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Research article

Further characterizations of the m-weak group inverse of a complex matrix

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Abstract: In this paper, we introduce certain different characterizations and several new properties of the m-weak group inverse of a complex matrix. Also, the relationship between the m-weak group inverse and a nonsingular bordered matrix is established as well as the Cramer's rule for the solution of the restricted matrix equation that depends on the m-weak group inverse.

Keywords: weak group inverse; m-weak group inverse; core-EP inverse; core-EP decomposition
Mathematics Subject Classification: 15A09

1. Introduction

Let $\mathbb{C}^{m\times n}$ and $\mathbb{Z}^{+}$ denote the set of all $m \times n$ complex matrices and the set of all positive integers, respectively. The symbols $r(A)$ and $\mathrm{Ind}(A)$ stand for the rank and the index of $A \in \mathbb{C}^{n\times n}$, respectively. For a matrix $A \in \mathbb{C}^{n\times n}$, we assume that $A^0 = I_n$. Let $\mathbb{C}_k^{n\times n}$ be the set of all $n \times n$ complex matrices with index k. By $\mathbb{C}_n^{\mathrm{CM}}$ we denote the set of all core matrices (or group invertible matrices), i.e.,

$$\mathbb{C}_n^{\mathrm{CM}} = \left\{A|A \in \mathbb{C}^{n\times n}, r(A) = r(A^2)\right\}.$$

The Drazin inverse [1] of $A \in \mathbb{C}_k^{n\times n}$, denoted by A^D, is the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying:

$$XA^{k+1} = A^k, XAX = X \text{ and } AX = XA. \tag{1.1}$$

Especially, when $A \in \mathbb{C}_n^{\mathrm{CM}}$, then X that satisfies (1.1) is called the group inverse of A and is denoted by $A^{\#}$. The Drazin inverse has been widely applied in different fields of mathematics and its applications. Here we will mention only some of them. The perturbation theory and additive results for the Drazin inverse were investigated in [2–5]. In [6], the algorithms for the computation of the Drazin inverse of a polynomial matrix are presented based on the discrete Fourier transformation. Karampetakis and Stanimirović [7] presented two algorithms for symbolic computation of the Drazin inverse of a given square one-variable polynomial matrix, which was effective with respect to CPU time and the elimination of redundant computations. Some representations of the W-weighted Drazin inverse were investigated and the computational complexities of the representations were also estimated in [8]. Kyrchei [9] generalized the weighted Drazin inverse, the weighted DMP-inverse, and the weighted dual DMP-inverse [10–12] for the matrices over the quaternion skew field and provided their determinantal representations by using noncommutative column and row determinants. In [13], the authors considered the quaternion two-sided restricted matrix equations and gave their unique solutions by the DMP-inverse and dual DMP-inverse. For interesting properties of different kinds of generalized inverses see [14].

In 2018, Wang [15] introduced the weak group inverse of complex square matrices using the core-EP decomposition [16] and gave its certain characterizations.

Definition 1.1. *Let $A \in \mathbb{C}_k^{n\times n}$. Then the unique solution of the system*

$$AX^2 = X, \quad AX = A^{\textcircled{\dagger}}A,$$

is the weak group inverse of A denoted by $A^{\textcircled{w}}$.

Recently, there has been a huge interest in the weak group inverse. For example, Wang et al. [17] compared the weak group inverse with the group inverse of a matrix. In [18], the weak group inverse was introduced in *-rings and characterized by three equations (see also [19, 20]). The weak group inverse in the setting of rectangular matrices was considered in [21]. In 2021, Zhou and Chen [19] introduced the m-weak group inverse in the ring and presented its different characterizations.

Definition 1.2. *Let R be a unitary ring with involution, $a \in R$ and $m \in \mathbb{Z}^+$. If there exist $x \in R$ and $k \in \mathbb{Z}^+$ such that*

$$xa^{k+1} = a^k, \quad ax^2 = x, \quad (a^k)^*a^{m+1}x = (a^m)^*a^k,$$

then x is called the m-weak group inverse of a and in this case, a is m-weak group invertible.

In general, the m-weak group inverse of a may not be unique. If the m-weak group inverse of a is unique, then it is denoted by $a^{\textcircled{w}_m}$.

In [22], we can find a relation between the weak core inverse and the m-weak group inverse as well as certain necessary and sufficient conditions that the Drazin inverse coincides with the m-weak group inverse of a complex matrix. It is interesting to note that X which satisfies (1.1) coincides with the m-weak group inverse on complex matrices, in which case X exists for every $A \in \mathbb{C}^{n\times n}$ and is unique. Now, we consider the system of equations

$$AX^2 = X, \; AX = (A^{\textcircled{\dagger}})^m A^m. \tag{1.2}$$

Motivated by the above discussion, we introduce a new characterization of the m-weak group inverse related with (1.2) and proved the existence and uniqueness of a solution of (1.2), for every $A \in \mathbb{C}^{n\times n}$. Some new characterizations of the m-weak group inverse are derived in terms of the range space, null space, rank equalities, and projectors. We present some representations of the m-weak group inverse involving some known generalized inverses and limit expressions as well as certain relations between the m-weak group inverse and other generalized inverses. Finally, we consider a relation between the

m-weak group inverse and the nonsingular bordered matrix, which is applied to the Cramer's rule for the solution of the restricted matrix equation.

The paper is organized as follows: In Section 2, we present some well-known definitions and lemmas. In Section 3, we provide a new characterization, as well as certain representations and properties of the m-weak group inverse of a complex matrix. In Section 4, we provide several expressions of the m-weak group inverse which are useful in computation. In Section 5, we present some properties of the m-weak group inverse as well as the relationships between the m-weak group inverse and other generalized inverses by core-EP decomposition. In Section 6, we show the applications of the m-weak group inverse concerned with the bordered matrices and the Cramer's rule for the solution of the restricted matrix equation.

2. Preliminaries

The symbols $\mathcal{R}(A)$, $\mathcal{N}(A)$ and A^* denote the range space, null space and conjugate transpose of $A \in \mathbb{C}^{m\times n}$, respectively. The symbol I_n denotes the identity matrix of order n. Let $P_{\mathcal{L},\mathcal{M}}$ be the projector on the space $\mathcal{L}$ along the $\mathcal{M}$, where $\mathcal{L}, \mathcal{M} \leq \mathbb{C}^n$ and $\mathcal{L} \oplus \mathcal{M} = \mathbb{C}^n$. For $A \in \mathbb{C}^{m\times n}$, P_A represents the orthogonal projection onto $\mathcal{R}(A)$, i.e., $P_A = P_{\mathcal{R}(A)} = AA^\dagger$. The symbols $\mathbb{C}_n^{\mathrm{P}}$ and $\mathbb{C}_n^{\mathrm{H}}$ represent the subsets of $\mathbb{C}^{n\times n}$ consisting of all idempotent and Hermitian matrices, respectively, i.e.,

$$\mathbb{C}_n^{\mathrm{P}} = \{A | A \in \mathbb{C}^{n\times n}, A^2 = A\},$$
$$\mathbb{C}_n^{\mathrm{H}} = \{A | A \in \mathbb{C}^{n\times n}, A = A^*\}.$$

Let $A \in \mathbb{C}^{m\times n}$. The MP-inverse $A^\dagger$ of A is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the following four Penrose equations (see [14, 23, 24]):

$$(1)\ AXA = A, \quad (2)\ XAX = X, \quad (3)\ (AX)^* = AX, \quad (4)\ (XA)^* = XA.$$

A matrix $X \in \mathbb{C}^{n\times m}$ that satisfies condition (1) above is called an inner inverse of A and the set of all inner inverses of A is denoted by $A\{1\}$, while a matrix $X \in \mathbb{C}^{n\times m}$ that satisfies condition (2) above is called an outer inverse of A. A matrix $X \in \mathbb{C}^{n\times m}$ that satisfies both conditions (1) and (2) is called a reflexive g-inverse of A. If a matrix $X \in \mathbb{C}^{n\times m}$ satisfies

$$X = XAX,\ \mathcal{R}(X) = \mathcal{T} \text{ and } \mathcal{N}(X) = \mathcal{S},$$

where $\mathcal{T}$ and $\mathcal{S}$ are the subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively, then X is an outer inverse of A with prescribed range and null space and it is denoted by $A^{(2)}_{\mathcal{T},\mathcal{S}}$. If $A^{(2)}_{\mathcal{T},\mathcal{S}}$ exists, then it is unique. The notion of the core inverse on the $\mathbb{C}_n^{\mathrm{CM}}$ was proposed and was denoted by $A^{\textcircled{\#}}$ [25–27]. The core inverse of $A \in \mathbb{C}_k^{n\times n}$ is the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$AX = P_A,\ \ \mathcal{R}(X) \subseteq \mathcal{R}(A).$$

In addition, it was proved that

$$A^{\textcircled{\#}} = A^\# AA^\dagger.$$

The core-EP inverse of $A \in \mathbb{C}_k^{n\times n}$, denoted by $A^{\textcircled{\dagger}}$ is given in [28–30]. The core-EP inverse of $A \in \mathbb{C}_k^{n\times n}$ is the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$XAX = X,\ \ \mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k).$$

Moreover, it was proved that

$$A^{\textcircled{\dagger}} = (A^{k+1}(A^k)^\dagger)^\dagger.$$

The DMP-inverse of $A \in \mathbb{C}_k^{n\times n}$, denoted by $A^{D,\dagger}$ was introduced in [10, 11]. The DMP-inverse of $A \in \mathbb{C}_k^{n\times n}$ is the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$XAX = X,\ \ XA = A^D A\ \ A^k X = A^k A^\dagger.$$

Moreover, it was shown that

$$A^{D,\dagger} = A^D A A^\dagger.$$

Also, the dual DMP-inverse of A was introduced in [10], as $A^{\dagger,D} = A^\dagger A A^D$.
The (B, C)-inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^{(B,C)}$ [31, 32], is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying

$$XAB = B, CAX = C, \mathcal{R}(X) = \mathcal{R}(B) \text{ and } \mathcal{N}(X) = \mathcal{N}(C),$$

where $B, C \in \mathbb{C}^{n\times m}$.

To discuss further properties of the m-weak group inverse, several auxiliary lemmas will be given. The first lemma gives the core-EP decomposition of a matrix $A \in \mathbb{C}_k^{n\times n}$ which will be a very useful tool throughout this paper.

Lemma 2.1. [16] Let $A \in \mathbb{C}_k^{n\times n}$. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$A = A_1 + A_2 = U \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^*, \tag{2.1}$$

$$A_1 = U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} U^*,\quad A_2 = U \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} U^*, \tag{2.2}$$

where $T \in \mathbb{C}^{t\times t}$ is nonsingular with $t = r(T) = r(A^k)$ and N is nilpotent of index k. The representation (2.1) is called the core-EP decomposition of A, while A_1 and A_2 are the core part and nilpotent part of A, respectively.

Following the representation (2.1) of a matrix $A \in \mathbb{C}_k^{n\times n}$, we have the following representations of certain generalized inverses (see [15, 16, 33]):

$$A^{\textcircled{\dagger}} = U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*, \tag{2.3}$$

$$A^{\textcircled{w}} = U \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^*, \tag{2.4}$$

$$A^D = U \begin{bmatrix} T^{-1} & (T^{k+1})^{-1}T_k \\ 0 & 0 \end{bmatrix} U^*, \tag{2.5}$$

where $T_k = \sum_{j=0}^{k-1} T^j S N^{k-1-j}$.

