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Research article

Some identities involving degenerate Stirling numbers arising from normal ordering

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Abstract: In this paper, we derive some identities and recurrence relations for the degenerate Stirling numbers of the first kind and of the second kind, which are degenerate versions of the ordinary Stirling numbers of the first kind and of the second kind. They are deduced from the normal orderings of degenerate integral powers of the number operator and their inversions, certain relations of boson operators and the recurrence relations of the Stirling numbers themselves. Here we note that, while the normal ordering of an integral power of the number operator is expressed with the help of the Stirling numbers of the second kind, that of a degenerate integral power of the number operator is represented by means of the degenerate Stirling numbers of the second kind.

Keywords: degenerate Stirling numbers of the first kind; degenerate Stirling numbers of the second kind; recurrence relations of Stirling numbers
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1. Introduction

The Stirling number of the second $S_2(n,k)$ is the number of ways to partition a set of n objects into k nonempty subsets (see (1.6)). The (signed) Stirling number of the first kind $S_1(n,k)$ is defined in such a way that the number of permutations of n elements having exactly k cycles is the nonnegative integer $(-1)^{n-k}S_1(n,k) = |S_1(n,k)|$ (see (1.5)). The degenerate Stirling numbers of the second kind $S_{2,\lambda}(n,k)$ (see (1.9)) and of the first kind $S_{1,\lambda}(n,k)$ (see (1.7)) appear most naturally when we replace the powers of x by the generalized falling factorial polynomials $(x)_{k,\lambda}$ in the defining equations (see (1.5)–(1.7), (1.9)).

Carlitz initiated a study of degenerate versions of some special numbers and polynomials in [4], where the degenerate Bernoulli and Euler numbers were investigated. It is remarkable that, in recent years, intensive studies have been done for degenerate versions of quite a few special polynomials and numbers and have yielded many interesting results (see [11,12] and the references therein). They have been explored by various methods, including mathematical physics, combinatorial methods, generating functions, umbral calculus techniques, p-adic analysis, differential equations, special functions, probability theory and analytic number theory. It turns out that the degenerate Stirling numbers play an important role in this exploration for degenerate versions of many special numbers and polynomials. The normal ordering of an integral power of the number operator $a^{+}a$ in terms of boson operators a and a^{+} can be written in the form

$$(a^{+}a)^{k}=\sum_{l=0}^{k}S_{2}(k,l)(a^{+})^{l}a^{l}.$$

The normal ordering of the degenerate kth power of the number operator $a^{+}a$, namely $(a^{+}a)_{k,\lambda}$, in terms of boson operators a,a^{+} can be written in the form

$$(a^{+}a)_{k,\lambda}=\sum_{l=0}^{k}S_{2,\lambda}(k,l)(a^{+})^{l}a^{l}, \tag{1.1}$$

where the generalized falling factorials $(x)_{n,\lambda}$ are given by (1.4).

By inversion, from (1.1) we obtain

$$(a^{+})^{k}a^{k}=\sum_{l=0}^{k}S_{1,\lambda}(k,l)(a^{+}a)_{l,\lambda}. \tag{1.2}$$

The aim of this paper is to derive some identities and recurrence relations for the degenerate Stirling numbers of the first kind and of the second kind by using the normal ordering in (1.1) and its inversion in (1.2), certain relations of boson operators and the recurrence relations of the Stirling numbers themselves. In more detail, our main results are as follows. First, we derive some recurrence relations and identities involving the degenerate Stirling numbers of the second kind in Theorems 3 and 5 by using (1.1) and certain relations for boson operators. Second, we find some recurrence relations and identities involving the degenerate Stirling numbers of the first kind in Theorems 4 and 6 by exploiting (1.2) and certain relations for boson operators. Third, we investigate some recurrence relations for the degenerate Stirling numbers of the second kind in Theorems 7 and 9 by using the recurrence relation in (2.3), and we investigate those for the degenerate Stirling numbers of the first kind in Theorems 8 and 11 by using the recurrence relation in (1.8). For the rest of this section, we recall the facts that are needed throughout this paper.

