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Research article

Compactness and connectedness via the class of soft somewhat open sets

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Abstract: This paper is devoted to study the concepts of compactness, Lindelöfness and connectedness via the class of soft somewhat open sets which represents one of the generalizations of soft open sets. Beside investigation the main properties of these concepts, it is demonstrated, with the help of examples, that some properties of their counterparts via soft open sets are invalid. Also, the relationships between these concepts and their counterparts defined in classical topology (which is studied herein under the name of parametric topology) are discussed in detail. Moreover, we provide the sufficient conditions that guarantee the equivalence between them. In this regard, it is proved that all introduced types of soft compact and Lindelöf spaces are transmitted to all parametric topologies without imposing any conditions, whereas the converse holds true under the conditions of a full soft topology and a finite (countable) set of parameters. These characterizations represent a unique behavior of these spaces compared to the other types defined by celebrated generalizations of soft open sets. Also, there is no relationship associating soft *sw*-connectedness with its counterparts via parametric topologies. We successfully describe soft *sw*-disconnectedness using soft open sets instead of soft *sw*-open sets and consequently prove that the concepts of soft *sw*-connected and soft hyperconnected spaces are identical. In conclusion, the obtained results show that the framework given in this manuscript enriches and generalizes the previous works, and has a good application prospect.

Keywords: soft somewhat open set; soft compact; soft Lindelöf; soft connected; soft topology; parametric topology
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1. Introduction

In our daily life, we face many problems involving uncertainty that we cannot cope with using traditional mathematical tools. To deal with these types of problems, it was proposed some scenarios such as fuzzy sets and soft sets. Molodtsov [35], in 1999, provided the concept of soft sets as a new mathematical approach to handle imprecise data and vague situations. Molodtsov elaborated how can be applied soft sets in several disciplines and gave their advantages compared to fuzzy sets. Then, this approach has attracted the attention of a huge number of researchers and scholars who are interested in uncertainty in both theoretical and applied issues. Maji et al. [34], in 2002, addressed decision-making problems by making use of soft sets, and then they [35] provided the initial conception for a set of operations between soft sets in 2003. Some of these operations suffered deficiencies which motivated some authors to adjust their definitions as well as adopted new types of them for different purposes as illustrated in the published literature [6, 39].

In 2011, it was introduced two techniques to define a topology over a family of soft sets by Shabir and Naz [43] and Çağman et al. [27]. The difference between their definition was in terms of the form of a set of parameters; Shabir and Naz familiarized a soft topology over a constant set of parameters whereas Çağman et al. formulated a soft topology over different subsets of parameters. According to the definition of Shabir and Naz, it can be characterized soft topological concepts in a similar way to their counterparts in classical topology, which promotes and enhances researchers to follow this line.

After the advent of soft topology, many contributions to discuss topological concepts and notions via soft settings have been conducted. The basic topological ideas were put forward by Shabir and Naz [43]. Min [36] shed light on soft separation axioms and proved that every soft T_3 space is soft T_2. El-Shafei et al. [28] defined new types of belonging and non-belonging relations and applied to present novel kinds of soft T_i-spaces. As illustrated in recent studies [15,21,44] soft separation axioms are still a hot topic. Compactness is one of the celebrated idea in soft topologies; it was introduced by Aygünoğlu and Aygün [25]. Hida [30] benefited from the variety of belonging relations in the soft set theory to initiate new kinds of soft compactness. Al-shami et al. [14] discussed new sorts of compactness namely, almost soft compact and approximately soft compact spaces. The classical behaviour of compact subset of Hausdorff spaces is invalid via soft topology as demonstrated in [28]. Al-jarrah et al. [7] and Al-shami et al. [16] studied soft compact and Lindelöf spaces via soft regular closed and soft somewhere dense sets. Selection principles and covering properties have been recently explored via soft topologies in [18,19,33]. The concept of soft connected spaces was probed in [42,46]. Asaad [23] scrutinized the main features of soft extremally disconnected spaces. Recently, Al-Ghour and Ameen [5] have discussed the idea of maximal soft connected topologies.

Kharal and Ahmad [32] formulated the concept of soft mappings by using two crisp mappings, one between the universal set and the other between the sets of parameters. To reduce calculation burden and its difficulty that arises from the definition of soft mappings, Al-shami [9] updated this definition by using the soft version of ordinary point. In [47,48], the authors studied continuity between soft topological spaces and gave its characterizations. Alcantud [1] examined the properties of soft countability axioms. The concept of soft expandable spaces was introduced in [40]. Generating soft topologies by operators was the main goal of some articles [24, 26]. The relationships between soft topological concepts and their classical counterparts were revealed by Al-shami and Kočinac [17]. They also researched sufficient conditions that guarantee keep properties between soft topologies and classical topologies. Alcantud [2] also discussed the relationships between soft topology and fuzzy topology. Topological-hybrid structures induced from fuzzy (intuitionistic) soft sets were discussed in [31,45].

Generalization of soft open sets is an interesting topic via soft topologies; they offered new frameworks to study the topological concepts such as separation axioms, compactness, etc. In [20], it was investigated some soft functions via soft γ-open sets. Al-Ghour [3] defined the concept of soft Q-sets which contains both soft nowhere dense and soft clopen sets by commutative between soft interior and closure operators. He also applied it to boolean algebra which is of importance to the information theory and theory of probability. He [4] also introduced the concepts of soft ω-regular open sets and investigated their main features. In this regard, Ameen et al. [22] defined some soft functions via the class of soft somewhat open sets. Then, Al-shami [13] applied this class to introduce four types of soft separation axioms with interesting properties. He also analyzed the nutrition systems of individuals and examined their suitability depending on the class of soft somewhat open sets defined on minimal soft structures.

This manuscript is a continuation of the previous works that are based on soft somewhat open sets. After this introduction, it is organized this article as follows. In Section 2, we review the main definitions and results that make this work self-contained. In Section 3, we define the concepts of soft *sw*-compact and soft *sw*-Lindelöf spaces and then display three types of their generalizations. With the aid of interesting example, we elucidate the relationships between them and probed their essential features. We dedicate Section 4 to presenting the concept of soft *sw*-connected spaces and elaborating its main characterizations. We show the equivalence between this concept and soft hyperconnected spaces and discuss the conditions under which this concept is preserved via soft topology and their parametric topologies. Finally, we offer some conclusions and open a door for some future works in Section 5.

2. Preliminaries

To make this manuscript self-contained, we will mention the concepts and findings introduced in published studies that we need to comprehend the results obtained herein.

2.1. Soft set theory

Definition 2.1. *[37] Let* $\mathbf{A}$ *be any nonempty set of parameters and* 2^E *be a power set of the universal set* E*. If* $H : \mathbf{A} \to 2^E$ *is a mapping, then the pair* $(H, \mathbf{A})$ *is called a soft set over* E *and it can be written as follows:* $(H, \mathbf{A}) = \{(a, H(a)) : a \in \mathbf{A}$ *and* $H(a) \in 2^E\}$*. Each* $H(a)$ *is called a component of* $(H, \mathbf{A})$*.*

Through this manuscript, $(H, \mathbf{A})$ will denote a soft set over E.

Definition 2.2. *[37,38] A soft set* $(H, \mathbf{A})$ *is said to be:*

(i) *absolute, denoted by* $\widetilde{E}$*, if* $H(a) = E$ *for each* $a \in \mathbf{A}$*;*

(ii) *null, denoted by* $\widetilde{\phi}$*, if* $H(a) = \emptyset$ *for each* $a \in \mathbf{A}$*;*

(iii) *a soft point if there are* $a \in \mathbf{A}$ *and* $e \in E$ *with* $H(a) = \{e\}$ *and* $H(b) = \emptyset$ *for each* $b \in \mathbf{A} - \{a\}$*. A soft point is briefly denoted by* P_a^e*;*

(iv) *pseudo constant if* $H(a) = E$ *or* $\emptyset$ *for each* $a \in \mathbf{A}$*;*

(v) *finite (resp., countable) if all components are finite (resp., countable); otherwise it is called infinite (resp., uncountable).*

Definition 2.3. *[29] We call* $(H, \mathbf{A})$ *a soft subset of* $(F, \mathbf{A})$ *(or* $(F, \mathbf{A})$ *a soft superset of* $(F, \mathbf{A})$*), denoted by* $(H, \mathbf{A})\widetilde{\subseteq}(F, \mathbf{A})$ *if* $H(a) \subseteq F(a)$ *for each* $a \in \mathbf{A}$.

Definition 2.4. *[6] If* $F(a) = E - H(a)$ *for each* $a \in \mathbf{A}$*, the we call* $(F, \mathbf{A})$ *a complement of* $(H, \mathbf{A})$*. The complement of* $(H, \mathbf{A})$ *is symbolized by* $(H, \mathbf{A})^c = (H^c, \mathbf{A})$.

Definition 2.5. *Let* $(H, \mathbf{A})$ *and* $(K, \mathbf{A})$ *be soft sets. Then:*

(i) $(H, \mathbf{A})\widetilde{\bigcup}(K, \mathbf{A}) = (G, \mathbf{A})$*, where* $G(a) = H(a) \bigcup K(a)$ *for each* $a \in \mathbf{A}$ *[6].*

(ii) $(H, \mathbf{A})\widetilde{\bigcap}(K, \mathbf{A}) = (G, \mathbf{A})$*, where* $G(a) = H(a) \bigcap K(a)$*, for each* $a \in \mathbf{A}$ *[35].*

Definition 2.6. *[28] Let* $(H, \mathbf{A})$ *be a soft set and* $e \in E$*. Then:*

(i) $e \in (H, \mathbf{A})$ *if* $e \in H(a)$ *for each* $a \in \mathbf{A}$.

(ii) $e \Subset (H, \mathbf{A})$ *if* $e \in H(a)$ *for some* $a \in \mathbf{A}$.

Definition 2.7. *Let* $\Omega : E \to X$ *and* $\pi : \mathbf{A} \to \mathbf{B}$ *be two crisp mappings. A soft mapping* Ω_π *from the domain (family of all soft points over E with* $\mathbf{A}$*) to the codomain (family of all soft points over X with* $\mathbf{B}$*) is a relation associated each soft point in the domain with one and only one soft point in codomain such that*

$$\Omega_\pi(P^e_a) = P^{\Omega(e)}_{\pi(a)} \text{ for each } P^e_a \in P(E_{\mathbf{A}}).$$

In addition, $\Omega^{-1}_\pi(P^x_b) = \bigsqcup_{\substack{a \in \pi^{-1}(b) \\ e \in \Omega^{-1}(x)}} P^e_a$ *for each* $P^x_b \in P(X_{\mathbf{B}})$.

2.2. Soft topology

Definition 2.8. *[43] A family* $\mathcal{T}$ *of soft sets defined over a universal set E with a parameters set* $\mathbf{A}$ *which contains absolute soft set* $\widetilde{E}$ *and null soft set* $\widetilde{\phi}$ *is said to be a soft topology on E provided that it is closed under arbitrary soft union and finite soft intersection.*

We call the triplet $(E, \mathcal{T}, \mathbf{A})$ *a soft topological space (briefly, soft*$_{TS}$*). A member of* $\mathcal{T}$ *is called soft open and its complement is called soft closed.*

Proposition 2.9. *[43] Let* $(E, \mathcal{T}, \mathbf{A})$ *be a soft*$_{TS}$*. Then* $\mathcal{T}_a = \{H(a) : (H, \mathbf{A}) \in \mathcal{T}\}$ *defines a classical topology on E for each* $a \in \mathbf{A}$*. This topology is called a parametric topology.*

Definition 2.10. *[18] Let* $(H, \mathbf{A})$ *be a soft subset of a soft*$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$*. Then* $(cl(H), \mathbf{A})$ *is defined by* $cl(H)(a) = cl(H(a))$*, where* $cl(H(a))$ *is the closure of* $H(a)$ *in* $(E, \mathcal{T}_a)$.

Definition 2.11. *[25, 38] Let* $(E, \mathcal{T}, \mathbf{A})$ *be a soft*$_{TS}$*. Then* $\mathcal{T}$ *is said to be:*

(i) *an enriched soft topology provided that* $\mathcal{T}$ *contains all pseudo constant soft sets;*

(ii) *an extended soft topology provided that* $(H, \mathbf{A}) \in \mathcal{T}$ *iff* $H(a) \in \mathcal{T}_a$ *for each* $a \in \mathbf{A}$.

It was proved in [18] the identity between enriched soft topology and extended soft topology. To unite terminology, we call this type of soft topology an extended soft topology, and call $(E, \mathcal{T}, \mathbf{A})$ an extended soft$_{TS}$. The next theorem is a key point to discovering the behaviours of soft topological concepts and keeping them via classical and soft topologies.