By direct computations, we get that $A \in \mathbb{C}_n^{\mathrm{CM}}$ is equivalent with $N = 0$, in which case

$$A^{\#} = U \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^*, \tag{2.6}$$

and

$$A^{\textcircled{\#}} = U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*. \tag{2.7}$$

Let $A \in \mathbb{C}_k^{n\times n}$ be of the form (2.1) and let $m \in \mathbb{Z}^+$. The notations below will be frequently used in this paper:

$$\begin{aligned} M &= S(I_{n-t} - N^\dagger N), \\ \triangle &= (TT^* + MS^*)^{-1}, \\ T_m &= \sum_{j=0}^{m-1} T^j S N^{m-1-j}. \end{aligned}$$

Lemma 2.2. [34, Lemma 6] *Let $A \in \mathbb{C}_k^{n\times n}$ be of the form* (2.1) *. Then*

$$A^\dagger = U \begin{bmatrix} T^*\triangle & T^*\triangle S N^\dagger \\ M^*\triangle & N^\dagger - M^*\triangle S N^\dagger \end{bmatrix} U^*. \tag{2.8}$$

From (2.8) and [16, Theorem 2.2], we get that

$$AA^\dagger = U \begin{bmatrix} I_t & 0 \\ 0 & NN^\dagger \end{bmatrix} U^*, \tag{2.9}$$

$$A^\dagger A = U \begin{bmatrix} T^*\triangle T & -T^*\triangle M \\ M^*\triangle T & N^\dagger N + M^*\triangle M \end{bmatrix} U^*, \tag{2.10}$$

$$A^k = U \begin{bmatrix} T^k & T_k \\ 0 & 0 \end{bmatrix} U^*, \tag{2.11}$$

$$A^m = U \begin{bmatrix} T^m & T_m \\ 0 & N^m \end{bmatrix} U^*, \tag{2.12}$$

$$P_{A^k} = A^k (A^k)^\dagger = U \begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix} U^*, \tag{2.13}$$

where $t = r(A^k)$.

Lemma 2.3. [29, 35, 36] *Let $A \in \mathbb{C}_k^{n\times n}$ and let $m \in \mathbb{Z}^+$. Then*
(*a*) $AA^{\textcircled{\dagger}} = P_{A^k}$;
(*b*) $A^{\textcircled{\dagger}}A = P_{\mathcal{R}(A^k),\mathcal{N}((A^{k+1})^*A)}$;
(*c*) $A^{\textcircled{\dagger}} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*)}$;
(*d*) $(A^{\textcircled{\dagger}})^m P_{A^k} = (A^{\textcircled{\dagger}})^m$.

Lemma 2.4. *Let $A \in \mathbb{C}_k^{n\times n}$ and let $m \in \mathbb{Z}^+$. Then $A^m (A^{\textcircled{\dagger}})^m = P_{A^k}$.*

Proof. Assume that $A \in \mathbb{C}_k^{n\times n}$ is of the form (2.1). By (2.3), (2.12) and (2.13), it follows that

$$A^m(A^{\textcircled{\dagger}})^m = U\begin{bmatrix} T^m & T_m \\ 0 & N^m \end{bmatrix}\begin{bmatrix} T^{-m} & 0 \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix}U^* = P_{A^k}.$$

□

3. m-weak group inverse on complex matrices

In this section, using the core-EP decomposition of a matrix $A \in \mathbb{C}_k^{n\times n}$ we will give another definition of the m-weak group inverse. Furthermore, some properties of the m-weak group inverse will be derived.

Theorem 3.1. *Let $A \in \mathbb{C}_k^{n\times n}$ be given by* (2.1) *and let $X \in \mathbb{C}^{n\times n}$ and $m \in \mathbb{Z}^+$. The system of equations*

$$AX^2 = X, \quad AX = (A^{\textcircled{\dagger}})^m A^m, \tag{3.1}$$

is consistent and has a unique solution X given by

$$X = (A^{\textcircled{\dagger}})^{m+1}A^m = U\begin{bmatrix} T^{-1} & (T^{m+1})^{-1}T_m \\ 0 & 0 \end{bmatrix}U^*. \tag{3.2}$$

Proof. If $m = 1$, then X coincides with $A^{\textcircled{W}}$. Clearly, X is the unique solution of (3.1) according to the definition of the weak group inverse. If $m \neq 1$, by (3.1), Lemmas 2.3 (d) and 2.4, it follows that

$$X = (AX)X = (A^{\textcircled{\dagger}})^m A^m X = (A^{\textcircled{\dagger}})^m A^{m-1}(A^{\textcircled{\dagger}})^m A^m = (A^{\textcircled{\dagger}})^m P_{A^k}A^{\textcircled{\dagger}}A^m = (A^{\textcircled{\dagger}})^m A^{\textcircled{\dagger}}A^m = (A^{\textcircled{\dagger}})^{m+1}A^m.$$

Thus, by (2.3) and (2.12), we have that

$$X = (A^{\textcircled{\dagger}})^{m+1}A^m = U\begin{bmatrix} T^{-(m+1)} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T^m & T_m \\ 0 & N^m \end{bmatrix}U^* = U\begin{bmatrix} T^{-1} & T^{-(m+1)}T_m \\ 0 & 0 \end{bmatrix}U^*.$$

□

Definition 3.2. *Let $A \in \mathbb{C}_k^{n\times n}$ and $m \in \mathbb{Z}^+$. The m-weak group inverse of A, denoted by $A^{\textcircled{W}_m}$, is the unique solution of the system* (3.1).

Remark 3.3. *The m-weak group inverse is in some sense a generalization of the weak group inverse and Drazin inverse. We have the following:*

(a) *If $m = 1$, then* 1*-weak group inverse of $A \in \mathbb{C}_k^{n\times n}$ coincides with the weak group inverse of A;*
(b) *If $m \geq k$, then m-weak group inverse of $A \in \mathbb{C}_k^{n\times n}$ coincides with the Drazin inverse of A.*

In the following example, we will show that the m-weak group inverse is different from some known generalized inverses.

Example 3.4. *Let* $A = \begin{bmatrix} I_3 & I_3 \\ 0 & N \end{bmatrix}$, *where* $N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. *It can be verified that* $\mathrm{Ind}(A) = 3$. *By computations, we can check the following:*

$$A^{\dagger} = \begin{bmatrix} H_1 & -N^{\dagger} \\ I_3 - H_1 & N^{\dagger} \end{bmatrix}, \quad A^D = \begin{bmatrix} I_3 & H_2 \\ 0 & 0 \end{bmatrix}, A^{\textcircled{\dagger}} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A^{D,\dagger} = \begin{bmatrix} I_3 & H_3 \\ 0 & 0 \end{bmatrix}, \quad A^{\dagger,D} = \begin{bmatrix} H_1 & H_4 \\ I_3 - H_1 & H_2 - H_4 \end{bmatrix}, A^{\textcircled{W}} = \begin{bmatrix} I_3 & I_3 \\ 0 & 0 \end{bmatrix},$$

where $H_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $H_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $H_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $H_4 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ *and* $N^{\dagger} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

It is clear that $A^{\textcircled{W}_2} = (A^{\textcircled{\dagger}})^3 A^2 = \begin{bmatrix} I_3 & I_3 + N \\ 0 & 0 \end{bmatrix}$.

Theorem 3.5. *Let* $A \in \mathbb{C}_k^{n\times n}$ *be decomposed by* $A = A_1 + A_2$ *as in* (2.1) *and let* $m \in \mathbb{Z}^+$. *Then*

(a) $A^{\textcircled{W}_m}$ *is an outer inverse of* A;
(b) $A^{\textcircled{W}_m}$ *is a reflexive g-inverse of* A_1.

Proof. (a) By Lemmas 2.3 (d), 2.4 and the definition of $A^{\textcircled{W}_m}$, it follows that

$$A^{\textcircled{W}_m} A A^{\textcircled{W}_m} = (A^{\textcircled{\dagger}})^{m+1} A^m A (A^{\textcircled{\dagger}})^{m+1} A^m = (A^{\textcircled{\dagger}})^{m+1} P_{A^k} A^m = A^{\textcircled{W}_m}.$$

(b) By (2.2) and (3.2), we get that

$$A_1 A^{\textcircled{W}_m} A_1 = U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^{-1} & T^{-(m+1)} T_m \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} U^* = A_1.$$

From [16, Theorem 3.4], we get $A_1 = AA^{\textcircled{\dagger}}A$. By the fact that $A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}$ and the statement (a) above, it follows that

$$A^{\textcircled{W}_m} A_1 A^{\textcircled{W}_m} = A^{\textcircled{W}_m} A A^{\textcircled{\dagger}} A (A^{\textcircled{\dagger}})^{m+1} A^m = A^{\textcircled{W}_m} A (A^{\textcircled{\dagger}})^{m+1} A^m = A^{\textcircled{W}_m} A A^{\textcircled{W}_m} = A^{\textcircled{W}_m}.$$

Hence $A^{\textcircled{W}_m}$ is a reflexive g-inverse of A_1. □

Theorem 3.6. *Let* $A \in \mathbb{C}_k^{n\times n}$ *and* $m \in \mathbb{Z}^+$. *Then*

(a) $r(A^{\textcircled{W}_m}) = r(A^k)$.
(b) $\mathcal{R}(A^{\textcircled{W}_m}) = \mathcal{R}(A^k)$, $\mathcal{N}(A^{\textcircled{W}_m}) = \mathcal{N}((A^k)^* A^m)$.
(c) $A^{\textcircled{W}_m} = A^{(2)}_{\mathcal{R}(A^k), \mathcal{N}((A^k)^* A^m)}$.

Proof. (a) Assume that A is given by (2.1). From (2.11) and (3.2), it is clear that $r(A^{\textcircled{W}_m}) = t = r(A^k)$.

(b) Since $A^{\textcircled{W}_m} = (A^{\textcircled{\dagger}})^{m+1} A^m$ implies that $\mathcal{R}(A^{\textcircled{W}_m}) = \mathcal{R}((A^{\textcircled{\dagger}})^{m+1} A^m) \subseteq \mathcal{R}(A^{\textcircled{\dagger}}) = \mathcal{R}(A^k)$ and since $r(A^{\textcircled{W}_m}) = r(A^k)$, we get $\mathcal{R}(A^{\textcircled{W}_m}) = \mathcal{R}(A^k)$. From $\mathcal{N}(A^{\textcircled{W}_m}) = \mathcal{N}((A^{\textcircled{\dagger}})^{m+1} A^m) \supseteq \mathcal{N}(A^{\textcircled{\dagger}} A^m)$ and $r(A^{\textcircled{W}_m}) = t = r(A^{\textcircled{\dagger}} A^m)$, we get $\mathcal{N}(A^{\textcircled{W}_m}) = \mathcal{N}(A^{\textcircled{\dagger}} A^m)$. If $x \in \mathcal{N}(A^{\textcircled{\dagger}} A^m)$, we get that $A^m x \in \mathcal{N}(A^{\textcircled{\dagger}}) = \mathcal{N}((A^k)^*)$. Then $\mathcal{N}(A^{\textcircled{W}_m}) = \mathcal{N}(A^{\textcircled{\dagger}} A^m) \subseteq \mathcal{N}((A^k)^* A^m)$, and by $r(A^{\textcircled{W}_m}) = r((A^k)^* A^m)$, it follows that $\mathcal{N}(A^{\textcircled{W}_m}) = \mathcal{N}(A^{\textcircled{\dagger}} A^m) = \mathcal{N}((A^k)^* A^m)$.

(c) It is a direct consequence from Theorems 3.5 (a) and 3.6 (b). □

Theorem 3.7. *Let $A \in \mathbb{C}_k^{n\times n}$ and $m \in \mathbb{Z}^+$. Then*

(*a*) $AA^{Ⓦ_m} = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)}$;
(*b*) $A^{Ⓦ_m}A = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^{m+1})}$.

