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Research article

Destruction of solutions for class of wave $p(x)-$bi-Laplace equation with nonlinear dissipation

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Abstract: An initial value problem is considered for the nonlinear dissipative wave equation containing the $p(x)$-bi-Laplacian operator. For this problem, sufficient conditions for the blow-up with nonpositive initial energy of a generalized solution are obtained in finite time where a wide variety of techniques are used.

Keywords: Sobolev spaces with variable exponents; $p(x)$-bi-Laplace; blow-up; initial energy
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1. Introduction and formulation of the problem

Let Ω is a bounded open set in $\mathbb{R}^n$ with a sufficiently smooth boundary $\partial\Omega$ and outward facing unit normal n, let $u(x,t) = u$. The purpose of this study is to obtain sufficient conditions to prove the global nonexistence result for initial boundary value problem of wave equation containing the $p(x)$-bi-Laplacian operator

$$\begin{cases} \partial_{tt}u + \Delta_x(div(|\Delta_x u|^{p(x)-2}\nabla_x u)) + \mu|\partial_t u|^{m-2}\partial_t u = b|u|^{r-2}u, & x \in \Omega, t > 0 \\ u = \Delta_x u = 0, & x \in \partial\Omega, t > 0 \\ u = u_0(x) \in \mathcal{V}(\Omega), \quad \partial_t u = u_1(x) \in L^{p(x)}(\Omega), & x \in \Omega, t = 0, \end{cases} \tag{1.1}$$

where μ, b are positive constants, the spaces $\mathcal{V}(\Omega)$ and $L^{p(x)}(\Omega)$ are defined in Definition 1 and (2.1).

This problem is a mathematical model of wave processes in mathematical physics, taking into account dissipation and the relationship between the different parameters. Recently, new strongly nonlinear dissipative wave equations of the hyperbolic type have been intensively considered in mathematical physics. It should be mentioned that many authors have studied the question of existence, uniqueness, regularity and blow-up of weak solutions for parabolic and elliptic equations involving the $p(x)$-Laplacian view of its applications in the fields of nonlinear elasticity, fluid dynamics, elastic mechanics etc, see [4, 6, 8, 12, 13, 15–17, 20, 21] and the references therein.

In [2], the author established the existence of weak solutions for $p(x,t)$-Laplacian equation with damping term

$$\partial_{tt}u = div(a(x,t)|\nabla_x u|^{p(x,t)-2}\nabla_x u) + \alpha\Delta_x u + b(x,t)u|u|^{\sigma(x,t)-2} + f(x,t),$$

and proved the blow-up of weak solutions with negative initial energy, where α is a nonnegative constant and a, b, p, σ are given functions. Such equations are usually referred as equations with nonstandard growth conditions. It is proved the blow-up result of weak solutions with negative initial energy as well as for certain solutions with positive initial energy to the following equation

$$\partial_{tt}u - div(|\nabla_x u|^{r(.)-2}\nabla_x u) + a\partial_t u|\partial_t u|^{m(.)-2} = bu|u|^{p(.)-2},$$

In particular case $p(x) = 2$, the problem (1.1) is reduced to the Petrovsky type equation

$$\begin{cases} \partial_{tt}u + \Delta_x^2 u + \mu|\partial_t u|^{m-2}\partial_t u = b|u|^{r-2}u \\ u = \frac{\partial u}{\partial n} = 0 \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x). \end{cases}$$

It is studied where, the author established an existence result and proved that the solution continues to exist globally if $m \geq r$ and blows up in finite time if $m < r$ and the initial energy is negative. Motivated by the above work, we obtain the blow-up results of solution to problem (1.1) for nonpositive initial energy. In order to state our result, we use some ideas introduced in the work of [7, 11, 14].

2. Main results

In this section, we recall some definitions and basic properties about the generalized Sobolev and Lebesgue spaces with variable exponents. The reader is referred to [3, 5, 9, 10] for more detailes.
Denote

$$C_+(\bar{\Omega}) = \{p(x) : p(x) \in C(\bar{\Omega}), p(x) > 2, for\ all\ x \in \bar{\Omega}\},$$

and

$$p^- = \operatorname{ess}\inf_{x\in\bar{\Omega}} p(x),\ \ p^+ = \operatorname{ess}\sup_{x\in\bar{\Omega}} p(x).$$