Theorem 2.12. *[18] A soft subset of a soft$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ is extended iff $(int(H), \mathbf{A}) = int(H, \mathbf{A})$ and $(cl(H), \mathbf{A}) = cl(H, \mathbf{A})$ for any subset $(H, \mathbf{A})$.*

Definition 2.13. *A soft$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ is said to be:*

(i) *soft compact (resp., soft Lindelöf) [25] if every soft open cover of $\widetilde{E}$ has a finite (resp., countable) subcover. If we replace "soft open cover" by "soft clopen cover", then we obtain the concepts of mildly soft compact and mildly soft Lindelöf;*

(ii) *almost soft compact (resp., almost soft Lindelöf) [14] if every soft open cover has a finite (resp., countable) subfamily such that the closure of whose members cover $\widetilde{E}$;*

(iii) *weakly soft compact (resp., weakly soft Lindelöf) [14] if every soft open cover has a finite (resp., countable) subfamily such that its closure covers $\widetilde{E}$.*

Definition 2.14. *[14] A family $C = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ is said to have the finite (resp., countable) intersection property if $\widetilde{\bigcap}_{\rho\in\delta}(H_\rho, \mathbf{A}) \neq \widetilde{\phi}$ for any finite (resp., countable) subset δ of I;*

Definition 2.15. *[42] A soft$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ which only contains the absolute and null soft sets as soft open and soft closed subsets is called soft connected.*

Definition 2.16. *[22] A soft subset $(H, \mathbf{A})$ of $(E, \mathcal{T}, \mathbf{A})$ is said to be soft somewhat open (briefly, soft sw-open) if $(H, \mathbf{A}) = \phi$ or $int(H, \mathbf{A}) \neq \phi$. The complement of a soft sw-open set is said to be soft somewhat closed (briefly, soft sw-closed).*

Definition 2.17. *[22] Let $(H, \mathbf{A})$ be a soft subset of soft$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$. Then*

(i) *The union of all soft somewhat open sets contained in $(H, \mathbf{A})$ is called the soft sw-interior and it is denoted by $swint(H, \mathbf{A})$.*

(ii) *The intersection of all soft sw-closed sets containing $(H, \mathbf{A})$ is called the soft sw-closure of $(H, \mathbf{A})$ and it is denoted by $swcl(H, \mathbf{A})$.*

Definition 2.18. *A soft$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ is said to be tt-soft swT_2 if for every $e \neq x \in E$, there are disjoint soft sw-open sets $(H, \mathbf{A})$ and $(F, \mathbf{A})$ such that $e \in (H, \mathbf{A})$ and $x \in (F, \mathbf{A})$.*

Theorem 2.19. *[13] A subset $(H, \mathbf{A})$ of an extended soft$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ is soft sw-open (resp., soft sw-closed) iff there is an sw-open (resp., sw-closed) subset G of $(E, \mathcal{T}_a)$ with $H(a) = G$.*

The mentioned-above result describes a unique character of this generalization of soft open sets via parametric topologies. For the other celebrated generalizations of soft open sets, the sufficient part of the above theorem must hold true for all attributes of **A**. The next example clarifies this fact.

Example 2.20. *Let $\mathcal{T} = \{\phi, \widetilde{E}, (G_\rho, \mathbf{A}) : \rho = 1, 2, ..., 7\}$ be a soft topology over the universal set $E = \{x, y\}$ with a set of parameters $\mathbf{A} = \{a, b\}$, where*
$(G_1, \mathbf{A}) = \{(a, E), (b, \emptyset)\}$;

$(G_2, \mathbf{A}) = \{(a, \emptyset), (b, E)\}$;
$(G_3, \mathbf{A}) = \{(a, \{x\}), (b, \emptyset)\}$;
$(G_4, \mathbf{A}) = \{(a, \emptyset), (b, \{x\})\}$;
$(G_5, \mathbf{A}) = \{(a, \{x\}), (b, \{x\})\}$;
$(G_6, \mathbf{A}) = \{(a, \{x\}), (b, E)\}$, and
$(G_7, \mathbf{A}) = \{(a, E), (b, \{x\})\}$.

It is clear that $(E, \mathcal{T}, \mathbf{A})$ is an extended soft$_{TS}$. For a soft set $(H, \mathbf{A}) = \{(a, \{x\}), (b, \{y\})\}$, we find that $cl(int(H, \mathbf{A})) = int(cl(H, \mathbf{A})) = \{(a, E), (b, \emptyset)\}$ which means that $(H, \mathbf{A})$ is neither soft semi-open nor soft pre-open. However, $\{x\}$ is both semi-open and pre-open subset of $(E, \mathcal{T}_a)$.

Definition 2.21. *[22] A soft mapping $\Omega_\pi : (E, \mathcal{T}_E, \mathbf{A}) \to (X, \mathcal{T}_X, \mathbf{A})$ is said to be soft sw-continuous (resp., soft sw-irresolute) if $\Omega_\pi^{-1}(H, \mathbf{A})$ is a soft sw-open set where $(H, \mathbf{A})$ is a soft open (resp., soft sw-open) set.*

Theorem 2.22. *[22] Let $\Omega_\pi : (E, \mathcal{T}_E, \mathbf{A}) \to (Z, \mathcal{T}_Z, \mathbf{A})$ be a soft mapping. Then the next properties are identical.*

(i) *Ω_π is soft sw-continuous.*

(ii) *$\Omega_\pi^{-1}(F, \mathbf{A})$ is soft sw-closed for every soft closed subset $(F, \mathbf{A})$ of $(Z, \mathcal{T}_Z, \mathbf{A})$.*

(iii) *$\Omega_\pi(swcl(H, \mathbf{A})) \widetilde{\subseteq} cl(\Omega_\pi(H, \mathbf{A}))$ for each $(H, \mathbf{A}) \widetilde{\subseteq} \widetilde{E}$.*

3. Novel types of compactness and Lindelöfness via soft somewhat open sets

In this section, we apply the family of *sw*-open sets to display four novel kinds of soft compact and Lindelöf spaces namely, soft *sw*-compact, soft *sw*-Lindelöf, almost soft *sw*-compact, almost soft *sw*-Lindelöf, weakly soft *sw*-compact, weakly soft *sw*-Lindelöf, mildly soft *sw*-compact and mildly soft *sw*-Lindelöf spaces. We explore the main properties of these spaces and elucidate the relationships between them with the assistance of counterexamples. Also, we supply a complete description for each one of these spaces and investigate them under specific kinds of soft mappings. Finally, we scrutinize how these types of compactness and Lindelöfness navigate between soft topology and their parametric topologies and then discuss the role of full and extended soft topologies in keeping this property.

3.1. Soft sw-compact and soft sw-Lindelöf spaces

Definition 3.1. *A family of soft sw-open subsets of $(E, \mathcal{T}, \mathbf{A})$ is called a soft somewhat open cover (briefly, sw-open cover) of $\widetilde{E}$ if $\widetilde{E}$ is a soft subset of this family.*

Definition 3.2. *A soft$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ is said to be soft sw-compact (resp., soft sw-Lindelöf) if for every soft sw-open cover $C = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ of $(E, \mathcal{T}, \mathbf{A})$, there is a finite (resp., countable) subset δ of I with $\widetilde{E} = \widetilde{\bigcup}_{\rho \in \delta}(H_\rho, \mathbf{A})$.*

In the next examples, we present two soft topological spaces, one of them is soft *sw*-compact and the other is not soft *sw*-Lindelöf.

Example 3.3. *Let $\mathcal{T} = \{\widetilde{\phi}, (E, \mathbf{A}) \widetilde{\subseteq} \widetilde{\mathbb{R}} : (E, \mathbf{A})$ is finite$\}$ be a soft topology on the set of real numbers $\mathbb{R}$, where $\mathbf{A}$ is a finite set of parameters. It is clear that every soft subset of $(\mathbb{R}, \mathcal{T}, \mathbf{A})$ is soft sw-open iff it is soft open. Then, $(\mathbb{R}, \mathcal{T}, \mathbf{A})$ is soft sw-compact.*

Example 3.4. *Let* $\mathbf{A} = \{a, b\}$ *be a set of parameters and* $(H, \mathbf{A}) = \{(a, \{1\}), (b, \mathbb{R} \setminus \{1\})\}$ *be a soft set over the set of real numbers* $\mathbb{R}$*. Then* $\mathcal{T} = \{\widetilde{\phi}, \widetilde{\mathbb{R}}, (H, \mathbf{A})\}$ *is a soft topology on* $\mathbb{R}$*. Now,* $\{\{(a, \{1\}), (b, \mathbb{R})\}, \{(a, \{1, r\}), (b, \mathbb{R} \setminus \{1\})\} : r \in \mathbb{R}\}$ *is a soft sw-open cover of* $\mathbb{R}$*. It is clear that this cover has not a countable subcover; hence,* $(\mathbb{R}, \mathcal{T}, \mathbf{A})$ *is not a soft sw-Lindelöf space.*

Proposition 3.5. **(i)** *Every soft sw-compact space is soft sw-Lindelöf.*

(ii) *Every soft sw-compact (resp., soft sw-Lindelöf) space is soft compact (resp., soft Lindelöf).*

(iii) *The family of soft sw-compact (resp., soft sw-Lindelöf) subsets is closed under finite (resp., countable) union.*

Proof. Straightforward. □

By replacing a word "finte" in Example 3.3 by "countable", it follows that the converse of (i) in Proposition 3.5 fails. Also, Example 3.4 elucidates that the converse of (ii) in Proposition 3.5 is not true in general.

Proposition 3.6. *Let* $(E, \mathcal{T}_1, \mathbf{A})$ *and* $(E, \mathcal{T}_2, \mathbf{A})$ *be soft*$_{TS\,s}$ *such that* $\mathcal{T}_1 \subseteq \mathcal{T}_2$*. If* $(E, \mathcal{T}_2, \mathbf{A})$ *is soft sw-compact (resp., soft sw-Lindelöf), then* $(E, \mathcal{T}_1, \mathbf{A})$ *is soft sw-compact (resp., soft sw-Lindelöf).*

Proof. Consider C is a soft *sw*-open cover of $(E, \mathcal{T}_1, \mathbf{A})$. It is clear that every soft *sw*-open subset of $\mathcal{T}_1$ is a soft *sw*-open subset of $\mathcal{T}_2$, so C is also a soft *sw*-open cover of $(E, \mathcal{T}_2, \mathbf{A})$. By soft *sw*-compactness (resp., soft *sw*-Lindelöfness) of $(E, \mathcal{T}_2, \mathbf{A})$ we obtain a finite (resp., countable) subcover for $\widetilde{E}$, as required. □

The indiscrete soft$_{TS}$ defined over the set of real numbers $\mathbb{R}$ with a set of parameters $\mathbf{A} = \{a, b\}$ is soft *sw*-compact. Obviously, this indiscrete soft topology is a subfamily of soft topology given in Example 3.4, which is not a soft *sw*-Lindelöf. Hence, the converse of Proposition 3.6 need not be true.

Proposition 3.7. *If a soft*$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ *is soft sw-compact (resp., soft sw-Lindelöf), then every soft sw-closed subset of* $(E, \mathcal{T}, \mathbf{A})$ *is soft sw-compact (resp., soft sw-Lindelöf).*

Proof. Assume that $C = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ is a soft *sw*-open cover of $(F, \mathbf{A})$ which is a soft *sw*-closed subset of $(E, \mathcal{T}, \mathbf{A})$. Now, $\{(H_\rho, \mathbf{A}) : \rho \in I\} \widetilde{\bigcup} (H^c, \mathbf{A})$ is a soft *sw*-open cover of $(E, \mathcal{T}, \mathbf{A})$ which is a soft *sw*-compact space. Therefore, $\widetilde{E} = \widetilde{\bigcup}_{\rho=1}^{n} (H_\rho, \mathbf{A}) \widetilde{\bigcup} (H^c, \mathbf{A})$. Thus, $(F, \mathbf{A}) \widetilde{\subseteq} \widetilde{\bigcup}_{\rho=1}^{n} (H_\rho, \mathbf{A})$, which ends the proof that $(F, \mathbf{A})$ is soft *sw*-compact. Similarly, it can be proved the case between parentheses. □

Corollary 3.8. *The soft intersection of soft sw-compact (resp., soft sw-Lindelöf) and soft sw-closed sets is soft sw-compact (resp., soft sw-Lindelöf).*

Theorem 3.9. *A soft*$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ *is soft sw-compact (resp., soft sw-Lindelöf) if and only if* $\widetilde{\bigcap}_{\rho \in I} (H_\rho, \mathbf{A}) \neq \widetilde{\phi}$ *for every family* $C = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ *of soft sw-closed sets has a finite (resp., countable) intersection property.*

Proof. Necessity: Let $C = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ be a family of soft *sw*-closed subsets of a soft *sw*-compact space $(E, \mathcal{T}, \mathbf{A})$. Suppose that $\widetilde{\bigcap}_{\rho \in I} (H_\rho, \mathbf{A}) = \widetilde{\phi}$. Then, $\widetilde{E} = \widetilde{\bigcup}_{\rho \in I} (H^c_\rho, \mathbf{A})$. By assumption, $\widetilde{E} = \widetilde{\bigcup}_{\rho=1}^{n} (H^c_\rho, \mathbf{A})$. This means that $\widetilde{\phi} = (\widetilde{\bigcup}_{\rho=1}^{n} (H^c_\rho, \mathbf{A}))^c = \widetilde{\bigcap}_{\rho=1}^{n} (H_\rho, \mathbf{A})$. Hence, C has has a finite intersection property, as required.

Sufficiency: Consider $\mathcal{C} = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ as a soft sw-open cover of $(E, \mathcal{T}, \mathbf{A})$. Then $\widetilde{\phi} = \widetilde{\bigcap}_{\rho\in I}(H_\rho^c, \mathbf{A})$ and so by the finite sw-intersection property of $\mathcal{C}$, we obtain $\widetilde{\phi} = \widetilde{\bigcap}_{\rho=1}^{n}(H_\rho^c, \mathbf{A})$. Thus, $\widetilde{E} = \widetilde{\bigcup}_{\rho=1}^{n}(H_\rho, \mathbf{A})$, which proves that $(E, \mathcal{T}, \mathbf{A})$ is soft sw-compact.