Proof. (*a*) From Theorem 3.5 (*a*), it follows that $AA^{Ⓦ_m} \in \mathbb{C}_n^{\mathrm{P}}$. By the definition of $A^{Ⓦ_m}$ and (3.2), it can be proved that $\mathcal{R}(AA^{Ⓦ_m}) = \mathcal{R}((A^{Ⓣ})^mA^m) \subseteq \mathcal{R}(A^{Ⓣ}) = \mathcal{R}(A^k)$ and $r(AA^{Ⓦ_m}) = r(A^{Ⓦ_m}) = r(A^k) = t$. Hence $\mathcal{R}(AA^{Ⓦ_m}) = \mathcal{R}(A^k)$. Similarly, we get that $\mathcal{N}(AA^{Ⓦ_m}) = \mathcal{N}(A^{Ⓦ_m}) = \mathcal{N}((A^k)^*A^m)$. Therefore, $AA^{Ⓦ_m} = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)}$.
(*b*) The proof follows similarly as for the part (*a*). □

4. Some characterizations of the *m*-weak group inverse

In this part, we represent some characterizations of the *m*-weak group inverse in terms of the range space, null space, rank equalities, and projectors.
The next theorem gives several characterizations of $A^{Ⓦ_m}$.

Theorem 4.1. *Let $A \in \mathbb{C}_k^{n\times n}$, $X \in \mathbb{C}^{n\times n}$ and let $m \in \mathbb{Z}^+$. Then the following hold:*

(*a*) $X = A^{Ⓦ_m}$.
(*b*) $\mathcal{R}(X) = \mathcal{R}(A^k)$, $AX = (A^{Ⓣ})^mA^m$.
(*c*) $\mathcal{R}(X) = \mathcal{R}(A^k)$, $A^{m+1}X = P_{A^k}A^m$.
(*d*) $\mathcal{R}(X) = \mathcal{R}(A^k)$, $(A^k)^*A^{m+1}X = (A^k)^*A^m$.

Proof. (*a*) ⇒ (*b*) : This follows directly by Theorem 3.6 (*b*) and the definition of $A^{Ⓦ_m}$.

(*b*) ⇒ (*c*) : Premultiplying $AX = (A^{Ⓣ})^mA^m$ by A^m, and by Lemma 2.4, it follows that

$$A^{m+1}X = (A)^m(A^{Ⓣ})^mA^m = P_{A^k}A^m.$$

(*c*) ⇒ (*d*) : Premultiplying $A^{m+1}X = P_{A^k}A^m$ by $(A^k)^*$, it follows that

$$(A^k)^*A^{m+1}X = (A^k)^*P_{A^k}A^m = (A^k)^*A^m.$$

(*d*) ⇒ (*a*) : Let A be of the form (2.1). By (2.11) and $\mathcal{R}(X) = \mathcal{R}(A^k)$, we obtain that

$$X = U\begin{bmatrix} X_1 & X_2 \\ 0 & 0 \end{bmatrix}U^*,$$

where $X_1 \in \mathbb{C}^{t\times t}$ and $X_2 \in \mathbb{C}^{t\times(n-t)}$. Thus $(A^k)^*A^{m+1}X = (A^k)^*A^m$ implies that

$$U\begin{bmatrix} (T^k)^*T^{m+1}X_1 & (T^k)^*T^{m+1}X_2 \\ (\tilde{T})^*T^{m+1}X_1 & (\tilde{T})^*T^{m+1}X_2 \end{bmatrix}U^* = U\begin{bmatrix} (T^k)^*T^m & (T^k)^*T_m \\ (\tilde{T})^*T^m & (\tilde{T})^*T_m \end{bmatrix}U^*,$$

i.e., $X_1 = T^{-1}$ *and* $X_2 = (T^{m+1})^{-1}T_m$, which imply $X = U\begin{bmatrix} T^{-1} & (T^{m+1})^{-1}T_m \\ 0 & 0 \end{bmatrix}U^* = A^{Ⓦ_m}$. □

By Theorem 3.5, it is known that $A^{Ⓦ_m}$ is an outer inverse of $A \in \mathbb{C}_k^{n\times n}$, i.e., $A^{Ⓦ_m}AA^{Ⓦ_m} = A^{Ⓦ_m}$. Using this result, we obtain some characterizations of $A^{Ⓦ_m}$.

Theorem 4.2. *Let $A \in \mathbb{C}_k^{n\times n}$, $X \in \mathbb{C}^{n\times n}$ and let $m \in \mathbb{N}^+$. Then the following conditions are equivalent:*

(a) $X = A^{\textcircled{W}_m}$.
(b) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A^k)$, $\mathcal{N}(X) = \mathcal{N}((A^k)^*A^m)$.
(c) $XAX = X$, $XA^{k+1} = A^k$, $AX = (A^{\textcircled{\dagger}})^m A^m$.
(d) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A^k)$, $(A^m)^*A^{m+1}X \in \mathbb{C}_n^{\mathrm{H}}$.

Proof. (a) $\Rightarrow$ (b) : It is a direct consequence of Theorem 3.6 (c).
(b) $\Rightarrow$ (c) : By $XAX = X$ and $\mathcal{R}(X) = \mathcal{R}(A^k)$, it follows that

$$\mathcal{R}(AX) = A\mathcal{R}(X) = \mathcal{R}(A^{k+1}) = \mathcal{R}(A^k) = \mathcal{R}((A^{\textcircled{\dagger}})^m A^m)$$

and

$$\mathcal{N}(AX) = \mathcal{N}(X) = \mathcal{N}((A^k)^*A^m) = \mathcal{N}((A^{\textcircled{\dagger}})^m A^m).$$

Since $AX, (A^{\textcircled{\dagger}})^m A^m \in \mathbb{C}_n^{\mathrm{P}}$, we have $AX = (A^{\textcircled{\dagger}})^m A^m$. By $\mathcal{R}(X) = \mathcal{R}(A^k)$ and $XAX = X$, we obtain that $XA^{k+1} = A^k$.
(c) $\Rightarrow$ (d) : We have that

$$r(X) = r(AX) = r((A^{\textcircled{\dagger}})^m A^m) = r(A^k),$$

and by $\mathcal{R}(A^k) = \mathcal{R}(XA^{k+1}) \subseteq \mathcal{R}(X)$, we get $\mathcal{R}(XA) = \mathcal{R}(A^k)$. Since $A^m(A^{\textcircled{\dagger}})^m = P_{A^k} \in \mathbb{C}_n^{\mathrm{H}}$, it follows that

$$(A^m)^*A^{m+1}X = (A^m)^*(A^m(A^{\textcircled{\dagger}})^m)A^m \in \mathbb{C}_n^{\mathrm{H}}.$$

(d) $\Rightarrow$ (a) : Assume that A is of the form (2.1). From $XAX = X$ and $\mathcal{R}(X) = \mathcal{R}(A^k)$, we get that $XA^{k+1} = A^k$. Then it is easy to conclude that

$$X = U\begin{bmatrix} T^{-1} & X_2 \\ 0 & 0 \end{bmatrix}U^*,$$

where $X_2 \in \mathbb{C}^{t\times(n-t)}$.
Since

$$\begin{aligned}(A^m)^*A^{m+1}X &= U\begin{bmatrix} (T^m)^* & 0 \\ (T_m)^* & (N^m)^* \end{bmatrix}\begin{bmatrix} T^{m+1} & TT_m + SN^m \\ 0 & N^{m+1} \end{bmatrix}\begin{bmatrix} T^{-1} & X_2 \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} (T^m)^*T^m & (T^m)^*T^{m+1}X_2 \\ (T_m)^*T^m & (T_m)^*T^{m+1}X_2 \end{bmatrix}U^* \in \mathbb{C}_n^{\mathrm{H}},\end{aligned}$$

we obtain that $X_2 = T^{-(m+1)}T_m$. Hence $X = U\begin{bmatrix} T^{-1} & T^{-(m+1)}T_m \\ 0 & 0 \end{bmatrix}U^* = A^{\textcircled{W}_m}$. □

Motivated by the first two matrix equations $XA^{k+1} = A^k$ and $XAX = X$, we provide several characterizations of $A^{\textcircled{W}_m}$.

Theorem 4.3. *Let $A \in \mathbb{C}_k^{n\times n}$, $X \in \mathbb{C}^{n\times n}$ and let $m \in \mathbb{Z}^+$. Then the following conditions are equivalent:*

(a) $X = A^{\textcircled{W}_m}$.
(b) $XA^{k+1} = A^k$, $AX^2 = X$, $(A^m)^*A^{m+1}X \in \mathbb{C}_n^{\mathrm{H}}$.

(*c*) $XA^{k+1} = A^k$, $AX^2 = X$, $A^{m+1}X = P_{A^k}A^m$.
(*d*) $XA^{k+1} = A^k$, $AX = (A^{\textcircled{\dagger}})^m A^m$, $r(X) = r(A^k)$.

Proof. $(a) \Leftrightarrow (b)$: This follows by Proposition 4.2 in [19].
$(a) \Rightarrow (c)$: It is a direct consequence of Theorems 4.1 (*c*) and 4.2 (*c*).
$(c) \Rightarrow (d)$: Assume that A is given by (2.1). By $XA^{k+1} = A^k$, we get that

$$X = U\begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix}U^*,$$

where $X_2 \in \mathbb{C}^{t\times(n-t)}$ and $X_4 \in \mathbb{C}^{(n-t)\times(n-t)}$.
By $AX^2 = X$, we have that $X_4 = N{X_4}^2$, which implies that

$$X_4 = N{X_4}^2 = N^2{X_4}^3 = \cdots = N^k{X_4}^{k+1} = 0.$$

Using (2.12) and (2.13) and that $A^{m+1}X = P_{A^k}A^m$, we get

$$X = U\begin{bmatrix} T^{-1} & T^{-(m+1)}T_m \\ 0 & 0 \end{bmatrix}U^*.$$

Now, the proof follows directly.
$(d) \Rightarrow (a)$: Since $XA^{k+1} = A^k$, it follows that $\mathcal{R}(A^k) = \mathcal{R}(XA^{k+1}) \subseteq \mathcal{R}(X)$ and by $r(X) = r(A^k)$, we get $\mathcal{R}(A^k) = \mathcal{R}(X)$. Hence, according to Theorem 4.1 (*b*), we get $X = A^{\textcircled{W}_m}$. □

According to Theorem 3.7, it follows that $AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)}$ and $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^{m+1})}$ when $X = A^{\textcircled{W}_m}$. Conversely, the implication does not hold. Here's an example below.

Example 4.4. *Let* $A = \begin{bmatrix} I_3 & L \\ 0 & N \end{bmatrix}$, $X = \begin{bmatrix} I_3 & L \\ 0 & L \end{bmatrix}$, *where* $N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $L = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. *Then it is clear that* $k = \mathrm{Ind}(A) = 3$ *and* $A^{\textcircled{W}_2} = \begin{bmatrix} I_3 & L \\ 0 & 0 \end{bmatrix}$. *It can be directly verified that* $AX = P_{\mathcal{R}(A^3),\mathcal{N}((A^3)^*A^2)}$, $XA = P_{\mathcal{R}(A^3),\mathcal{N}((A^3)^*A^3)}$. *However,* $X \neq A^{\textcircled{W}_2}$.

Based on the example above, the next theorem, we consider other characterizations of $A^{\textcircled{W}_m}$ by using $AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)}$ and $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^{m+1})}$.

Theorem 4.5. *Let* $A \in \mathbb{C}_k^{n\times n}$ *be of the form* (2.1), $X \in \mathbb{C}^{n\times n}$ *and* $m \in \mathbb{Z}^+$. *Then the following statements are equivalent:*

(*a*) $X = A^{\textcircled{W}_m}$;
(*a*) $AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)}$, $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^{m+1})}$ and $r(X) = r(A^k)$;
(*a*) $AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)}$, $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^{m+1})}$ and $XAX = X$;
(*a*) $AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)}$, $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^{m+1})}$ and $AX^2 = X$.

Proof. $(a) \Rightarrow (b)$: It is a direct consequence of Theorems 3.6 (*a*) and 3.7.
$(b) \Rightarrow (c)$: Since $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^{m+1})}$ and $r(X) = r(A^k)$, we get that $\mathcal{R}(X) = \mathcal{R}(A^k)$ and by $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^{m+1})}$, we obtain $XAX = X$.

