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Research article

Existence of solutions for Caputo fractional iterative equations under several boundary value conditions

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Abstract: In this paper, we investigate the existence and uniqueness of solutions for nonlinear quadratic iterative equations in the sense of the Caputo fractional derivative with different boundary conditions. Under a one-sided-Lipschitz condition on the nonlinear term, the existence and uniqueness of a solution for the boundary value problems of Caputo fractional iterative equations with arbitrary order is demonstrated by applying the Leray-Schauder fixed point theorem and topological degree theory, where the solution for the case of fractional order greater than 1 is monotonic. Then, the existence and uniqueness of a solution for the period and integral boundary value problems of Caputo fractional quadratic iterative equations in R^N are also demonstrated. Furthermore, the well posedness of the control problem of a nonlinear iteration system with a disturbance is established by applying set-valued theory, and the existence of solutions for a neural network iterative system is guaranteed. As an application, an example is provided at the end.

Keywords: existence; uniqueness; Caputo fractional; iterative equation
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1. Introduction

Boundary value and periodic problems for iterative differential equations have attracted significant interest in recent years. The investigation of iterative differential equations dates back to 1965, when Petuhov [1] first proposed the existence and uniqueness of solutions to the following equation:

$$z'' = \lambda z(z(t)),\ \forall t \in [-b, b],$$

with $z(0) = z(b) = k$ under the different ranges of parameters λ and k. In general, compared with ordinary differential equations, it is more difficult to deal with such iterative differential equations since the existence of iterative terms leads to considerable difficulties of analysis and the failure of some methods, such as the monotone method [2], the measure of noncompactness [3], the upper and lower solutions method [4], the coincidence degree theory [5, 6], etc. However, the fixed point method is one of the most powerful and fruitful tools studying the boundary value or periodic problems of this kind of iterative differential equation. It is worth mentioning that Ke [7] considered the existence of a solution for the first order iterative differential equation

$$z' = f(z(z(t))),\ z(a) = a,$$

where a is the end point of an interval. Recently, applying Schauder's fixed point theorem, Kaufmann [8] further investigated the boundary value problems of the second-order iterative differential equation

$$z'' = f(t, z(t), z^{[2]}(t)),$$

where $z^{[2]}(t) = z(z(t))$, with $z(a) = a, z(b) = b$ or $z(a) = b, z(b) = a$. In [9], the authors studied a third-order differential equations with linear iterative source for which the existence and uniqueness of a periodic solution are established by using Banach's and Krasnoselskii's fixed point theorems. The same method is used to deal with a first-order iterative differential equations together with some properties of Green's functions in [10]. Periodic problems were investigated in [11–13] by Schauder's or Banach's fixed point theorem. In recent years, fractional calculus has attracted the attention of many mathematicians, as well as researchers in a number of other fields, such as engineering, chemistry and physics. It is recognized that the usage of fractional calculus in various modeling applications is quite outstanding in the process. The main reason for the broad application of fractional operators is the fact that, distinct from "integer" operators, these operators have a non-local behavior which enables us to trace the past impact of the involved phenomena [14–18]. Along with the recent advances in fractional differential equations, researchers have contributed many investigations that discuss the behaviors of solutions of different types of fractional differential equations [19–25]. In addition, since fractional derivatives and integrals have different forms, other types of fractional differential systems are also investigated in [26–32], to name a few. To the best of our knowledge, there are few results on boundary value problems for fractional iterative equations. It should be noted that, the rotational periodic problem of some Caputo-fractional iterative systems is investigated in [33]. The existence and uniqueness of solutions for fractional iterative equations was studied in [34], in which the nonlinear function satisfies the Lipschitz condition.

In this paper, however, under weaker conditions of one-sided Lipschitz, we consider the existence and uniqueness for a fractional quadratic iterative differential equation with arbitrary order in the sense of Caputo. It is worth emphasizing that the solution we obtained the case of fractional order greater than 1 is monotonically continuous for the first time. Then, the existence and uniqueness for a fractional vector iterative differential equation in R^N are established by the Leray-Schauder fixed point theorem and topological degree theory under periodic and integral boundary value conditions, respectively. Furthermore, applying set-value theory, we prove the existence of solutions for a fractional nonlinear control system with a disturbance. Eventually, motivated by [35], the existence of a solution for a fractional neural networks iterative system where neuron activations are continuous or discontinuous.

The framework of the present paper is arranged as follows. Some basic definitions and auxiliary results on fractional calculus are provided in section 2. In section 3, the existence and uniqueness of solutions of Caputo-fractional quadratic iterative equations with arbitrary order are addressed by applying the Leray-Schauder fixed point theorem and topological degree theory. In section 4, the existence and uniqueness of periodic solutions of fractional vector iterative equations are established in R^N. Then, the existence of an integral boundary value solution for this vector iterative equation follows in section 5. In section 6, the well-posed result for a nonlinear control system is studied by applying set-valued theory, followed the existence of solutions for a neural network iterative system in section 7 and the presentation of an example in section 8.

2. Preliminaries

In this section, let R^N be N dimensional Euclid space, $\langle\cdot,\cdot\rangle$ be the inner product in R^N, and $\|\cdot\|$ be the norm of R^N space. Let $T = [0, b]$ and $C(T; R^N)$ denote the space composed of all continuous functions from T to R^N, with norm $\|\cdot\|_C = \max_{t\in T}\|\cdot\|$. Some basic definitions and fundamental facts of fractional calculus which will be used later need to be provided in the sequel. For details, we refer interest readers to [16–18].

Definition 2.1. *Assume that g is a function defined on the interval $[a, b]$. The Caputo fractional integral of order $\alpha > 0$ of g is defined by*

$$\mathcal{I}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s)ds, \quad t > a, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. *Let g be a function defined on the interval $[a, b]$. Then, the Caputo fractional derivative of order $\alpha > 0$ of g is defined by*

$${}^{C}\mathcal{D}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} g^{(n)}(s)ds, \ (n-1 \le \alpha < n, \ t > a),$$

where $g^{(n)}(t) := \frac{d^n g}{dt^n}$, $n = [\alpha] + 1$, and $[\alpha]$ is defined as the maximum positive integer not exceeding the number α.

Definition 2.3. *If $g \in C^n[a, b]$, then*

$$\mathcal{I}^{\alpha C}\mathcal{D}^{\alpha} g(t) = g(t) - \sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{k!} t^k,$$

where $\alpha \in [n-1, n), \forall n \in \mathbb{N}$. In particular, if $g \in C^1[0, b]$ and $\alpha \in (0, 1]$, then for any $t \in [0, b]$,

$$\mathcal{I}^{\alpha C}\mathcal{D}^{\alpha} g(t) = g(t) - g(0).$$

Next, the following lemmas given here are of significant completing the proof of the main results.

Lemma 2.1. *([36]) Let $\alpha > 0$ and $\beta(t)$ be locally integrable on $[a, b]$, where $\beta(t)$ is a nondecreasing and nonnegative function. Assume that h is a nonnegative, nondecreasing and continuous function, and U is a nonnegative continuous function satisfying*

$$U(t) \le \beta(t) + h(t) \int_a^t (t-s)^{\alpha-1} U(s)ds.$$

Then, it holds that

$$U(t) \leq \beta(t)E_\alpha(h(t)\Gamma(\alpha)t^\alpha),$$

where E_α is the single parameter Mittag-Leffler function given by $E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha+1)}$.

Now, we introduce several inequalities on the fractional derivative which will be used later.

Lemma 2.2. *([37])Let $U : T \to R^N$ be a continuous differentiable function, and $A \in R^{N\times N}$ is a positive definite matrix. Then, it holds that*

$$\frac{1}{2}{}^C\mathcal{D}^\alpha[U^\top(t)AU(t)] \leq U^\top(t)A{}^C\mathcal{D}^\alpha U(t),$$

for any $\alpha \in (0, 1]$.

Lemma 2.3. *([38]) Assume that $U : \mathbb{R} \to \mathbb{R}_+$ is a continuous function such that*

$${}^C\mathcal{D}^\alpha U(t) \leq -\omega U(t), \ \ \forall\alpha \in (0, 1],$$

where $\omega > 0$ is a constant. Then, the following estimate holds:

$$U(t) \leq U(0)E_\alpha(-\omega t^\alpha), \ \ \forall t \geq 0.$$

Lemma 2.4. *([39]) Suppose that $\mathcal{W}$ is a Banach space, the set $C \subseteq \mathcal{W}$ is nonempty and convex with $0 \in C$, and $G : C \to C$ is an upper semicontinuous multifunction with compact convex value, which maps bounded sets to relatively compact sets. Then, one of the following statements is true:*
(i) $\Xi = \{z \in C : z \in \epsilon G(z), \epsilon \in (0, 1)\}$ is an unbounded set;
(ii) The multifunction $G(\cdot)$ has a fixed point, i.e., there exists $z \in C$ such that $z \in G(z)$.