Similarly, it can be proved the case between parentheses. □

Lemma 3.10. **(i)** *The inverse image of a soft sw-open set under a surjective soft sw-continuous mapping is soft sw-open.*

(ii) *The image of a soft sw-open set under a soft sw-open mapping is soft sw-open.*

Proof. To prove (i), let $\Omega_\pi : (E, \mathcal{T}_E, \mathbf{A}) \to (X, \mathcal{T}_X, \mathbf{A})$ be a soft sw-continuous mapping and let $(F, \mathbf{A})$ be a soft sw-open subset of $(X, \mathcal{T}_X, \mathbf{A})$. Then, there is a non-null soft open subset $(G, \mathbf{A})$ of $(Z, \mathcal{T}_Z, \mathbf{A})$ such that $(G, \mathbf{A})\widetilde{\subseteq}(F, \mathbf{A})$. Now, $\Omega_\pi^{-1}(G, \mathbf{A})\widetilde{\subseteq}\Omega_\pi^{-1}(F, \mathbf{A})$. By hypothesis, $\Omega_\pi^{-1}(G, \mathbf{A})$ is a non-null soft open subset of $(E, \mathcal{T}_E, \mathbf{A})$. Hence, $\Omega_\pi^{-1}(F, \mathbf{A})$ is soft sw-open. Following similar technique, it can be proved (ii). □

According to the above lemma, we see that the property of being a soft sw-open set is a soft topological property. Also, the concepts of soft sw-continuity and soft sw-irresolute are identical under a surjective soft mapping.

Proposition 3.11. *The surjective soft continuous image of a soft sw-compact (resp., soft sw-Lindelöf) set is soft sw-compact (resp., soft sw-Lindelöf).*

Proof. Consider $\Omega_\pi : (E, \mathcal{T}_E, \mathbf{A}) \to (X, \mathcal{T}_X, \mathbf{A})$ is a soft continuous mapping and let $(F, \mathbf{A})$ be a soft sw-Lindelöf subset of $(E, \mathcal{T}_E, \mathbf{A})$. Let us consider $\mathcal{C} = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ is a soft sw-open cover of $\Omega_\pi(F, \mathbf{A})$. Obviously, $(F, \mathbf{A})\widetilde{\subseteq}\widetilde{\bigcup}_{\rho\in I}\Omega_\pi^{-1}(H_\rho, \mathbf{A})$. It follows from Lemma 3.10 that $\Omega_\pi^{-1}(H_\rho, \mathbf{A})$ is a soft sw-open set for each $\rho \in I$. According to soft sw-Lindelöfness of $(F, \mathbf{A})$, there is a countable set δ such that $(F, \mathbf{A})\widetilde{\subseteq}\widetilde{\bigcup}_{\rho\in\delta}\Omega_\pi^{-1}(H_\rho, \mathbf{A})$. Now, we obtain $\Omega_\pi(F, \mathbf{A})\widetilde{\subseteq}\widetilde{\bigcup}_{\rho\in\delta}\Omega_\pi(\Omega_\pi^{-1}(H_\rho, \mathbf{A}))\widetilde{\subseteq}(H_\rho, \mathbf{A})$, which means that $\Omega_\pi(F, \mathbf{A})$ is soft sw-Lindelöf. It can be prove the case of soft sw-compact in a similar way. □

One can prove the next two propositions following a similar arguments.

Proposition 3.12. *The soft sw-continuous image of a soft sw-compact (resp., soft sw-Lindelöf) set is soft compact (resp., soft Lindelöf).*

Proposition 3.13. *The soft sw-irresolute image of a soft sw-compact (resp., soft sw-Lindelöf) set is soft sw-compact (resp., soft sw-Lindelöf).*

Now, we explore the relationship between soft topology and its parametric topologies with respect to possessing the properties of sw-compactness and sw-Lindelöfness. First, we demonstrate that possessing the properties of soft sw-compactness and sw-Lindelöfness by soft topologies leads to possess these properties by their parametric topologies. In fact, this is one of the unique properties of these types of covering properties which is invalid for the other types of compactness and Lindelöfness produced by soft α-open, soft semi-open, soft pre-open and soft b-open sets.

Theorem 3.14. *Every parametric topological space $(E, \mathcal{T}_a)$ inspired by a soft sw-compact (resp., soft sw-Lindelöf) space $(E, \mathcal{T}, \mathbf{A})$ is sw-compact (resp., sw-Lindelöf) for each $a \in \mathbf{A}$.*

Proof. Suppose that $\{H_\rho : \rho \in I\}$ is an *sw*-open cover of a parametric topological space $(E, \mathcal{T}_a)$. Then, for each $\rho \in I$ there exists a nonempty open subset G_ρ of $\mathcal{T}_a$ such that $G_\rho \subseteq H_\rho$. Accordingly, for each $\rho \in I$ there exists a soft open subset $(V_\rho, \mathbf{A})$ of $\mathcal{T}$ such that $V_\rho = G_\rho$. Now, defining a family of soft open sets $(W_\rho, \mathbf{A})$ as follows $W_\rho(a) = H_\rho$ and $W_\rho(a') = E$ for $a' \neq a$. Then, $\{(W_\rho, \mathbf{A}) : \rho \in I\}$ forms a soft *sw*-open cover of $(E, \mathcal{T}, \mathbf{A})$. By hypothesis of soft *sw*-compactness of $(E, \mathcal{T}, \mathbf{A})$, we obtain $\widetilde{E} = \bigcup_{\rho=1}^{n} (W_\rho, \mathbf{A})$. This implies that $E = \bigcup_{\rho=1}^{n} W_\rho(a) = \bigcup_{\rho=1}^{n} H_\rho$. Hence, $(E, \mathcal{T}_a)$ is *sw*-compact, as required. Following similar argument, one prove the case between parentheses. □

We point out by the next example that the converse of Theorem 3.14 is false.

Example 3.15. *Let $\mathcal{T} = \{\phi, \widetilde{\mathbb{R}}, (G_1, \mathbf{A}), (G_2, \mathbf{A})\}$ be a soft topology over the set of real numbers $\mathbb{R}$ with a set of parameters $\mathbf{A} = \{a, b\}$, where $(G_1, \mathbf{A}) = \{(a, \mathbb{R}), (b, \emptyset)\}$ and $(G_1, \mathbf{A}) = \{(a, \emptyset), (b, \mathbb{R})\}$. Now, both parametric topologies inspired by $(\mathbb{R}, \mathcal{T}, \mathbf{A})$ are the indiscrete topology, so they are sw-compact. On the other hand, a soft$_{TS}$ $(\mathbb{R}, \mathcal{T}, \mathbf{A})$ is not soft sw-Lindelöf.*

It should be noted that Theorem 3.14 does not hold true for compactness and Lindelöfness defined by the other kinds of generalizations of soft open sets as the next example elucidates.

Example 3.16. *Let $\mathbf{A} = \{a, b\}$ be a set of parameters and $\mathcal{T}$ be a soft topology over the set of real numbers $\mathbb{R}$ consists of the absolute soft sets and all soft sets $(H, \mathbf{A})$ satisfying that $1 \notin H(a)$. It can be noted that a soft subset of $(\mathbb{R}, \mathcal{T}, \mathbf{A})$ is soft open iff it is soft pre-open. This automatically leads to that $(\mathbb{R}, \mathcal{T}, \mathbf{A})$ is soft pre-compact. On the other hand, a parametric topology $\mathcal{T}_b$ is the discrete topology, so $(\mathbb{R}, \mathcal{T}_b)$ is not pre-Lindelöf.*

Second, we investigate under which conditions the converse of Theorem 3.14 holds true.

Definition 3.17. *A soft$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ is said to be full if all components of every non-null soft open set are nonempty.*

Proposition 3.18. *Let $(H, \mathbf{A})$ be a soft sw-open subset of a full soft$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$. Then, $H(a)$ is an sw-open subset of $(E, \mathcal{T}_a)$ for each $a \in \mathbf{A}$.*

Proof. Since $(H, \mathbf{A})$ is a soft *sw*-open set, there exists a soft open set $(G, \mathbf{A})$ such that

$$(G, \mathbf{A}) \widetilde{\subseteq} (H, \mathbf{A}). \tag{3.1}$$

By hypothesis, $(E, \mathcal{T}, \mathbf{A})$ is a full soft$_{TS}$, so $G(a)$ is a nonempty open subset of $(E, \mathcal{T}_a)$ for each $a \in \mathbf{A}$. It follows from (3.1) that $G(a) \subseteq H(a)$ for each $a \in \mathbf{A}$. This means that $H(a)$ is an *sw*-open subset of $(E, \mathcal{T}_a)$ for each $a \in \mathbf{A}$. Hence, the proof is complete. □

To elucidate that the converse of the above proposition fails, we build the next example.

Example 3.19. *Let $\mathcal{T} = \{\phi, \widetilde{E}, (G_\rho, \mathbf{A}) : \rho = 1, 2, ..., 6\}$ be a soft topology over the universal set $E = \{x, y, z\}$ with a set of parameters $\mathbf{A} = \{a, b\}$, where*
$(G_1, \mathbf{A}) = \{(a, \{x\}), (b, \{x\})\}$;
$(G_2, \mathbf{A}) = \{(a, \{y\}), (b, \{y\})\}$;
$(G_3, \mathbf{A}) = \{(a, \{z\}), (b, \{z\})\}$;
$(G_4, \mathbf{A}) = \{(a, \{x, y\}), (b, \{x, y\})\}$;

$(G_5, \mathbf{A}) = \{(a, \{x, z\}), (b, \{x, z\}\}$;
$(G_6, \mathbf{A}) = \{(a, \{y, z\}), (b, \{y, z\})\}$.

It is clear that $(E, \mathcal{T}, \mathbf{A})$ *is a full soft*$_{TS}$. *For a soft set* $(H, \mathbf{A}) = \{(a, \{y\}), (b, \{x\})\}$, $int(H(a)) \neq \emptyset$ *and* $int(H(b)) \neq \emptyset$. *But* $(H, \mathbf{A})$ *is not soft sw-open.*

Remark 3.20. *Note that a soft*$_{TS}$ *given in Example 3.15 is extended but not full, whereas a soft*$_{TS}$ *given in Example 3.19 is full but not extended. This points out that the concepts of extended soft topology and full soft topology are independent of each other. Moreover, if* $\mathcal{T}$ *is a non-indiscrete soft topology, then a full soft*$_{TS}$ *is not extended, and an extended soft*$_{TS}$ *is not full. This means that the equivalence between them is obtained in the case of* $\mathcal{T}$ *is indiscrete soft topology.*

Theorem 3.21. *Let* $\mathbf{A}$ *be a finite (resp., countable) parameter set and* $(E, \mathcal{T}, \mathbf{A})$ *be a full soft*$_{TS}$. *Then, all parametric topological spaces* $(E, \mathcal{T}_a)$ *inspired by* $(E, \mathcal{T}, \mathbf{A})$ *is sw-compact (resp., sw-Lindelöf) if and only if* $(E, \mathcal{T}, \mathbf{A})$ *is soft sw-compact (resp., soft sw-Lindelöf).*

Proof. We prove the theorem for the compactness property and the Lindelöf property is proved in a similar way.

Necessity: Assume that $C = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ is a soft sw-open cover of $(E, \mathcal{T}, \mathbf{A})$ and let $|\mathbf{A}| = m$. Then $E = \bigcup_{\rho \in I} H_\rho(a)$ for each $a \in \mathbf{A}$. According to Proposition 3.18, $H_\rho(a)$ is an sw-open set for each $a \in \mathbf{A}$. By hypothesis, $(E, \mathcal{T}_a)$ is sw-compact for each $a \in \mathbf{A}$, so we obtain $E = \bigcup_{\rho=1}^{n_1} H_\rho(a_1)$, $E = \bigcup_{\rho=n_1+1}^{n_2} H_\rho(a_2), \ldots, E = \bigcup_{\rho=n_{m-1}+1}^{n_m} H_\rho(a_m)$. This implies that $\widetilde{E} = \widetilde{\bigcup}_{\rho=1}^{n_m}(H_\rho, \mathbf{A})$. Hence, $(E, \mathcal{T}, \mathbf{A})$ is soft sw-compact.

Sufficiency: Follows from Theorem 3.14. □

To see that the condition of "full soft topology" in the above theorem is not superfluous, see, Example 3.15.

Now, we review some properties and results that are invalid for soft sw-compact and soft sw-Lindelöf spaces. The property reports that every finite (resp., countable) topological space is sw-compact (resp., sw-Lindelöf) is not valid via soft topologies. Also, the property says that every compact subset of T_2-space is closed does not hold true via soft sw-compact spaces. It can be illustrated these facts by the next example.