For any $\lambda\in\mathbb{R}$, the degenerate exponential functions are defined by (see [9–12])

$$e_{\lambda}^{x}(t)=\sum_{k=0}^{\infty}(x)_{n,\lambda}\frac{t^{k}}{k!}, \tag{1.3}$$

where the generalized falling factorials are given by

$$(x)_{0,\lambda}=1,\quad (x)_{k,\lambda}=x(x-\lambda)\cdots\big(x-(k-1)\lambda\big),\quad (k\geq 1). \tag{1.4}$$

When $x=1$, we use the notation $e_\lambda(t)=e_\lambda^1(t)$. It is well known that the Stirling numbers of the first kind are defined by (see [1,4,5,14,15])

$$(x)_n=\sum_{k=0}^{n}S_1(n,k)x^k,\quad (n\geq 0),\tag{1.5}$$

where $(x)_0=1,\ (x)_n=x(x-1)\cdots(x-n+1),\quad (n\geq 1)$.

As the inversion formula of (1.5), the Stirling numbers of the second kind are defined by

$$x^n=\sum_{k=0}^{n}S_2(n,k)(x)_k,\quad (n\geq 0).\tag{1.6}$$

In [11], the degenerate Stirling numbers of the first kind are defined by

$$(x)_n=\sum_{k=0}^{n}S_{1,\lambda}(n,k)(x)_{k,\lambda},\quad (n\geq 0).\tag{1.7}$$

Here we note that the degenerate Stirling numbers of the first kind and of the second kind are special cases of the multiparameter non-central Stirling numbers of the first kind and of the second kind in [6].

From (1.7), we note that

$$S_{1,\lambda}(k+1,l)=S_{1,\lambda}(k,l-1)-(k-l\lambda)S_{1,\lambda}(k,l),\tag{1.8}$$

where $k,l\in\mathbb{N}$ with $k\geq l$.

As the inverse formula of (1.7), the degenerate Stirling numbers of the second kind are defined by (see [11])

$$(x)_{n,\lambda}=\sum_{k=0}^{n}S_{2,\lambda}(n,k)(x)_k,\quad (n\geq 0).\tag{1.9}$$

Note that $\lim_{\lambda\to 0}S_{2,\lambda}(n,k)=S_2(n,k)$, and $\lim_{\lambda\to 0}S_{1,\lambda}(n,k)=S_1(n,k)$. We also observe that $S_{1,\lambda}(n,0)=0$, and $S_{2,\lambda}(n,0)=0$, for any positive integer n. This fact will be used repeatedly in the next section.

By (1.7) and (1.9), we easily obtain the next proposition.

Proposition 1. *The following orthogonality and inverse relations hold true.*

$$\begin{aligned}
&\sum_{k=l}^{n}S_{1,\lambda}(n,k)S_{2,\lambda}(k,l)=\delta_{n,l},\ \sum_{k=l}^{n}S_{2,\lambda}(n,k)S_{1,\lambda}(k,l)=\delta_{n,l},\ (0\leq l\leq n),\\
&a_n=\sum_{k=0}^{n}S_{2,\lambda}(n,k)b_k\Longleftrightarrow b_n=\sum_{k=0}^{n}S_{1,\lambda}(n,k)a_k,\\
&a_n=\sum_{k=n}^{m}S_{2,\lambda}(k,n)b_k\Longleftrightarrow b_n=\sum_{k=n}^{m}S_{1,\lambda}(k,n)a_k,
\end{aligned}$$

where $\delta_{n,l}$ is the Kronecker delta.

Let a and a^+ be the boson annihilation and creation operators satisfying the commutation relation (see [2,3,13])

$$[a,a^+] = aa^+ - a^+a = 1. \tag{1.10}$$

It has been known for some time that the normal ordering of $(a^+a)^n$ has a close relation to the Stirling numbers of the second kind ([2,3,13]). Indeed, the normal ordering of an integral power of the number operator a^+a in terms of boson operators a and a^+, which satisfy the commutation $[a,a^+] = aa^+ - a^+a = 1$, can be written in the form (see [2,3,9,10,13])

$$(a^+a)^k = \sum_{l=0}^{k} S_2(k,l)(a^+)^l a^l. \tag{1.11}$$

The number states $|m\rangle$, $m = 1,2,\ldots$, are defined as

$$a|m\rangle = \sqrt{m}|m-1\rangle, \quad a^+|m\rangle = \sqrt{m+1}|m+1\rangle. \tag{1.12}$$