Let $E_{\alpha,\gamma}(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\gamma)}$ be the two parameter (α, γ)-Mittag-Leffler function for $x \in \mathbb{R}$. For notational convenience, set $M_A = \max_{t\in T} \|E_\alpha(-At^\alpha)\|$, $\hat{M}_A = \max_{t\in T} \|E_{\alpha,\alpha}(-At^\alpha)\|$.

3. Boundary value solution of nonlinear fractional quadratic iterative equations

This subsection investigates the existence and uniqueness of solutions of the Caputo-fractional quadratic iterative equation represented as follows:

$${}^C\mathcal{D}^\alpha z(t) = f(t, z(t), z^{[2]}(t)), \ a < t < b, \tag{3.1}$$

where $z^{[2]}(t) = z(z(t))$ for any $\alpha > 0$. In order to obtain our desired results, the following hypothesis on the nonlinear function f is needed:
(H_1) $f \in [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that
(i) there exists a function $\theta_1 \in L_+^\infty[a, b]$ such that $|f(t, \mu, \eta)| \leq \theta_1(t)$, for all $t \in [a, b]$, $\mu, \eta \in \mathbb{R}$;
(ii) there exists a constant $W > 0$, for any $(t_1, u_1, v_1), (t_2, u_2, v_2) \in [a, b] \times \mathbb{R} \times \mathbb{R}$, such that

$$[f(t_1, u_1, v_1) - f(t_2, u_2, v_2)](u_1 - u_2) \leq W|u_1 - u_2|^2.$$

Theorem 3.1. *Let $0 < \alpha < 1$, $z(a) = a$ and $b > \left(\frac{\|\theta_1\|_\infty}{\Gamma(\alpha)\alpha}\right)^{\frac{1}{1-\alpha}}$. If hypothesis (H_1) holds, then the Caputo-fractional quadratic iterative equation (3.1) admits a unique solution.*

Proof. Following Proposition 2.1, Eq (3.1) is equivalent to the following integral equation:

$$z(t) - z(a) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, z(s), z^{[2]}(s))ds.$$

Define the operator $T_1 : C[a,b] \to C[a,b]$ by

$$T_1(z(t)) := z(a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, z(s), z^{[2]}(s))ds, \quad \forall t \in [a,b].$$

Now, we divide the proof into two steps.

First step. We claim the existence of a solution for Eq (3.1).

From the definition of the operator T_1 and the assumption (H_1)(i), we can deduce

$$\begin{aligned} |T_1(z(t))| &\le |a| + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} |f(s, z(s), z^{[2]}(s))| ds \\ &\le |a| + \frac{\|\theta_1\|_\infty}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} ds \\ &\le |a| + \frac{\|\theta_1\|_\infty}{\alpha\Gamma(\alpha)} (b-a)^\alpha \le b, \end{aligned}$$

which means that T_1 is uniformly bounded in $C[a,b]$. Now, we show the equicontinuity of T_1. For any $t, t+\delta \in [a,b]$ and $\delta > 0$, it follows that

$$\begin{aligned} & |T_1(z(t+\delta)) - T_1(z(t))| \\ = & \left| \frac{1}{\Gamma(\alpha)} \int_a^{t+\delta} (t+\delta-s)^{\alpha-1} f(s, z(s), z^{[2]}(s))ds - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, z(s), z^{[2]}(s))ds \right| \\ \le & \frac{1}{\Gamma(\alpha)} \mid \int_t^{t+\delta} (t+\delta-s)^{\alpha-1} f(s, z(s), z^{[2]}(s))ds \\ & + \int_a^t [(t+\delta-s)^{\alpha-1} - (t-s)^{\alpha-1}] f(s, z(s), z^{[2]}(s))ds \mid \\ \le & \frac{\|\theta_1\|_\infty}{\Gamma(\alpha)} \left| \int_t^{t+\delta} (t+\delta-s)^{\alpha-1} ds \right| + \left| \int_a^t (t+\delta-s)^{\alpha-1} - (t-s)^{\alpha-1} ds \right| \\ \le & \frac{2\|\theta_1\|_\infty}{\alpha\Gamma(\alpha)} \mid \delta^\alpha \mid + \frac{\|\theta_1\|_\infty}{\alpha\Gamma(\alpha)} |(t+\delta-a)^\alpha - (t-a)^\alpha|. \end{aligned}$$

Let $\delta \to 0$, and then $|T_1(z(t+\delta)) - T_1(z(t))| \to 0$. Applying the Arzela-Ascoli theorem, it is easy to find that the operator $T_1 : \Omega \to \Omega$ is completely continuous where

$$\Omega := \{u \in C[a,b] : \|u\|_C < b+1\}.$$

Then, the existence of a solution for the system (3.1) is transformed into the fixed point problem of T_1, i.e., $z = T_1(z)$. For this, let $\mathbb{Q}_\lambda(z) := z - \lambda T_1(z)$ with $\lambda \in [0,1]$, and then the definition of Ω implies that $p \notin \mathbb{Q}(\partial\Omega)$. Thus, for any $\lambda \in [0,1]$, we have

$$\deg(\mathbb{Q}_\lambda, \Omega, p) = \deg(\mathbb{Q}_1, \Omega, p) = \deg(I - T_1, \Omega, p) = \deg(\mathbb{Q}_0, \Omega, p) = \deg(I, \Omega, p) = 1 \neq 0,$$

where I is the identity map. Consequently, T_1 has a fixed point in Ω, i.e. $z = T_1(z)$, which leads to the existence of a solution for the system (3.1).

Second step. We show the uniqueness of a solution for the system (3.1).

Assume $z_1(t), z_2(t)$ are two solutions of system (3.1) with the same initial value. Then, for all $t \in [a, b]$, we can deduce from (3.1) that

$$^{C}\mathcal{D}^{\alpha}(z_1(t) - z_2(t)) = f(t, z_1(t), z_1^{[2]}(t)) - f(t, z_2(t), z_2^{[2]}(t)),$$

which, with multiplying by $z_1(t) - z_2(t)$, leads to

$$\begin{aligned} ^{C}\mathcal{D}^{\alpha}|z_1(t) - z_2(t)|^2 &\leq \frac{1}{2}(z_1(t) - z_2(t))^{C}\mathcal{D}^{\alpha}(z_1(t) - z_2(t)) \\ &\leq \frac{1}{2}[f(t, z_1(t), z_1^{[2]}(t)) - f(t, z_2(t), z_2^{[2]}(t))](z_1(t) - z_2(t)). \end{aligned} \tag{3.2}$$

Integrate on both sides with Proposition 2.1 and apply (H_1)(ii) to get

$$\begin{aligned} &|z_1(t) - z_2(t)|^2 \\ &\leq \frac{1}{2\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}[(f(s, z_1(s), z_1^{[2]}(s)) - f(s, z_2(s), z_2^{[2]}(s)))(z_1(s) - z_2(s))]ds \\ &\leq \frac{W}{2\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}|z_1(s) - z_2(s)|^2 ds, \quad \forall t \in [a, b], \end{aligned}$$

which, by using Lemma 2.1, yields

$$|z_1(t) - z_2(t)|^2 = 0,$$

for any $t \in [a, b]$. This gives $z_1 \equiv z_2$, and therefore, a unique solution for system (3.1) is guaranteed. □

Now, we consider the case $1 < \alpha < 2$ for system (3.1), and then the hypothesis on the nonlinear function f needs to be improved as follows:

(H_2) $f \in [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that

(i) there exist two constants $\mathbb{N}, Q > 0$ such that $-\mathbb{N} < f(t, \mu, \eta) < Q$ for all $(t, \mu, \eta) \in [a, b] \times \mathbb{R}^2$;

(ii) there exist two constants $M, F > 0$ such that

$$[f(t_1, u_1, v_1) - f(t_2, u_2, v_2)](u_1 - u_2) \leq M|u_1 - u_2|^2 + F|v_1 - v_2|^2$$

for any $(t, u_1, v_1), (t, u_2, v_2) \in [a, b] \times \mathbb{R}^2$.