Example 3.22. *Let* $(E, \mathcal{T}, \mathbb{R})$ *be a soft*$_{TS}$, *where* $E = \{x, y\}$ *is the universal set, the set of real numbers* $\mathbb{R}$ *is a set of parameters and* $\mathcal{T}$ *is the discrete soft topology. Obviously, every parametric topological space* $(E, \mathcal{T}_r)$ *inspired by* $(E, \mathcal{T}, \mathbb{R})$ *is sw-compact. On the other hand, a soft*$_{TS}$ $(E, \mathcal{T}, \mathbb{R})$ *is not soft sw-Lindelöf.*

Example 3.23. *A soft*$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ *given in Example 3.19 is a tt-soft sw*T_2*-space. Now,* $\{(a, \{y\}), (b, \{z\})\}$ *is a soft sw-compact subset of* $(E, \mathcal{T}, \mathbf{A})$ *but it is not soft sw-closed.*

The following properties of soft compactness and Lindelöfness and some of their generalizations in soft topologies are invalid for soft sw-compact and sw-Lindelöf spaces as it can be seen from Example 3.15 in cases of compactness and Lindelöfness, α-compactness and α-Lindelöfness, and semi-compactness and semi-Lindelöfness.

(i) Let $\mathbf{A}$ be a finite (resp., countable) set of parameters. If $(E, \mathcal{T}_a)$ is compact (resp., Lindelöf) for each $a \in \mathbf{A}$, then $(E, \mathcal{T}, \mathbf{A})$ is soft compact (resp., soft Lindelöf).

(ii) Let $(E, \mathcal{T}, \mathbf{A})$ be an extended soft$_{TS}$. Then,

- if every parametric topological space $(E, \mathcal{T}_a)$ inspired by $(E, \mathcal{T}, \mathbf{A})$ is compact (resp., α-compact, semi-compact, pre-compact, b-compact), then $(E, \mathcal{T}, \mathbf{A})$ is soft compact (resp., soft α-compact, soft semi-compact, soft pre-compact, soft b-compact).
- if every parametric topological space $(E, \mathcal{T}_a)$ inspired by $(E, \mathcal{T}, \mathbf{A})$ is Lindelöf (resp., α-Lindelöf, semi-Lindelöf, pre-Lindelöf, b-Lindelöf), then $(E, \mathcal{T}, \mathbf{A})$ is soft Lindelöf (resp., soft α-Lindelöf, soft semi-Lindelöf, soft pre-Lindelöf, soft b-Lindelöf).

It can be easily proved the next result that associates the concepts of soft sw-compact and soft sw-Lindelöf spaces with the cardinality number of the universal set and a set of parameters.

Lemma 3.24. *An extended soft$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ is soft sw-compact (resp., soft sw-Lindelöf) if and only if the universal set E and set of parameters $\mathbf{A}$ are finite (resp., countable).*

We close this subsection by the next remark which gives a unique characteristic of soft sw-compact and soft sw-Lindelöf spaces.

Remark 3.25. *Let $(E, \mathcal{T}, \mathbf{A})$ be a soft$_{TS}$ such that E is infinite (resp., uncountable) and $\mathcal{T}$ is a non-indiscrete soft topology. If $\mathcal{T}$ contains a finite (resp., countable) proper non-null soft set, then $(E, \mathcal{T}, \mathbf{A})$ is not soft sw-compact (resp., not soft sw-Lindelöf).*

3.2. Almost soft sw-compact and almost soft sw-Lindelöf spaces

Definition 3.26. *A soft$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ is said to be almost soft sw-compact (resp., almost soft sw-Lindelöf) if every soft sw-open cover has a finite (resp., countable) subfamily such that the soft sw-closure of whose members covers $\widetilde{E}$. That is, for every soft sw-open cover $C = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ there is a finite (resp., countable) subset δ of I with $\widetilde{E} = \widetilde{\bigcup}_{\rho \in \delta} swcl(H_\rho, \mathbf{A})$.*

In what follows, we provide two soft topological spaces, one of them is almost soft sw-compact and the other is not almost soft sw-Lindelöf.

Example 3.27. *In a soft$_{TS}$ $(\mathbb{R}, \mathcal{T}, \mathbf{A})$ given in Example 3.4, note that every soft sw-open cover of $\widetilde{\mathbb{R}}$ contains a soft sw-open set $\{(a, \{1\}), (b, \mathbb{R} \setminus \{1\})\}$. Since $swcl(\{(a, \{1\}), (b, \mathbb{R} \setminus \{1\})\}) = \widetilde{\mathbb{R}}$, $(\mathbb{R}, \mathcal{T}, \mathbf{A})$ is almost soft sw-compact.*

Example 3.28. *Let $(\mathbb{R}, \mathcal{T}, \mathbf{A})$ be a soft$_{TS}$, where $\mathbb{R}$ is the set of real numbers, $\mathbf{A} = \{a, b\}$ and $\mathcal{T} = \{\phi, \mathbb{R}, \{(a, \{1\}), (b, \phi)\}, \{(a, \phi), (b, \{1\})\}, \{(a, \{1\}), (b, \{1\})\}\}$. Note that the family $\{\{(a, \{1, r\}), (b, \phi)\}, \{(a, \phi), (b, \{1, r\})\} : r \in \mathbb{R}\}$ forms a soft sw-open cover of $\widetilde{\mathbb{R}}$. Note that every member of this cover is soft sw-open and soft sw-closed set, so $(\mathbb{R}, \mathcal{T}, \mathbf{A})$ is not almost soft sw-Lindelöf.*

The proof of the following result is easy, so we remove it.

Proposition 3.29. **(i)** *Every almost soft sw-compact space is almost soft sw-Lindelöf.*

(ii) *Every soft sw-compact (resp., soft sw-Lindelöf) space is almost soft sw-compact (resp., almost soft sw-Lindelöf).*

(iii) *The family of almost soft sw-compact (resp., almost soft sw-Lindelöf) subsets is closed under finite (resp., countable) union.*

If we replace the universal set $\mathbb{R}$ in Example 3.28 by the set of natural numbers $\mathbb{N}$, then we obtain a soft$_{TS}$ which is almost soft *sw*-Lindelöf but not almost soft *sw*-compact. Also, it is illustrated in Example 3.27 that a soft$_{TS}$ given in Example 3.4 is almost soft *sw*-compact but this soft$_{TS}$ is not soft *sw*-Lindelöf. This elucidates that the converse of Proposition 3.29 fails.

Proposition 3.30. *If a soft$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ is almost soft sw-compact (resp., almost soft sw-Lindelöf), then every soft sw-clopen subset of $(E, \mathcal{T}, \mathbf{A})$ is almost soft sw-compact (resp., almost soft sw-Lindelöf).*

Proof. Let $C = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ be a soft *sw*-open cover of a soft *sw*-clopen subset $(F, \mathbf{A})$ of $(E, \mathcal{T}, \mathbf{A})$. Then, $\{(H_\rho, \mathbf{A}) : \rho \in I\}\widetilde{\bigcup}(H^c, \mathbf{A})$ is a soft *sw*-open cover of an almost soft *sw*-compact space $(E, \mathcal{T}, \mathbf{A})$. By hypothesis, $\widetilde{E} = \widetilde{\bigcup}_{\rho=1}^{n} swcl(H_\rho, \mathbf{A})\widetilde{\bigcup}(H^c, \mathbf{A})$, which implies that $(F, \mathbf{A})\widetilde{\subseteq}\widetilde{\bigcup}_{\rho=1}^{n} swcl(H_\rho, \mathbf{A})$. This finishes the proof that $(F, \mathbf{A})$ is almost soft *sw*-compact. Similarly, it can be proved the case between parentheses. □

Corollary 3.31. *The soft intersection of almost soft sw-compact (resp., almost soft sw-Lindelöf) and soft sw-clopen sets is almost soft sw-compact (resp., almost soft sw-Lindelöf).*

Definition 3.32. *A family of soft sets $C = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ is said to have the first kind of finite (resp., countable) sw-intersection property if $\widetilde{\bigcap}_{\rho\in\delta} swint(H_\rho, \mathbf{A}) \neq \widetilde{\phi}$ for any finite (resp., countable) subset δ of I.*

Theorem 3.33. *A soft$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ is almost soft sw-compact (resp., almost soft sw-Lindelöf) if and only if $\widetilde{\bigcap}_{\rho\in I}(H_\rho, \mathbf{A}) \neq \widetilde{\phi}$ for every family $C = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ of soft sw-closed sets has the first kind of finite (resp., countable) sw-intersection property.*

Proof. We prove the theorem in case of compactness and one can prove the case of Lindelöfness in a similar way.

Necessity: Let us consider $C = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ as a family of soft *sw*-closed subsets of an almost soft *sw*-compact $(E, \mathcal{T}, \mathbf{A})$. Suppose that $\widetilde{\bigcap}_{\rho\in I}(H_\rho, \mathbf{A}) = \widetilde{\phi}$. Then, $\widetilde{E} = \widetilde{\bigcup}_{\rho\in I}(H_\rho^c, \mathbf{A})$. Therefore, $\widetilde{E} = \widetilde{\bigcup}_{\rho=1}^{n} swcl(H_\rho^c, \mathbf{A})$. Thus, $\widetilde{\phi} = (\widetilde{\bigcup}_{\rho=1}^{n} swcl(H_\rho^c, \mathbf{A}))^c = \widetilde{\bigcap}_{\rho=1}^{n} swint(H_\rho, \mathbf{A})$, as required.

Sufficiency: Suppose that $C = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ is a soft *sw*-open cover of $(E, \mathcal{T}, \mathbf{A})$. Then $\widetilde{\phi} = \widetilde{\bigcap}_{\rho\in I}(H_\rho^c, \mathbf{A})$. According to the first kind of finite *sw*-intersection property, we obtain $\widetilde{\phi} = \widetilde{\bigcap}_{\rho=1}^{n} swint(H_\rho^c, \mathbf{A})$. This automatically means that $\widetilde{E} = \widetilde{\bigcup}_{\rho=1}^{n} swcl(H_\rho, \mathbf{A})$, which proves that $(E, \mathcal{T}, \mathbf{A})$ is almost soft *sw*-compact. □

Theorem 3.34. *A soft sw-continuous image of an almost soft sw-compact (resp., almost soft sw-Lindelöf) set is almost soft compact (resp., almost soft Lindelöf).*

Proof. Consider $\Omega_\pi : (E, \mathcal{T}_E, \mathbf{A}) \to (Z, \mathcal{T}_Z, \mathbf{A})$ is a soft *sw*-continuous mapping and let $(F, \mathbf{A})$ be an almost soft *sw*-Lindelöf subset of $(E, \mathcal{T}, \mathbf{A})$. Suppose that $C = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ is a soft open cover of $\Omega_\pi(F, \mathbf{A})$. Obviously, $\Omega_\pi^{-1}(H_\rho, \mathbf{A})$ is a soft *sw*-open set for each $\rho \in I$ such that $(F, \mathbf{A})\widetilde{\subseteq}\widetilde{\bigcup}_{\rho\in I}\Omega_\pi^{-1}(H_\rho, \mathbf{A})$. By hypothesis of almost soft *sw*-Lindelöfness of $(F, \mathbf{A})$ there is a countable set δ with $(F, \mathbf{A})\widetilde{\subseteq}\widetilde{\bigcup}_{\rho\in\delta} swcl(\Omega_\pi^{-1}(H_\rho, \mathbf{A}))$. Now, $\Omega_\pi(F, \mathbf{A})\widetilde{\subseteq}\widetilde{\bigcup}_{\rho\in\delta}\Omega_\pi(swcl(\Omega_\pi^{-1}(H_\rho, \mathbf{A})))$; it follows from

(iii) of Theorem 2.22 that $\Omega_\pi(swcl(\Omega_\pi^{-1}(H_\rho,\mathbf{A})))\ \widetilde{\subseteq} cl(\Omega_\pi(\Omega_\pi^{-1}(H_\rho,\mathbf{A})))\widetilde{\subseteq} cl(H_\rho,\mathbf{A})$. Hence, $\Omega_\pi(F,\mathbf{A})\widetilde{\subseteq}\ \widetilde{\bigcup}_{\rho\in\delta} cl(H_\rho,\mathbf{A})$ which proves that $\Omega_\pi(F,\mathbf{A})$ is almost soft Lindelöf.

Following similar technique, the case of compactness is proved. □

Corollary 3.35. *The soft sw-irresolute image of an almost soft sw-compact (resp., almost soft sw-Lindelöf) set is almost soft sw-compact (resp., almost soft sw-Lindelöf).*

Definition 3.36. *For each soft subset $(H,\mathbf{A})$ of a soft$_{TS}$ $(E,\mathcal{T},\mathbf{A})$, define $(swcl(H),\mathbf{A})$ as $swcl(H)(a) = swcl(H(a))$, where $swcl(H(a))$ is the sw-closure of $H(a)$ in a parametric topological space $(E,\mathcal{T}_a)$ for each $a\in\mathbf{A}$.*

Lemma 3.37. *Let $(H,\mathbf{A})$ be an extended soft subset of a soft$_{TS}$ $(E,\mathcal{T},\mathbf{A})$. Then $swcl(H,\mathbf{A}) = (swcl(H),\mathbf{A})$.*

Proof. It is clear that $(H,\mathbf{A})\widetilde{\subseteq}(swcl(H),\mathbf{A})$. Since $\mathcal{T}$ is extended, $(swcl(H),\mathbf{A})$ is a soft *sw*-closed set. As it is well known that $swcl(H,\mathbf{A})$ is the smallest soft *sw*-closed set containing $(H,\mathbf{A})$, so $swcl(H,\mathbf{A})\widetilde{\subseteq}(swcl(H),\mathbf{A})$. □

Note that $(E,\mathcal{T},\mathbf{A})$ given in Example 2.20 is an extended soft$_{TS}$. For a soft set $(H,\mathbf{A}) = \{(a,\{x\}),(b,\{y\})\}$, we find that $swcl(H,\mathbf{A}) = (H,\mathbf{A})$ whereas $(swcl(H),\mathbf{A}) = \{(a,E),(b,\{y\})\}$. This confirms that the converse of the above lemma need not be true in general.