By (1.12), we get $a^+a|m\rangle = m|m\rangle$, (see [9,10,13]). The coherent states $|z\rangle$, where z is a complex number, satisfy $a|z\rangle = z|z\rangle$, $z|z\rangle = 1$. To show a connection to coherent states, we recall that the harmonic oscillator has Hamiltonian $\hat{n} = a^+a$ (neglecting the zero point energy) and the usual eigenstates $|n\rangle$ (for $n \in \mathbb{N}$) satisfying $\hat{n}|n\rangle = n|n\rangle$, and $\langle m|n\rangle = \delta_{m,n}$, where $\delta_{m,n}$ is the Kronecker delta. In this paper, we derive some identities involving the degenerate Stirling numbers arising from the normal ordering of degenerate integral powers of the number operator a^+a in terms of boson operators a and a^+, which are given by

$$(a^+a)_{k,\lambda} = \sum_{l=0}^{k} S_{2,\lambda}(k,l)(a^+)^l a^l.$$

The reader may refer to the papers [6–8] for some related results on generalized Stirling numbers.

2. Some identities involving degenerate Stirling numbers

We recall that the bosonic commutation relation $[a,a^+] = aa^+ - a^+a = 1$ can be realized formally in a suitable space of functions by letting $a = \frac{d}{dx}$ and $a^+ = x$ (the operator of multiplication by x). It is known that (see [12])

$$\left(x\frac{d}{dx}\right)_{n,\lambda} f(x) = \sum_{k=0}^{n} S_{2,\lambda}(n,k)x^k \left(\frac{d}{dx}\right)^k f(x), \tag{2.1}$$

where n is a nonnegative integer.

The Eq (2.1) can be written as

$$(a^+a)_{n,\lambda} = \sum_{k=0}^{n} S_{2,\lambda}(n,k)(a^+)^k a^k. \tag{2.2}$$

From (1.9), we note that

$$S_{2,\lambda}(k+1,l) = S_{2,\lambda}(k,l-1) + (l-k\lambda)S_{2,\lambda}(k,l), \tag{2.3}$$

where $k, l \in \mathbb{N}$ with $k \geq l$. It is easy to show that

$$[a, \hat{n}] = a, \quad [\hat{n}, a^+] = a^+.$$

Inverting (2.1) by using Proposition 1, we have

$$x^k \left(\frac{d}{dx}\right)^k f(x) = \sum_{m=0}^{k} S_{1,\lambda}(k,m) \left(x\frac{d}{dx}\right)_{m,\lambda} f(x). \tag{2.4}$$

In view of (2.4), the normal ordering of a degenerate integral power of the number operator a^+a in terms of boson operators a, a^+ can be rewritten in the form

$$\begin{aligned} (a^+)^k a^k &= \sum_{m=0}^{k} S_{1,\lambda}(k,m)(a^+a)_{m,\lambda} \\ &= \sum_{m=0}^{k} S_{1,\lambda}(k,m)(\hat{n})_{m,\lambda} = (\hat{n})_k, \end{aligned} \tag{2.5}$$

where k is a nonnegative integer.

Proposition 2. *For $k \in \mathbb{N}$, we have*

$$(\hat{n})_k = (a^+)^k a^k = \sum_{m=0}^{k} S_{1,\lambda}(k,m)(a^+a)_{m,\lambda} = \sum_{m=0}^{k} S_{1,\lambda}(k,m)(\hat{n})_{m,\lambda}.$$

We note that

$$\begin{aligned} a^+(\hat{n}+1-\lambda)_{k,\lambda}a &= a^+ \sum_{l=0}^{k} \binom{k}{l} (\hat{n})_{l,\lambda}(1-\lambda)_{k-l,\lambda}a \\ &= \sum_{l=0}^{k} \binom{k}{l} (1-\lambda)_{k-l,\lambda} \sum_{m=0}^{l} S_{2,\lambda}(l,m)(a^+)^{m+1}a^{m+1} \\ &= \sum_{m=0}^{k} \left(\sum_{l=m}^{k} \binom{k}{l} (1-\lambda)_{k-l,\lambda} S_{2,\lambda}(l,m) \right) (a^+)^{m+1}a^{m+1}. \end{aligned} \tag{2.6}$$