Then, the mesurable function

$$p : \bar{\Omega} \to [p^-, p^+] \subset (2, \infty),$$

satisfies the log-Hölder continuity condition

$$|p(x) - p(y)| \leq \frac{C}{\ln(e + |x-y|^{-1})} \quad , for\ all\ x, y \in \Omega.$$

For some $\lambda > 0$ the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined as the set of mesurable functions $u : \Omega \to \mathbb{R}$ such that $\mathcal{P}_{p(.)}(\lambda u) < \infty$ with respect to the Luxemburg norm

$$\|u\|_{p(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx < \infty\right\}, \tag{2.1}$$

where

$$\mathcal{P}_{p(.)}(u) = \int_\Omega |u(x)|^{p(x)} dx, \quad \|u\|_{p(x)} := \|u\|_{L^{p(x)}(\Omega)}.$$

The space $(L^{p(x)}(\Omega), \|.\|_{p(x)})$ is separable, uniformly convex, reflexive and its dual space is $L^{q(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1, for\ all\ x \in \Omega$.
Morever if $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ then $uv \in L^{s(x)}(\Omega)$ and we have the generalised Hölder's type inequality

$$\|uv\|_{s(.)} \leq 2\|u\|_{p(.)}.\|v\|_{q(.)}, \quad \frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}.$$

Lemma 1. *If p is a mesurable function on Ω then for any $u \in L^{p(x)}(\Omega)$ we have*

$$\min\left(\|u\|_{p(x)}^{p-}, \|u\|_{p(x)}^{p+}\right) \leq \mathcal{P}_{p(.)}(u) \leq \max\left(\|u\|_{p(x)}^{p-}, \|u\|_{p(x)}^{p+}\right).$$

For any nonnegative integer k the variable exponent Sobolev space is defined

$$W^{k,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : |\alpha| \leq K \Longrightarrow D^\alpha u \in L^{p(x)}(\Omega)\right\},$$

endowed with the norm

$$\|u\|_{W^{k,p(x)}} = \sum_{|\alpha| \leq k} |D^\alpha u|_{L^{p(x)}(\Omega)}.$$

Then $W^{k,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{k,p(x)}}$. In this way $L^{p(x)}(\Omega)$, $W^{k,p(x)}(\Omega)$ and $W_0^{k,p(x)}(\Omega)$ are separable and reflexive Banach spaces.
We shall frequently use the generalized Poincaré's inequality in $W_0^{1,p(x)}(\Omega)$ given by

$$\exists C > 0, \|u\|_{p(x)} \leq C\|\nabla_x u\|_{p(x)}, \quad , for\ all\ u \in W_0^{1,p(x)}(\Omega).$$

Definition 1. *We define the function space of our problem and its norm as follows*

$$\mathcal{V}(\Omega) = \left\{u | u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega), |\Delta_x u| \in W_0^{1,p(x)}(\Omega)\right\},$$

with the norm

$$\|u\|_{\mathcal{V}(\Omega)} = \|u\|_{W_0^{1,p(x)}(\Omega)} + \|u\|_{W^{2,p(x)}(\Omega)} + \|\Delta_x u\|_{W_0^{1,p(x)}(\Omega)}.$$

Lemma 2. *[18, Theorem 4.4] Let Ω is a bounded domain with Lipschitz boundary. In the space $X = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ the norm $\|.\|_X$ and $\|\Delta_x.\|_{L^{p(x)}(\Omega)}$ are equivalent norms.*

Lemma 3. *[1, Theorem 5.4] Let Ω be a domain in $\mathbb{R}^n$ that has the cone property then for $n > p$ and $p \leq q \leq \frac{np}{n-p}$ there exist the following imbeddings*

$$W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \hookrightarrow w_0^{1,q}(\Omega) \hookrightarrow L^q(\Omega). \tag{2.2}$$

Lemma 4. *[19, Lemma 2.1] Assume that $L(t)$ is is a twice continuously differentiable function satisfying*

$$\begin{cases} L''(t) + L'(t) \geq C_0(t+\theta)^{\beta} L^{1+\alpha}(t), & t > 0 \\ L(0) > 0, L'(0) \geq 0, \end{cases}$$

where $C_0, \theta > 0$, $-1 < \beta \leq 0$, $\alpha > 0$ are constants. Then $L(t)$ blows up in finite time.