Theorem 3.38. *Let $(E,\mathcal{T},\mathbf{A})$ be extended. Then, every parametric topological space $(E,\mathcal{T}_a)$ inspired by almost soft sw-compact (resp., almost soft sw-Lindelöf) space $(E,\mathcal{T},\mathbf{A})$ is almost sw-compact (resp., almost sw-Lindelöf) for each $a\in\mathbf{A}$.*

Proof. Let $\{H_\rho : \rho\in I\}$ be an *sw*-open cover of a parametric topological space $(E,\mathcal{T}_a)$. Consider a family of soft sets $\{(V_\rho,\mathbf{A}) : \rho\in I\}$, where $V_\rho(a) = H_\rho$ and $V_\rho(a') = E$ for $a'\neq a$. Obviously, this family forms a soft *sw*-open cover of $(E,\mathcal{T},\mathbf{A})$. By almost soft *sw*-compactness of $(E,\mathcal{T},\mathbf{A})$, we obtain $\widetilde{E} = \bigcup_{\rho=1}^{n} swcl(V_\rho,\mathbf{A})$. Since $\mathcal{T}$ is extended, it follows from Lemma 3.37 that $swcl(V_\rho,\mathbf{A})\widetilde{\subseteq}(swcl(V_\rho),\mathbf{A})$. So, $\widetilde{E} = \bigcup_{\rho=1}^{n}(swcl(V_\rho),\mathbf{A})$. Thus, $E = \bigcup_{\rho=1}^{n} swcl(V_\rho(a)) = \bigcup_{\rho=1}^{n} swcl(H_\rho)$ which ends the proof that $(E,\mathcal{T}_a)$ is almost *sw*-compact. One can prove the case between parentheses following similar technique. □

Example 3.22 shows that the converse of Theorem 3.38 is false in general. In the next example, we clarify that the topological condition of extended in the mentioned-above theorem is indispensable.

Example 3.39. *Let $\mathbf{A} = \{a,b\}$ be a set of parameters and $\mathcal{T}$ be a soft topology over the set of real numbers $\mathbb{R}$ consists of the null soft sets and all soft sets $(H,\mathbf{A})$ satisfying that $1\in H(a)$. Now, $\{(a,\emptyset),(b,\{2\})\}\notin\mathcal{T}$ despite $\emptyset\in\mathcal{T}_a$ and $\{2\}\notin\mathcal{T}_b$, so $(\mathbb{R},\mathcal{T},\mathbf{A})$ is not extended. Note that every soft sw-open cover of $\widetilde{\mathbb{R}}$ contains a soft sw-open set containing $\{(a,\{1\}),(b,\emptyset)\}$. Since $swcl(\{(a,\{1\}),(b,\emptyset)\}) = \widetilde{\mathbb{R}}$, $(\mathbb{R},\mathcal{T},\mathbf{A})$ is almost soft sw-compact. On the other hand, a parametric topology $\mathcal{T}_b$ is the discrete topology, so $(\mathbb{R},\mathcal{T}_b)$ is not almost sw-Lindelöf.*

Lemma 3.40. *Let $(H,\mathbf{A})$ be a soft subset of a full soft$_{TS}$ $(E,\mathcal{T},\mathbf{A})$. Then $(swcl(H),\mathbf{A})\widetilde{\subseteq} swcl(H,\mathbf{A})$.*

Proof. Let $P_a^e\ \widetilde{\notin}\ swcl(H,\mathbf{A})$. Then, we can find a soft *sw*-open set $(G,\mathbf{A})$ containing P_a^e such that $(G,\mathbf{A})\widetilde{\bigcap}(H,\mathbf{A}) = \phi$. This means that $G(a)\cap H(a) = \emptyset$ for each $a\in\mathbf{A}$. Since $\mathcal{T}$ is full, $G(a)$ is an

sw-open subset of $\mathcal{T}_a$. This implies that $G(a) \cap swcl(H(a)) = \emptyset$. Therefore, $P_a^e \notin (swcl(H), \mathbf{A})$. Hence, $(swcl(H), \mathbf{A}) \widetilde{\subseteq} swcl(H, \mathbf{A})$, as required. □

To point out that the converse of the above lemma is generally false, consider a full soft$_{TS}$ displayed in Example 3.19 and take a soft set $(H, \mathbf{A}) = \{(a, \{y\}), (b, \{x\})\}$. Now, $swcl(H, \mathbf{A}) = \{(a, \{x, y\}), (b, \{x, y\})\}$ but $(swcl(H), \mathbf{A}) = \{(a, \{y\}), (b, \{x\})\}$.

Theorem 3.41. *Let* $\mathbf{A}$ *be a finite (resp., countable) parameter set and* $(E, \mathcal{T}, \mathbf{A})$ *be a full soft*$_{TS}$*. Then, all parametric topological spaces* $(E, \mathcal{T}_a)$ *inspired by* $(E, \mathcal{T}, \mathbf{A})$ *is almost sw-compact (resp., almost sw-Lindelöf) if and only if* $(E, \mathcal{T}, \mathbf{A})$ *is almost soft sw-compact (resp., almost soft sw-Lindelöf).*

Proof. We prove the theorem for the compactness property and the Lindelöf property is proved in a similar way.

Necessity: Assume that $C = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ is a soft sw-open cover of $(E, \mathcal{T}, \mathbf{A})$ and let $\mid \mathbf{A} \mid= m$. Then $E = \bigcup\limits_{\rho \in I} H_\rho(a)$ for each $a \in \mathbf{A}$. According to Proposition 3.18, $H_\rho(a)$ is an sw-open set for each $a \in \mathbf{A}$. By hypothesis, $(E, \mathcal{T}_a)$ is almost sw-compact for each $a \in \mathbf{A}$, so we obtain $E = \bigcup\limits_{\rho=1}^{n_1} swcl(H_\rho(a_1))$, $E = \bigcup\limits_{\rho=n_1+1}^{n_2} swcl(H_\rho(a_2))$,..., $E = \bigcup\limits_{\rho=n_{m-1}+1}^{n_m} swcl(H_\rho(a_m))$. This implies that $\widetilde{E} = \widetilde{\bigcup}_{\rho=1}^{n_m}(swcl(H_\rho), \mathbf{A})$. According to Lemma 3.40, we obtain $(swcl(H), \mathbf{A}) \widetilde{\subseteq} swcl(H, \mathbf{A})$, so $\widetilde{E} = \widetilde{\bigcup}_{\rho=1}^{n_m} swcl(H_\rho, \mathbf{A})$. Hence, $(E, \mathcal{T}, \mathbf{A})$ is almost soft sw-compact.

Sufficiency: Follows from Theorem 3.38. □

The next lemma associates these types of spaces with the cardinality numbers of the universal set and a set of parameters.

Lemma 3.42. *An extended soft*$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ *is almost soft sw-compact (resp., almost soft sw-Lindelöf) if and only if the universal set E and set of parameters* $\mathbf{A}$ *are finite (resp., countable).*

Proof. The necessary part follows from the fact that all soft sets which their components are the universal set or empty set are soft sw-open and soft sw-closed sets. The sufficient part is obvious. □

Finally, we draw attention of the readers to that Remark 3.25 does not hold true for almost soft sw-compact and almost soft sw-Lindelöf spaces as illustrated in the next example.

Example 3.43. *Let* $(\mathbb{R}, \mathcal{T}, \mathbf{A})$ *be a soft*$_{TS}$*, where* $\mathbb{R}$ *is the set of real numbers,* $\mathbf{A} = \{a, b\}$ *and* $\mathcal{T} = \{\phi, \mathbb{R}, \{(a, \{1\}), (b, \phi)\}\}$*. Then,* $(\mathbb{R}, \mathcal{T}, \mathbf{A})$ *satisfies the conditions of a soft*$_{TS}$ *given in Remark 3.25. On the other hand, note that* $\{(a, \{1\}), (b, \phi)$ *is a soft subset of a member of any soft sw-open cover of* $\widetilde{\mathbb{R}}$*. Since* $swcl(\{(a, \{1\}), (b, \phi)) = \widetilde{\mathbb{R}}$*,* $(\mathbb{R}, \mathcal{T}, \mathbf{A})$ *is almost soft sw-compact.*

3.3. Weakly soft sw-compact and weakly soft sw-Lindelöf spaces

Definition 3.44. *A soft*$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ *is said to be weakly soft sw-compact (resp., weakly soft sw-Lindelöf) if every soft sw-open cover has a finite (resp., countable) subfamily such that its soft sw-closure covers* $\widetilde{E}$*. That is, for every soft sw-open cover* $C = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ *there is a finite (resp., countable) subset* δ *of I with* $\widetilde{E} = swcl(\widetilde{\bigcup}_{\rho \in \delta}(H_\rho, \mathbf{A}))$*.*

In what follows, we provide two soft topological spaces, one of them is weakly soft *sw*-compact and the other is not weakly soft *sw*-compact.

Example 3.45. *In a soft$_{TS}$ $(\mathbb{R}, \mathcal{T}, \mathbf{A})$ given in Example 3.28, note that every soft sw-open cover of* $\widetilde{\mathbb{R}}$ *contains a soft sw-open set* $\{(a, \{1\}), (b, \{1\})\}$. *Since* $swcl(\{(a, \{1\}), (a, \{1\})\}) = \widetilde{\mathbb{R}}$, $(\mathbb{R}, \mathcal{T}, \mathbf{A})$ *is weakly soft sw-compact.*

Example 3.46. *Let* $\mathcal{T}$ *be the soft discrete over the set of real numbers* $\mathbb{R}$ *with nay set of parameters* $\mathbf{A}$. *Then, a soft$_{TS}$ $(\mathbb{R}, \mathcal{T}, \mathbf{A})$ is not weakly soft sw-Lindelöf.*

It can be easily prove the next result, so we remove it.

Proposition 3.47. **(i)** *Every weakly soft sw-compact is weakly soft sw-Lindelöf.*

(ii) *Every almost soft sw-compact (resp., almost soft sw-Lindelöf) space is weakly soft sw-compact (resp., weakly soft sw-Lindelöf).*

(iii) *The family of weakly soft sw-compact (resp., weakly soft sw-Lindelöf) subsets is closed under finite (resp., countable) union.*

If we replace the universal set $\mathbb{R}$ in Example 3.46 by any infinite countable set, then we obtain a soft$_{TS}$ which is weakly soft *sw*-Lindelöf but not weakly soft *sw*-compact. Also, it follows from illustration given in Example 3.45 that a soft$_{TS}$ given in Example 3.28 is weakly soft *sw*-compact but this soft$_{TS}$ is not almost soft *sw*-Lindelöf. Hence, the converse of Proposition 3.47 is incorrect in general.

Definition 3.48. *A family of soft sets* $C = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ *is said to have the second kind of finite (resp., countable) sw-intersection property if* $swint(\widetilde{\bigcap}_{\rho\in\delta}(H_\rho, \mathbf{A})) \neq \widetilde{\phi}$ *for any finite (resp., countable) subset* δ *of* I.

Theorem 3.49. *A soft$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ is weakly soft sw-compact (resp., weakly soft sw-Lindelöf) if and only if* $\widetilde{\bigcap}_{\rho\in I}(H_\rho, \mathbf{A}) \neq \widetilde{\phi}$ *for every family* $C = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ *of soft sw-closed sets has the second kind of finite (resp., countable) sw-intersection property.*

Proof. We prove the theorem in case of compactness and one can prove the case of Lindelöfness in a similar way.

Necessity: Let us consider $C = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ as a family of soft *sw*-closed subsets of a weakly soft *sw*-compact $(E, \mathcal{T}, \mathbf{A})$. Suppose that $\widetilde{\bigcap}_{\rho\in I}(H_\rho, \mathbf{A}) = \widetilde{\phi}$. Then, $\widetilde{E} = \widetilde{\bigcup}_{\rho\in I}(H^c_\rho, \mathbf{A})$. Therefore, $\widetilde{E} = swcl(\widetilde{\bigcup}^n_{\rho=1}(H^c_\rho, \mathbf{A}))$. Thus, $\widetilde{\phi} = (swcl(\widetilde{\bigcup}^n_{\rho=1}(H^c_\rho, \mathbf{A})))^c = swint(\widetilde{\bigcap}^n_{\rho=1}(H_\rho, \mathbf{A}))$, as required.