Theorem 3.2. *Let $1 < \alpha < 2$, $z(a) = a$ and $z(b) = b$. Suppose that $b \leq a + \left(\frac{\Gamma(\alpha)\alpha}{\alpha N + Q}\right)^{\frac{1}{\alpha-1}}$ is satisfied, and (H_2) holds. Then Eq (3.1) admits a unique solution which is monotonically increasing and continuous.*

Proof. Similar to (3.2), by Proposition 2.1, the solution of (3.1) is equivalent to

$$z(t) = z(a) + z'(a)(t - a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, z(s), z^{[2]}(s)) ds. \tag{3.3}$$

Due to $z(a) = a,\ z(b) = b$, it follows from (3.3) that

$$b = a + z'(a)(b - a) + \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} f(s, z(s), z^{[2]}(s)) ds,$$

which implies that

$$z'(a) = 1 - \frac{1}{\Gamma(\alpha)(b-a)} \int_a^b (b-s)^{\alpha-1} f(s, z(s), z^{[2]}(s))ds.$$

Now, substitute $z'(a)$ into (3.3), and use Green's function to turn problem (3.3) into the following integral equation:

$$\begin{aligned} z(t) &= a + t - a - \frac{t-a}{\Gamma(\alpha)(b-a)} \int_a^b (b-s)^{\alpha-1} f(s, z(s), z^{[2]}(s))ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, z(s), z^{[2]}(s))ds \\ &= t + \frac{1}{\Gamma(\alpha)(b-a)} \int_a^b (a-t)(b-s)^{\alpha-1} f(s, z(s), z^{[2]}(s))ds \\ &\quad + \frac{1}{\Gamma(\alpha)(b-a)} \int_a^t (b-a)(t-s)^{\alpha-1} f(s, z(s), z^{[2]}(s))ds \\ &= t + \frac{1}{\Gamma(\alpha)(b-a)} \int_a^b G(t,s) f(s, z(s), z^{[2]}(s))ds, \end{aligned} \tag{3.4}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)(b-a)} \begin{cases} (a-t)(b-s)^{\alpha-1} + (b-a)(t-s)^{\alpha-1}, & a \le s \le t \le b, \\ (a-t)(b-s)^{\alpha-1}, & a \le t \le s \le b. \end{cases}$$

Similar to the proof of Theorem 3.1, define the operator $T_2 : C[a,b] \to C[a,b]$, by

$$T_2(z(t)) := t + \frac{1}{\Gamma(\alpha)(b-a)} \int_a^b G(t,s) f(s, z(s), z^{[2]}(s))ds.$$

First step. The existence of a solution to the problem (3.3).

In light of the definition of T_2, we can derive from (H_2)(i) that

$$\begin{aligned} \frac{d(T_2(z(t)))}{dt} &= 1 - \frac{1}{\Gamma(\alpha)(b-a)} \int_a^b (b-s)^{\alpha-1} f(s, z(s), z^{[2]}(s))ds \\ &\quad + \frac{\alpha-1}{\Gamma(\alpha)(b-a)} \int_a^t (b-a)(t-s)^{\alpha-2} f(s, z(s), z^{[2]}(s))ds \\ &\ge 1 - \frac{Q}{\Gamma(\alpha)(b-a)} \int_a^b (b-s)^{\alpha-1} ds - \frac{N(\alpha-1)}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-2} ds \\ &\ge 1 - \frac{(\alpha\mathbb{N} + Q)(b-a)^{\alpha-1}}{\alpha\Gamma(\alpha)}, \end{aligned}$$

and

$$
\begin{aligned}
\frac{d(T_2(z(t)))}{dt} &= 1-\frac{1}{\Gamma(\alpha)(b-a)}\int_a^b (b-s)^{\alpha-1} f(s,z(s),z^{[2]}(s))ds \\
&\quad +\frac{\alpha-1}{\Gamma(\alpha)(b-a)}\int_a^t (b-a)(t-s)^{\alpha-2} f(s,z(s),z^{[2]}(s))ds \\
&\leq 1+\frac{N}{\Gamma(\alpha)(b-a)}\int_a^b (b-s)^{\alpha-1}ds+\frac{Q(\alpha-1)}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-2}ds \\
&\leq 1+\frac{(\mathbb{N}+\alpha Q)(b-a)^{\alpha-1}}{\alpha\Gamma(\alpha)}.
\end{aligned}
$$

Notice that $b \leq a+\left(\frac{\Gamma(\alpha)\alpha}{\alpha\mathbb{N}+Q}\right)^{\frac{1}{\alpha-1}}$, and then $(\mathbb{N}+\alpha Q)(b-a)^{\alpha-1} \leq \alpha\Gamma(\alpha)$, which gives $\frac{d(T_2(z(t)))}{dt} > 0$, $\frac{d(T_2(z(t)))}{dt} < 2$ for any $t \in [a,b]$. Consequently, T_2 is bounded and monotone increasing. Since $T_2(z(a)) = a$, $T_2(z(b)) = b$, then we have $a \leq T_2(z(t)) \leq b$, for all $t \in [a,b]$. The application of Schauder's theorem yields that there exists at least one fixed point of T_2 such that $z = T_2(z)$, i.e., z is the solution of system (3.1).

Second step. The uniqueness of the solution of Eq (3.1).

Let z_1, z_2 be two solutions to Eq (3.1) with same initial data, and then it follows from $(T_2(z_i(t)))' < 2 (i = 1,2)$, (3.4) and (H_2)(ii) that

$$
\begin{aligned}
|z_1(t)-z_2(t)|^2 &= \int_a^b G(t,s)[f(s,z_1(s),z_1^{[2]}(s))-f(s,z_2(s),z_2^{[2]}(s))][z_1(s)-z_2(s)]ds \\
&\leq \int_a^b G(t,s)(\mathbb{M}|z_1(s)-z_2(s)|^2+F|z_1^{[2]}(s)-z_2^{[2]}(s)|^2)ds \\
&\leq (\mathbb{M}+2F)\int_a^b G(t,s)|z_1(s)-z_2(s)|^2 ds,
\end{aligned}
$$

which together with Lemma 2.1 yields

$$
|z_1(t)-z_2(t)|^2 = 0,
$$

for any $t \in [a,b]$. Hence, $z_1 \equiv z_2$, i.e., the uniqueness of the solution is proved, which completes the proof of Theorem 3.2. □

Theorem 3.3. *Let $n-1 < \alpha < n$ $(n \geq 3)$, $z(a) = a$, $z(b) = b$, and $z^{(i)}(a) = \xi_i > 0$ $(i = 1,2,\cdots,n-1)$ with $\xi_1 > \frac{\mathbb{N}(b-a)^{\alpha-1}}{\Gamma(\alpha)}$. Assume that the hypothesis (H_2) is fulfilled, and then there is a unique solution of system (3.1) which is monotone.*

Proof. Integrating on both sides of system (3.1) and applying Proposition 2.1 gives

$$
z(t) = \sum_{k=0}^{n-1}\frac{z^{(k)}(a)}{k!}(t-a)^k+\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s,z(s),z^{[2]}(s))ds. \tag{3.5}
$$

Define the operator $T_3 : C[a,b] \to C[a,b]$, by

$$
T_3(z(t)) := \sum_{k=0}^{n-1}\frac{z^{(k)}(a)}{k!}(t-a)^k+\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s,z(s),z^{[2]}(s))ds.
$$

First step. We establish the existence of a solution for Eq (3.1).