$(c) \Rightarrow (d)$: From $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^{m+1})}$ and $r(X) = r(A^k)$, we have that $\mathcal{R}(X) = \mathcal{R}(A^k)$ and by $AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)}$, it follows that $AX^2 = X$.
$(d) \Rightarrow (a)$: By $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^{m+1})}$ and $AX^2 = X$, it follows that

$$\mathcal{R}(A^k) = \mathcal{R}(XA) \subseteq \mathcal{R}(X) = \mathcal{R}(AX^2) = \cdots = \mathcal{R}(A^kX^{k+1}) \subseteq \mathcal{R}(A^k),$$

which implies $\mathcal{R}(X) = \mathcal{R}(A^k)$. By $AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)}$, we get that

$$(A^k)^*A^{m+1}X = (A^k)^*A^m.$$

According to Theorem 4.1 (d), we have that $X = A^{\textcircled{W}_m}$. □

Analogously, we characterize $A^{\textcircled{W}_m}$ using that $AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)}$ or $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^{m+1})}$ as follows:

Theorem 4.6. *Let $A \in \mathbb{C}_k^{n\times n}$, $X \in \mathbb{C}^{n\times n}$ and let $m \in \mathbb{Z}^+$. Then*

(a) *$X = A^{\textcircled{W}_m}$ is the unique solution of the system of equations:*

$$AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)}, \ \mathcal{R}(X) = \mathcal{R}(A^k). \tag{4.1}$$

(b) *$X = A^{\textcircled{W}_m}$ is the unique solution of the system of equations:*

$$XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^{m+1})}, \ \mathcal{N}(X) = \mathcal{N}((A^k)^*A^m). \tag{4.2}$$

Proof. (a) By Theorems 3.6 (b) and 3.7 (a), it follows that $X = A^{\textcircled{W}_m}$ is a solution of the system of Eq (4.1). Conversely, if the system (4.1) is consistent, it follows that $(A^k)^*A^mAX = (A^k)^*A^m$. Hence by Theorem 4.1, $X = A^{\textcircled{W}_m}$ (d).
(b) By Theorems 3.6 (b) and 3.7 (b), it is evident that $X = A^{\textcircled{W}_m}$ is a solution of (4.2) . Next, we prove the uniqueness of the solution.
Assume that X_1, X_2 satisfy the system of Eq (4.2). Then $X_1A = X_2A$ and $\mathcal{N}(X_1) = \mathcal{N}(X_2) = \mathcal{N}((A^k)^*A^m)$. Thus, we get that $\mathcal{R}(X_1^* - X_2^*) \subseteq \mathcal{N}(A^*) \subseteq \mathcal{N}((A^k)^*)$ and $\mathcal{R}(X_1^* - X_2^*) \subseteq \mathcal{R}((A^m)^*A^k)$. For any $\eta \in \mathcal{N}((A^k)^*) \cap \mathcal{R}((A^m)^*A^k)$, we obtain that $(A^k)^*\eta = 0$, $\eta = (A^m)^*A^k\xi$ for some $\xi \in \mathbb{C}^n$. Since $\mathrm{Ind}(A) = k$, we derive that $\mathcal{R}(A^k) = \mathcal{R}(A^{k+m})$, and it follows that $A^k\xi = A^{k+m}\xi_0$ for some $\xi_0 \in \mathbb{C}^{n\times n}$. Then we have that

$$0 = (A^k)^*\eta = (A^{k+m})^*A^{k+m}\xi_0.$$

Premultiplying the equation above by ξ_0^*, we derive that $(A^{k+m}\xi_0)^*A^{k+m}\xi_0 = 0$, which implies $A^{k+m}\xi_0 = 0$. Hence $\eta = 0$, i.e., $\mathcal{R}(X_1^* - X_2^*) = \{0\}$, which implies $X_1 = X_2$. □

Remark 4.7. *Notice that the condition $\mathcal{R}(X) = \mathcal{R}(A^k)$ in Theorem 4.6 (a) can be replaced by $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$. Also the condition $\mathcal{N}(X) = \mathcal{N}((A^k)^*A^m)$ in Theorem 4.6 (b) can be replaced by $\mathcal{N}(X) \supseteq \mathcal{N}((A^k)^*A^m)$.*

5. Representations of the m-weak group inverse

From Theorem 3.1, we get an expression of $A^{\textcircled{W}_m}$ in terms of $A^{\textcircled{\dagger}}$. In the next results, we present several expressions of $A^{\textcircled{W}_m}$ in terms of certain generalized inverses.

Theorem 5.1. *Let $A \in \mathbb{C}_k^{n\times n}$ and let $m \in \mathbb{Z}^+$. Then the following statements hold:*

(a) $A^{\textcircled{W}_m} = (A^D)^{m+1} P_{A^k} A^m$.
(b) $A^{\textcircled{W}_m} = A^{k-m} (A^{k+1})^{\textcircled{\#}} A^m$ $(k \geq m)$.
(c) $A^{\textcircled{W}_m} = (A^k)^{\#} A^{k-m-1} P_{A^k} A^m$ $(k \geq m+1)$.
(d) $A^{\textcircled{W}_m} = (A^{m+1} P_{A^k})^{\dagger} A^m$.
(e) $A^{\textcircled{W}_m} = A^{m-1} P_{A^k} (A^m)^{\textcircled{W}}$.

Proof. Assume that A is given by (2.1). By (2.3)–(2.7) and (2.11)–(2.13), we get that

$$A^{m+1} P_{A^k} = U \begin{bmatrix} T^{m+1} & 0 \\ 0 & 0 \end{bmatrix} U^*, \tag{5.1}$$

$$(A^{m+1} P_{A^k})^{\dagger} = U \begin{bmatrix} T^{-m-1} & 0 \\ 0 & 0 \end{bmatrix} U^*, \tag{5.2}$$

$$(A^D)^{m+1} = U \begin{bmatrix} T^{-m-1} & T^{-2-m-k} T_k \\ 0 & 0 \end{bmatrix} U^*, \tag{5.3}$$

$$(A^k)^{\#} = U \begin{bmatrix} T^{-k} & T^{-2k} T_k \\ 0 & 0 \end{bmatrix} U^*, \tag{5.4}$$

$$(A^{k+1})^{\textcircled{\#}} = U \begin{bmatrix} T^{-k-1} & 0 \\ 0 & 0 \end{bmatrix} U^*, \tag{5.5}$$

$$(A^m)^{\textcircled{W}} = U \begin{bmatrix} T^{-m} & T^{-2m} T_m \\ 0 & 0 \end{bmatrix} U^*. \tag{5.6}$$

(a) By (2.12), (2.13) and (5.3), it follows that

$$\begin{aligned} (A^D)^{m+1} P_{A^k} A^m &= U \begin{bmatrix} T^{-1} & T^{-k-1} T_k \\ 0 & 0 \end{bmatrix}^{m+1} \begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^m & T_m \\ 0 & N^m \end{bmatrix} U^* \\ &= U \begin{bmatrix} T^{-1} & T^{-(m+1)} T_m \\ 0 & 0 \end{bmatrix} U^*. \end{aligned}$$

Hence $A^{\textcircled{W}_m} = (A^D)^{m+1} P_{A^k} A^m$.
The proofs of (b)–(e) are analogous to that of (a). □

Next, we consider the accuracy of the expression in Theorem 5.1 (a) for computing the m-weak group inverse.

Example 5.2. *Let*

$$A=\begin{bmatrix} 0.8485+0.1676i & 0.2540+0.5983i & 0.6425+0.9363i & 0.8275+0.4257i & 0.6969+0.8590i & 0.5510+0.6352i & 0.2347+0.9504i \\ 0.1680+0.6196i & 0.3756+0.1390i & 0.2441+0.4382i & 0.7763+0.2670i & 0.2739+0.3483i & 0.4470+0.4406i & 0.7125+0.1951i \\ 0.4884+0.8135i & 0.1611+0.8553i & 0.3944+0.4832i & 0.7271+0.4400i & 0.1657+0.8773i & 0.6828+0.6348i & 0.1984+0.7051i \\ 0.1033+0.3637i & 0.5945+0.2874i & 0.1809+0.4247i & 0.5422+0.2813i & 0.2855+0.1739i & 0.5710+0.6704i & 0.6415+0.3145i \\ 0.4750+0.7706i & 0.5137+0.4274i & 0.4025+0.1004i & 0.4356+0.3288i & 0.1589+0.1206i & 1.0058+0.7174i & 0.6422+0.1015i \\ 0.3229+0.7518i & 0.5552+0.4735i & 0.4742+0.2084i & 0.2175+0.6228i & 0.2705+0.1671i & 0.7580+0.5195i & 0.1824+0.6410i \\ 0.2069+0.0437i & 0.6633+0.5112i & 0.3382+0.8101i & 0.6209+0.2514i & 0.5148+0.5723i & 0.9051+0.5467i & 0.3012+0.3692i \end{bmatrix}.$$

Assume that A is given by (2.1). *Then*

$$U=\begin{bmatrix} 0.4825-0.0849i & 0.0469-0.1756i & 0.5890+0.3816i & -0.2856+0.1754i & -0.2172-0.1097i & -0.1298-0.0671i & 0.0533+0.1967i \\ 0.2893-0.0752i & -0.1794-0.3491i & -0.5178+0.1008i & -0.0098+0.1153i & -0.1657+0.2133i & -0.5416+0.2449i & 0.1310-0.1466i \\ 0.4536+0.0132i & 0.2822+0.1450i & -0.0509-0.0460i & 0.0355-0.1626i & 0.0535+0.7433i & 0.2983-0.1107i & 0.0663+0.0081i \\ 0.2796-0.0694i & -0.2498-0.2853i & -0.2797-0.1038i & 0.0927+0.0482i & 0.0299-0.2892i & 0.5633+0.0729i & 0.3269+0.3991i \\ 0.3355-0.1257i & -0.1231+0.3086i & -0.0681+0.0045i & 0.6071+0.0994i & 0.0793-0.1390i & -0.2447-0.2386i & -0.3902+0.2889i \\ 0.3535-0.0406i & 0.3291+0.4917i & -0.1044-0.1231i & -0.0927+0.2237i & 0.1346-0.3811i & -0.0086+0.2839i & 0.2660-0.3518i \\ 0.3446-0.0896i & -0.3255-0.0457i & 0.0798-0.3137i & -0.2170-0.5928i & -0.0771-0.1818i & 0.0470+0.0421i & -0.3504-0.3101i \end{bmatrix},$$

$$T=\begin{bmatrix} 3.1562+3.3883i & -0.4572+0.0272i & -0.3260-0.7290i & 0.1393+0.2261i \\ 0.0000+0.0000i & 0.4103-0.7005i & 0.3751+0.2669i & -0.0078+0.2713i \\ 0.0000+0.0000i & 0.0000+0.0000i & 0.6433-0.3382i & 0.4614+0.3799i \\ 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & -0.8309-0.2692i \end{bmatrix},$$

$$S=\begin{bmatrix} 0.2397-0.7281i & -0.1778+0.2997i & 0.6513+0.2162i \\ 0.1628-0.2162i & 0.0120+0.0676i & -0.0751-0.0869i \\ -0.1639+0.2880i & -0.0551-0.4207i & 0.2045+0.1170i \\ 0.3038-0.3764i & -0.1479+0.0719i & 0.0834+0.0878i \end{bmatrix}, N=\begin{bmatrix} 0.0000+0.0000i & -0.4926-0.4478i & 0.0000+0.0000i \\ 0.0000+0.0000i & 0.0000+0.0000i & 0.1329-0.0269i \\ 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i \end{bmatrix}.$$