Sufficiency: Suppose that $C = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ is a soft *sw*-open cover of $(E, \mathcal{T}, \mathbf{A})$. Then $\widetilde{\phi} = \widetilde{\bigcap}_{\rho\in I}(H^c_\rho, \mathbf{A})$. According to the second kind of finite *sw*-intersection property, we obtain $\widetilde{\phi} = swint(\widetilde{\bigcap}^n_{\rho=1}(H^c_\rho, \mathbf{A}))$. This automatically means that $\widetilde{E} = swcl(\widetilde{\bigcup}^n_{\rho=1}(H_\rho, \mathbf{A}))$, which proves that $(E, \mathcal{T}, \mathbf{A})$ is weakly soft *sw*-compact. □

Theorem 3.50. *A soft sw-continuous image of a weakly soft sw-compact (resp., weakly soft sw-Lindelöf) set is weakly soft compact (resp., weakly soft Lindelöf).*

Proof. Consider $\Omega_\pi : (E, \mathcal{T}_E, \mathbf{A}) \to (Z, \mathcal{T}_Z, \mathbf{A})$ is a soft *sw*-continuous mapping and let $(F, \mathbf{A})$ be a weakly soft *sw*-Lindelöf subset of $(E, \mathcal{T}, \mathbf{A})$. Suppose that $C = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ is a soft open cover of $\Omega_\pi(F, \mathbf{A})$. Obviously, $\Omega_\pi^{-1}(H_\rho, \mathbf{A})$ is a soft *sw*-open set for each $\rho \in I$ such that $(F, \mathbf{A}) \widetilde{\subseteq} \widetilde{\bigcup}_{\rho\in I} \Omega_\pi^{-1}(H_\rho, \mathbf{A})$. By hypothesis of weakly soft *sw*-Lindelöfness of $(F, \mathbf{A})$ there is a countable set δ with $(F, \mathbf{A}) \widetilde{\subseteq} swcl(\widetilde{\bigcup}_{\rho\in\delta}(\Omega_\pi^{-1}(H_\rho, \mathbf{A})))$. Now, $\Omega_\pi(F, \mathbf{A}) \widetilde{\subseteq} \Omega_\pi(swcl(\widetilde{\bigcup}_{\rho\in\delta}(\Omega_\pi^{-1}(H_\rho, \mathbf{A})))$; it follows from (iii) of Theorem 2.22 that $\Omega_\pi(swcl(\widetilde{\bigcup}_{\rho\in\delta}\Omega_\pi^{-1}(H_\rho, \mathbf{A}))) \widetilde{\subseteq} cl(\widetilde{\bigcup}_{\rho\in\delta}\Omega_\pi(\Omega_\pi^{-1}(H_\rho, \mathbf{A}))) \widetilde{\subseteq} cl(\widetilde{\bigcup}_{\rho\in\delta}(H_\rho, \mathbf{A})$. Hence, $\Omega_\pi(F, \mathbf{A}) \widetilde{\subseteq} \widetilde{\bigcup}_{\rho\in\delta} cl(H_\rho, \mathbf{A})$ which proves that $\Omega_\pi(F, \mathbf{A})$ is weakly soft Lindelöf.

Following similar technique, the case of compactness is proved. □

Corollary 3.51. *The soft sw-irresolute image of a weakly soft sw-compact (resp., weakly soft sw-Lindelöf) set is weakly soft sw-compact (resp., weakly soft sw-Lindelöf).*

Theorem 3.52. *Let $(E, \mathcal{T}, \mathbf{A})$ be extended. Then, every parametric topological space $(E, \mathcal{T}_a)$ inspired by weakly soft sw-compact (resp., weakly soft sw-Lindelöf) space $(E, \mathcal{T}, \mathbf{A})$ is weakly sw-compact (resp., weakly sw-Lindelöf) for each $a \in \mathbf{A}$.*

Proof. Let $\{H_\rho : \rho \in I\}$ be an *sw*-open cover of a parametric topological space $(E, \mathcal{T}_a)$. Consider a family of soft sets $\{(V_\rho, \mathbf{A}) : \rho \in I\}$, where $V_\rho(a) = H_\rho$ and $V_\rho(a') = E$ for $a' \neq a$. Obviously, this family forms a soft *sw*-open cover of $(E, \mathcal{T}, \mathbf{A})$. By weakly soft *sw*-compactness of $(E, \mathcal{T}, \mathbf{A})$, we obtain $\widetilde{E} = swcl(\bigcup_{\rho=1}^{n}(V_\rho, \mathbf{A}))$. Since $\mathcal{T}$ is extended, it follows from Lemma 3.37 that $E = swcl(\bigcup_{\rho=1}^{n}(V_\rho(a))) = swcl(\bigcup_{\rho=1}^{n}(H_\rho))$. Hence, $(E, \mathcal{T}_a)$ is weakly *sw*-compact. One can prove the case between parentheses following similar technique. □

Example 3.22 sets forth that the converse of Theorem 3.52 is false, and Example 3.39 demonstrates that the topological condition of extended in the mentioned-above result is indispensable.

Theorem 3.53. *Let $\mathbf{A}$ be a finite (resp., countable) parameter set and $(E, \mathcal{T}, \mathbf{A})$ be a full soft$_{TS}$. If all parametric topological spaces $(E, \mathcal{T}_a)$ inspired by $(E, \mathcal{T}, \mathbf{A})$ is weakly sw-compact (resp., weakly sw-Lindelöf), then $(E, \mathcal{T}, \mathbf{A})$ is weakly soft sw-compact (resp., weakly soft sw-Lindelöf).*

Proof. We prove the theorem for the compactness property and the Lindelöf property is proved in a similar way. To do this, assume that $C = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ is a soft *sw*-open cover of $(E, \mathcal{T}, \mathbf{A})$ and let $|\ \mathbf{A}\ |= m$. Then $E = \bigcup_{\rho\in I} H_\rho(a)$ for each $a \in \mathbf{A}$. According to Proposition 3.18, $H_\rho(a)$ is an *sw*-open set for each $a \in \mathbf{A}$. By hypothesis, $(E, \mathcal{T}_a)$ is weakly *sw*-compact for each $a \in \mathbf{A}$, so we obtain $E = swcl(\bigcup_{\rho=1}^{n_1}(H_\rho(a_1)))$, $E = swcl(\bigcup_{\rho=n_1+1}^{n_2}(H_\rho(a_2)))$,..., $E = swcl(\bigcup_{\rho=n_{m-1}+1}^{n_m}(H_\rho(a_m))$. This implies that $\widetilde{E} = (swcl(\widetilde{\bigcup}_{\rho=1}^{n_m}(H_\rho)), \mathbf{A})$. According to Lemma 3.40, we obtain $\widetilde{E} = swcl(\widetilde{\bigcup}_{\rho=1}^{n_m}(H_\rho, \mathbf{A}))$. Hence, $(E, \mathcal{T}, \mathbf{A})$ is weakly soft *sw*-compact. □

The next result points out the cardinality numbers of the universal set and a set of parameters in the weakly soft *sw*-compact and weakly soft *sw*-Lindelöf spaces.

Lemma 3.54. *An extended soft$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ is weakly soft sw-compact (resp., weakly soft sw-Lindelöf) if and only if the universal set E and set of parameters $\mathbf{A}$ are finite (resp., countable).*

Proof. The necessary part follows from the fact that all soft sets which their components are the universal set or empty set are soft *sw*-open and soft *sw*-closed sets. The sufficient part is obvious. □

3.4. Mildly soft sw-compact and mildly soft sw-Lindelöf spaces

Definition 3.55. *We call a soft subset of a soft$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ a soft swoc-set if it is both soft sw-open and soft sw-closed.*

Definition 3.56. *We call a family of soft swoc-subsets of a soft$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ a soft swoc-cover of $\widetilde{E}$ if it covers $\widetilde{E}$.*

Definition 3.57. *A soft$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ is said to be mildly soft sw-compact (resp., mildly soft sw-Lindelöf) if every soft swoc-cover has a finite (resp., countable) subcover.*

In a soft$_{TS}$ displayed in Example 3.4, it can be noted that the only soft *swoc*-sets are absolute and null soft sets. So, this soft$_{TS}$ is weakly soft *sw*-compact. Correspondingly, in a soft$_{TS}$ displayed in Example 3.28, it is illustrated that $\{\{(a, \{1, r\}), (b, \phi)\}, \{(a, \phi), (b, \{1, r\})\} : r \in \mathbb{R}\}\}$ is a soft *swoc*-cover of the universal set which has not a countable subcover. So, this soft$_{TS}$ is not weakly soft *sw*-Lindelöf. These two examples clarify the existence and uniqueness of weakly soft *sw*-compact and weakly soft *sw*-Lindelöf spaces.

Proposition 3.58. **(i)** *Mildly soft sw-compact spaces are mildly soft sw-Lindelöf.*

(ii) *Almost soft sw-compact (resp., almost soft sw-Lindelöf) spaces are mildly soft sw-compact (resp., mildly soft sw-Lindelöf).*

(iii) *The family of mildly soft sw-compact (resp., mildly soft sw-Lindelöf) subsets is closed under finite (resp., countable) union.*

Proof. Straightforward. □

Corollary 3.59. *Soft sw-compact (resp., soft sw-Lindelöf) spaces are mildly soft sw-compact (resp., mildly soft sw-Lindelöf).*

A discrete soft$_{TS}$ defined over any infinite countable set with any finite set of parameters is mildly soft *sw*-Lindelöf but not mildly soft *sw*-compact, so the converse of (i) in Proposition 3.58 fails. To show that the converse of (ii) in Proposition 3.58 is also incorrect, we suffice by an example introduced in general topology. In Example 3.3 if we consider a set of parameters any uncountable set, then it can be checked that this soft$_{TS}$ is not soft *sw*-Lindelöf. On the other hand, note that the soft *swoc*-subsets of this soft$_{TS}$ are the absolute and null soft sets, which means that this soft$_{TS}$ is mildly soft *sw*-compact. Hence, the converse of Corollary 3.59 is generally false as well.

One can prove the next results following similar arguments in the proofs of their counterparts given in Subsection 3.1

Proposition 3.60. *If a soft$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ is mildly soft sw-compact (resp., mildly soft sw-Lindelöf), then every soft swoc-subset of $(E, \mathcal{T}, \mathbf{A})$ is mildly soft sw-compact (resp., mildly soft sw-Lindelöf).*

Corollary 3.61. *The intersection of a mildly soft sw-compact (resp., mildly soft sw-Lindelöf) set and a soft swoc-set is mildly soft sw-compact (resp., mildly soft sw-Lindelöf).*

Theorem 3.62. *A soft$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ is mildly soft sw-compact (resp., mildly soft sw-Lindelöf) if and only if $\widetilde{\bigcap}_{\rho \in I}(H_\rho, \mathbf{A}) \neq \widetilde{\phi}$ for every family $C = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ of soft swoc-sets has a finite (resp., countable) intersection property.*

Proposition 3.63. *The property of being a mildly soft sw-compact (resp., mildly soft sw-Lindelöf) set is preserved by a surjective soft continuous mapping.*

Proposition 3.64. *The soft sw-continuous image of a mildly soft sw-compact (resp., mildly soft sw-Lindelöf) set is mildly soft compact (resp., mildly soft Lindelöf).*

Proposition 3.65. *The property of being a mildly soft sw-compact (resp., mildly soft sw-Lindelöf) set is preserved by a soft sw-irresolute mapping.*

Lemma 3.66. *An extended soft$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ is mildly soft sw-compact (resp., mildly soft sw-Lindelöf) if and only if the universal set E and set of parameters $\mathbf{A}$ are finite (resp., countable).*

Theorem 3.67. *Every parametric topological space $(E, \mathcal{T}_a)$ inspired by a mildly soft sw-compact (resp., mildly soft sw-Lindelöf) space $(E, \mathcal{T}, \mathbf{A})$ is mildly sw-compact (resp., mildly sw-Lindelöf) for each $a \in \mathbf{A}$.*

Example 3.15 also demonstrates that the converse of Theorem 3.67 is not true in general.

Theorem 3.68. *Let $\mathbf{A}$ be a finite (resp., countable) parameter set and $(E, \mathcal{T}, \mathbf{A})$ be a full soft$_{TS}$. Then, all parametric topological spaces $(E, \mathcal{T}_a)$ inspired by $(E, \mathcal{T}, \mathbf{A})$ is mildly sw-compact (resp., mildly sw-Lindelöf) if and only if $(E, \mathcal{T}, \mathbf{A})$ is mildly soft sw-compact (resp., mildly soft sw-Lindelöf).*

Definition 3.69. *We call $(E, \mathcal{T}, \mathbf{A})$ a soft sw-locally indiscrete soft$_{TS}$ if every soft sw-open set is soft sw-closed.*

Proposition 3.70. *The next concepts are identical under a soft sw-locally indiscrete soft$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$.*

(i) $(E, \mathcal{T}, \mathbf{A})$ *is soft sw-compact (resp., soft sw-Lindelöf).*

(ii) $(E, \mathcal{T}, \mathbf{A})$ *is almost soft sw-compact (resp., almost soft sw-Lindelöf).*

(iii) $(E, \mathcal{T}, \mathbf{A})$ *is weakly soft sw-compact (resp., weakly soft sw-Lindelöf).*

(iv) $(E, \mathcal{T}, \mathbf{A})$ *is mildly soft sw-compact (resp., mildly soft sw-Lindelöf).*

Proof. We prove the proposition for compactness and the case of Lindelöfness is proved in a similar way.