By (1.10), we also have

$$\begin{aligned} (\hat{n})_{k+1,\lambda} &= (\hat{n}-\lambda)_{k,\lambda}\hat{n} = a^+\big((\hat{n}+1-\lambda)\cdots(\hat{n}+1-k\lambda)\big)a \\ &= a^+(\hat{n}+1-\lambda)_{k,\lambda}a. \end{aligned} \tag{2.7}$$

By (2.7) and (2.2), we get

$$\begin{aligned} a^+(\hat{n}+1-\lambda)_{k,\lambda}a = (\hat{n})_{k+1,\lambda} &= (a^+a)_{k+1,\lambda} \\ &= \sum_{m=0}^{k+1} S_{2,\lambda}(k+1,m)(a^+)^m a^m \\ &= \sum_{m=1}^{k+1} S_{2,\lambda}(k+1,m)(a^+)^m a^m \\ &= \sum_{m=0}^{k} S_{2,\lambda}(k+1,m+1)(a^+)^{m+1}a^{m+1}. \end{aligned} \tag{2.8}$$

Therefore, by (2.6) and (2.8), we obtain the following theorem.

Theorem 3. *For $m,k \in \mathbb{Z}$ with $k \geq m \geq 0$, we have*

$$\sum_{l=m}^{k} \binom{k}{l} (1-\lambda)_{k-l,\lambda} S_{2,\lambda}(l,m) = S_{2,\lambda}(k+1,m+1).$$

From Proposition 2, we note that

$$\begin{aligned} (a^+)^{k+1} a^{k+1} &= \sum_{m=0}^{k+1} S_{1,\lambda}(k+1,m)(a^+ a)_{m,\lambda} \\ &= \sum_{m=1}^{k+1} S_{1,\lambda}(k+1,m)(\hat{n})_{m,\lambda} \\ &= \sum_{m=0}^{k} S_{1,\lambda}(k+1,m+1)(\hat{n})_{m+1,\lambda}, \quad (k \geq 0). \end{aligned} \tag{2.9}$$

The degenerate rising factorial sequence is defined by

$$\langle x \rangle_{0,\lambda} = 1, \quad \langle x \rangle_{n,\lambda} = x(x+\lambda)\cdots(x+(n-1)\lambda), \quad (n \geq 1). \tag{2.10}$$

Now, from (1.10) we note that

$$a^+ (\hat{n})_{k,\lambda} a = (\hat{n}-1)_{k,\lambda} \hat{n} = \hat{n}(\hat{n}-1)_{k,\lambda}, \quad (k \in \mathbb{N}). \tag{2.11}$$

On the other hand, by (2.5) and (2.11), we get

$$\begin{aligned} (a^+)^{k+1} a^{k+1} &= \sum_{l=0}^{k} S_{1,\lambda}(k,l) a^+ (\hat{n})_{l,\lambda} a = \sum_{l=0}^{k} S_{1,\lambda}(k,l)(\hat{n}-1)_{l,\lambda} \hat{n} \\ &= \sum_{l=0}^{k} S_{1,\lambda}(k,l) \sum_{m=0}^{l} \binom{l}{m} (-1)^{l-m} \langle 1 \rangle_{l-m,\lambda} (\hat{n})_{m,\lambda} \left(\hat{n} - m\lambda + m\lambda\right) \\ &= \sum_{l=0}^{k} S_{1,\lambda}(k,l) \sum_{m=0}^{l} \binom{l}{m} (-1)^{l-m} \langle 1 \rangle_{l-m,\lambda} (\hat{n})_{m+1,\lambda} \\ &\quad + \lambda \sum_{l=0}^{k} S_{1,\lambda}(k,l) \sum_{m=1}^{l+1} \binom{l}{m} (-1)^{l-m} \langle 1 \rangle_{l-m,\lambda} (\hat{n})_{m,\lambda} m \\ &= \sum_{m=0}^{k} \sum_{l=m}^{k} S_{1,\lambda}(k,l) \binom{l}{m} (-1)^{l-m} \langle 1 \rangle_{l-m,\lambda} (\hat{n})_{m+1,\lambda} \\ &\quad + \sum_{m=0}^{k} \sum_{l=m}^{k} S_{1,\lambda}(k,l)(m+1)\lambda \binom{l}{m+1} (-1)^{l-m-1} \langle 1 \rangle_{l-m-1,\lambda} (\hat{n})_{m+1,\lambda}. \end{aligned} \tag{2.12}$$

Therefore, by (2.9) and (2.12), we obtain the following theorem.