It is clear that $k=\mathrm{Ind}(A)=3$. *According to* (2.12), (2.13), (3.2) *and* (5.3), *a straightforward computation shows that*

$$A^{\textcircled{W}_2}=\begin{bmatrix} 0.5500+0.0731i & -0.3216-0.5308i & 0.3542+0.2685i & 0.0240-0.0992i & 0.1347+0.1272i & -0.2551-0.5635i & -0.5893+0.1896i \\ -0.5670+0.5317i & 0.8843+0.3216i & -0.4879-0.7221i & -0.2015+0.4591i & -0.1916-0.3123i & 0.4252-0.0781i & 0.5168-0.0711i \\ 0.0613-0.3227i & -0.3442+0.1506i & 0.3922+0.1385i & -0.0511-0.3461i & 0.1347+0.1018i & 0.1026+0.0907i & -0.2209-0.0521i \\ -0.3141+0.2405i & 0.6100+0.3344i & -0.4033-0.5827i & -0.1523+0.3461i & -0.0966-0.1740i & 0.1994-0.0641i & 0.3864+0.0751i \\ 0.4276+0.1334i & -0.2170-0.1890i & -0.3404+0.1071i & 0.3855-0.1231i & -0.3630+0.1546i & 0.0708+0.1494i & 0.1807-0.5088i \\ -0.0347-0.3614i & -0.7335-0.1810i & 0.6827+0.3219i & 0.1600-0.4144i & 0.1112-0.1606i & -0.1645+0.4531i & 0.0694-0.0409i \\ 0.1126-0.5274i & 0.3706+0.0871i & -0.1810+0.2321i & -0.1211+0.0487i & 0.4600+0.2349i & -0.2612-0.3429i & -0.2482+0.4110i \end{bmatrix},$$

$$A^{D}=\begin{bmatrix} 0.5519+0.0637i & -0.3151-0.5243i & 0.3573+0.2679i & 0.0382-0.1188i & 0.1155+0.1148i & -0.2416-0.5478i & -0.6048+0.2051i \\ -0.5868+0.5393i & 0.8886+0.3016i & -0.4926-0.7271i & -0.2550+0.4561i & -0.1919-0.2619i & 0.4384-0.1220i & 0.5643-0.0607i \\ 0.0712-0.3336i & -0.3406+0.1643i & 0.3968+0.1401i & -0.0170-0.3611i & 0.1191+0.0704i & 0.1082+0.1221i & -0.2536-0.0436i \\ -0.3314+0.2454i & 0.6152+0.3178i & -0.4069-0.5873i & -0.1973+0.3394i & -0.1009-0.1314i & 0.2140-0.1002i & 0.4257+0.0876i \\ 0.4426+0.1457i & -0.2348-0.1834i & -0.3423+0.1131i & 0.4074-0.0792i & -0.3228+0.1320i & 0.0300+0.1588i & 0.1674-0.5512i \\ -0.0182-0.3618i & -0.7419-0.1674i & 0.6848+0.3269i & 0.1986-0.3982i & 0.1247-0.1976i & -0.1858+0.4820i & 0.0371-0.0610i \\ 0.1003-0.5340i & 0.3824+0.0806i & -0.1805+0.2276i & -0.1428+0.0207i & 0.4347+0.2565i & -0.2336-0.3553i & -0.2329+0.4392i \end{bmatrix},$$

$$P_{A^3}=\begin{bmatrix} 0.8779+0.0000i & -0.0446-0.1662i & 0.1193-0.1338i & -0.0432+0.0514i & -0.0818+0.1456i & 0.0604-0.0110i & 0.0519+0.0811i \\ -0.0446+0.1662i & 0.5351+0.0000i & 0.0316-0.1369i & 0.3696-0.0357i & 0.0621+0.1759i & -0.0570-0.1243i & 0.0416-0.0797i \\ 0.1193+0.1338i & 0.0316+0.1369i & 0.3389+0.0000i & 0.0285+0.0703i & 0.1692-0.1423i & 0.2953-0.0623i & 0.1557+0.0476i \\ -0.0432-0.0514i & 0.3696+0.0357i & 0.0285-0.0703i & 0.3267+0.0000i & 0.1249+0.1524i & -0.0766-0.0331i & 0.1585+0.0310i \\ -0.0818-0.1456i & 0.0621-0.1759i & 0.1692+0.1423i & 0.1249-0.1524i & 0.6219+0.0000i & 0.2074-0.0227i & -0.0447+0.1980i \\ 0.0604+0.0110i & -0.0570+0.1243i & 0.2953+0.0623i & -0.0766+0.0331i & 0.2074+0.0227i & 0.5614+0.0000i & -0.0864-0.2733i \\ 0.0519-0.0811i & 0.0416+0.0797i & 0.1557-0.0476i & 0.1585-0.0310i & -0.0447-0.1980i & -0.0864+0.2733i & 0.7381+0.0000i \end{bmatrix},$$

$$A^{2}=\begin{bmatrix} -0.7769+3.6386i & -0.5344+4.1046i & -0.4095+3.6111i & 0.7045+4.1930i & -0.6174+3.1904i & 0.1077+5.9761i & -0.2581+3.9581i \\ -0.4248+2.4587i & 0.2472+2.2693i & -0.2516+2.5356i & 0.6811+2.4977i & -0.2256+2.0334i & 0.6321+3.4819i & -0.0733+2.0920i \\ -1.3540+3.2996i & -0.9052+3.4946i & -1.2506+3.2411i & -0.2180+4.0436i & -1.1420+2.8767i & -0.9506+5.1147i & -1.1106+3.6001i \\ -0.5540+2.3649i & 0.2360+2.2815i & -0.2197+2.3588i & 0.5680+2.4478i & -0.1974+1.8996i & 0.5955+3.4055i & -0.0409+2.0276i \\ 0.0321+2.9089i & 0.5594+2.7931i & 0.0854+2.9168i & 0.9166+2.9714i & 0.1421+2.6006i & 1.0418+3.6923i & -0.1839+2.8256i \\ -0.3304+2.9125i & -0.3410+2.8783i & -0.8216+2.5238i & 0.3039+3.2251i & -0.7160+2.5512i & -0.0904+3.8501i & -0.6645+2.7485i \\ -0.8272+3.1061i & 0.1375+2.8116i & 0.0451+2.5280i & 0.6730+3.1529i & -0.3009+2.0044i & 0.6401+4.3322i & 0.2366+2.7781i \end{bmatrix}.$$

Let $K=(A^D)^3P_{A^3}A^2$. *Then*

$$K=\begin{bmatrix} 0.5500+0.0731i & -0.3216-0.5308i & 0.3542+0.2685i & 0.0240-0.0992i & 0.1347+0.1272i & -0.2551-0.5635i & -0.5893+0.1896i \\ -0.5670+0.5317i & 0.8843+0.3216i & -0.4879-0.7221i & -0.2015+0.4591i & -0.1916-0.3123i & 0.4252-0.0781i & 0.5168-0.0711i \\ 0.0613-0.3227i & -0.3442+0.1506i & 0.3922+0.1385i & -0.0511-0.3461i & 0.1347+0.1018i & 0.1026+0.0907i & -0.2209-0.0521i \\ -0.3141+0.2405i & 0.6100+0.3344i & -0.4033-0.5827i & -0.1523+0.3461i & -0.0966-0.1740i & 0.1994-0.0641i & 0.3864+0.0751i \\ 0.4276+0.1334i & -0.2170-0.1890i & -0.3404+0.1071i & 0.3855-0.1231i & -0.3630+0.1546i & 0.0708+0.1494i & 0.1807-0.5088i \\ -0.0347-0.3614i & -0.7335-0.1810i & 0.6827+0.3219i & 0.1600-0.4144i & 0.1112-0.1606i & -0.1645+0.4531i & 0.0694-0.0409i \\ 0.1126-0.5274i & 0.3706+0.0871i & -0.1810+0.2321i & -0.1211+0.0487i & 0.4600+0.2349i & -0.2612-0.3429i & -0.2482+0.4110i \end{bmatrix}$$

and

$$r_1=\| A^{\textcircled{W}_2}-K \|=6.6885\times 10^{-14},$$

where $\|\cdot\|$ *is the Frobenius norm.*

Hence, Theorem 5.1 (*a*) *gives a good result in terms of computational accuracy.*

In the following theorem, we present a connection between the (B, C)-inverse and the m-weak group inverse showing that the m-weak group inverse of $A \in \mathbb{C}_k^{n\times n}$ is its $(A^k, (A^k)^* A^m)$-inverse.

Theorem 5.3. *Let $A \in \mathbb{C}_k^{n\times n}$ and let $m \in \mathbb{Z}^+$. Then $A^{\textcircled{w}_m} = A^{(A^k,(A^k)^*A^m)}$.*

Proof. By Theorem 3.7 we have that $A^{\textcircled{w}_m} AA^k = A^k$ and $((A^k)^* A^m)AA^{\textcircled{w}_m} = (A^k)^* A^m$. From Theorem 3.6 (b), we derive that $\mathcal{R}(A^{\textcircled{w}_m}) = \mathcal{R}(A^k)$ and $\mathcal{N}(A^{\textcircled{w}_m}) = \mathcal{N}((A^k)^* A^m)$. Evidently, $A^{\textcircled{w}_m} = A^{(A^k,(A^k)^*A^m)}$. □

Now we will give some limit expressions of $A^{\textcircled{w}_m}$, but before we need the next auxiliary lemma:

Lemma 5.4. [37] *Let $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times p}$ and $Y \in \mathbb{C}^{p\times m}$. Then the following hold:*

(a) $\lim_{\lambda\to 0} X(\lambda I_p + YAX)^{-1} Y$ *exists;*
(b) $r(XYAXY) = r(XY)$;
(c) $A^{(2)}_{\mathcal{R}(XY),\mathcal{N}(XY)}$ *exists,*

in which case,

$$\lim_{\lambda\to 0} X(\lambda I_p + YAX)^{-1} Y = A^{(2)}_{\mathcal{R}(XY),\mathcal{N}(XY)}.$$

Theorem 5.5. *Let $A \in \mathbb{C}_k^{n\times n}$ be given by* (2.1) *and let $m \in \mathbb{N}^+$. Then the following statements hold:*

(a) $A^{\textcircled{w}_m} = \lim_{\lambda\to 0} A^k(\lambda I_n + (A^k)^* A^{k+m+1})^{-1}(A^k)^* A^m$;
(b) $A^{\textcircled{w}_m} = \lim_{\lambda\to 0} A^k(A^k)^*(\lambda I_n + A^{k+m+1}(A^k)^*)^{-1} A^m$;
(c) $A^{\textcircled{w}_m} = \lim_{\lambda\to 0} A^k(A^k)^* A^m(\lambda I_n + A^{k+1}(A^k)^* A^m)^{-1}$;
(d) $A^{\textcircled{w}_m} = \lim_{\lambda\to 0} (\lambda I_n + A^k(A^k)^* A^{m+1})^{-1} A^k(A^k)^* A^m$.

Proof. (a) It is easy to check that $r(A^k(A^k)^* A^m) = r((A^k)^* A^m) = r(A^k) = t$. By Theorem 3.6, we get that $\mathcal{R}(A^k) = \mathcal{R}(A^k(A^k)^* A^m)$, $\mathcal{N}((A^k)^* A^m) = \mathcal{N}(A^k(A^k)^* A^m)$. From Theorem 3.6, we get

$$A^{\textcircled{w}_m} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)} = A^{(2)}_{\mathcal{R}(A^k(A^k)^*A^m),\mathcal{N}(A^k(A^k)^*A^m)}.$$

Let $X = A^k$, $Y = (A^k)^* A^m$. By Lemma 5.4, we get that

$$A^{\textcircled{w}_m} = \lim_{\lambda\to 0} A^k(\lambda I_n + (A^k)^* A^{k+m+1})^{-1}(A^k)^* A^m.$$

The statements (b)–(d) can be similarly proved. □

The following example will test the accuracy of expression in Theorem 5.5 (a) for computing the m-weak group inverse.