The proofs of the directions (i) → (ii) and (ii) → (iii) follow from Propositions 3.5 and 3.29, respectively. To prove that (iii) → (iv), let $C = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ be a soft *swoc*-cover of a weakly soft *sw*-compact space $(E, \mathcal{T}, \mathbf{A})$. Then there exists a finite subset δ of I such that $\widetilde{E} = swcl(\widetilde{\bigcup}_{\rho\in\delta}(H_\rho, \mathbf{A}))$. Since $(E, \mathcal{T}, \mathbf{A})$ is soft *sw*-locally indiscrete, $swcl(\widetilde{\bigcup}_{\rho\in\delta}(H_\rho, \mathbf{A})) = \widetilde{\bigcup}_{\rho\in\delta}(H_\rho, \mathbf{A})$. Thus, $(E, \mathcal{T}, \mathbf{A})$ is mildly soft *sw*-compact. Finally, we prove that (iv) → (i). To do this, let $C = \{(H_\rho, \mathbf{A}) : \rho \in I\}$ be a soft *sw*-open cover of a soft *sw*-locally indiscrete soft$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$. Then C is also a soft *swoc*-cover. By hypothesis of mildly soft *sw*-compactness, there exists a finite subset δ of I such that $\widetilde{E} = \widetilde{\bigcup}_{\rho\in\delta}(H_\rho, \mathbf{A})$. Thus, $(E, \mathcal{T}, \mathbf{A})$ is soft *sw*-compact. Hence, the proof is complete. □

4. Soft *sw*-connected spaces

In this section, we define the concept of soft *sw*-connected spaces and discuss its characterizations induced from soft *sw*-open sets. Then, we describe this concept by disjoint soft open sets, which implies the equivalence between soft sw-connected and soft hyperconnected spaces. Also, we explore the main properties of this concept; especially, those related to specific types of soft compactness, soft separation axioms, and soft continuous mappings. Finally, with the aid of counterexamples, we demonstrate that the property of this type of connectedness is not commutative between soft topologies and their parametric topologies. We also prove that possessing this property by a full soft topology implies possessing by its parametric topologies and vice versa.

Definition 4.1. *We call the soft subsets* $(H, \mathbf{A})$ *and* $(K, \mathbf{A})$ *of a soft*$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ *sw-separated soft sets if* $(H, \mathbf{A})\widetilde{\bigcap} swcl(K, \mathbf{A}) = \widetilde{\phi}$ *and* $swcl(H, \mathbf{A})\widetilde{\bigcap}(K, \mathbf{A}) = \widetilde{\phi}$.

It is clear that a condition of sw-separated soft sets is proper stronger than a condition of disjoint.

Definition 4.2. *A soft*$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ *is called soft sw-disconnected if there are two non-null sw-separated soft sets* $(H, \mathbf{A})$ *and* $(K, \mathbf{A})$ *which their soft union is* $\widetilde{E}$. *Otherwise, we call* $(E, \mathcal{T}, \mathbf{A})$ *a soft sw-connected soft*$_{TS}$.

In this case, we call $(H, \mathbf{A})$ *and* $(K, \mathbf{A})$ *a soft sw-disconnection of* $\widetilde{E}$.

Proposition 4.3. *Every soft disconnected soft*$_{TS}$ *is soft sw-disconnected.*

Proof. Follows from the fact that $swcl(H, \mathbf{A})\widetilde{\subseteq}cl(H, \mathbf{A})$ for any soft subset $(H, \mathbf{A})$ of $\widetilde{E}$. □

A soft$_{TS}$ displayed in Example 3.28 is soft connected. But it is soft sw-disconnected because $\{(a, \widetilde{\mathbb{R}}), (b, \emptyset)\}$ and $\{(a, \emptyset), (b, \widetilde{\mathbb{R}})\}$ are sw-separated soft sets in which their soft union is the absolute soft set $\widetilde{\mathbb{R}}$. So, the converse side of Proposition 4.3 is not always true.

Proposition 4.4. *The next properties are equivalent.*

(i) $(E, \mathcal{T}, \mathbf{A})$ *is soft sw-connected.*

(ii) *It cannot exist two proper disjoint soft sw-closed sets whose soft union is* $\widetilde{E}$.

(iii) *It cannot exist two proper disjoint soft sw-open sets whose soft union is* $\widetilde{E}$.

(iv) *The absolute and null soft sets are the only soft sw-open and soft sw-closed.*

Proof. $(i) \Rightarrow (ii)$: If we consider there exist two disjoint proper soft sw-closed sets $(H, \mathbf{A})$ and $(K, \mathbf{A})$ such that $(H, \mathbf{A})\widetilde{\bigcup}(K, \mathbf{A}) = \widetilde{E}$. Then we obtain $(H, \mathbf{A})$ and $(K, \mathbf{A})$ are sw-separated soft sets because $swcl(H, \mathbf{A}) = (H, \mathbf{A})$ and $swcl(K, \mathbf{A}) = (K, \mathbf{A})$. But this contradicts (i). Hence, (ii) hold true.
$(ii) \Rightarrow (iii)$: Obvious.
$(iii) \Rightarrow (iv)$: Let us consider $(H, \mathbf{A})$ is both soft sw-open and soft sw-closed set. This leads to that $(H^c, \mathbf{A})$ is soft sw-open such that $(H, \mathbf{A})\widetilde{\bigcap}(H^c, \mathbf{A}) = \widetilde{\phi}$ and $(H, \mathbf{A})\widetilde{\bigcup}(H^c, \mathbf{A}) = \widetilde{E}$, which contradicts (iii). Hence, (iv) hold true.
$(iv) \Rightarrow (i)$: Suppose, by contrary, that there is a soft set $(H, \mathbf{A})$ which is both soft sw-open and soft sw-closed. Then $swcl(H, \mathbf{A}) = (H, \mathbf{A})$ and $swcl(H^c, \mathbf{A}) = (H^c, \mathbf{A})$. So, $(H, \mathbf{A})$ and $(H^c, \mathbf{A})$ are two sw-separated soft sets such that their union is $\widetilde{E}$, thus $(E, \mathcal{T}, \mathbf{A})$ is soft sw-disconnected. Hence, $(iv) \Rightarrow (i)$ hold true. □

Now, we describe the concept of soft sw-disconnectedness using soft open sets instead of soft sw-open sets.

Theorem 4.5. *A soft$_{TS}$ $(E,\mathcal{T},\mathbf{A})$ is soft sw-disconnected if and only if $\mathcal{T}$ contains two non-null disjoint soft open sets.*

Proof. To prove the necessary part, let $(E,\mathcal{T},\mathbf{A})$ be soft sw-disconnected. Then, it follows from (iii) of Proposition 4.4 that there exist two proper non-null disjoint soft sw-open sets; say, $(H,\mathbf{A})$ and $(K,\mathbf{A})$. Thus, there exist non-null disjoint soft open sets $(G_1,\mathbf{A}),(G_2,\mathbf{A})$ such that $(G_1,\mathbf{A})\widetilde{\subseteq}(H,\mathbf{A})$ and $(G_2,\mathbf{A})\widetilde{\subseteq}(K,\mathbf{A})$, as required. To prove the sufficient part, let $(U,\mathbf{A})$ and $(V,\mathbf{A})$ be two non-null disjoint soft open sets. Then, $(U,\mathbf{A})\widetilde{\subseteq}(V^c,\mathbf{A})$, which means that $(V^c,\mathbf{A})$ is a proper soft sw-open set. Accordingly, we obtain $(V,\mathbf{A})$ and $(V^c,\mathbf{A})$ are non-null disjoint soft sw-open subsets of $\widetilde{E}$ such that their soft union is $\widetilde{E}$. Hence, $(E,\mathcal{T},\mathbf{A})$ is soft sw-disconnected. □

The above theorem informs us that the concepts of soft sw-connected and soft hyperconnected spaces are identical.

Corollary 4.6. *The next properties are equivalent.*

(i) *$(E,\mathcal{T},\mathbf{A})$ is soft sw-connected.*

(ii) *There do not exist two non-null soft open sets are disjoint.*

(iii) *There do not exist two proper soft closed sets such that their union is $\widetilde{E}$.*

(iv) *Every non-null soft open set is soft dense.*

(v) *The soft interior of any proper soft closed set is the null soft set.*

(vi) *Every soft subset is soft dense or soft nowhere dense.*

(vii) *There do not exist two soft points separated by disjoint soft neighbourhoods.*

Proposition 4.7. *A non-null proper subset $(K,\mathbf{A})$ of a soft sw-connected space $(E,\mathcal{T},\mathbf{A})$ has a non-null soft sw-boundary points.*

Proof. Suppose, by contrary, that $swb(K,\mathbf{A}) = \widetilde{\phi}$. Then $swcl(K,\mathbf{A}) = swint(K,\mathbf{A})$, which means that $(K,\mathbf{A})$ is both soft sw-open and soft sw-closed. But this contradicts the soft sw-connectedness of $(E,\mathcal{T},\mathbf{A})$. Hence, $swb(K,\mathbf{A}) \neq \widetilde{\phi}$, as required. □

Definition 4.8. *A soft subset $(K,\mathbf{A})$ of $(E,\mathcal{T},\mathbf{A})$ is called a soft sw-connected set if there do not exist two non-null sw-separated soft sets $(G,\mathbf{A})$ and $(H,\mathbf{A})$ such that their union is $(K,\mathbf{A})$. Otherwise $(K,\mathbf{A})$ is called a soft sw-disconnected set.*

Lemma 4.9. *Let $(H,\mathbf{A})$ and $(G,\mathbf{A})$ be two soft sw-disconnection sets of $(E,\mathcal{T},\mathbf{A})$. If $(K,\mathbf{A})$ is a soft sw-connected subset of $(E,\mathcal{T},\mathbf{A})$, then $(K,\mathbf{A})\widetilde{\subseteq}(H,\mathbf{A})$ or $(K,\mathbf{A})\widetilde{\subseteq}(G,\mathbf{A})$.*

Proof. Since $(H,\mathbf{A})$ and $(G,\mathbf{A})$ are soft sw-disconnection sets of $(E,\mathcal{T},\mathbf{A})$, we obtain $(H,\mathbf{A})\widetilde{\bigcup}(G,\mathbf{A}) = \widetilde{E}$ and $[(H,\mathbf{A})\widetilde{\bigcap}swcl(G,\mathbf{A})]\widetilde{\bigcup}[swcl(H,\mathbf{A})\widetilde{\bigcap}(G,\mathbf{A})] = \widetilde{\phi}$. Now, $(K,\mathbf{A}) = [(K,\mathbf{A})\widetilde{\bigcap}(H,\mathbf{A})]\widetilde{\bigcup}[(K,\mathbf{A})\widetilde{\bigcap}(G,\mathbf{A})]$. It is clear that

$$[((K,\mathbf{A})\widetilde{\bigcap}(H,\mathbf{A}))\widetilde{\bigcap}swcl((K,\mathbf{A})\widetilde{\bigcap}(H,\mathbf{A}))]\widetilde{\bigcup}[((K,\mathbf{A})\widetilde{\bigcap}(G,\mathbf{A}))\widetilde{\bigcap}swcl((K,\mathbf{A})\widetilde{\bigcap}(G,\mathbf{A}))]$$

$$\widetilde{\subseteq}[(K,\mathbf{A})\widetilde{\bigcap}swcl(H,\mathbf{A})]\widetilde{\bigcup}[(K,\mathbf{A})\widetilde{\bigcap}swcl(G,\mathbf{A})] = \widetilde{\phi}.$$

So, we infer that $(K,\mathbf{A})\widetilde{\bigcap}(H,\mathbf{A})$ and $(K,\mathbf{A})\widetilde{\bigcap}(G,\mathbf{A})$ are soft sw-disconnection sets of $(K,\mathbf{A})$. But this contradicts the soft sw-connectedness of $(K,\mathbf{A})$. Hence, $(K,\mathbf{A})\widetilde{\bigcap}(H,\mathbf{A}) = \widetilde{\phi}$ or $(K,\mathbf{A})\widetilde{\bigcap}(G,\mathbf{A}) = \widetilde{\phi}$, which means that $(K,\mathbf{A})\widetilde{\subseteq}(H,\mathbf{A})$ or $(K,\mathbf{A})\widetilde{\subseteq}(G,\mathbf{A})$. □

Theorem 4.10. *Let* $(K,\mathbf{A})$ *be a soft subset of* $(E,\mathcal{T},\mathbf{A})$ *such that for each* $P^e_a, P^x_b \in (K,\mathbf{A})$ *there is a soft sw-connected subset* $(F,\mathbf{A})$ *of* $(K,\mathbf{A})$ *containing* P^e_a, P^x_b. *Then* $(K,\mathbf{A})$ *is soft sw-connected.*

Proof. Suppose, by contrary, that $(K,\mathbf{A})$ is soft sw-disconnected. Then there exist soft sw-disconnection sets $(H,\mathbf{A})$ and $(G,\mathbf{A})$ of $(K,\mathbf{A})$. So there are soft points P^e_a, P^x_b such that $P^e_a \in (H,\mathbf{A})$ and $P^x_b \in (G,\mathbf{A})$. By the given, there exists a soft sw-connected set $(F,\mathbf{A})$ containing P^e_a, P^y_e such that $(F,\mathbf{A})\widetilde{\subseteq}(K,\mathbf{A}) = (H,\mathbf{A})\widetilde{\bigcup}(G,\mathbf{A})$. By Lemma 4.9, we get $(F,\mathbf{A})\widetilde{\subseteq}(H,\mathbf{A})$ or $(F,\mathbf{A})\widetilde{\subseteq}(G,\mathbf{A})$. Consequentially, $(H,\mathbf{A})\widetilde{\bigcap}(G,\mathbf{A}) \neq \widetilde{\phi}$, which contradicts that $(H,\mathbf{A})$ and $(G,\mathbf{A})$ are soft sw-disconnection of $(K,\mathbf{A})$. This means that $(K,\mathbf{A})$ is soft sw-connected. □

