, \mathcal{W}))$ for $\mathcal{X}, \mathcal{W} \in \mathcal{Z}$ with $d(Q\mathcal{X}, Q\mathcal{W}) \geq 0$. Therefore, Q is an α-type F-contraction. From ($\mathcal{G}$3), we have

$$\begin{aligned} \alpha(\mathcal{X}, \mathcal{W}) \geq 1 \quad &\Rightarrow \quad \mathcal{G}(\mathcal{X}, \mathcal{W}) \geq 0 \quad \Rightarrow \quad \mathcal{G}(Q\mathcal{X}, Q\mathcal{W}) \\ &\Rightarrow \quad \alpha(Q\mathcal{X}, Q\mathcal{W}) \geq 1, \end{aligned}$$

for all $x, t \in [0, 1]$. Thus, Q is α-admissible. From ($\mathcal{G}$2), there exists $\mathcal{X}_\circ \in \mathcal{Z}$ with $\alpha(\mathcal{X}_\circ, Q\mathcal{X}_\circ) \geq 1$. From ($\mathcal{G}$4) and Theorem [29], there exists $\mathcal{X}^* \in \mathcal{Z}$ such that $Q\mathcal{X}^*$. Hence, $\mathcal{X}^*$ is the solution of the IVP (3.2).

□

4. Modified double Laplace transform decomposition method (MDLDM)

Here, we briefly introduce the suggested technique MDLDM, which combines the DL transformation and Adomian decomposition method (ADM) for obtaining a series solution of non-linear ordinary DEs and PDEs. It is a very effective approach for obtaining the approximate values of dynamical problems such as KdV-Burgers, Sine-Gordon and KdV type equations. Here, we first introduce the technique and then the application of the given method on Eq (1.1).
Take the following form:

$$L\mathcal{U} + R\mathcal{U} + N\mathcal{U} = f(x,t), \quad \forall \ \ t \in \mathbb{R}, \tag{4.1}$$

where L is linear, R and N are operators containing linear and non-linear terms, respectively, and $f(x,t)$ is some external function. A_n, the well-known Adomian polynomials [31] of the functions $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \ldots$, can be described as

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[\sum_{k=0}^{n} \lambda^k \mathcal{U}_k \mathcal{U}_{kx} \right]_\lambda = 0. \tag{4.2}$$

4.1. The suggested model in ABC sense

Here, first take Eq (1.1) in the form of ABC sense as:

$${}^{ABC}D_t^\alpha \mathcal{U} + \sigma \mathcal{U}_{xxx} + \zeta \mathcal{U}_{xx} + \eta \mathcal{U}\mathcal{U}_x = 0\ , \tag{4.3}$$

with initial conditions

$$\mathcal{U}(x, 0) = R(x).$$

By applying the DL transform method on both sides of Eq (4.3), we obtain

$$\mathrm{L}_x\mathrm{L}_t{}^{ABC}D_t^{\alpha}\mathcal{U} = -1\Big[\sigma\mathrm{L}_x\mathrm{L}_t\mathcal{U}_{xxx} + \zeta\mathrm{L}_x\mathrm{L}_t\mathcal{U}_{xx} + \eta\mathrm{L}_x\mathrm{L}_t\mathcal{U}\mathcal{U}_x\Big].$$

Using initial condition and after some calculation, we have

$$\mathrm{L}_x\mathrm{L}_t\mathcal{U} = \frac{1}{s}\mathrm{L}_xR(x) - \Big(1-\alpha+\frac{\alpha}{s^{\alpha}}\Big)\mathrm{L}_x\mathrm{L}_t\Big[\sigma\mathcal{U}_{xxx} + \zeta\mathcal{U}_{xx} + \eta\mathcal{U}\mathcal{U}_x\Big].$$

Consider the series form

$$\mathcal{U} = \sum_{n=0}^{\infty}\mathcal{U}_n. \tag{4.4}$$

The non-linear term $\mathcal{U}\mathcal{U}_x$ can be calculated using "ADM"; we obtain

$$\mathcal{U}\mathcal{U}_x = \sum_{n=0}^{\infty}A_n,$$

$$\mathrm{L}_x\mathrm{L}_t\sum_{n=0}^{\infty}\mathcal{U}_n = \frac{1}{s}\mathrm{L}_xR(x) - \Big(1-\alpha+\frac{\alpha}{s^{\alpha}}\Big)\Big[\mathrm{L}_x\mathrm{L}_t\Big(\sigma\sum_{n=0}^{\infty}\mathcal{U}_{nxxx} + \zeta\sum_{n=0}^{\infty}\mathcal{U}_{nxx} + \eta\sum_{n=0}^{\infty}A_n\Big)\Big]. \tag{4.5}$$