Example 5.6. *Let*

$$A = \begin{bmatrix} 4.8990 + 7.3786i & 6.8197 + 3.0145i & 7.2244 + 1.2801i & 4.5380 + 1.9043i & 8.3138 + 3.7627i & 6.2797 + 3.8462i & 3.7241 + 9.8266i \\ 1.6793 + 2.6912i & 0.4243 + 7.0110i & 1.4987 + 9.9908i & 4.3239 + 3.6892i & 8.0336 + 1.9092i & 2.9198 + 5.8299i & 1.9812 + 7.3025i \\ 9.7868 + 4.2284i & 0.7145 + 6.6634i & 6.5961 + 1.7112i & 8.2531 + 4.6073i & 0.6047 + 4.2825i & 4.3165 + 2.5181i & 4.8969 + 3.4388i \\ 7.1269 + 5.4787i & 5.2165 + 5.3913i & 5.1859 + 0.3260i & 0.8347 + 9.8164i & 3.9926 + 4.8202i & 0.1549 + 2.9044i & 3.3949 + 5.8407i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 9.8406 + 6.1709i & 0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 9.2033 + 9.0631i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \end{bmatrix}$$

with $k =\mathrm{Ind}(A) = 3$. By $A^{\textcircled{\dagger}} = (A^{k+1}(A^k)^\dagger)^\dagger$, we get

$$A^{\textcircled{\dagger}} \;=\; (A^4(A^3)^\dagger)^\dagger$$

$$= \begin{bmatrix} -0.0032-0.1411i & 0.0709+0.0429i & 0.0106-0.0097i & 0.0187+0.0560i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i \\ 0.0878+0.0001i & 0.0438+0.0326i & -0.1096-0.0524i & 0.0490-0.0158i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i \\ -0.0896+0.0173i & -0.0314-0.1473i & 0.0445+0.0629i & 0.0118+0.0056i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i \\ 0.0431+0.0788i & -0.0348+0.0182i & 0.0536+0.0096i & -0.0658-0.0844i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i \\ 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i \\ 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i \\ 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i \end{bmatrix}.$$

Together with (3.2)*, it follows that*

$$\begin{aligned} A^{\textcircled{w}_2} &= (A^{\textcircled{\dagger}})^3A^2 \\ &= \begin{bmatrix} -0.0032-0.1411i & 0.0709+0.0429i & 0.0106-0.0097i & 0.0187+0.0560i & -0.0151-0.0310i & -0.2502+0.2087i & -0.1223+0.0627i \\ 0.0878+0.0001i & 0.0438+0.0326i & -0.1096-0.0524i & 0.0490-0.0158i & 0.1788+0.2332i & -0.3818+0.4296i & -0.1105+0.0067i \\ -0.0896+0.0173i & -0.0314-0.1473i & 0.0445+0.0629i & 0.0118+0.0056i & -0.0415-0.2864i & 0.7974-0.1768i & 0.2687+0.0926i \\ 0.0431+0.0788i & -0.0348+0.0182i & 0.0536+0.0096i & -0.0658-0.0844i & 0.0125-0.0262i & 0.0972-0.3512i & 0.0454-0.1317i \\ 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i \\ 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i \\ 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i \end{bmatrix}. \end{aligned}$$

Let $L = \lim_{\lambda\to 0} A^3(\lambda I_n + (A^3)^*A^6)^{-1}(A^3)^*A^2$*. Then*

$$L = \begin{bmatrix} -0.003189-0.1411i & 0.07088+0.04294i & 0.01062-0.009663i & 0.01868+0.05603i & -0.01512-0.03101i & -0.2502+0.2087i & -0.1223+0.06266i \\ 0.08776+5.907e-5i & 0.04379+0.03259i & -0.1096-0.05243i & 0.04899-0.01576i & 0.1788+0.2332i & -0.3818+0.4296i & -0.1105+0.006749i \\ -0.08964+0.01729i & -0.03136-0.1473i & 0.04449+0.06285i & 0.01184+0.005646i & -0.04154-0.2864i & 0.7974-0.1768i & 0.2687+0.09263i \\ 0.04314+0.07879i & -0.03482+0.01822i & 0.05364+0.009637i & -0.06583-0.08442i & 0.01248-0.02618i & 0.09717-0.3512i & 0.04544-0.1317i \\ 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i \\ 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i \\ 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i \end{bmatrix}$$

and

$$r_2 = \| A^{\textcircled{w}_2} - L \| = 6.136 \times 10^{-11},$$

where $\|\cdot\|$ *is the Frobenius norm. Hence, the representation in Theorem 5.5 (a) is efficient for computing the m-weak group inverse.*

6. Relationships between the m-weak group inverse and other generalized inverses

In this section, we consider some relations between the m-weak group inverse and other generalized inverses as well as certain matrix classes. The symbols $\mathbb{C}_n^{\mathrm{OP}}$, $\mathbb{C}_n^{\mathrm{EP}}$, $\mathbb{C}_n^{i-\mathrm{EP}}$ and $\mathbb{C}_n^{k,\textcircled{\dagger}}$ stand for the subsets of $\mathbb{C}^{n\times n}$ consisting of orthogonal projectors (Hermitian idempotent matrices), EP (Range-Hermitian) matrices, i-EP matrices and k-core-EP matrices, respectively, i.e.,

$$\begin{aligned} \mathbb{C}_n^{\mathrm{OP}} &= \{A | A \in \mathbb{C}^{n\times n}, A^2 = A = A^*\} = \{A | A \in \mathbb{C}^{n\times n}, A^2 = A = A^\dagger\}, \\ \mathbb{C}_n^{\mathrm{EP}} &= \{A | A \in \mathbb{C}^{n\times n}, AA^\dagger = A^\dagger A\} = \{A | A \in \mathbb{C}^{n\times n}, \mathcal{R}(A) = \mathcal{R}(A^*)\}, \\ \mathbb{C}_n^{i-\mathrm{EP}} &= \{A | A \in \mathbb{C}_k^{n\times n}, A^k(A^k)^\dagger = (A^k)^\dagger A^k\}, \\ \mathbb{C}_n^{k,\textcircled{\dagger}} &= \{A | A \in \mathbb{C}_k^{n\times n}, A^kA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}A^k\}. \end{aligned}$$

First, we will state the following lemma auxiliary lemma:

Lemma 6.1. *Let* $A \in \mathbb{C}_k^{n\times n}$ *be given by* (2.1)*. Then* $T_m = 0$ *if and only if* $S = 0$*.*

Proof. Notice that $T_m = 0$ can be equivalently expressed by the equation below:

$$T^{m-1}S + T^{m-2}SN + \cdots + TSN^{m-2} + SN^{m-1} = 0. \tag{6.1}$$

Multiplying the equation above from the right side by N^{k-1}, we get $SN^{k-1} = 0$. Then multiplying from the right by N^{k-2}, we get $SN^{k-1} = 0$,. Similarly, we get $SN^{k-3} = 0, \cdots, SN = 0$. Now by (6.1), it follows that $T^{m-1}S = 0$, i.e, $S = 0$. □

The next theorem provides some necessary and sufficient conditions for $A^{\textcircled{W}_m}$ to be equal to various transformations of $A \in \mathbb{C}_k^{n\times n}$.

Theorem 6.2. *Let $A \in \mathbb{C}_k^{n\times n}$ and let $m \in \mathbb{Z}^+$. Then the following statements hold:*

(a) $A^{\textcircled{W}_m} \in A\{1\}$ *if and only if* $A \in \mathbb{C}_n^{\mathrm{CM}}$.
(b) $A^{\textcircled{W}_m} \in \mathbb{C}_n^{\mathrm{CM}}$.
(c) $A^{\textcircled{W}_m} = A$ *if and only if* $A = A^3$.
(d) $A^{\textcircled{W}_m} = A^*$ *if and only if* $AA^* \in \mathbb{C}_n^{\mathrm{OP}}$ *and* $A \in \mathbb{C}_n^{\mathrm{EP}}$.
(e) $A^{\textcircled{W}_m} = P_A$ *if and only if* $A \in \mathbb{C}_n^{\mathrm{OP}}$.

Proof. Let A be given by (2.1).

(a) By (3.2), it follows that

$$\begin{aligned} A^{\textcircled{W}_m} \in A\{1\} &\iff AA^{\textcircled{W}_m}A = A \\ &\iff U\begin{bmatrix} T & S + T^{-m}T_mN \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^* \\ &\iff N = 0 \\ &\iff A \in \mathbb{C}_n^{\mathrm{CM}}. \end{aligned}$$

(b) By (3.2), it is clear that $r(A^{\textcircled{W}_m}) = r((A^{\textcircled{W}_m})^2) = t$, which implies $A^{\textcircled{W}_m} \in \mathbb{C}_n^{\mathrm{CM}}$.

(c) From (3.2), we get that

$$\begin{aligned} A^{\textcircled{W}_m} = A &\iff U\begin{bmatrix} T^{-1} & (T^{m+1})^{-1}T_m \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^* \\ &\iff T^2 = I_t \text{ and } N = 0 \\ &\iff A = A^3. \end{aligned}$$

(d) According to (3.2), we obtain that

$$\begin{aligned} A^{\textcircled{W}_m} = A^* &\iff U\begin{bmatrix} T^{-1} & (T^{m+1})^{-1}T_m \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^* & 0 \\ S^* & N^* \end{bmatrix}U^* \\ &\iff T^{-1} = T^*, S = 0 \text{ and } N = 0 \\ &\iff AA^* \in \mathbb{C}_n^{\mathrm{OP}} \text{and } A \in \mathbb{C}_n^{\mathrm{EP}}. \end{aligned}$$

(e) By (2.9) and (3.2), it follows that

$$\begin{aligned} A^{\textcircled{W}_m} = P_A &\iff U\begin{bmatrix} T^{-1} & (T^{m+1})^{-1}T_m \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} I_t & 0 \\ 0 & NN^\dagger \end{bmatrix}U^* \\ &\iff T = I_t, NN^\dagger = 0 \text{ and } T_m = 0 \\ &\iff T = I_t, S = 0 \text{ and } N = 0. \end{aligned}$$

Hence $A^{\textcircled{W}_m} = P_A$ if and only if $A \in \mathbb{C}_n^{\mathrm{OP}}$. □

Using the core-EP decompositio, we proved that $A \in \mathbb{C}_n^{i-\mathrm{EP}}$ if and only if $A^{\textcircled{W}_m} \in \mathbb{C}_n^{\mathrm{EP}}$. Therefore, we will consider certain equivalent conditions for $A^{\textcircled{W}_m} \in \mathbb{C}_n^{\mathrm{EP}}$.

Lemma 6.3. [17] *Let $A \in \mathbb{C}_k^{n\times n}$ be of the form* (2.1). *Then $A \in \mathbb{C}_n^{i\text{–EP}}$ if and only if $S = 0$.*

Moreover, $S = 0$ if and only if $A \in \mathbb{C}_n^{k,\textcircled{\dagger}}$.

Theorem 6.4. *Let $A \in \mathbb{C}_k^{n\times n}$ and let $m \in \mathbb{Z}^+$. The following statements are equivalent:*

(a) $A^{\textcircled{W}_m} \in \mathbb{C}_n^{\mathrm{EP}}$;
(b) $A \in \mathbb{C}_n^{i\text{–EP}}$;
(c) $A^{\textcircled{W}} \in \mathbb{C}_n^{\mathrm{EP}}$;
(d) $A^{\textcircled{W}_m} = A^{\textcircled{\dagger}}$;
(e) $AA^{\textcircled{W}_m} = AA^{\textcircled{\dagger}}$.