3. Blow up result

Theorem 1. *Let u be an energy weak solution to problem (1.1). Suppose that*

$$2 \leq m \leq r \quad \text{and} \quad 2 \leq p(x) \leq \frac{2n}{n-2}.$$

Assume further that

$$E(0) = \frac{1}{2}\int_{\Omega} |u_1|^2 dx + \int_{\Omega} \frac{1}{p(x)} |\Delta_x u_0|^{p(x)} dx - \frac{b}{r}\int_{\Omega} |u_0|^r dx \leq 0,$$

and

$$\int_{\Omega} u_0 u_1 dx \geq 0, \tag{3.1}$$

then the solution u blows up on the finite interval $(0, t_{max})$.

Proof. Multiplying Eq (1.1) by $\partial_t u$, and integration by parts over Ω, one has

$$\begin{aligned} &\partial_t \int_{\Omega} \frac{1}{2} |\partial_t u|^2 dx - \int_{\Omega} div(\Delta_x \partial_t u \nabla_x u) |\Delta_x u|^{p(x)-2} dx \\ &+ \partial_t \int_{\Omega} \frac{1}{p(x)} |\Delta_x u|^{p(x)} dx + \mu \int_{\Omega} |\partial_t u|^m dx = \partial_t \int_{\Omega} \frac{b}{r} |u|^r dx. \end{aligned}$$

So the corresponding energy of solution to (1.1) is defined by

$$E(t) = \frac{1}{2}\int_{\Omega} |\partial_t u|^2 dx + \int_{\Omega} \frac{1}{p(x)} |\Delta_x u|^{p(x)} dx - \frac{b}{r}\int_{\Omega} |u|^r dx. \tag{3.2}$$

In addition

$$\partial_t E(t) = \int_{\Omega} div(\Delta_x \partial_t u \nabla_x u) |\Delta_x u|^{p(x)-2} dx - \mu \int_{\Omega} |\partial_t u|^m dx. \tag{3.3}$$

Which gives in turn the following energy identity

$$E(t) + \mu \int_0^t \int_{\Omega} |\partial_t u|^m dx ds = E(0) + \int_0^t \int_{\Omega} div(\Delta_x \partial_t u \nabla_x u) |\Delta_x u|^{p(x)-2} dx ds. \tag{3.4}$$

We define the sets

$$\Omega_- = \{x \in \Omega : |\Delta_x u| < 1\},$$

and

$$\Omega_+ = \{x \in \Omega : |\Delta_x u| \geq 1\}.$$

So by applying Hölder and Young inequality we arrive at

$$\begin{aligned}
&\left|\int_\Omega div(\Delta_x \partial_t u \nabla_x u)|\Delta_x u|^{p(x)-2} dx\right| \\
&= \left|\int_\Omega \nabla_x(\Delta_x \partial_t u)\nabla_x u |\Delta_x u|^{p(x)-2} dx + \int_\Omega \Delta_x \partial_t u \Delta_x u |\Delta_x u|^{p(x)-2} dx\right| \\
&\leq \|\nabla_x(\Delta_x \partial_t u)\|_2 . \|\nabla_x u\|_{\frac{2p^-}{4-p^-}} . \|\Delta_x u\|_{p^-}^{p^- -2} \\
&+ \|\nabla_x(\Delta_x \partial_t u)\|_2 . \|\nabla_x u\|_{\frac{2p^+}{4-p^+}} . \|\Delta_x u\|_{p^+}^{p^+ -2} \\
&+ \frac{1}{p^-}\|\Delta_x \partial_t u\|_{p^-}^{p^-} + \frac{p^- - 1}{p^-}\|\Delta_x u\|_{p^-}^{p^-} + \frac{1}{p^+}\|\Delta_x \partial_t u\|_{p^+}^{p^+} + \frac{p^+ - 1}{p^-}\|\Delta_x u\|_{p^+}^{p^+}.
\end{aligned}$$

Clearly since $2 \leq p^- \leq p(x) \leq p^+ \leq \frac{2n}{n-2}$ then by exploiting lemma 3, we have

$$\begin{aligned}
&\left|\int_\Omega div(\Delta_x \partial_t u \nabla_x u)|\Delta_x u|^{p(x)-2} dx\right| \\
&\leq C_0\|\nabla_x(\Delta_x \partial_t u)\|_{p^-} . \|\Delta_x u\|_{p^-}^{p^- -1} \\
&+ C_1\|\nabla_x(\Delta_x \partial_t u)\|_{p^+} . \|\Delta_x u\|_{p^+}^{p^+ -1} \\
&+ \frac{1}{p^-}\|\Delta_x \partial_t u\|_{p^-}^{p^-} + \frac{p^- - 1}{p^-}\|\Delta_x u\|_{p^-}^{p^-} \\
&+ \frac{1}{p^+}\|\Delta_x \partial_t u\|_{p^+}^{p^+} + \frac{p^+ - 1}{p^-}\|\Delta_x u\|_{p^+}^{p^+}.
\end{aligned}$$