Applying the double Laplace inverse transform and Equating terms of both sides,

$$\begin{aligned}
\mathcal{U}_0 &= R(x),\\
\mathcal{U}_1 &= -\mathrm{L}_x^{-1}\mathrm{L}_t^{-1}\Big(1-\alpha+\frac{\alpha}{s^{\alpha}}\Big)\mathrm{L}_x\mathrm{L}_t\Big(\sigma\mathcal{U}_{0xxx} + \zeta\mathcal{U}_{0xx} + \eta A_0\Big),\\
\mathcal{U}_2 &= -\mathrm{L}_x^{-1}\mathrm{L}_t^{-1}\Big(1-\alpha+\frac{\alpha}{s^{\alpha}}\Big)\mathrm{L}_x\mathrm{L}_t\Big(\sigma\mathcal{U}_{1xxx} + \zeta\mathcal{U}_{1xx} + \eta A_1\Big),\\
\mathcal{U}_3 &= -\mathrm{L}_x^{-1}\mathrm{L}_t^{-1}\Big(1-\alpha+\frac{\alpha}{s^{\alpha}}\Big)\mathrm{L}_x\mathrm{L}_t\Big(\sigma\mathcal{U}_{2xxx} + \zeta\mathcal{U}_{2xx} + \eta A_2\Big),\\
&\vdots
\end{aligned}$$

The final solution can be calculated as follows:

$$\mathcal{U} = \sum_{n=0}^{\infty}\mathcal{U}_n. \tag{4.6}$$

4.2. The proposed model in CF Form

Now, we take Eq (1.1) in CF fractional derivative sense as:

$${}^{CF}D_t^{\alpha}\mathcal{U} + \sigma\mathcal{U}_{xxx} + \zeta\mathcal{U}_{xx} + \eta\mathcal{U}\mathcal{U}_x = 0\,, \tag{4.7}$$

with initial conditions

$$\mathcal{U}(x,0) = R(x).$$

Using (3.1), we obtain the following series solutions:

$$\begin{aligned}
\mathcal{U}_0 &= R(x).\\
\mathcal{U}_1 &= -\mathrm{L}_x^{-1}\mathrm{L}_t^{-1}\left(\frac{(1-s)\alpha+s}{s}\right)\mathrm{L}_x\mathrm{L}_t\Big(\sigma\mathcal{U}_{0xxx}+\zeta\mathcal{U}_{0xx}+\eta A_0\Big),\\
\mathcal{U}_2 &= -\mathrm{L}_x^{-1}\mathrm{L}_t^{-1}\left(\frac{(1-s)\alpha+s}{s}\right)\mathrm{L}_x\mathrm{L}_t\Big(\sigma\mathcal{U}_{1xxx}+\zeta\mathcal{U}_{1xx}+\eta A_1\Big),\\
\mathcal{U}_3 &= -\mathrm{L}_x^{-1}\mathrm{L}_t^{-1}\left(\frac{(1-s)\alpha+s}{s}\right)\mathrm{L}_x\mathrm{L}_t\Big(\sigma\mathcal{U}_{2xxx}+\zeta\mathcal{U}_{2xx}+\eta A_2\Big),\\
&\vdots
\end{aligned}$$

The final form can be expressed as

$$\mathcal{U}=\sum_{n=0}^{\infty}\mathcal{U}_n. \tag{4.8}$$

5. Applications

Here, we consider a numerical example in the form of time fractional derivatives in ABC and CF operators.

Example 1. *Consider Eq (1.1) in the following form:*

$$\frac{\partial^\alpha\mathcal{U}}{\partial t^\alpha}+\sigma\frac{\partial^3\mathcal{U}}{\partial x^3}+\zeta\frac{\partial^2\mathcal{U}}{\partial x^2}+\eta\mathcal{U}\frac{\partial\mathcal{U}}{\partial x}=0,\quad 0<\alpha\le 1,\quad t>0, \tag{5.1}$$

with initial condition

$$\mathcal{U}_0=\mathrm{a}_0-\frac{3\zeta^2\tanh^2\left(\frac{\zeta x}{10\sigma}\right)}{25\eta\sigma}+\frac{6\zeta^2\tanh\left(\frac{\zeta x}{10\sigma}\right)}{25\eta\sigma}.$$

The exact solution of Eq (5.1) is:

$$\mathcal{U}=\mathrm{a}_0+\frac{6\zeta^2}{25\eta\sigma}\tanh\left(\frac{\zeta}{10\sigma}\left(x+\frac{\left(3\zeta^2-25\mathrm{a}_0\eta\sigma\right)t}{25\sigma}\right)\right)-\frac{3\zeta^2}{25\eta\sigma}\tanh^2\left(\frac{\zeta}{10\sigma}\left(x+\frac{\left(3\zeta^2-25\mathrm{a}_0\eta\sigma\right)t}{25\sigma}\right)\right).$$

Here, we will discuss two cases.