Theorem 4. *For $m,k \in \mathbb{Z}$ with $k \geq m \geq 0$, we have*

$$\begin{aligned} &S_{1,\lambda}(k+1,m+1) \\ &\quad = \sum_{l=m}^{k} S_{1,\lambda}(k,l) \left\{ \binom{l}{m} \langle 1 \rangle_{l-m,\lambda} (-1)^{l-m} + l\lambda \binom{l-1}{m} (-1)^{l-m-1} \langle 1 \rangle_{l-m-1,\lambda} \right\}. \end{aligned}$$

From (2.11), we get

$$a^{+}(\hat{n})_{k,\lambda}a=\hat{n}(\hat{n}-1)_{k,\lambda}=\sum_{m=0}^{k}\binom{k}{m}(-1)^{k-m}\langle 1\rangle_{k-m,\lambda}\hat{n}(\hat{n})_{m,\lambda} \tag{2.13}$$
$$=\sum_{m=0}^{k}\binom{k}{m}(-1)^{k-m}\langle 1\rangle_{k-m,\lambda}(\hat{n}-m\lambda+m\lambda)(\hat{n})_{m,\lambda}$$
$$=\sum_{m=0}^{k}\binom{k}{m}(-1)^{k-m}\langle 1\rangle_{k-m,\lambda}\big((\hat{n})_{m+1,\lambda}+m\lambda(\hat{n})_{m,\lambda}\big)$$
$$=\sum_{m=0}^{k}\binom{k}{m}(-1)^{k-m}\langle 1\rangle_{k-m,\lambda}(\hat{n})_{m+1,\lambda}$$
$$+\sum_{m=0}^{k}\binom{k}{m+1}(-1)^{k-m-1}\langle 1\rangle_{k-m-1,\lambda}(m+1)\lambda(\hat{n})_{m+1,\lambda}$$
$$=\sum_{m=0}^{k}\binom{k}{m}(-1)^{k-m}\langle 1\rangle_{k-m,\lambda}\sum_{p=1}^{m+1}S_{2,\lambda}(m+1,p)(a^{+})^{p}a^{p}$$
$$+\lambda\sum_{m=0}^{k}\binom{k}{m+1}(-1)^{k-m-1}\langle 1\rangle_{k-m-1,\lambda}(m+1)\sum_{p=1}^{m+1}S_{2,\lambda}(m+1,p)(a^{+})^{p}a^{p}.$$
$$=\sum_{p=1}^{k+1}\left(\sum_{m=p-1}^{k}\binom{k}{m}(-1)^{k-m}\langle 1\rangle_{k-m,\lambda}S_{2,\lambda}(m+1,p)\right)(a^{+})^{p}a^{p}$$
$$+\lambda\sum_{p=1}^{k+1}\sum_{m=p-1}^{k}\binom{k}{m+1}(-1)^{k-m-1}\langle 1\rangle_{k-m-1,\lambda}(m+1)S_{2,\lambda}(m+1,p)(a^{+})^{p}a^{p}$$
$$=\sum_{p=0}^{k}\left\{\sum_{m=p}^{k}\binom{k}{m}(-1)^{k-m}\langle 1\rangle_{k-m,\lambda}S_{2,\lambda}(m+1,p+1)\right\}(a^{+})^{p+1}a^{p+1}$$
$$+\lambda\sum_{p=0}^{k}\sum_{m=p}^{k}k\binom{k-1}{m}(-1)^{k-m-1}\langle 1\rangle_{k-m-1,\lambda}S_{2,\lambda}(m+1,p+1)(a^{+})^{p+1}a^{p+1}.$$

We observe that

$$a^{+}(\hat{n})_{k,\lambda}a=a^{+}\left(\sum_{p=0}^{k}S_{2,\lambda}(k,p)(a^{+})^{p}a^{p}\right)a \tag{2.14}$$
$$=\sum_{p=0}^{k}S_{2,\lambda}(k,p)(a^{+})^{p+1}a^{p+1}.$$

Therefore, by (2.13) and (2.14), we obtain the following theorem.