Because $\partial_t u$ is regular and by Young inequality we obtain

$$\begin{aligned}
\left|\int_\Omega div(\Delta_x \partial_t u \nabla_x u)|\Delta_x u|^{p(x)-2} dx\right| &\leq k_0\left(\|\nabla_x(\Delta_x \partial_t u)\|_{p^-}^{p^-} + \|\nabla_x(\Delta_x u)\|_{p^-}^{p^-}\right) \\
&+ k_1\left(\|\nabla_x(\Delta_x \partial_t u)\|_{p^+}^{p^+} + \|\nabla_x(\Delta_x u)\|_{p^+}^{p^+}\right).
\end{aligned}$$

At this step we will assume that

$$\sup_{0\leq t\leq t_{max}}\left(\|\nabla_x(\Delta_x \partial_t u)\|_{p^-}^{p^-} + \|\nabla_x(\Delta_x u)\|_{p^-}^{p^-} + \|\nabla_x(\Delta_x \partial_t u)\|_{p^+}^{p^+} + \|\nabla_x(\Delta_x u)\|_{p^+}^{p^+}\right) \leq \frac{|E(0)|}{kt_{max}}, \tag{3.5}$$

where $k = \max(k_0, k_1)$. We notice that estimate (3.5) will be important to prove the blow-up result. Therfore

$$\left|\int_0^t \int_\Omega div(\Delta_x \partial_t u \nabla_x u)|\Delta_x u|^{p(x)-2} dxds\right| \leq |E(0)|, \quad 0 \leq t \leq t_{max}.$$

Consequently by virtue of (3.4) we derive the following estimate for the energy functional

$$E(t) + \mu \int_0^t \int_\Omega |\partial_t u|^m dxds \leq E(0) + |E(0)|. \tag{3.6}$$

Suppose that $E(0) \leq 0$ then it follows from (3.6) that $E(t) \leq 0$. Define the auxiliary function $L(t)$ by the following formula

$$L(t) = \frac{1}{2}\int_{\Omega} |u(x,t)|^2 dx + N\int_0^t H(s)ds + N(t + t_{max}), \tag{3.7}$$

where $N > 0$ is to be specified later and $H(t)$ is given by

$$H(t) = \alpha|E(0)|t - E(t), \quad \theta \geq \frac{1}{kt_{max}}. \tag{3.8}$$

We differentiate (3.8) and use the Eq (3.4) to arrive at

$$\partial_t H(t) = \mu\int_0^t \|\partial_t u\|_m^m - \int_0^t\int_{\Omega} div(\Delta_x\partial_t u\nabla_x u)|\Delta_x u|^{p(x)-2}dx - (1+\theta t)E(0). \tag{3.9}$$

Therfore

$$\partial_t H(t) \geq \|\partial_t u\|_m^m + \left(\frac{1}{kt_{max}} - \theta\right)E(0). \tag{3.10}$$

From (3.8) we see that H is a nondecreasing function and

$$H(0) = -E(0) > 0.$$

Differentiating (3.7) twice leads to

$$\begin{aligned} L'(t) &= \int_{\Omega} u\partial_t u dx + NH(t) + N \\ L''(t) &= \int_{\Omega} u\partial_{tt} u dx + \int_{\Omega} |\partial_t u|^2 dx + N\partial_t H(t). \end{aligned} \tag{3.11}$$

It's clear from (3.1) and (3.11) that

$$L(0) > 0, \quad \partial_t L(0) > 0.$$

Now, by using Young's inequality we have

$$\int_{\Omega} |\Delta_x u|^{p(x)-2}|\nabla_x u||\nabla_x(\Delta_x u)|dx \leq C\left(\|\nabla_x(\Delta_x u)\|_{p^-}^{p^-} + \|\nabla_x(\Delta_x u)\|_{p^+}^{p^+}\right).$$