Case I:

Consider Eq (5.1) in ABC form as follows:

$${}^{ABC}D_t^\alpha\mathcal{U}+\sigma\mathcal{U}_{xxx}+\zeta\mathcal{U}_{xx}+\eta\mathcal{U}\mathcal{U}_x=0. \tag{5.2}$$

The approximate solutions of Eq (5.2) by using the techniques discussed in Section 4 are obtained in the series up to $O(3)$ and given by

$$\mathcal{U}_0=\mathrm{a}_0-\frac{3\zeta^2\tanh^2\left(\frac{\zeta x}{10\sigma}\right)}{25\eta\sigma}+\frac{6\zeta^2\tanh\left(\frac{\zeta x}{10\sigma}\right)}{25\eta\sigma},$$

$$\mathcal{U}_1 = -\frac{3\zeta^3\left(t^\alpha+\Gamma(\alpha)-\alpha\Gamma(\alpha)\right)\left(3\zeta^2-25a_0\eta\sigma\right)\left(\tanh\left(\frac{\zeta x}{10\sigma}\right)-1\right)\operatorname{sech}^2\left(\frac{\zeta x}{10\sigma}\right)}{3125\eta\sigma^3\Gamma(\alpha)},$$

$$\mathcal{U}_2 = \frac{3\zeta^4}{781250\eta\sigma^5\Gamma(\alpha)\Gamma(1+2\alpha)}\left((t^{2\alpha}\alpha+2(\alpha-1)^2\Gamma(2\alpha))\Gamma(1+\alpha)-2t^\alpha(\alpha-1)\Gamma(1+2\alpha)\right)$$
$$\left(3\zeta^2-25a_0\eta\sigma\right)^2\left(\tanh\left(\frac{\zeta x}{10\sigma}\right)-1\right)\left(3\tanh\left(\frac{\zeta x}{10\sigma}\right)+1\right)\operatorname{sech}^2\left(\frac{\zeta x}{10\sigma}\right),$$

$$\begin{aligned}
\mathcal{U}_3 &= \frac{-3\zeta^5(3\zeta^2-25a_0\eta\sigma)^2\operatorname{sech}^6\left(\frac{x\zeta}{10\sigma}\right)\left(\tanh\left(\frac{\zeta x}{10\sigma}\right)-1\right)}{781250000\sigma^7\eta(\Gamma(\alpha))^2\Gamma(1+\alpha)\Gamma(1+2\alpha)\Gamma(1+3\alpha)}[24\,(-1+\alpha)^2\,(\zeta)^2(\Gamma(\alpha))^2(t^\alpha\alpha-\\
&\quad (-1+\alpha)\Gamma(1+\alpha))\Gamma(1+2\alpha)\Gamma(1+3\alpha)\left(\cosh\left(\frac{x\zeta}{10\sigma}\right)-\sinh\left(\frac{x\zeta}{10\sigma}\right)\right)\left(\cosh\left(\frac{x\zeta}{10\sigma}\right)+3\sinh\left(\frac{x\zeta}{10\sigma}\right)\right)\\
&- 24t^{2\alpha}\zeta^2\Gamma(1+\alpha)\Gamma(1+2\alpha)(t^\alpha\alpha\Gamma(1+2\alpha)-(-1+\alpha)\Gamma(1+3\alpha))\left(-2+\cosh\left(\frac{x\zeta}{5\sigma}\right)-\sinh\left(\frac{x\zeta}{5\sigma}\right)\right)\\
&+ \Gamma(\alpha)\Gamma(1+\alpha)(2t^\alpha(-1+\alpha)^2\Gamma(1+3\alpha)(\alpha\Gamma(2\alpha)(-123\zeta^2+225a_0\eta\sigma+4(9\zeta^2+25a_0\eta\sigma)\cosh\left(\frac{x\zeta}{5\sigma}\right)\\
&+ 5(3\zeta^2-25a_0\eta\sigma)\cosh\left(\frac{2x\zeta}{5\sigma}\right)-30(\zeta^2+5a_0\eta\sigma)\sinh\left(\frac{x\zeta}{5\sigma}\right)+9\zeta^2\sinh\left(\frac{2x\zeta}{5\sigma}\right)-75a_0\eta\sigma\sinh\left(\frac{2x\zeta}{5\sigma}\right))\\
&+ \Gamma(1+2\alpha)(-75\zeta^2+225a_0\eta\sigma+4(3\zeta^2+25a_0\eta\sigma)\cosh\left(\frac{x\zeta}{5\sigma}\right)+5(3\zeta^2-25a_0\eta\sigma)\cosh\left(\frac{2x\zeta}{5\sigma}\right)\\
&- 6(\zeta^2+25a_0\eta\sigma)\sinh\left(\frac{x\zeta}{5\sigma}\right)+9\zeta^2\sinh\left(\frac{2x\zeta}{5\sigma}\right)-75a_0\eta\sigma\sinh\left(\frac{2x\zeta}{5\sigma}\right)))+\Gamma(1+\alpha)(t^{3\alpha}\alpha^2\Gamma(1+2\alpha)(-123\zeta^2\\
&+ 225a_0\eta\sigma+4(9\zeta^2+25a_0\eta\sigma)\cosh\left(\frac{x\zeta}{5\sigma}\right)+5(3\zeta^2-25a_0\eta\sigma)\cosh\left(\frac{2x\zeta}{5\sigma}\right)-30(\zeta^2+5a_0\eta\sigma)\sinh\left(\frac{x\zeta}{5\sigma}\right)\\
&+ 9\zeta^2\sinh\left(\frac{2x\zeta}{5\sigma}\right)-75a_0\eta\sigma\sinh\left(\frac{2x\zeta}{5\sigma}\right))-(-1+\alpha)\Gamma(1+3\alpha)(3t^{2\alpha}\alpha(-91\zeta^2+225a_0\eta\sigma\\
&+ 20(\zeta^2+5a_0\eta\sigma)\cosh\left(\frac{x\zeta}{5\sigma}\right)+5(3\zeta^2-25a_0\eta\sigma)\cosh\left(\frac{2x\zeta}{5\sigma}\right)-14\zeta^2\sinh\left(\frac{x\zeta}{5\sigma}\right)-150a_0\eta\sigma\sinh\left(\frac{x\zeta}{5\sigma}\right)\\
&+ 9\zeta^2\sinh\left(\frac{2x\zeta}{5\sigma}\right)-75a_0\eta\sigma\sinh\left(\frac{2x\zeta}{5\sigma}\right))+2(-1+\alpha)^2\Gamma(2\alpha)(-123\zeta^2+225a_0\eta\sigma\\
&+ 4(9\zeta^2+25a_0\eta\sigma)\cosh\left(\frac{x\zeta}{5\sigma}\right)+5(3\zeta^2-25a_0\eta\sigma)\cosh\left(\frac{2x\zeta}{5\sigma}\right)-30(\zeta^2+5a_0\eta\sigma)\sinh\left(\frac{x\zeta}{5\sigma}\right)\\
&+ 9\zeta^2\sinh\left(\frac{2x\zeta}{5\sigma}\right)-75a_0\eta\sigma\sinh\left(\frac{2x\zeta}{5\sigma}\right)))))]
\end{aligned}$$