The derivation rule, (H_2)(i) and the boundary value condition in (3.1) show that

$$\begin{aligned}\frac{d(T_3(z(t)))}{dt} &= \xi_1+\cdots+\frac{\xi_{n-1}}{(n-2)!}(t-a)^{n-2}+\frac{\alpha-1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-2}f(s,z(s),z^{[2]}(s))ds\\ &\geq \xi_1-\frac{(\alpha-1)\mathbb{N}}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-2}ds\\ &\geq \xi_1-\frac{\mathbb{N}(b-a)^{\alpha-1}}{\Gamma(\alpha)}>0,\end{aligned}\tag{3.6}$$

for any $t\in[a,b]$. Also, we can find

$$\begin{aligned}\frac{d(T_3(z(t)))}{dt} &= \xi_1+\cdots+\frac{\xi_{n-1}}{(n-2)!}(t-a)^{n-2}+\frac{\alpha-1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-2}f(s,z(s),z^{[2]}(s))ds\\ &\leq \xi_1+\cdots+\frac{\xi_{n-1}}{(n-2)!}(b-a)^{n-2}+\frac{(\alpha-1)Q}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-2}ds\\ &\leq C:=\sum_{i=1}^{n-1}\frac{\xi_i}{(i-1)!}(b-a)^{i-1}+\frac{Q(b-a)^{\alpha-1}}{\Gamma(\alpha)}.\end{aligned}\tag{3.7}$$

Therefore, $T_3(z(t))$ is a bounded and monotone increasing function. By taking $T_3(z(a))=a$, $T_3(z(b))=b$ into account, one has $a\leq T_3(z(t))\leq b$ for all $t\in[a,b]$. Thanks to Schauder's theorem, we can conclude that z is a fixed point of T_3. Therefore, z is the solution of Eq (3.1).

Second step. We claim the uniqueness of a solution of Eq (3.1).

Assume that $z_1,z_2\in C^{n-1}[a,b]$ are two solutions of (3.1) with the same initial data. Similar to (3.2), and from assumption (H_2)(ii), (3.5) and (3.7), one obtains that

$$\begin{aligned}|z_1(t)-z_2(t)|^2 &= \frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}\langle f(s,z_1(s),z_1^{[2]})-f(s,z_2(s),z_2^{[2]}),z_1(s)-z_2(s)\rangle ds\\ &\leq \int_a^t(t-s)^{\alpha-1}(\mathbb{M}|z_1(t)-z_2(t)|^2+F|z_1^{[2]}(s)-z_2^{[2]}(s)|^2)ds\\ &\leq \frac{\mathbb{M}+CF}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}|z_1(s)-z_2(s)|^2ds,\end{aligned}$$

where

$$C=\sum_{i=1}^{n-1}\frac{\xi_i}{(i-1)!}(b-a)^{i-1}+\frac{Q(b-a)^{\alpha-1}}{\Gamma(\alpha)}.\tag{3.8}$$

Lemma 2.1 and $z_1(0)=z_2(0)$ show $z_1(t)=z_2(t)$ for any $t\in[a,b]$, and then $z_1\equiv z_2$. The proof of Theorem 3.3 is completed. □

4. Boundary value periodic solution of Caputo-fractional quadratic iterative equations in R^N

Consider the following boundary value periodic problem of the fractional iterative vector differential equation

$$^C\mathcal{D}^\alpha z(t)+Bz(t) = f(t,z(t),z^{[2]}(t))+g(t),\ t\in T:=[0,b],$$

$$z(0) \;=\; z(b), \tag{4.1}$$

where $\alpha \in (0,1)$, $z^{[2]}(t) = (z_1(\|z\|), z_2(\|z\|), \cdots, z_n(\|z\|)) \in R^N$ for any $t \in T$, the linear operator $B : R^N \to R^N$ is positive definite, $f : T \times R^N \times R^N \to R^N$ is a continuous function, and $g \in L^\infty(T; R^N)$. Throughout this section, we assume that b is greater than some constant $M^{\frac{1}{1-\alpha}}$ to be determined later.

We need the following assumptions:

(H_3) The linear operator $B : R^N \to R^N$ is bounded and positive definite, that is, for any $z \in R^N$, there exists a constant $\xi \in \mathbb{R}_+$, such that $\langle Az, z\rangle \geq \xi\|z\|^2$.

(H_4) $f : T \times R^N \times R^N \to R^N$ is a continuous function satisfying

(i) for any $x, y \in R^N$, there exists a function $\theta \in L_+^\infty(T)$, such that $\|f(t,x,y)\| \leq \theta(t)$ for any $t \in T$;

(ii) for any $t \in T$, $(u_1, v_1), (u_2, v_2) \in R^N \times R^N$, there exists a function $\mu \in L_+^\infty(T)$, such that

$$\langle f(t,u_1,v_1) - f(t,u_2,v_2), u_1 - u_2\rangle \leq \mu(t)\|u_1 - u_2\|^2,$$

where $\|\mu\|_\infty < \xi$, and ξ is a constant in (H_3).

Theorem 4.1. *If assumptions* $(H_3), (H_4)$ *hold, then the fractional iterative differential system* (4.1) *has a unique solution.*

Proof. In view of Corollary 7.1 in [40], problem (4.1) is equivalent to the following integral iterative equation:

$$z(t) = E_\alpha(-At^\alpha)z(0) + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-A((t-\tau)^\alpha))[f(\tau, z(\tau), z^{[2]}(\tau)) + g(\tau)]d\tau. \tag{4.2}$$

Then, it suffices to show the existence of a solution for problem (4.2). For this, the operator T_4: $C(T; R^N) \to C(T; R^N)$ is defined by

$$T_4(z(t)) := E_\alpha(-At^\alpha)z(0) + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-A((t-\tau)^\alpha))[f(\tau, z(\tau), z^{[2]}(\tau)) + g(\tau)]d\tau. \tag{4.3}$$

First, we show a priori boundedness of the solution. From the definition of operator T_4 and the assumption (H_4)(i), it can be deduced that

$$\begin{aligned}
\|T_4(z(t))\| &\leq \|z(0)\| \max_{s\in T}\|E_\alpha(-As^\alpha)\| \\
&\quad + \frac{\max_{s\in T}\|E_{\alpha,\alpha}(-As^\alpha)\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\|[f(\tau, z(\tau), z^{[2]}(\tau)) + g(\tau)]\|ds \\
&\leq \|z(0)\|M_A + \frac{(\|\theta\|_\infty + \|g\|_\infty)\hat{M}_A}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}ds \\
&\leq \|z(0)\|M_A + \frac{(\|\theta\|_\infty + \|g\|_\infty)\hat{M}_A}{\alpha\Gamma(\alpha)} b^\alpha,
\end{aligned} \tag{4.4}$$

where

$$M_A = \max_{s\in T}\|E_\alpha(-As^\alpha)\|, \quad \hat{M}_A = \max_{s\in T}\|E_{\alpha,\alpha}(-As^\alpha)\|.$$

Now, let us estimate the initial value $\|z(0)\|$. Let $t = b$ in Eq (4.2), and then it follows that

$$z(b) = E_\alpha(-Ab^\alpha)z(0) + \int_a^b (b-\tau)^{\alpha-1}E_{\alpha,\alpha}(-A(b-\tau)^\alpha)[f(\tau, z(\tau), z^{[2]}(\tau)) + g(\tau)]d\tau. \tag{4.5}$$

Invoking $z(0) = z(b)$ and the assumption (H_3), it is easy to see the determinant $|E - E_\alpha(-Ab^\alpha)| \neq 0$, where E represents the identity matrix. Thus, we have from (4.4)

$$z(0) = (E - E_\alpha(-Ab^\alpha))^{-1}\int_a^b (b-\tau)^{\alpha-1}E_{\alpha,\alpha}(-A(b-\tau)^\alpha)[f(\tau, z(\tau), z^{[2]}(\tau)) + g(\tau)]d\tau.$$

By means of hypothesis H_4(i) and similar to estimate (4.4), one can show

$$\|z(0)\| \leq \frac{M_E\hat{M}_A(\|\theta\|_\infty + \|g\|_\infty)b^\alpha}{\alpha\Gamma(\alpha)}, \tag{4.6}$$

where

$$M_E = \|(E - E_\alpha(-Ab^\alpha))^{-1}\|.$$

Substitute estimate (4.6) into estimate (4.4) to obtain

$$\|T_4(z(t))\| \leq \frac{(M_E M_A + 1)\hat{M}_A(\|\theta\|_\infty + \|g\|_\infty)}{\alpha\Gamma(\alpha)}b^\alpha, \quad \forall t \in T.$$