Again Young's inequality yields

$$\int_{\Omega} u\partial_t u|\partial_t u|^{m-2}dx \leq \frac{\beta^m}{m}\|u\|_m^m + \frac{m-1}{m}\beta^{-m/m-1}\|\partial_t u\|_m^m, \tag{3.12}$$

where β in an arbitrary nonnegative constant to be specified later. By combining (3.3) and (3.5) we get

$$\begin{aligned} \mu\|\partial_t u\|_m^m &= -\partial_t E(t) - \int_{\Omega} div(\Delta_x\partial_t u\nabla_x u)|\Delta_x u|^{p(x)-2}dx \\ &\leq -\partial_t E(t) - \frac{E(0)}{t_{max}} \\ &\leq \partial_t H(t) + \alpha E(0) + \frac{H(0)}{t_{\max}} \\ &\leq \partial_t H(t) + \frac{H(t)}{t_{\max}}. \end{aligned} \tag{3.13}$$

Inserting (3.13) into (3.12) leads to

$$\int_\Omega u\partial_t u|\partial_t u|^{m-2}dx \leq \frac{\beta^m}{m}\|u\|_m^m + \frac{m-1}{m}\beta^{-m/m-1}\left(\partial_t H(t) + \frac{H(t)}{t_{\max}}\right). \tag{3.14}$$

By virtue of (3.5) we have

$$-\left(\|\nabla_x(\Delta_x u)\|_{L^{p^-}(\Omega)}^{p^-} + \|\nabla_x(\Delta_x u)\|_{L^{p^+}(\Omega)}^{p^+}\right) \geq \frac{E(0)}{kt_{max}} \geq -\frac{H(t)}{kt_{max}}. \tag{3.15}$$

We define the sets

$$\Omega_- = \{x \in \Omega : |u| < 1\},$$

and

$$\Omega_+ = \{x \in \Omega : |u| \geq 1\}.$$

So

$$\int_\Omega |u|^m dx = \int_{\Omega_-} |u|^m dx + \int_{\Omega_+} |u|^m dx \leq \int_{\Omega_-} |u|^2 dx + \int_{\Omega_+} |u|^r dx. \tag{3.16}$$

We first note that

$$\int_\Omega |u|^2 dx \leq C_0 \int_\Omega \left(|u|^{\frac{2p^+}{4-p^+}} dx\right)^{\frac{4-p^+}{p^+}} \leq C_1\left(1 + \|\nabla_x \Delta_x u\|_{L^{p^+}(\Omega)}^{p^+}\right).$$

Therfore from (3.15) we have

$$\begin{aligned}\int_\Omega |u|^m dx &\leq \Delta_x\left(1 + \|\nabla_x\Delta_x u\|_{p^-}^{p^-} + \|\nabla_x\Delta_x u\|_{p^+}^{p^+} + \|u\|_r^r\right)\\ &\leq \Delta_x\left(1 + \frac{H(t)}{kt_{max}} + \|u\|_r^r\right).\end{aligned} \tag{3.17}$$

Consequently

$$\begin{aligned}L''(t) + L'(t) = &\int_\Omega u\Delta_x(div(|\Delta_x u|^{p(x)-2}\nabla_x u))dx - \mu|\partial_t u|^{m-2}\partial_t uu + b|u|^r dx\\ &+ \|\partial_t u\|_2^2 + \int_\Omega u\partial_t u dx + NH(t) + N\partial_t H(t) + N\\ &\geq -C\left(\|\nabla_x(\Delta_x u)\|_{p^-}^{p^-} + \|\nabla_x(\Delta_x u)\|_{p^+}^{p^+}\right)\\ &- \mu\left(\frac{\beta^m}{m}\|u\|_m^m + \frac{m-1}{m}\beta^{-m/m-1}\|\partial_t u\|_m^m\right) + b\|u\|_r^r\\ &+ \|\partial_t u\|_2^2 + \int_\Omega u\partial_t u dx + NH(t) + N\partial_t H(t) + N.\end{aligned} \tag{3.18}$$

Combination of (3.15) and (3.2) leads to

$$\begin{aligned}\int_\Omega u\partial_t u dx &\leq \frac{1}{2}\|\partial_t u\|_2^2 + \sigma\left(1 + \|\nabla_x\Delta_x u\|_{p^-}^{p^-}\right)\\ &\leq \frac{1}{2}\|\partial_t u\|_2^2 + \sigma\left(1 + \frac{H(t)}{kt_{max}}\right).\end{aligned} \tag{3.19}$$