$$\vdots$$

The simplified form of $\mathcal{U}_3$ is:

$$\begin{aligned}
\mathcal{U}_3 = \frac{3\zeta^5(3\zeta^2-25a_0\eta\sigma)^2\operatorname{sech}^4\left(\frac{x\zeta}{10\sigma}\right)\left(\tanh\left(\frac{\zeta x}{10\sigma}\right)-1\right)}{1562500000\sigma^7\eta}&\left[\frac{4}{3(1+\exp^{\frac{x\zeta}{5\sigma}})^2}\left(\exp^{\frac{-x\zeta}{5\sigma}}\left(6(\alpha-1)^3(3(1+3\exp^{\frac{x\zeta}{5\sigma}}-25\exp^{\frac{2x\zeta}{5\sigma}}\right.\right.\right.\\
+\exp^{\frac{3x\zeta}{5\sigma}}+4\exp^{\frac{4x\zeta}{5\sigma}})\zeta^2-25a_0(1+\exp^{\frac{x\zeta}{5\sigma}})^2(1-7\exp^{\frac{x\zeta}{5\sigma}}+4\exp^{\frac{2x\zeta}{5\sigma}})\eta\sigma)&+t^{2\alpha}\alpha\left(\frac{9(\alpha-1)}{\Gamma(2\alpha)}((3+17\exp^{\frac{x\zeta}{5\sigma}}-91\exp^{\frac{2x\zeta}{5\sigma}}\right.\\
+3\exp^{\frac{3x\zeta}{5\sigma}}+12\exp^{\frac{4x\zeta}{5\sigma}})\zeta^2-25a_0(1+\exp^{\frac{x\zeta}{5\sigma}})^2(1-7\exp^{\frac{x\zeta}{5\sigma}}+4\exp^{\frac{2x\zeta}{5\sigma}})\eta\sigma)&-\frac{2t^\alpha\alpha}{\Gamma(3\alpha)}(3(1+11\exp^{\frac{x\zeta}{5\sigma}}-41\exp^{\frac{2x\zeta}{5\sigma}}+\exp^{\frac{3x\zeta}{5\sigma}}\\
\left.\left.+4\exp^{\frac{4x\zeta}{5\sigma}})\zeta^2-25a_0(1+\exp^{\frac{x\zeta}{5\sigma}})^2(1-7\exp^{\frac{x\zeta}{5\sigma}}+4\exp^{\frac{2x\zeta}{5\sigma}})\eta\sigma)\right)\right)&-\left.\frac{48(2\exp^{\frac{x\zeta}{5\sigma}}-1)t^{2\alpha}\zeta^2(-3(\alpha-1)\Gamma(3\alpha)+t^\alpha\Gamma(1+2\alpha))}{(\Gamma(\alpha))^2\Gamma(3\alpha)}\right)
\end{aligned}$$

$$-\frac{24t^{\alpha}(\alpha-1)}{\Gamma(\alpha-1)}\Big(-9\zeta^2+175a_0\eta\sigma+5(3\zeta^2-25a_0\eta\sigma)\cosh\left[\frac{x\zeta}{5\sigma}\right]-18\zeta^2 sech^2\left[\frac{x\zeta}{5\sigma}\right]+3(3\zeta^2-25a_0\eta\sigma)\sinh\left[\frac{x\zeta}{5\sigma}\right]-12\zeta^2\tanh\left[\frac{x\zeta}{5\sigma}\right]\Big)\Big].$$

Therefore,

$$\mathcal{U}=\mathcal{U}_0+\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3..... \tag{5.3}$$

Case II:

Consider Eq (1.2) in the CF sense as

$${}^{CF}D_t^{\alpha}\mathcal{U}+\sigma\mathcal{U}_{xxx}+\zeta\mathcal{U}_{xx}+\eta\mathcal{U}\mathcal{U}_x=0\,. \tag{5.4}$$

The approximate solutions of Eq (5.4) obtained by using the approaches discussed in Section 4 up to $O(3)$ are given by

$$\begin{aligned}
\mathcal{U}_0 &= = a_0-\frac{\left(3\zeta^2\right)\tanh^2\left(\frac{\zeta x}{10\sigma}\right)}{25\eta\sigma}+\frac{\left(6\zeta^2\right)\tanh\left(\frac{\zeta x}{10\sigma}\right)}{25\eta\sigma},\\
\mathcal{U}_1 &= -\frac{3\zeta^3(1+(t-1)\alpha)\left(3\zeta^2-25a_0\eta\sigma\right)\left(\tanh\left(\frac{\zeta x}{10\sigma}\right)-1\right)\operatorname{sech}^2\left(\frac{\zeta x}{10\sigma}\right)}{3125\eta\sigma^3},\\
\mathcal{U}_2 &= \frac{3\zeta^4\left(\alpha^2(t^2-4t+2)+4\alpha(t-1)+2\right)\left(3\zeta^2-25a_0\eta\sigma\right)^2\left(\tanh\left(\frac{\zeta x}{10\sigma}\right)-1\right)\left(3\tanh\left(\frac{\zeta x}{10\sigma}\right)+1\right)\operatorname{sech}^2\left(\frac{\zeta x}{10\sigma}\right)}{1562500\eta\sigma^5},
\end{aligned}$$

$$\begin{aligned}
\mathcal{U}_3 &= \frac{\zeta^5(3\zeta^2-25a_0\eta\sigma)^2\operatorname{sech}^6(\frac{x\zeta}{10\sigma})}{1562500000\eta\sigma^7}[4(3(-6-18(-1+t)\alpha-3(6-12t+t^2)\alpha^2+(6-18t+3t^2+t^3)\alpha^3)\zeta^2\\
&- 25a_0(6+18(-1+t)\alpha+9(2-4t+t^2)\alpha^2+(-6+18t-9t^2+t^3)\alpha^3)\eta\sigma)\cosh(\frac{x\zeta}{5\sigma})-5(6+18(-1+t)\alpha\\
&+ 9(2-4t+t^2)\alpha^2+(-6+18t-9t^2+t^3)\alpha^3)(3\zeta^2-25a_0\eta\sigma)\cosh(\frac{2x\zeta}{5\sigma})-3(-3(50+150(-1+t)\alpha\\
&+ (150-300t+59t^2)\alpha^2+(-50+150t-59t^2+3t^3)\alpha^3)\zeta^2+75a_0(6+18(-1+t)\alpha+9(2-4t+t^2)\alpha^2\\
&+ (-6+18t-9t^2+t^3)\alpha^3)\eta\sigma+2(3(-2-6(-1+t)\alpha+(-6+12t+t^2)\alpha^2+(2-6t-t^2+t^2)\alpha^3)\zeta^2\\
&- 25a_0(6+18(-1+t)\alpha+9(2-4t+t^2)\alpha^2+(-6+18t-9t^2+t^3)\alpha^3)\eta\sigma)\sinh(\frac{x\zeta}{5\sigma})+(6+18(-1+t)\alpha\\
&+ 9(2-4t+t^2)\alpha^2+(-6+18t-9t^2+t^3)\alpha^3)(3\zeta^2-25a_0\eta\sigma)\sinh(\frac{2x\zeta}{5\sigma}))](-1+\tanh(\frac{x\zeta}{10\sigma}))\\
&\vdots
\end{aligned}$$

The simplified form is

$$\mathcal{U}_3=\frac{\zeta^5(3\zeta^2-25a_0\eta\sigma)^2\operatorname{sech}^6(\frac{x\zeta}{10\sigma})(\tanh(\frac{x\zeta}{10\sigma})-1)}{1562500000\eta\sigma^7}\Big[9\Big(50+\alpha\Big(150t(\alpha-1)^2-59t^2(\alpha-1)\alpha+3t^3\alpha^2-50(3+(\alpha-3)\alpha)\Big)\Big)\zeta^2-225a_0\Big(6+18(t-1)\alpha+9(2+(t-4)t)\alpha^2+(-6+(t-6)(t-3)t)\alpha^3\Big)\eta\sigma+12\Big(-6-18(t-1)\alpha-3(6+(t-12)t)\alpha^2+(6+(t-3)t(t+6))\alpha^3\Big)\zeta^2\cosh\left[\frac{x\zeta}{5\sigma}\right]+\Big(6+18(t-1)\alpha$$

$$+ 9(2 + (t-4)t)\alpha^2 + (-6 + (t-6)(t-3)t)\alpha^3\Big)(3\zeta^2 - 25\mathrm{a}_0\eta\sigma)\Big(-5\cosh\left[\frac{2x\zeta}{5\sigma}\right] - 3\sinh\left[\frac{2x\zeta}{5\sigma}\right]\Big)$$
$$- 18\Big(-2 + \alpha(6 - 6t + (-6 + t(t+12))\alpha + (2 + (t-3)t(t+2))\alpha^2)\Big)\zeta^2\sinh\left[\frac{x\zeta}{5\sigma}\right] + 50\mathrm{a}_0\Big(6 + 18(-1+t)\alpha$$
$$+ 9(2 - 4t + t^2)\alpha^2 + (-6 + 18t - 9t^2 + t^3)\alpha^3\Big)\sigma\eta\Big(-2\cosh\left[\frac{x\zeta}{5\sigma}\right] + 3\sinh\left[\frac{x\zeta}{5\sigma}\right]\Big)\Big]$$