Let $M := \frac{(M_E M_A+1)\hat{M}_A(\|\theta\|_\infty+\|g\|_\infty)}{\alpha\Gamma(\alpha)}$. From $b > M^{\frac{1}{1-\alpha}}$, we therefore have

$$\|T_4(z(t))\| \leq Mb^\alpha < b, \ \forall t \in T. \tag{4.7}$$

Second, it will be shown that the existence of the solution is obtained. For this, we shall prove that the nonlinear operator T_4 is completely continuous. For any $z \in C(T; R^N)$, we claim $T_4(z(t)) \in C(T; R^N)$. For all $t, t+\delta \in [0, b]$ and $\delta > 0$, from (4.2) one can show

$$\begin{aligned}
&\|T_4(z(t+\delta)) - T_4(z(t))\| \\
&\leq |\frac{1}{\Gamma(\alpha)}\int_0^{t+\delta}(t+\delta-s)^{\alpha-1}E_{\alpha,\alpha}(-A(t+\delta-\tau)^\alpha)f(s, z(s), z^{[2]}(s))ds \\
&\quad - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-A(t+\tau)^\alpha)f(s, z(s), z^{[2]}(s))ds| \\
&\quad + |[E_\alpha(-A(t+\delta)^\alpha) - E_\alpha(-At^\alpha)]z(0)\| \\
&\quad + \|\int_0^{t+\delta}(t+\delta-s)^{\alpha-1}E_{\alpha,\alpha}(-A(t+\delta-s)^\alpha)g(s)ds \\
&\quad - \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-A(t-s)^\alpha)g(s)ds\| \\
&\leq \frac{(\|\theta\|_\infty + \|g\|_\infty)\hat{M}_A}{\Gamma(\alpha)}\left|\int_0^{t+\delta}(t+\delta-s)^{\alpha-1}ds + \int_0^t (t+\delta-s)^{\alpha-1} - (t-s)^{\alpha-1}ds\right| \\
&\quad + |[E_\alpha(-A(t+\delta)^\alpha) - E_\alpha(-At^\alpha)]z(0)|
\end{aligned}$$

$$\leq \frac{2(\|\theta\|_\infty + \|g\|_\infty)\hat{M}_A}{\alpha\Gamma(\alpha)}\delta^\alpha + \frac{2(\|\theta\|_\infty + \|g\|_\infty)\hat{M}_A}{\alpha\Gamma(\alpha)}|(t+\delta-a)^\alpha - (t-a)^\alpha|$$
$$+ |[E_\alpha(-A(t+\delta)^\alpha) - E_\alpha(-At^\alpha)]z(0)|. \tag{4.8}$$

As $\delta \to 0$, it is easy to get that

$$\|T_4(z(t+\delta)) - T_4(z(t))\| \to 0.$$

Therefore, $T_4(z(t)) \in C(T; R^N)$. Taking $z_n \to z$ in $C(T; R^N)$, we can deduce $|T_4(z_n) - T_4(z)| \to 0$, so T_4: $C(T; R^N) \to C(T; R^N)$ is continuous. According to the above prior estimate, by applying the Arzela-Ascoli theorem, it is easy to find that the operator T_4: $\Omega \to \Omega$ is completely continuous, where

$$\Omega := \{u \in C(T; R^N) : \|u\|_C \leq b + 1\}.$$

Thus, to get the existence of the solution, it suffices to show the fixed point problem of T_4. Define the mapping $H_\varepsilon(z) = z - \varepsilon T_4(z)$, where $\varepsilon \in [0, 1]$. Take $p \notin H(\partial\Omega)$, and then for any $\varepsilon \in [0, 1]$, we get

$$\begin{aligned} \deg(H_\varepsilon, \Omega, p) = \deg(H_1, \Omega, p) &= \deg(I - T_4, \Omega, p) \\ = \deg(H_0, \Omega, p) &= \deg(I, \Omega, p) \neq 0. \end{aligned}$$

So, T_4 has a fixed point on Ω, i.e., $z = T_4(z)$, which leads to the existence of the solution.

Finally, we will establish the uniqueness of the solution of differential iterative system (4.1). If $z_1, z_2 \in C(T; R^N)$ are the two solutions to problem (4.1) with the same initial data, and take the difference between the two solutions and take the inner product with $z_1 - z_2$ to get

$$\begin{aligned} &\langle z_1(t) - z_2(t), {}^C\mathcal{D}^\alpha(z_1(t) - z_2(t))\rangle + \langle z_1(t) - z_2(t), A(z_1(t) - z_2(t))\rangle \\ &= \langle z_1(t) - z_2(t), f(t, z_1(t), z_1^{[2]}(t)) - f(t, z_2(t), z_2^{[2]}(t))\rangle. \end{aligned}$$

Based on the assumptions (H_3) and (H_4)(ii), and using Lemma 2.2, we can derive

$$\begin{aligned} {}^C\mathcal{D}^\alpha\|z_1(t) - z_2(t)\|^2 &\leq 2\langle z_1(t) - z_2(t), {}^C\mathcal{D}^\alpha(z_1(t) - z_2(t))\rangle \\ &\leq 2\mu(t)\|z_1(t) - z_2(t)\|^2 - 2\xi\|z_1(t) - z_2(t)\|^2. \end{aligned} \tag{4.9}$$

Let $S(t) = \|z_1(t) - z_2(t)\|^2$, and inequality (4.9) can be simplified as

$${}^C\mathcal{D}^\alpha S(t) \leq 2(\mu(t) - \xi)S(t),$$

which, by applying Lemma 2.3, presents

$$S(t) \leq S(0)E_\alpha(2(\|\mu\|_\infty - \xi)t^\alpha), \quad \forall t \in [0, b]. \tag{4.10}$$

Take $t = b$ in (4.10) to obtain

$$S(b) \leq S(0)E_\alpha(2(\|\mu\|_\infty - \xi)b^\alpha). \tag{4.11}$$

Boundary condition $z(b) = z(0)$ shows $S(b) = S(0)$. Hence, we find from (4.11) that

$$S(0)\{1 - E_\alpha[2(\|\mu\|_\infty - \xi)b^\alpha]\} \leq 0.$$

From the monotonicity of Mittag-Leffler function $E_\alpha(t)$ ($\alpha \in (0,1)$) and $\|\mu\|_\infty < \xi$, it holds that

$$E_\alpha[(2(\|\mu\|_\infty - \xi)b^\alpha)] < 1.$$

Due to

$$S(0) = \|z_1(0) - z_2(0)\|^2 \geq 0,$$

one gets $S(0) = 0$, which with (4.11) implies $S(t) \equiv 0$. This means $z_1 \equiv z_2$, so the uniqueness of the solution follows. □

5. Integral boundary value solution of Caputo-fractional quadratic iterative equations in R^N

Consider the following integral boundary value problem of Caputo-fractional quadratic iterative differential equations:

$$\begin{aligned} &{}^C\mathcal{D}^\alpha z(t) + Az(t) = f(t, z(t), z^{[2]}(t)) + g(t), \quad t \in T := [0, b], \\ &z(0) = \frac{1}{\lambda b}\int_0^b z(s)ds, \end{aligned} \tag{5.1}$$

where $z^{[2]}(t) = (z_1(\|z\|), z_2(\|z\|), \cdots, z_n(\|z\|))$, $\lambda > 1$, linear operator $A : R^N \to R^N$ is positive definite satisfying (H_3), $f : T \times R^N \times R^N \to R^N$ is a Carathéodory function satisfying (H_4), and $g \in L^\infty(T)$. Throughout this section, we assume that $b > M^{\frac{1}{1-\alpha}}$ with

$$M := \frac{(\lambda - 1)(M_E M_A + 1)\hat{M}_A(\|\theta\|_\infty + \|g\|_\infty)}{\lambda\alpha\Gamma(\alpha)}.$$

Theorem 5.1. *If the assumptions* (H_3) *and* (H_4) *are fulfilled, then the fractional quadratic iterative differential system* (5.1) *admits a unique solution.*

Proof. Similar to Theorem 4.1, it is sufficient to investigate the following integral iterative equation:

$$z(t) = E_\alpha(-At^\alpha)z(0) + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-A(t-\tau)^\alpha)[f(\tau, z(\tau), z^{[2]}(\tau)) + g(\tau)]d\tau, \ \forall t \in T.$$

Due to $z(0) = \frac{1}{\lambda b}\int_0^b z(s)ds$, define an operator $O_1 : C(T; R^N) \to C(T; R^N)$ by

$$O_1 z(t) = \frac{E_\alpha(-At^\alpha)}{\lambda b}\int_0^b z(s)ds + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-A(t-\tau)^\alpha)[f(\tau, z(\tau), z^{[2]}(\tau)) + g(\tau)]d\tau.$$

Then, the integral boundary value problem of Eq (5.1) can be transformed into a fixed point problem:

$$z = O_1 z. \tag{5.2}$$

The proof process is divided into three steps.

Step 1. The a priori boundedness of the solutions for problem (5.1).