Proof. Let $A \in \mathbb{C}_k^{n\times n}$ be of the form (2.1). According to Lemma 6.3, we will prove that each of the statements (a), (c), (d) and (e) is equivalent to $S = 0$.
(a) According to (3.2) and Lemma 6.1, it follows that

$$\begin{aligned} A^{\textcircled{W}_m} \in \mathbb{C}_n^{\mathrm{EP}} &\iff \mathcal{R}(A^{\textcircled{W}_m}) = \mathcal{R}((A^{\textcircled{W}_m})^*) \\ &\iff (T^{m+1})^{-1}T_m = 0 \\ &\iff S = 0. \end{aligned}$$

(c) By (2.4), we get that

$$\begin{aligned} A^{\textcircled{W}} \in \mathbb{C}_n^{\mathrm{EP}} &\iff \mathcal{R}(A^{\textcircled{W}}) = \mathcal{R}((A^{\textcircled{W}})^*) \\ &\iff T^{-2}S = 0 \\ &\iff S = 0. \end{aligned}$$

(d) By (2.3), (3.2) and Lemma 6.1, it follows that

$$\begin{aligned} A^{\textcircled{W}_m} = A^{\textcircled{\dagger}} &\iff U\begin{bmatrix} T^{-1} & (T^{m+1})^{-1}T_m \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* \\ &\iff (T^{m+1})^{-1}T_m = 0 \\ &\iff S = 0. \end{aligned}$$

(e) From (2.3), (3.2) and Lemma 6.1, we get that

$$\begin{aligned} AA^{\textcircled{W}_m} = AA^{\textcircled{\dagger}} &\iff U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\begin{bmatrix} T^{-1} & (T^{m+1})^{-1}T_m \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* \\ &\iff T^{-m}T_m = 0 \\ &\iff S = 0. \end{aligned}$$

□

In [22], the authors proved that $A^{\textcircled{W}_m} = A^D$ if and only if $SN^m = 0$. In the following results, we investigate the relation between the m-weak group inverse and other generalized inverses such as the MP-inverse, group inverse, core inverse, DMP-inverse, dual DMP-inverse, weak group inverse by core-EP decomposition.

Theorem 6.5. *Let $A \in \mathbb{C}_k^{n\times n}$ be given by* (2.1) *and let $m \in \mathbb{Z}^+$. Then the following statements hold:*

(a) $A^{\textcircled{W}_m} = A^\dagger \Longleftrightarrow A \in \mathbb{C}_n^{\mathrm{EP}}$;
(b) $A^{\textcircled{W}_m} = A^\# \Longleftrightarrow A \in \mathbb{C}_n^{\mathrm{CM}}$;
(c) $A^{\textcircled{W}_m} = A^{\textcircled{\#}} \Longleftrightarrow A \in \mathbb{C}_n^{\mathrm{CM}}$;
(d) $A^{\textcircled{W}_m} = A^{D,\dagger} \Longleftrightarrow T^{k-m}T_m = T_k NN^\dagger$;
(e) $A^{\textcircled{W}_m} = A^{\dagger,D} \Longleftrightarrow SN^m = 0$ *and* $S = SN^\dagger N$;
(f) $A^{\textcircled{W}_m} = A^{\textcircled{W}} \Longleftrightarrow SN = 0$ ($m > 1$).

Proof. (a) It follows from (2.8) and (3.2) that

$$\begin{aligned} A^{\textcircled{W}_m} = A^\dagger \quad &\Longleftrightarrow \quad U\begin{bmatrix} T^{-1} & (T^{m+1})^{-1}T_m \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^*\triangle & -T^*\triangle SN^\dagger \\ M^*\triangle & N^\dagger - M^*\triangle SN^\dagger \end{bmatrix}U^* \\ &\Longleftrightarrow \quad M^* = 0, N^\dagger = 0, T^{-1} = T^*\triangle \text{ and } (T^{m+1})^{-1}T_m = -T^*\triangle SN^\dagger \\ &\Longleftrightarrow \quad S = 0 \text{ and } N = 0 \\ &\Longleftrightarrow \quad A \in \mathbb{C}_n^{\mathrm{EP}}. \end{aligned}$$

(b) Since $A^\#$ exits if and only if $A \in \mathbb{C}_n^{\mathrm{CM}}$, which is equivalent to $N = 0$, we get by (2.6) and (3.2) the following:

$$\begin{aligned} A^{\textcircled{W}_m} = A^\# \quad &\Longleftrightarrow \quad U\begin{bmatrix} T^{-1} & (T^{m+1})^{-1}T_m \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^* \text{ and } N = 0 \\ &\Longleftrightarrow \quad (T^{m+1})^{-1}T_m = T^{-2}S \text{ and } N = 0 \\ &\Longleftrightarrow \quad N = 0 \\ &\Longleftrightarrow \quad A \in \mathbb{C}_n^{\mathrm{CM}}. \end{aligned}$$

(c) The proof follows similarly as in (b).

(d) Using (2.5) and (2.9) to $A^{D,\dagger} = A^D AA^\dagger$, we derive

$$A^{D,\dagger} = \begin{bmatrix} T^{-1} & (T^{k+1})^{-1}T_k NN^\dagger \\ 0 & 0 \end{bmatrix},$$

and by (3.2), it follows that

$$\begin{aligned} A^{\textcircled{W}_m} = A^{D,\dagger} \quad &\Longleftrightarrow \quad U\begin{bmatrix} T^{-1} & (T^{m+1})^{-1}T_m \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^{-1} & (T^{k+1})^{-1}T_k NN^\dagger \\ 0 & 0 \end{bmatrix}U^* \\ &\Longleftrightarrow \quad T^{k-m}T_m = T_k NN^\dagger. \end{aligned}$$

(e) Using (2.5) and (2.10) and the faact that $A^{\dagger,D} = A^\dagger AA^D$, we obtain that

$$A^{\dagger,D} = \begin{bmatrix} T^*\triangle & -T^*\triangle T^{-k}T_k \\ M^*\triangle & M^*\triangle T^{-k}T_k \end{bmatrix},$$

which together with (3.2), gives

$$A^{\textcircled{W}_m} = A^{\dagger,D} \quad \Longleftrightarrow \quad U\begin{bmatrix} T^{-1} & (T^{m+1})^{-1}T_m \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^*\triangle & T^*\triangle T^{-k}T_k \\ M^*\triangle & M^*\triangle T^{-k}T_k \end{bmatrix}U^*$$

$$
\begin{aligned}
&\iff \quad M^* = 0, T^{-1} = T^*\triangle \text{ and } (T^{m+1})^{-1}T_m = T^*\triangle T^{-k}T_k\\
&\iff \quad S = SN^\dagger N \text{ and } T^{k-m}T_m = T_k\\
&\iff \quad S = SN^\dagger N \text{ and } SN^m = 0.
\end{aligned}
$$

(f) If $m > 1$, from (2.4) and (3.2), we get

$$
\begin{aligned}
A^{\textcircled{W}_m} = A^{\textcircled{W}} \quad &\iff \quad U\begin{bmatrix} T^{-1} & (T^{m+1})^{-1}T_m \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^*\\
&\iff \quad (T^{m+1})^{-1}T_m = T^{-2}S.
\end{aligned}
$$

Clearly, $(T^{m+1})^{-1}T_m = T^{-2}S$ is equivalent to $T^{-3}SN + \cdots + (T^{m+1})^{-1}SN^{m-1} = 0$, which is further equivalent to $SN = 0$. Hence $A^{\textcircled{W}_m} = A^{\textcircled{W}}$ if and only if $SN = 0$. □

7. Applications of the m-weak group inverse

In this section, we consider a relation between the m-weak group inverse and the nonsingular bordered matrix, which will be applied to the Cramer's rule for the solution of the restricted matrix equation.

Theorem 7.1. *Let $A \in \mathbb{C}_k^{n\times n}$ be such that $r(A^k) = t$ and let $m \in \mathbb{Z}^+$. Let $B \in \mathbb{C}^{n\times(n-t)}$ and $C^* \in \mathbb{C}^{n\times(n-t)}$ be of full column rank such that $\mathcal{N}((A^k)^*A^m) = \mathcal{R}(B)$ and $\mathcal{R}(A^k) = \mathcal{N}(C)$. Then the bordered matrix*

$$
K = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}
$$

is invertible and its inverse is given by

$$
K^{-1} = \begin{bmatrix} A^{\textcircled{W}_m} & (I_n - A^{\textcircled{W}_m}A)C^\dagger \\ B^\dagger(I_n - AA^{\textcircled{W}_m}) & B^\dagger(AA^{\textcircled{W}_m}A - A)C^\dagger \end{bmatrix}.
$$

Proof. Let $X = \begin{bmatrix} A^{\textcircled{W}_m} & (I_n - A^{\textcircled{W}_m}A)C^\dagger \\ B^\dagger(I_n - AA^{\textcircled{W}_m}) & B^\dagger(AA^{\textcircled{W}_m}A - A)C^\dagger \end{bmatrix}$. Since $\mathcal{R}(A^{\textcircled{W}_m}) = \mathcal{R}(A^k) = \mathcal{N}(C)$, we have that $CA^{\textcircled{W}_m} = 0$. Since C is a full row rank matrix, it is right invertible and $CC^\dagger = I_{n-t}$. From

$$
\mathcal{R}(I_n - AA^{\textcircled{W}_m}) = \mathcal{N}(AA^{\textcircled{W}_m}) = \mathcal{N}((A^k)^*A^m) = \mathcal{R}(B) = \mathcal{R}(BB^\dagger),
$$

we get $BB^\dagger(I_n - AA^{\textcircled{W}_m}) = I_n - AA^{\textcircled{W}_m}$. Hence,

$$
\begin{aligned}
KX &= \begin{bmatrix} AA^{\textcircled{W}_m} + BB^\dagger(I_n - AA^{\textcircled{W}_m}) & A(I_n - A^{\textcircled{W}_m}A)C^\dagger + BB^\dagger(AA^{\textcircled{W}_m}A - A)C^\dagger \\ CA^{\textcircled{W}_m} & C(I_n - A^{\textcircled{W}_m}A)C^\dagger \end{bmatrix}\\
&= \begin{bmatrix} AA^{\textcircled{W}_m} + I_n - AA^{\textcircled{W}_m} & A(I_n - A^{\textcircled{W}_m}A)C^\dagger - (I_n - AA^{\textcircled{W}_m})AC^\dagger \\ 0 & CC^\dagger \end{bmatrix}\\
&= \begin{bmatrix} I_n & 0 \\ 0 & I_{n-t} \end{bmatrix}.
\end{aligned}
$$

Thus, $X = K^{-1}$. □

In the next result, we will discuss the solution of the restricted matrix equation

$$AX = D, \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^k), \tag{7.1}$$

using the m-weak group inverse.

Theorem 7.2. *Let $A \in \mathbb{C}_k^{n\times n}$, $X \in \mathbb{C}^{n\times p}$ and $D \in \mathbb{C}^{n\times p}$. If $\mathcal{R}(D) \subseteq \mathcal{R}(A^k)$, then the restricted matrix equation*

$$AX = D, \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^k) \tag{7.2}$$

has a unique solution $X = A^{\textcircled{w}_m} D$.

Proof. Since $\mathcal{R}(A^k) = \mathcal{R}(AA^k)$ and $\mathcal{R}(D) \subseteq \mathcal{R}(A^k)$, we get that $\mathcal{R}(D) \subseteq A\mathcal{R}(A^k)$, which implies solvability of the matrix Eq (7.1). Obviously, $X = A^{\textcircled{w}_m} D$ is a solution of (7.1). Then we prove the uniqueness of X. If X_1 also satisfies (7.1), then

$$X = A^{\textcircled{w}_m} D = A^{\textcircled{w}_m} AX_1 = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)} X_1 = X_1.$$

□

Based on the nonsingularity of the bordered matrix given in Theorem 7.1, we will show in the next theorem how the Cramer's rule can be used for solving the restricted matrix Eq (7.1).