The final result in the series form up to $O(3)$ is given by

$$\mathcal{U} = \mathcal{U}_0 + \mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3..... \tag{5.5}$$

The errors between the exact and approximate solutions in the ABC and CF senses are shown in the Tables 1 and 2, respectively.

Table 1. The comparison of approximated ABC and exact solutions for $\alpha = 1$ at various points for Example 1.

(x,t)	$\mathcal{U}$	Exact	| Exact−$\mathcal{U}$|	(x,t)	$\mathcal{U}$	Exact	| Exact−$\mathcal{U}$|
(−4,0.01)	−3.5967	−3.5967	4.437×10^{-11}	(−2,0.01)	−3.42484	−3.42484	1.488×10^{-9}
(−1,0.01)	−2.50239	−2.50239	7.202×10^{-9}	(0,0.01)	0.02862	0.02862	1.666×10^{-8}
(1,0.01)	1.13462	1.13462	4.376×10^{-9}	(2,0.01)	1.19852	1.19852	2.930×10^{-10}
(4,0.01)	1.2000	1.2000	1.184×10^{-13}	(−4,0.1)	−3.5959	−3.5959	4.635×10^{-7}
(−2,0.1)	−3.3840	−3.3840	1.515×10^{-5}	(−1,0.1)	−2.2941	−2.2942	7.761×10^{-5}
(0,0.1)	0.2693	0.2695	1.725×10^{-4}	(1,0.1)	1.1556	1.1556	4.302×10^{-5}
(2,0.1)	1.1990	1.1990	2.715×10^{-6}	(4,0.1)	1.2000	1.2000	1.085×10^{-9}
(−4,0.05)	−3.5963	−3.5963	2.827×10^{-8}	(−2,0.05)	−3.4077	−3.4077	9.378×10^{-7}
(−1,0.05)	−2.4131	−2.4131	4.656×10^{-6}	(0,0.05)	0.1395	0.1395	1.06×10^{-5}
(1,0.05)	1.1448	1.1448	2.716×10^{-6}	(2,0.05)	1.1987	1.1987	1.769×10^{-7}
(4,0.05)	1.2000	1.2000	7.099×10^{-11}	(−4,0.2)	−3.5948	−3.5948	7.795×10^{-6}
(−2,0.2)	−3.3282	−3.3279	2.4641×10^{-4}	(−1,0.2)	−2.0304	−2.0317	1.3403×10^{-3}
(0,0.2)	0.4958	0.4986	2.8191×10^{-3}	(1,0.2)	1.1720	1.1713	6.7111×10^{-4}
(2,0.2)	1.1994	1.1994	4.007×10^{-5}	(4,0.2)	1.2000	1.2000	1.5911×10^{-8}

Table 2. The comparison of approximated the CF and exact solution for $\alpha = 1$ at various points for Example 1.

(x,t)	$\mathcal{U}$	Exact	| Exact−$\mathcal{U}$|	(x,t)	$\mathcal{U}$	Exact	| Exact−$\mathcal{U}$|
(−4,0.01)	−3.5967	−3.5967	4.437×10^{-11}	(−2,0.01)	−3.4248	−3.4248	1.488×10^{-9}
(−1,0.01)	−2.5023	−2.5023	7.202×10^{-9}	(0,0.01)	0.02862	0.02862	1.666×10^{-8}
(1,0.01)	1.13462	1.13462	4.376×10^{-9}	(2,0.01)	1.19852	1.19852	2.930×10^{-10}
(4,0.01)	1.2000	1.2000	1.184×10^{-13}	(−4,0.1)	−3.5959	−3.5959	4.635×10^{-7}
(−2,0.1)	−3.3840	−3.3840	1.515×10^{-5}	(−1,0.1)	−2.2941	−2.2942	7.761×10^{-5}
(0,0.1)	0.2693	0.2695	1.725×10^{-4}	(1,0.1)	1.1556	1.1556	4.302×10^{-5}
(2,0.1)	1.1990	1.1990	2.715×10^{-6}	(4,0.1)	1.2000	1.2000	1.085×10^{-9}
(−4,0.05)	−3.5963	−3.5963	2.827×10^{-8}	(−2,0.05)	−3.4077	−3.4077	9.378×10^{-7}
(−1,0.05)	−2.4131	−2.4131	4.656×10^{-6}	(0,0.05)	0.1395	0.1395	1.06×10^{-5}
(1,0.05)	1.1448	1.1448	2.716×10^{-6}	(2,0.05)	1.1987	1.1987	1.769×10^{-7}
(4,0.05)	1.2000	1.2000	7.099×10^{-11}	(−4,0.2)	−3.5948	−3.5948	7.795×10^{-6}
(−2,0.2)	−3.3282	−3.3279	2.4641×10^{-4}	(−1,0.2)	−2.0304	−2.0317	1.3403×10^{-3}
(0,0.2)	0.4958	0.4986	2.8191×10^{-3}	(1,0.2)	1.1720	1.1713	6.7111×10^{-4}
(2,0.2)	1.1994	1.1994	4.007×10^{-5}	(4,0.2)	1.2000	1.2000	1.5911×10^{-8}