Let z be the solution of operator equation $z = O_1 z$. Apply the hypothesis (H_4)(i), to deduce

$$\begin{aligned}\|z(t)\| &\leq \frac{1}{\lambda}\max_{t\in T}\|z(t)\|\|E_\alpha(-At^\alpha)\| + \max_{t\in T}\|E_{\alpha,\alpha}(-At^\alpha)\|\int_0^t (t-s)^{\alpha-1}|f(s,z(s),z^{[2]}(s))|ds \\ &\quad +\max_{t\in T}\|E_{\alpha,\alpha}(-At^\alpha)\|\int_0^t (t-s)^{\alpha-1}|g(s)|ds \\ &\leq \frac{1}{\lambda}\|z\|_C M_A + (\|g\|_\infty + \|\theta\|_\infty)\hat{M}_A\int_0^t (t-s)^{\alpha-1}ds, \qquad (5.3)\end{aligned}$$

where

$$M_A = \max_{t\in T}\|E_\alpha(-At^\alpha)\|,\ \hat{M}_A = \max_{t\in T}\|E_{\alpha,\alpha}(-At^\alpha)\|$$

for any $t \in T$. Due to the monotonicity of Mittag-Leffler function $E_\alpha(-t)(t \geq 0)$ and because A is positive definite, it is easy to get $M_A < 1$. It follows from (5.3) that

$$\|z\|_C \leq \frac{\lambda}{\lambda-1}\frac{(\|g\|_\infty + \|\theta\|_\infty)\hat{M}_A}{\Gamma(\alpha)\alpha}b^\alpha. \qquad (5.4)$$

Notice

$$M = \frac{\lambda}{\lambda-1}\frac{(\|g\|_\infty + \|\theta\|_\infty)\hat{M}_A}{\Gamma(\alpha)\alpha}$$

and $b > M^{\frac{1}{1-\alpha}}$, and then (5.6) yields

$$\|z\|_C \leq b. \qquad (5.5)$$

Step 2. The existence of the solution for problem (5.1).

To begin with, we claim that $O_1 z \in C(T;R^N)$ for any $z \in C(T;R^N)$. For any $t, t+\delta \in T$, and $\delta > 0$, it follows from (5.2) that

$$\begin{aligned}&|O_1 z(t+\delta) - O_1 z(t)| \\ &\leq \|\int_0^{t+\delta}(t+\delta-s)^{\alpha-1}E_{\alpha,\alpha}(-A(t+\delta-s)^\alpha)f(s,z(s),z^{[2]}(s))ds \\ &\quad - \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-A(t-s)^\alpha)f(s,z(s),z^{[2]}(s))ds\| \\ &\quad + \|\int_0^{t+\delta}(t+\delta-s)^{\alpha-1}E_{\alpha,\alpha}(-A(t+\delta-s)^\alpha)g(s)ds \\ &\quad - \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-A(t-s)^\alpha)g(s)ds\| \\ &\quad + \|[E_\alpha(-A(t+\delta)^\alpha) - E_\alpha(-At^\alpha)]z(0)\| \\ &\leq (\|\theta\|_\infty + \|g\|_\infty)\hat{M}_A|\int_0^{t+\delta}(t+\delta-s)^{\alpha-1}ds + \int_0^t (t+\delta-s)^{\alpha-1} - (t-s)^{\alpha-1}ds| \\ &\quad + \|[E_\alpha(-A(t+\delta)^\alpha) - E_\alpha(-At^\alpha)]z(0)\| \\ &\leq \frac{2(\|\theta\|_\infty + \|g\|_\infty)\hat{M}_A}{\alpha}\delta^\alpha + \frac{2(\|\theta\|_\infty + \|g\|_\infty)\hat{M}_A}{\alpha}|(t+\delta-a)^\alpha - (t-a)^\alpha|\end{aligned}$$

$$+ \|[E_\alpha(-A(t+\delta)^\alpha) - E_\alpha(-At)^\alpha]z(0)\|.$$

As $\delta \to 0$, one has $|O_1 z(t+\delta) - O_1 z(t)| \to 0$, and therefore $O_1 z \in C(T; R^N)$. Taking $z_n \to z$ in $C(T; R^N)$, we have $z_{in}(\|z\|) \to z_i(\|z\|)$ for each $i = 1, 2, \cdots, n$, which together with the continuity of $(s, v) \to f(t, s, v)$, yields $|O_1 z_n - O_1 z| \to 0$. Thus, $O_1 : C(T; R^N) \to C(T; R^N)$ is continuous. By taking the prior estimation (Step 1) into account and applying the Arzela-Ascoli theorem, it holds that the operator $O_1 : \Omega \to \Omega$ is completely continuous, where

$$\Omega = \{z \in C(T; R^N) : \|z\|_C \leq b + 1\}.$$

Hence, the existence of solutions for the differential iterative system (5.1) can be transformed into a fixed point problem of O_1. Define the mapping $H_\varepsilon(z) = z - \varepsilon O_1(z)$ for $z \in C(T; R^N)$, with $\varepsilon \in [0, 1]$. Let $p \notin H_\varepsilon(\partial\Omega)$, for any $\varepsilon \in [0, 1]$, and this leads to

$$\deg(H_\varepsilon, \Omega, p) = \deg(H_1, \Omega, p) = \deg(I - O_1, \Omega, p) = \deg(H_0, \Omega, p) = \deg(I, \Omega, p) = 1 \neq 0.$$

Therefore, the operator O_1 has a fixed point on Ω, i.e., $z = O_1 z$, so the existence of the solution z for differential system (5.1) follows.

Step 3. The uniqueness of the solution for problem (5.1).

Let $z_1, z_2 \in C(T; R^N)$ be two solutions of problem (5.1). Substitute z_1 and z_2 into (5.1), respectively, and then take a difference and the inner product with $z_1 - z_2$ to get

$$\begin{aligned} &\langle z_1(t) - z_2(t), {}^C\mathcal{D}^\alpha(z_1(t) - z_2(t))\rangle + \langle z_1(t) - z_2(t), A(z_1(t) - z_2(t))\rangle \\ &\quad = \langle z_1(t) - z_2(t), f(t, z_1(t), z_1^{[2]}(t)) - f(t, z_2(t), z_2^{[2]}(t))\rangle. \end{aligned} \tag{5.6}$$

By means of the hypotheses (H_3) and (H_4)(ii), applying Lemma 2.2, it results that

$$\begin{aligned} D^\alpha\|z_1(t) - z_2(t)\|^2 &\leq 2\langle z_1(t) - z_2(t), D^\alpha(z_1(t) - z_2(t))\rangle \\ &\leq 2\mu(t)\|z_1(t) - z_2(t)\|^2 - 2\xi\|z_1(t) - z_2(t)\|^2. \end{aligned}$$

Let $Q(t) := \|z_1(t) - z_2(t)\|^2$ for brevity, and the above inequality can be simplified as

$${}^C\mathcal{D}^\alpha Q(t) \leq 2(\mu(t) - \xi)Q(t).$$

Invoke Lemma 2.3 to show

$$Q(t) \leq Q(0)E_\alpha(2(\|\mu\|_\infty - \xi)t^\alpha), \quad \forall t \in T. \tag{5.7}$$

In light of $z_1(0) - z_2(0) = \frac{1}{\lambda b}\int_0^b [z_1(t) - z_2(t)]dt$ in (5.7), one obtains

$$\|z_1(t) - z_2(t)\|_C^2 \leq \frac{1}{\lambda b}\int_0^b [z_1(t) - z_2(t)]^2 dt E_\alpha((2\|\mu\|_\infty - \xi)b^\alpha), \tag{5.8}$$

which leads to

$$\|z_1(t) - z_2(t)\|_C^2 \left(1 - \frac{E_\alpha((2\|\mu\|_\infty - \xi)t^\alpha)}{\lambda}\right) \leq 0, \tag{5.9}$$

with $\lambda > 1$. Since the Mittag-Leffler function $E_\alpha(-t)(t \geq 0)$ is monotonically decreasing, and $\|\mu\|_\infty < \xi$, we can conclude that $z_1 \equiv z_2$, so the iterative differential equation (5.1) has a unique solution, which yields our desired result. □

6. Control problem for a fractional iterative differential system with a disturbance

The aim of this section is to study the following Control problem for an iterative differential system with a disturbance:

$$\begin{cases} {}^C\mathcal{D}^\alpha z(t) + Az(t) = f(t, z(t), z^{[2]}(t)) + u(t) + d(t), \quad t \in T := [0, b], \\ u(t) \in U(t, x), \\ z(0) = \frac{1}{\lambda b}\int_0^b z(s)ds, \end{cases} \tag{6.1}$$

where A, f are shown as in (5.1), $u : T \to R^N$ is a control input, $U : T \times R^N \to 2^{R^N} \setminus \{\emptyset\}$ is a multifunction of observation value, and $d \in L^\infty(T)$ is a disturbance function. The hypothesis on U is presented as follows.