Substituting (3.14), (3.17) and (3.19) into (3.18) we obtain

$$\begin{aligned} L''(t) + L'(t) \geq & \left(N - \frac{C}{kt_{max}} - \mu\beta^{-m/m-1}\frac{m-1}{mt_{max}} - \mu\Delta_x\frac{\beta^m}{mkt_{max}} - \frac{\sigma}{kt_{max}}\right)H(t) \\ & + \frac{1}{2}\|\partial_t u\|_2^2 + \left(N - \mu\frac{m-1}{m}\beta^{-m/m-1}\right)H'(t) \\ & + \left(b - \mu\frac{\beta^m}{m}\Delta_x\right)\|u\|_r^r + N - \mu\Delta_x\frac{\beta^m}{m} - \sigma. \end{aligned} \tag{3.20}$$

Now we pick β so small that

$$b - \mu\frac{\beta^m}{m}\Delta_x > 0. \tag{3.21}$$

Once β is chosen we select N large enough that

$$\begin{aligned} N - \frac{C}{kt_{max}} - \mu\beta^{-m/m-1}\frac{m-1}{mt_{max}} - \mu\Delta_x\frac{\beta^m}{mkt_{max}} - \frac{\sigma}{kt_{max}} > 0 \\ N - \mu\frac{m-1}{m}\beta^{-m/m-1} > 0 \\ N - \mu\Delta_x\frac{\beta^m}{m} - \sigma > 0. \end{aligned} \tag{3.22}$$

Therfore from (3.21) and (3.22) there exists a constant γ such that (3.20) takes the form

$$L''(t)L(t) + L'(t)L(t) \geq \gamma\|u\|_{L^r(\Omega)}^r. \tag{3.23}$$

Now we use Hölder inequality to estimate the term $\|u\|_{L^r(\Omega)}^r$ as follows

$$\begin{aligned} \int_\Omega |u|^2 dx & \leq |\Omega|^{r-2/r}.\|u\|_r^2 \\ & \leq (N(t+t_{max}))^{r-2/r}\, |\Omega|^{r-2/r}.\|u\|_r^2. \end{aligned} \tag{3.24}$$

Hence

$$\|u\|_r^r \geq |\Omega|^{2-r/2}.\, (N(t+t_{max}))^{2-r/2}\, .\|u\|_2^r, \tag{3.25}$$

and from the definition of $L(t)$ in (3.7) we have

$$\begin{aligned} (2L(t))^{r/2} & \leq \|u\|_2^r + \left(N\int_0^t H(s)ds + N(t+t_{max})\right)^{r/2} \\ & \leq 2^{r-2/2}\left(\|u\|_2^r + \left(N\int_0^t H(s)ds + N(t+t_{max})\right)^{r/2}\right). \end{aligned} \tag{3.26}$$

This gives

$$\|u\|_2^r \geq 2(L(t))^{r/2} - \left(N\int_0^t H(s)ds + N(t+t_{max})\right)^{r/2} \geq (L(t))^{r/2}. \tag{3.27}$$

Combining (3.23) and (3.27) yields

$$L''(t) + L'(t) \geq \gamma|\Omega|^{2-r/2}\, (N(t+t_{max}))^{2-r/2}\, (L(t))^{r/2}. \tag{3.28}$$

We see that the requirements of theorem 1 are satisfied with

$$-1 < \frac{2-r}{2} \leq 0, \quad \alpha = \frac{r-2}{2} > 0, \quad C_0 = \gamma|\Omega|^{2-r/2}N^{2-r/2} > 0. \tag{3.29}$$

Therefore, L blows up in finite time. This completes the proof. □

4. Conclusions

Let us pass to a survey of the results and methods of proving non-existence and blow-up theorems applicable to equations of hyperbolic type. Here it is necessary to clarify what is meant by the term "destruction of the solution". By this term, we understand the existence of a finite time moment at which the solution of the evolutionary problem leaves the smoothness class to which this solution belonged on the interval $(0, T_{max})$ (the smoothness class for which the local solvability theorem is formulated and proved). Looking ahead, we note that in all problems for nonlinear equations considered in the literature, the destruction of the solution is accompanied by the inversion of the norm of the latter to infinity (in the space where we are looking for a solution), however, such behavior of solution is not at all necessary in the concept of destruction.

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Conflict of interest

The authors declare there is no conflict of interest.

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