Corollary 4.11. *If* $(K,\mathbf{A})$ *is a soft union of soft sw-connected sets* $(H_\rho,\mathbf{A})$ *such that their soft intersection is non-null, then* $(K,\mathbf{A})$ *is soft sw-connected.*

Proof. Suppose, by contrary, that $(K,\mathbf{A})$ is soft sw-disconnected. Then there are two soft sw-disconnection sets $(F,\mathbf{A})$ and $(G,\mathbf{A})$ of $(K,\mathbf{A})$. Since $\widetilde{\bigcap}(H_\rho,\mathbf{A}) \neq \widetilde{\phi}$, then there is a soft point P^e_a such that $P^e_a \in (H_\rho,\mathbf{A})$ for each ρ. Now, either $P^e_a \in (F,\mathbf{A})$ or $P^e_a \in (G,\mathbf{A})$. Say, $P^e_a \in (F,\mathbf{A})$. Then $[\widetilde{\bigcap}(H_\rho,\mathbf{A})]\widetilde{\bigcap}(F,\mathbf{A}) \neq \widetilde{\phi}$. According to the above theorem we obtain $(H_\rho,\mathbf{A})\widetilde{\subseteq}(F,\mathbf{A})$ for each ρ. This means that $(K,\mathbf{A})\widetilde{\subseteq}(F,\mathbf{A})$. But this is a contradiction. Hence, $(K,\mathbf{A})$ is soft sw-connected. □

Proposition 4.12. *Let* Ω_π *be a soft continuous mapping of a soft sw-connected space* $(E,\mathcal{T}_E,\mathbf{A})$ *onto a soft*$_{TS}$ $(X,\mathcal{T}_X,\mathbf{A})$. *Then* $\Omega_\pi(\widetilde{E})$ *soft sw-connected.*

Proof. Suppose that $\Omega_\pi(\widetilde{E}) = \widetilde{X}$ is soft sw-disconnected. Then, by Theorem 4.5 we get $\mathcal{T}_X$ contains two non-null disjoint soft open sets $(F,\mathbf{A})$ and $(H,\mathbf{A})$. By hypothesis, $\Omega^{-1}_\pi(F,\mathbf{A})$ and $\Omega^{-1}_\pi(H,\mathbf{A})$ are disjoint soft open subsets in $\mathcal{T}_E$. Since Ω_π is surjective, these soft open sets are non-null. Again it follows from Theorem 4.5 that $(E,\mathcal{T}_E,\mathbf{A})$ is soft sw-disconnected. But this is a contradiction. Hence, $(X,\mathcal{T}_X,\mathbf{A})$ is soft sw-connected. □

Proposition 4.13. **(i)** *Any type of soft sw-Hausdorff space is soft sw-disconnected.*

(ii) *Every soft sw-connected is almost soft sw-compact.*

A soft$_{TS}$ displayed in Example 3.19 is almost soft sw-compact, but it is soft sw-disconnected. The next example provides a soft$_{TS}$ which is soft sw-disconnected, but not pp-soft swT_2.

Example 4.14. *Let* $\mathcal{T} = \{\widetilde{\phi}, \widetilde{E}, (G_1,\mathbf{A}), (G_2,\mathbf{A})\}$ *be a soft topology over the universal set* $E = \{x,y,z\}$ *with a set of parameters* $\mathbf{A} = \{a,b\}$, *where* $(G_1,\mathbf{A}) = \{(a,\{x\}),(b,\{x\})\}$ *and* $(G_2,\mathbf{A}) = \{(a,\{y,z\}),(b,\{y,z\})\}$. *It is clear that* $(E,\mathcal{T},\mathbf{A})$ *is soft sw-disconnected, but it is not pp-soft* swT_2.

Proposition 4.15. *Let* $(E,\mathcal{T},\mathbf{A})$ *be a soft*$_{TS}$ *such that* E *or* $\mathbf{A}$ *is infinite countable (resp., uncountable). Then every soft sw-disconnected is not soft sw-compact (resp., soft sw-Lindelöf).*

The converse of the above proposition need not be true as illustrated by Example 3.4.

Proposition 4.16. *The next properties are identical under a soft sw-connected space* $(E,\mathcal{T},\mathbf{A})$.

(i) $(E, \mathcal{T}, \mathbf{A})$ *is almost soft sw-Lindelöf.*

(ii) $(E, \mathcal{T}, \mathbf{A})$ *is almost soft sw-compact.*

(iii) $(E, \mathcal{T}, \mathbf{A})$ *is weakly soft sw-Lindelöf.*

(iv) $(E, \mathcal{T}, \mathbf{A})$ *is weakly soft sw-compact.*

(v) $(E, \mathcal{T}, \mathbf{A})$ *is mildly soft sw-Lindelöf.*

(vi) $(E, \mathcal{T}, \mathbf{A})$ *is mildly soft sw-compact.*

Proof. The proof follows from the fact that $swcl(H, \mathbf{A}) = \widetilde{E}$ for any non-null soft open subset $(H, \mathbf{A})$ of a soft *sw*-connected space $(E, \mathcal{T}, \mathbf{A})$. □

In rest of this section, we study the relationships between soft topologies and their parametric topologies with respect to this type of connectedness. A soft$_{TS}$ displayed in Example 3.15 is soft *sw*-disconnected, whereas its parametric topological spaces are *sw*-connected. This means the property of being *sw*-connectedness does not navigate from parametric topologies to soft topology. On the other hand, the property of being soft *sw*-connectedness does not transmit from soft topology to its parametric topologies as we elaborate by the next example.

Example 4.17. *Let $E = \{x, y, z\}$ be the universal set and the set of natural numbers $\mathbb{N}$ be a set of parameters. Then, the family $\{\widetilde{\phi}, \widetilde{E}, (H_\rho, \mathbb{N}) : H_\rho(n) = E$ for all but finitely many $n \in \mathbb{N}\}$ forms a soft topology over E with $\mathbb{N}$. It can be checked that the families of soft open sets and soft sw-open sets are equal. Since we cannot get two disjoint soft open sets, $(E, \mathcal{T}, \mathbb{N})$ is soft sw-connected. Whereas all parametric topological spaces inspired by $(E, \mathcal{T}, \mathbb{N})$ is sw-disconnected.*

Theorem 4.18. *Let $\mathbf{A}$ be a finite set of parameters. If all parametric topological spaces inspired by a soft$_{TS}$ $(E, \mathcal{T}, \mathbf{A})$ is sw-disconnected, then $(E, \mathcal{T}, \mathbf{A})$ is soft sw-disconnected.*

Proof. Without loss of generality, take $\mathbf{A} = \{a, b\}$. By hypothesis, $(E, \mathcal{T}_a)$ and $(E, \mathcal{T}_b)$ are *sw*-disconnected. Then, there exist two disjoint nonempty open sets V, W in $\mathcal{T}_a$ and two disjoint nonempty open sets X, Y in $\mathcal{T}_b$. This implies that there exist four non-null soft open subsets $(H_1, \mathbf{A}), (H_2, \mathbf{A}), (H_3, \mathbf{A}), (H_4, \mathbf{A})$ of $(E, \mathcal{T}, \mathbf{A})$ such that $H_1(a) = V, H_2(a) = W$ and $H_3(b) = X, H_4(b) = Y$. Now, we have two cases:

(i) there exist two soft open sets of them are disjoint.

(ii) there does not exist two soft open sets of them are disjoint. Then, $(H_5, \mathbf{A}) = (H_1, \mathbf{A}) \widetilde{\bigcap} (H_2, \mathbf{A})$ and $(H_6, \mathbf{A}) = (H_3, \mathbf{A}) \widetilde{\bigcap} (H_4, \mathbf{A})$ are non-null soft open sets. Since $H_5(a) = \emptyset$ and $H_6(b) = \emptyset$, $(H_5, \mathbf{A})$ and $(H_6, \mathbf{A})$ are disjoint.

Hence, according to Theorem 4.5, $(E, \mathcal{T}, \mathbf{A})$ is soft *sw*-disconnected. □

By Example 3.15, we see that the converse of the above theorem fails.

Theorem 4.19. *Let $(E, \mathcal{T}, \mathbf{A})$ be a full soft$_{TS}$. Then, $(E, \mathcal{T}, \mathbf{A})$ is soft sw-disconnected if and only if all parametric topological spaces inspired by $(E, \mathcal{T}, \mathbf{A})$ is sw-disconnected.*

Proof. Necessity: Let $(E, \mathcal{T}, \mathbf{A})$ be soft sw-disconnected. Then, according to Theorem 4.5, $\mathcal{T}$ contains two disjoint non-null soft open sets $(F, \mathbf{A})$ and $(H, \mathbf{A})$. Since $(E, \mathcal{T}, \mathbf{A})$ is full, $F(a)$ and $H(a)$ are both nonempty disjoint open subsets of $(E, \mathcal{T}_a)$ for each $a \in \mathbf{A}$. This implies that $(E, \mathcal{T}_a)$ is sw-disconnected for each $a \in \mathbf{A}$.

Sufficiency: Let a parametric topological space $(E, \mathcal{T}_a)$ be sw-disconnected. Then, there exist two disjoint nonempty open sets V, W in $\mathcal{T}_a$. So, there exist two non-null soft open subsets $(F, \mathbf{A})$ and $(H, \mathbf{A})$ of $(E, \mathcal{T}, \mathbf{A})$ such that $F(a) = V$ and $H(a) = W$. Obviously, $(F, \mathbf{A}) \widetilde{\bigcap} (H, \mathbf{A})$ is a soft open set with an empty component; i.e. $(F \bigcap H)(a) = \emptyset$. Since $(E, \mathcal{T}, \mathbf{A})$ is full, we obtain $(F \bigcap H)(a) = \emptyset$ for all $a \in \mathbf{A}$. This means that $(F, \mathbf{A})$ and $(H, \mathbf{A})$ are also disjoint. Hence, $(E, \mathcal{T}, \mathbf{A})$ is soft sw-disconnected. □

5. Conclusion remarks and future works

Soft set theory has been widely used in many real-world applications for its tremendous practical value. Typologists have exploited soft sets to define soft topological spaces and studied classical topological concepts via soft frame. As we noted in the published literature the abstract topological concepts in soft settings such as soft compactness [8], soft separation axioms [12] and generalized soft open sets [13] have been successfully applied to address practical issues in the information systems, economic and medical science. Also, the variety of belonging and non-belonging relations between ordinary points and soft sets produces a fruitful environment to display classical topological concepts from different standpoints as illustrated to soft compact spaces and soft separation axioms.

In this manuscript, we have benefited from one of the generalizations of soft open sets called "soft sw-open sets" to introduce the concepts of compactness, Lindelöfness and connectedness via the frame of soft topologies. First, we have presented novel types of soft compact and Lindelöf spaces namely, soft sw-compact, soft sw-Lindelöf, almost soft sw-compact, almost soft sw-Lindelöf, weakly soft sw-compact, weakly soft sw-Lindelöf, mildly soft sw-compact and mildly soft sw-Lindelöf spaces. We have established the main properties of these spaces and provided some interesting examples to show the relationships between them. Some exciting results described the behaviours of these spaces via soft topology and its parametric topologies have been obtained and illustrated with some counterexamples.

Then, we have introduced the concept of soft sw-connected spaces and described by soft sw-open and sw-closed sets. One of the important results that we have proved is the identity between soft sw-connected and soft hyperconnected spaces. We have also scrutinized the relationships that associate this concept with some types of soft compactness and soft separation axioms. Moreover, we have elaborated that the property of being sw-connectedness is commutative between soft topology and its parametric topological spaces under a condition of full soft topology.

We draw attention to that the concepts introduced herein have some unique characterizations that are invalid for their counterparts defined by the other types of famous generalizations of soft open sets such as soft α-open, soft semi-open, soft pre-open, soft b-open and soft β-open sets. For example, the properties of being soft sw-compact and soft sw-Lindelöf (mildly soft sw-compact and mildly soft sw-Lindelöf) spaces are transmitted to all their parametric topological spaces; see, Theorems 3.14 and 3.67. But it is imposed a condition of extended soft topology to preserve this characteristic for the other types given in the published literature. Also, some descriptions of the previous types of compactness and Lindelöfness are preserved under a condition of extended soft topology, whereas these descriptions are satisfied for the current types of compactness and Lindelöfness under a condition of full soft topology; see, Theorems 3.21, 3.41, 3.53 and 3.68. With respect to soft *sw*-connectedness, it has been shown that the property of being a soft *sw*-connected space is navigated to all its parametric topological spaces provided that a set of parameters is finite; see, Theorem 4.18. Also, it has been demonstrated that this property is interchangeable between soft topology and its parametric topologies under a condition of full soft topology; see, Theorem 4.19.

Regarding the relationships between the current covering properties and the previous ones, it can be noted that the covering properties produced by soft α-open and soft semi-open sets are special cases of their counterparts introduced herein. This fact can be seen from Example 3.15. On the other hand, any indiscrete soft$_{TS}$ defined over an uncountable universal set and a soft$_{TS}$ displayed in Example 3.16 clarify that the current covering properties and those defined by soft pre-open, soft b-open and soft β-open sets are independent of each other.

In future work, we are going to study further topological concepts via the family of soft *sw*-open sets such as paracompactness, local compactness, and local connectedness. Also, we will research the possibility of applying the concepts introduced herein to handle some real-life issues as presented in [8]. Moreover, we analysis rough set models via soft frame using the concepts of soft *sw*-open sets as given in the classical frame by [10, 11].

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