5.1. Discussion

Take the parameteric values as $\eta = 1$, $a_0 = 0$ and ζ and $\sigma = 0.1$ for numerical computations. Tables 1 and 2 show the errors between the exact and approximated values in the ABC and CF senses, while the exact and computational values are also shown in these tables. Figure 1 depicts the comparison of approximated solutions in ABC and CF forms for different values of α at $t = 1$. The surfaces in Figure 2 show the exact and approximated solutions of the ABC and the CF forms, respectively, for Eq (1.2) at $\alpha = 1$. Figures 3 and 4 present the approximate solutions when $\alpha = 0.95$ and $\alpha = 0.90$, respectively. These graphical representations show the behaviour of ABC and CF operators respectively for the proposed problem. Figures 5 and 6 depict the behaviours of surfaces for various values of α at different time levels. The first representation depicts the ABC form, while the second shows the CF form. It is straightforward to deduce that, as the fractional parameter α decreases, the wave response bifurcates into a wave but only for small values of x, and we observe that the amplitude of the wave grows over time t.

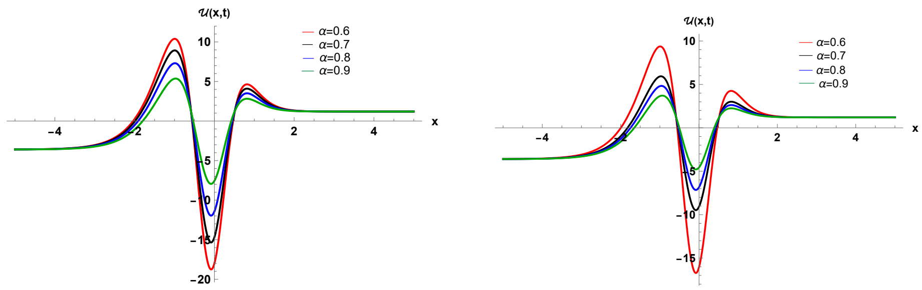

Figure 1. Comparison plots of approximated solutions of ABC and CF, respectively, for various values α.

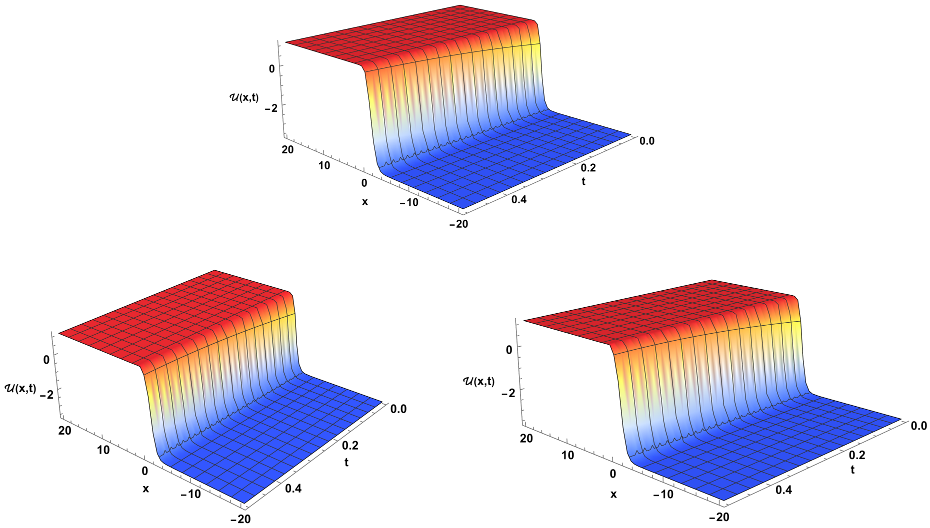

Figure 2. Comparison plots of exact solutions, approximated solutions of ABC and CF for $\alpha = 1$ respectively.