$H(U)$: $U : T \times R^N \to 2^{R^N} \setminus \{\emptyset\}$ is a multivalued function with closed, convex value such that

(i) $(t, y) \to U(t, y)$ is graph measurable for every $(t, y) \in T \times R^N$;

(ii) for almost all $t \in T$, $y \to U(t, y)$ has a closed graph;

(iii) for every $y \in R^N$ and all $t \in T$, there exists a function $\Phi \in L^\infty_+(T)$ such that

$$|U| = \sup\{\|u\|; u \in U\} \le \Phi(t).$$

In this section, we assume that $b > M^{\frac{1}{1-\alpha}}$ with

$$M := \frac{(\lambda - 1)(M_E M_A + 1)\hat{M}_A(\|\theta\|_\infty + \|\Phi\|_\infty + \|d\|_\infty)}{\lambda\alpha\Gamma(\alpha)}.$$

Theorem 6.1. *If the assumptions $(H_3), (H_4)$ and $H(U)$ hold, then the problem (6.1) admits at least one solution $z \in C(T; R^N)$.*

Proof. We first construct a closed convex subset $\mathcal{K}$ in $L^\infty(T; R^N)$ given by

$$\mathcal{K} := \left\{h \in L^\infty(T; R^N);\ \|h\|_\infty \le \|\Phi\|_\infty + \|d\|_\infty\right\}.$$

Due to Theorem 5.1, it is straightforward to derive that the iterative equation

$$\begin{cases} {}^C\mathcal{D}^\alpha z(t) - f(t, z(t), z^{[2]}(t)) + Az(t) = h(t), \quad t \in T, \\ z(0) = \frac{1}{\lambda b}\int_0^b z(s)ds, \end{cases} \tag{6.2}$$

admits a unique solution $z_h \in C(T; R^N)$ for every $h \in \mathcal{K}$. An operator

$$O_2 : D(O_2) \subset C(T; R^N) \to L^\infty(T; R^N),$$

is defined by

$$O_2 z = \mathcal{D}^\alpha z - f(t, z(t), z^{[2]}(t)) + Az, \quad z \in D(O_2), \tag{6.3}$$

where

$$D(O_2) := \{z \in C(T; R^N),\ z(0) = \frac{1}{\lambda b}\int_0^b z(s)ds\}.$$

Since $O_2 : D(O_2) \to \mathcal{K}(\subset L^\infty(T;R^N))$ is a one-to-one mapping, it results that $O_2^{-1} : \mathcal{K} \to D(O_2)$ exists. Now, we claim that the operator

$$O_2^{-1} : \mathcal{K} \to D(O_2),$$

is completely continuous. For this, we will claim that $O_2^{-1} : \mathcal{K} \to D(O_2)$ is continuous. Let $h_m \to h$ in $\mathcal{K}$ as $m \to \infty$, and it remains to prove that $z_m = O_2^{-1}(h_m) \to z = O_2^{-1}(h)$ in $D(O_2)(\subset C(T;R^N))$. Replacing z with z_m in (6.2) and subtracting (6.2) implies

$${}^C\mathcal{D}^\alpha(z_m(t)-z(t)) + A(z_m(t)-z(t)) = f(t,z_m(t),z_m^{[2]}(t)) - f(t,z(t),z^{[2]}(t)) + h_m(t) - h(t).$$

Then, take the inner product with $z_m - z$ on the above equation and apply Lemma 2.2 to obtain

$$\begin{aligned}\frac{1}{2}{}^C\mathcal{D}^\alpha\|z_m-z\|^2 &\le \langle z_m(t)-z(t), {}^C\mathcal{D}^\alpha(z_m(t)-z(t))\rangle \\ &\le \langle z_1(t)-z_2(t), f(t,z_1(t),z_1^{[2]}(t)) - f(t,z_2(t),z_2^{[2]}(t)) \\ &\quad -\langle z_m(t)-z(t), A(z_m(t)-z(t))\rangle \\ &\quad -\langle z_m(t)-z(t), h_m(t)-h(t)\rangle. \end{aligned} \tag{6.4}$$

Now, integrating in time and invoking Proposition 2.1 yields

$$\begin{aligned} &\frac{1}{2}\|z_m-z\|^2 \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\langle z_m(\tau)-z(\tau), f(t,z_m(\tau),z_m^{[2]}(\tau)) - f(t,z(\tau),z^{[2]}(\tau)\rangle d\tau \\ &\quad - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\langle z_m(\tau)-z(\tau), A(z_m(\tau)-z(\tau))\rangle d\tau \\ &\quad - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\langle z_m(\tau)-z(\tau), h_m(\tau)-h(\tau)\rangle d\tau \\ &\quad + \frac{1}{2}\|z_m(0)-z(0)\|^2. \end{aligned} \tag{6.5}$$

Similar to the a priori estimate of the solution of Theorem 5.1, it is easy to verify that $\|z_m\|_C \le b$, which together with $z_m \in C(T;R^N)$ and the Arzela-Ascoli theorem reveals that there exists a subsequence z_m (still denoted by itself) such that $z_m \to \hat{z}$ in $D(O_2)$ as $m \to \infty$. By passing the limit in (6.5), this enables us to obtain

$$\begin{aligned}\frac{1}{2}\|\hat{z}-z\|^2 &\le \|\hat{z}(0)-z(0)\|^2 - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\langle \hat{z}(\tau)-z(\tau), A(\hat{z}(\tau)-z(\tau))\rangle d\tau \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\langle \hat{z}(\tau)-z(\tau), f(\tau,\hat{z}(\tau),\hat{z}^{[2]}(\tau)) - f(\tau,z(\tau),z^{[2]}(\tau))\rangle d\tau. \end{aligned} \tag{6.6}$$

Analogous to the analysis of (5.6), set $Y = \|\hat{z}-z\|^2$, and then we can conclude that $Y(t) \equiv 0$, i.e., $\hat{z} \equiv z$ in $D(O_2)$. This implies the continuity of operator O_2^{-1}. In view of the a priori estimate of the solution, it is easy to deduce that $O_2^{-1}(\mathcal{K})$ is a bounded set in $C(T;R^N)$. Invoking the Arzela-Ascoli theorem, $O_2^{-1}(\mathcal{K}) \subset L^\infty(T;R^N)$ is relatively compact. As a result, $O_2^{-1} : \mathcal{K} \to L^\infty(T;R^N)$ is completely continuous.

Now, a multivalued Nemitsky operator $\mathcal{N} : L^\infty(T; R^N) \to 2^{L^\infty(T;R^N)}$ corresponding to $U(t, z)$ is defined by

$$\mathcal{N}(z) = \{u \in L^\infty(T; R^N);\ u(t) \in U(t, z),\ a.e.\ t \in T\}.$$

From hypothesis $H(U)$, it holds that the multivalued Nemitsky operator $\mathcal{N}(\cdot)$ is nonempty, closed convex value and upper hemicontinuous (Theorem 3.2, [41]). Thus, we can find that $O_2^{-1} \circ \mathcal{N} : \mathcal{K} \to L^\infty(T; R^N)$ is an upper hemicontinuous multifunction with closed, convex value, which maps a bounded set into a relatively compact set. Then, the control problem for a iterative differential system (6.1) is turned into the following fixed points problem:

$$z \in O_2^{-1} \circ \mathcal{N}(z). \tag{6.7}$$

For this aim, via Lemma 2.4, it remains to show that the set

$$\Xi := \{z \in L^\infty(T; R^N) : z \in \epsilon O_2^{-1} \circ \mathcal{N}(z),\ \epsilon \in (0, 1)\}$$

is bounded. Let $z \in \Xi$, and then $O_2(\frac{z}{\epsilon}) \in \mathcal{N}(z)$, which gives

$$^C\mathcal{D}^\alpha \frac{z}{\epsilon} - f(t, \frac{z(t)}{\epsilon}, \frac{z_\epsilon^{[2]}(t)}{\epsilon}) + \frac{Az}{\epsilon} = h(t) + d(t), \tag{6.8}$$

where $h(t) \in U(z, t)$ for all $t \in T$. Similar to (5.2), Eq (6.8) can be rewritten as

$$\begin{aligned} z(t) \ = \ & E_\alpha(-At^\alpha)z(0) + \epsilon \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-A(t-\tau)^\alpha) f(\tau, \frac{z(\tau)}{\epsilon}, \frac{z_\epsilon^{[2]}(\tau)}{\epsilon}) d\tau \\ & + \epsilon \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-A(t-\tau)^\alpha)(h(\tau) + d(\tau)) d\tau. \end{aligned} \tag{6.9}$$