Theorem 7.3. *Let $A \in \mathbb{C}_k^{n\times n}$ be such that $r(A^k) = t$ and let $X \in \mathbb{C}^{n\times p}$ and $D \in \mathbb{C}^{n\times p}$. Let $B \in \mathbb{C}^{n\times(n-t)}$ and $C^* \in \mathbb{C}^{n\times(n-t)}$ be full column rank matrices such that $\mathcal{N}((A^k)^*A^m) = \mathcal{R}(B)$ and $\mathcal{R}(A^k) = \mathcal{N}(C)$. Then the unique solution of the restricted matrix Eq (7.1) is given by $X = [x_{ij}]$, where*

$$x_{ij} = \frac{det\begin{bmatrix} A(i \to d_j) & B \\ C(i \to 0) & 0 \end{bmatrix}}{det\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}}, \ i = 1, 2, ..., n, \ j = 1, 2, ..., m, \tag{7.3}$$

where d_j denotes the j-th column of D.

Proof. Since X is the solution of the restricted matrix Eq (7.1), we get that $\mathcal{R}(X) \subseteq \mathcal{R}(A^k) = \mathcal{N}(C)$, which implies $CX = 0$. Then the restricted matrix Eq (7.1) can be rewritten as

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\begin{bmatrix} X \\ 0 \end{bmatrix} = \begin{bmatrix} AX \\ CX \end{bmatrix} = \begin{bmatrix} D \\ 0 \end{bmatrix}.$$

By Theorem 7.1, we have that $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ is invertible. Consequently, (7.2) follows from the Cramer's rule for the above equation. □

Example 7.4. *Let*

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 10 & 14 & 24 & 28 \\ 6 & 19 & 20 & 22 \\ 4 & 10 & 14 & 15 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 3 & 6 \\ -2 & -4 & -7 \\ 1 & 2 & 4 \\ -1 & -3 & -6 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

It can be verified that $\mathrm{Ind}(A) = 3$. *Then we get that*

$$A^3 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, A^{\textcircled{\dagger}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, A^{\textcircled{w}_2} = (A^{\textcircled{\dagger}})^3 A^2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to check that

$$X = A^{\textcircled{w}_2} D = \begin{bmatrix} 10 & 14 & 24 & 28 \\ 6 & 19 & 20 & 22 \\ 4 & 10 & 14 & 15 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

satisfies the restricted matrix equation $AX = D$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}(A^3)$. *By simple calculations, we can also get that the components of* X *can be expressed by* (7.2).

8. Conclusions

This paper gives a new definition of the m-weak group inverse for the complex matrices, which extends the Drazin inverse and the weak group inverse. Some characterizations of the m-weak group inverse in terms of the range space, null space, rank, and projectors are presented. Several representations of the m-weak group inverse involving some known generalized inverses as well as limitations are also derived. The representation in Theorem 5.1 gives a better result in terms of the computational accuracy (see Examples 5.2 and 5.6). The m-weak group inverses are concerned with the solution of a restricted matrix Eq (7.1). The solution of (7.1) can also be expressed by the Cramer's rule (see Theorem 7.3). In [38–40], there are some iterative methods and algorithms to compute the outer inverses. Motivated by these, further investigations deserve more attention as follows:

(1) The applications of the m-weak group inverse in linear equations and matrix equations;
(2) Perturbation formulae as well as perturbation bounds for the m-weak group inverse;
(3) Iterative algorithm, a splitting method for computing the m-weak group inverse;
(4) Other representations of the m-weak group inverse.

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Conflict of interest

The authors declare no conflict of interest.

References

1. M. P. Drazin, Pseudo-inverses in associative rings and semigroups, *Am. Math. Mon.*, **65** (1958), 506–514. https://doi.org/10.2307/2308576
2. Y. Wei, G. Wang, The perturbation theory for the Drazin inverse and its applications, *Linear Algebra Appl.*, **258** (1997), 179–186. https://doi.org/10.1016/s0024-3795(96)00159-0
3. R. E. Hartwig, G. Wang, Y. Wei, Some additive results on Drazin inverse, *Linear Algebra Appl.*, **322** (2001), 207–217. https://doi.org/10.1016/j.amc.2009.05.021
4. D. S. Cvetković-Ilić, Xiaoji Liu, Y. Wei, Some additive results for the generalized Drazin inverse in a Banach algebra, *Electronic J. Linear Algebra*, **22** (2011), 1049–1058.
5. D. S. Cvetković-Ilić, New additive results on Drazin inverse and its applications, *Appl. Math. Comp.*, **218** (2011), 3019–3024. https://doi.org/10.1016/j.amc.2011.08.083
6. N. P. Karampetakis, S. Vologiannidis, DFT calculation of the generalized and Drazin inverse of a polynomial matrix, *Appl. Math. Comput.*, **143** (2003), 501–521. https://doi.org/10.1016/S0096-3003(02)00377-6
7. N. P. Karampetakis, P. S. Stanimirović, M. B. Tasić, On the computation of the Drazin inverse of a polynomial matrix, *Far East J. Math. Sci.*, **26** (2007), 1–24. https://doi.org/10.1016/s1474-6670(17)38994-2
8. P. S. Stanimirović, V. N. Katsikis, H. Ma, Representations and properties of the W-Weighted Drazin inverse, *Linear Multilinear A.*, **65** (2017), 1080–1096. https://doi.org/10.1080/03081087.2016.1228810
9. I. I. Kyrchei, Weighted quaternion core-EP, DMP, MPD, and CMP inverses and their determinantal representations, *RACSAM Rev. R. Acad. A*, **114** (2020), 198. https://doi.org/10.1007/s13398-020-00930-3
10. S. B. Malik, N. Thome, On a new generalized inverse for matrices of an arbitrary index, *Appl. Math. Comput.*, **226** (2014), 575–580. https://doi.org/10.1016/j.amc.2013.10.060
11. K. Zuo, D. S. Cvetković-Ilić, Y. Cheng, Different characterizations of DMP-inverse of matrices, *Linear Multilinear A.*, **70** (2022), 411–418 . https://doi.org/10.1080/03081087.2020.1729084
12. L. Meng, The DMP inverse for rectangular matrices, *Filomat*, **31** (2017), 6015–6019. https://doi.org/10.2298/FIL1719015M
13. I. I. Kyrchei, D. Mosić, P. S. Stanimirović, MPD-DMP-solutions to quaternion two-sided restricted matrix equations, *Comput. Appl. Math.*, **40** (2021), 177. https://doi.org/10.1007/s40314-021-01566-8
14. D. S. Cvetković-Ilić, Y. Wei, *Algebraic properties of generalized inverses*, In: Developments on Mathematics, Springer, Singapore, **52** (2017). http://dx.doi.org/10.1007/978-981-10-6349-7

15. H. Wang, J. Chen, Weak group inverse, *Open Math.*, **16** (2018), 1218–1232. https://doi.org/10.1515/math-2018-0100

16. H. Wang, Core-EP decomposition and its applications, *Linear Algebra Appl.*, **508** (2016), 289–300. https://doi.org/10.1016/j.laa.2016.08.008

17. H. Wang, X. Liu, The weak group matrix, *Aequationes Math.*, **93** (2019), 1261–1273. https://doi.org/10.1007/s00010-019-00639-8

18. M. Zhou, J. Chen, Y. Zhou, Weak group inverses in proper *-rings, *J. Algebra Appl.*, **19** (2020). https://doi.org/10.1142/S0219498820502382

19. Y. Zhou, J. Chen, M. Zhou, m-weak group inverses in a ring with involution, *RACSAM Rev. R. Acad. A*, **115** (2021). https://doi.org/10.1007/s13398-020-00932-1

20. M. Zhou, J. Chen, Y. Zhou, N. Thome, Weak group inverses and partial isometries in proper *-rings, *Linear Multilinear A.*, 2021. https://doi.org/10.1080/03081087.2021.1884639

21. D. E. Ferreyra, V. Orquera, N. Thome, A weak group inverse for rectangular matrices, *RACSAM Rev. R. Acad. A*, **113** (2019), 3727–3740. https://doi.org/10.1007/s13398-019-00674-9

22. Y. Zhou, J. Chen, Weak core inverses and pseudo core inverses in a ring with involution, *Linear Multilinear A.*, 2021. https://doi.org/10.1080/03081087.2021.1971151

23. R. Penrose, A generalized inverse for matrices, *Math. Proc. Cambridge Philos. Soc.*, **51** (1955), 406–413. https://doi.org/10.1017/S0305004100030401

24. G. Wang , Y. Wei , S. Qiao, *Generalized inverses: Theory and computations*, In: Developments on Mathematics, Springer and Beijing, Science Press, **53** (2018). http://dx.doi.org/10.1007/978-981-13-0146-9

25. O. M. Baksalary, G. Trenkler, Core inverse of matrices, *Linear Multilinear A.*, **58** (2010), 681–697. http://dx.doi.org/10.1080/03081080902778222

26. H. Kurata, Some theorems on the core inverse of matrices and the core partial ordering, *Appl. Math. Comput.*, **316** (2018), 43–51. http://dx.doi.org/10.1016/j.amc.2017.07.082

27. G. Luo, K. Zuo, L. Zhou, Revisitation of core inverse, *Wuhan Univ. J. Nat. Sci.*, **20** (2015), 381–385. https://doi.org/10.1007/s11859-015-1109-6

28. K. Prasad, K. S Mohana, Core-EP inverse, *Linear Multilinear A.*, **62** (2014), 792–802. http://dx.doi.org/10.1080/03081087.2013.791690

29. D. E. Ferreyra, F. E. Levis, N. Thome, Revisiting the core-EP inverse and its extension to rectangular matrices, *Quaest. Math.*, **41** (2018), 1–17. http://dx.doi.org/10.2989/16073606.2017.1377779

30. K. Zuo, Y. Cheng, The new revisitation of core EP inverse of matrices, *Filomat*, **33** (2019), 3061–3072. https://doi.org/10.2298/FIL1910061Z

31. J. Benítez, E. Boasso, H. Jin, On one-sided (B, C)-inverse of arbitrary matrices, *Electron. J. Linear Al.*, **32** (2017), 391–422. https://arxiv.org/pdf/1701.09054v1.pdf

32. M. P. Drazin, A class of outer generalized inverses, *Linear Algebra Appl.*, **436** (2012), 1909–1923. https://doi.org/10.1016/j.laa.2011.09.004

33. X. Wang, C. Deng, Properties of m-EP operators, *Linear Multilinear A.*, **65** (2017), 1349–1361. https://doi.org/10.1080/03081087.2016.1235131

34. C. Deng, H. Du, Representation of the Moore-Penrose inverse of 2×2 block operator valued matrices, *J. Korean Math. Soc.*, **46** (2009), 1139–1150. https://doi.org/10.4134/JKMS.2009.46.6.1139

35. D. E. Ferreyra, F. E. Levis, N. Thome, Maximal classes of matrices determining generalized inverses, *Appl. Math. Comput.*, **333** (2018), 42–52. https://doi.org/10.1016/j.amc.2018.03.102

36. D. E. Ferreyra, F. E. Levis, N. Thome, Characterizations of k-commutative equalities for some outer generalized inverses, *Linear Multilinear A.*, **68** (2020), 177–192. https://doi.org/10.1080/03081087.2018.1500994

37. Y. Yuan, K. Zuo, Compute $\lim_{\lambda\to 0} X(\lambda I_p + YAX)^{-1}Y$ be the product singular value decomposition, *Linear Multilinear A.*, **64** (2016), 269–278. https://doi.org/10.1080/03081087.2015.1034641

38. P. S. Stanimirović, M. Śirić, I. Stojanović, D. Gerontitis, Conditions for existence, representations, and computation of matrix generalized inverses, *Complexity*, **2017** (2017). https://doi.org/10.1155/2017/6429725

39. Y. Wei, P. S. Stanimirović, M. D. Petković, *Numerical and symbolic computations of generalized inverses*, World Scientific Publishing Co. Pte. Ltd., Hackensack, 2018. http://dx.doi.org/10.1142/10950

40. P. S. Stanimirović, M. D. Petković, D. Mosić, Exact solutions and convergence of gradient based dynamical systems for computing outer inverses, *Appl. Math. Comput.*, **412** (2022), 126588. https://doi.org/10.1016/j.amc.2021.126588

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