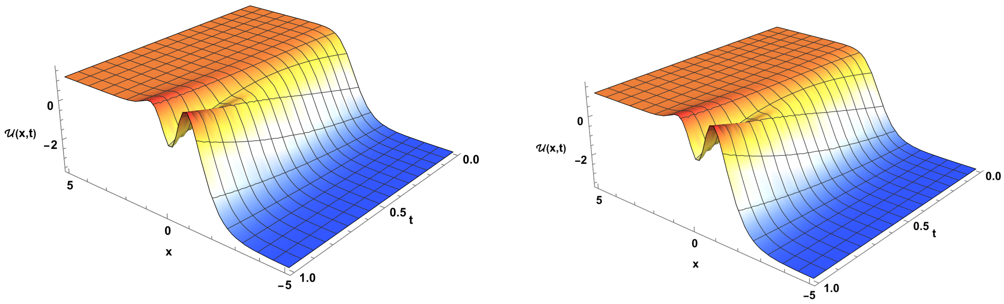

Figure 3. Comparison plots of approximated solutions of ABC and CF, respectively, for $\alpha = 0.95$.

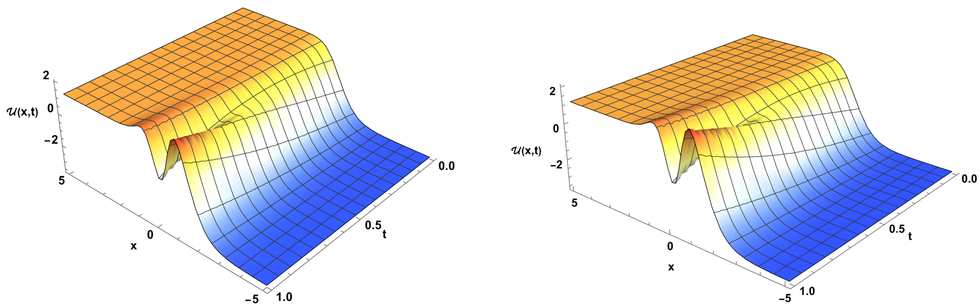

Figure 4. Comparison plots of approximated solutions of ABC and CF, respectively, for $\alpha = 0.9$.

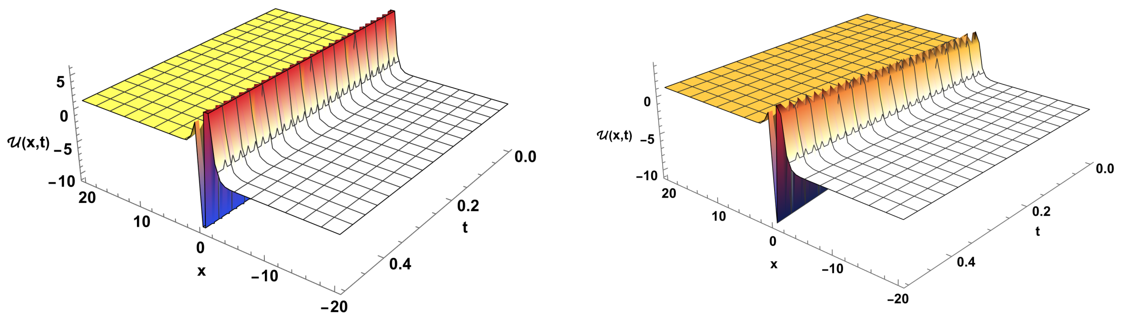

Figure 5. Comparison plots of approximated solutions of ABC and CF, respectively, for $\alpha = 0.5$.

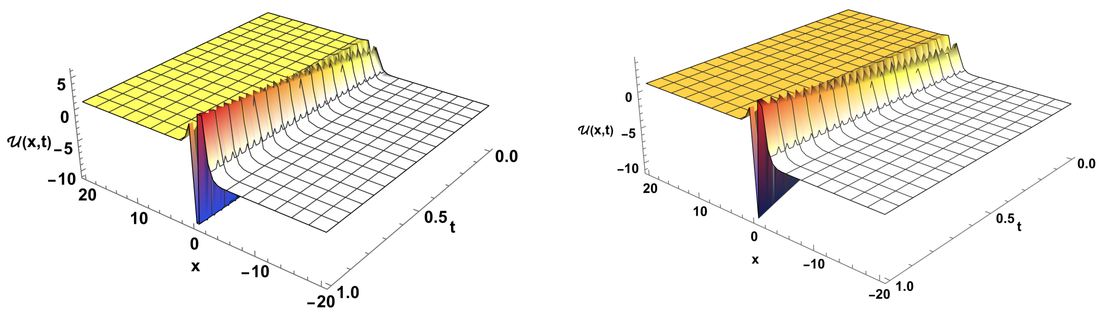

Figure 6. Comparison plots of approximated solutions of ABC and CF, respectively, for $\alpha = 0.25$.

6. Conclusions

In this manuscript, the time fractional KdV-Burger's model with initial conditions under two non-local operators with exponential kernel and Mittag-Leffler kernel has been investigated. The existence of the solution for both operators has been demonstrated through fixed point results of α-type F contraction. The MDLDM was utilized to compute a series solutions that tends to the exact values for the special case $\alpha = 1$. As a result, we found highly accurate computed solutions to the fractional KdV-Burger's equation. The solutions show how consistently accurate the technique is and how broadly applicable it is to fractional nonlinear evolution problems. Quick convergence is seen when solutions are simulated numerically. Furthermore, neither linearization nor perturbation are needed. Therefore, it offers more authentic series solutions that typically converge quickly in actual physical problems.

Conflict of interest

The authors declare that they have no conflict of interest.

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