Likewise as in (5.3) and by taking (H_4)(ii) into account, it results from (6.9) that

$$\begin{aligned} \|z\|_C \ \leq \ & \|z(0)\| M_A + (\|\theta\|_\infty + \|\Phi\|_\infty + \|d\|_\infty)\hat{M}_A \max_{t \in I} \int_0^t (t-s)^{\alpha-1} ds \\ \leq \ & \|z(0)\| M_A + \frac{(\|\theta\|_\infty + \|\Phi\|_\infty + \|d\|_\infty)\hat{M}_A}{\alpha} b^\alpha. \end{aligned} \tag{6.10}$$

Similar to the estimate (5.3), it holds that $\|z(t)\|$ is uniformly bounded for any $t \in T$. Thanks to Lemma 2.4, there exists $z \in D(O_2)$, such that $z \in O_2^{-1} \circ \mathcal{N}(z)$. It is obvious that z is the solution of problem (5.1). This proof is thus complete. □

7. Integral boundary value problem of a fractional iterative neural network system

Consider the fractional iterative neural network system described as follows:

$$\begin{aligned} & D^\alpha z(t) + Bz(t) = \mathcal{F}(z(t), z^{[2]}(t)) + I(t),\ t \in T = [0, b], \\ & z(0) = \frac{1}{\lambda b} \int_0^b z(s) ds, \end{aligned} \tag{7.1}$$

where $B = \mathrm{diag}(d_1, d_2, \cdots, d_N)$ is a diagonal matrix with $d_i > 0$ $(i = 1, 2, \cdots, N)$, $z : T \to R^N$ denotes the vector element of the neuron system, $I \in L^\infty(T; R^N)$ is the mapping of neuron inputs, and $\mathcal{F} : R^N \times R^N \to R^N$ stands for the neuron input-output continuous activation function such that

(i) For any $x, y \in R^N$, there exists a function $\widehat{\omega} \in L_+^\infty(T)$ such that $\|\mathcal{F}(x, y)\| \le \widehat{\omega}(t)$, $\forall t \in T$;

(ii) For any $x_1, x_2, y_1, y_2 \in R^N$, there exists a function $\widehat{\mu} \in L_+^\infty(T)$ such that

$$\langle \mathcal{F}(x_1, y_1) - \mathcal{F}(x_2, y_2), x_1 - x_2 \rangle \le \widehat{\mu}(t)\|x_1 - x_2\|^2,$$

where $\|\widehat{\mu}\|_\infty < \min\{d_i : i = 1, 2, \cdots, N\}$, for almost all $t \in T$.

The initial value problem of the differential system (7.1) without iteration was discussed in [35], where the existence and uniqueness of the solution were demonstrated. Here, however, considering the iterative term and the integral boundary value condition, the existence of a unique integral boundary value solution to system (7.1) is guaranteed by using our results. Let $b > M^{\frac{1}{1-\alpha}}$ with

$$M := \frac{(M_E M_A + 1)\hat{M}_A(\|\widehat{\omega}\|_\infty + \|I\|_\infty)}{\alpha\Gamma(\alpha)}.$$

It is easy to show that all assumptions of Theorem 5.1 hold, and then the existence result for system (7.1) is provided as follows.

Theorem 7.1. *If the above assumptions are fulfilled, problem (7.1) admits a unique integral boundary value solution.*

It should be noted that the fractional system (7.1) without iteration was considered in Song et al. [35], where input function $I(t)$ is assumed to be continuous. Automatically, a question is whether the fractional iteration system (7.1) has a solution, provided that $I(t) = (I_1(t), \cdots, I_N(t))$ is discontinuous. The following work is to deal with this question. For this, it is assumed that $I_i \in \Lambda$ $(i = 1, 2, \cdots, N)$ are nondecreasing monotone, bounded mappings, where $\Lambda : \mathbb{R} \to \mathbb{R}$ denotes the class of functions which have a finite number of jumping discontinuities in the closed interval. If only isolated jump discontinuities for any I_i $(i = 1, 2, \cdots, N)$ appear, then we can deduce

$$\mathcal{Y}(I(t)) := ([\underline{I_1}, \overline{I_1}], [\underline{I_2}, \overline{I_2}], \cdots, [\underline{I_N}, \overline{I_N}])$$

with

$$\underline{I_i} \le I_i \le \overline{I_i}, \underline{I_i} = \underline{\lim}_{\varepsilon \to t_i} I_i(\varepsilon), \overline{I_i} = \overline{\lim}_{\varepsilon \to t_i} I_i(\varepsilon)(i = 1, 2, \cdots, N).$$

Hence, in this case, the iteration problem (7.1) can be rewritten as the following iteration differential inclusion:

$$D^\alpha z(t) + Bz(t) \in \mathcal{F}(z(t), z^{[2]}(t)) + \mathcal{Y}(I(t)). \tag{7.2}$$

Here, $\mathcal{Y}(I(t))$ can be handled as a multivalued control item of problem (6.1). Then, the argument is similar to that of Theorem 6.1, and we can conclude the following result.

Theorem 7.2. *If the given assumptions are satisfied, then the solution set of iteration differential system (7.2) is nonempty.*

8. Example

As an application of the previous results, we introduce an example. Let us consider the boundary value problem of the following iterative differential equation:

$$\begin{cases} {}^{C}\mathcal{D}^{\frac{3}{2}} z(t) = \cos(t)[\frac{1}{\sqrt{1+z^2(t)}} + \sin(z^{[2]}(t))], \quad t \in T := [0, b], \\ z(0) = 0, \ z(b) = b. \end{cases} \tag{8.1}$$

From (3.1), it is easy to see that $\alpha = \frac{3}{2}$ and

$$f(t, z(t), z^{[2]}(t)) = \cos(t)[\frac{1}{\sqrt{1 + z^2(t)}} + \sin(z^{[2]}(t))].$$

Then we can directly calculate $-2 \leq f(t, s, w) \leq 2$ for any $(t, s, w) \in T \times \mathbb{R} \times \mathbb{R}$, so the condition (H_2)(i) holds. For any $(s_1, w_1), (s_2, w_2) \in \mathbb{R} \times \mathbb{R}$ and $t \in T$, it is easy to deduce that

$$\begin{aligned} |f(t, s_1, w_1) - |f(t, s_2, w_2)| &\leq |\cos(t)| \left[|\frac{1}{\sqrt{1 + s_1^2}} - \frac{1}{\sqrt{1 + s_2^2}}| + |\sin w_1 - \sin w_2| \right] \\ &\leq |s_1 - s_2| + |w_1 - w_2|, \end{aligned}$$

which yields

$$\begin{aligned} [f(t, s_1, w_1) - |f(t, s_2, w_2)][s_1 - s_2] &\leq |s_1 - s_2|^2 + |w_1 - w_2||s_1 - s_2| \\ &\leq \frac{3}{2}|s_1 - s_2| + \frac{1}{2}|w_1 - w_2|. \end{aligned}$$

This implies that the condition (H_2)(ii) holds. Assume $b < (\frac{3\Gamma(\frac{3}{2})}{10})^2$, and then apply Theorem 3.2 to conclude that Eq (8.1) admits a unique solution which is monotonically increasing continuous.

9. Conclusions

In this article, we established the existence and uniqueness of solutions for nonlinear quadratic iterative equations in the sense of the Caputo fractional derivative with several boundary conditions by using several fixed point theorems, and we finally applied it to a neural network iterative system and a control problem of a nonlinear iteration system with a disturbance. Furthermore, when the fractional order of the differential system is greater than 1, it was shown for the first time that the solution for a nonlinear quadratic iterative equation is monotonic. Since different definitions of the fractional integral lead to different fractional order differential systems, the method applied in this article can also be applied to systems of many other fractional order iterative differential equations.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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