Theorem 5. *For $p,k\in\mathbb{Z}$ with $0\leq p\leq k$, we have*

$$S_{2,\lambda}(k,p)=\sum_{m=p}^{k}\left\{\binom{k}{m}(-1)^{k-m}\langle 1\rangle_{k-m,\lambda}S_{2,\lambda}(m+1,p+1)\right.$$
$$\left.+\lambda k\binom{k-1}{m}(-1)^{k-m-1}\langle 1\rangle_{k-m-1,\lambda}S_{2,\lambda}(m+1,p+1)\right\}.$$

By (2.5), we obtain

$$
\begin{aligned}
(a^+)^{k+1}a^{k+1} &= \sum_{p=0}^{k+1} S_{1,\lambda}(k+1,p)(a^+a)_{p,\lambda} = \sum_{p=1}^{k+1} S_{1,\lambda}(k+1,p)(\hat{n})_{p,\lambda} \\
&= \sum_{p=1}^{k+1} S_{1,\lambda}(k+1,p)\hat{n}(\hat{n}-1+1-\lambda)_{p-1,\lambda} \\
&= \sum_{p=1}^{k+1} S_{1,\lambda}(k+1,p)\hat{n}\sum_{l=0}^{p-1}\binom{p-1}{l}(1-\lambda)_{p-1-l,\lambda}(\hat{n}-1)_{l,\lambda} \\
&= \sum_{l=0}^{k}\left(\sum_{p=l+1}^{k+1} S_{1,\lambda}(k+1,p)\binom{p-1}{l}(1-\lambda)_{p-1-l,\lambda}\right)\hat{n}(\hat{n}-1)_{l,\lambda} \\
&= \sum_{l=0}^{k}\left(\sum_{p=l}^{k} S_{1,\lambda}(k+1,p+1)\binom{p}{l}(1-\lambda)_{p-l,\lambda}\right)\hat{n}(\hat{n}-1)_{l,\lambda}.
\end{aligned}
\tag{2.15}
$$

Therefore, from the first line of (2.12) and (2.15), we obtain the following theorem.

Theorem 6. *For $k,l \in \mathbb{Z}$ with $k \geq l \geq 0$, we have*

$$
S_{1,\lambda}(k,l) = \sum_{p=l}^{k} S_{1,\lambda}(k+1,p+1)\binom{p}{l}(1-\lambda)_{p-l,\lambda}.
$$

From (1.9), we note that

$$
\frac{1}{k}\big(e_\lambda(t)-1\big)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}, \quad (k \geq 0). \tag{2.16}
$$

Thus, by (2.16), we get

$$
\begin{aligned}
\sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!} &= \frac{1}{k!}\big(e_\lambda(t)-1\big)^k = \frac{1}{k!}\sum_{m=0}^{k}\binom{k}{m}(-1)^{k-m}e_\lambda^m(t) \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{k!}\sum_{m=0}^{k}\binom{k}{m}(-1)^{k-m}(m)_{n,\lambda}\right)\frac{t^n}{n!}.
\end{aligned}
\tag{2.17}
$$

Comparing the coefficients on both sides of (2.17), we have

$$
\sum_{m=0}^{k}\binom{k}{m}(-1)^{k-m}(m)_{n,\lambda} = \begin{cases} k!S_{2,\lambda}(n,k), & \text{if } n \geq k, \\ 0, & \text{if } 0 \leq n < k. \end{cases}
$$

By (2.3), we get

$$
\begin{aligned}
S_{2,\lambda}(k+1,l+1) &= S_{2,\lambda}(k,l) + (l+1-k\lambda)S_{2,\lambda}(k,l+1) \\
&= S_{2,\lambda}(k,l) + (l+1-k\lambda) \\
&\quad \times \big\{S_{2,\lambda}(k-1,l) + (l+1-(k-1)\lambda)S_{2,\lambda}(k-1,l+1)\big\}
\end{aligned}
\tag{2.18}
$$

$$
\begin{aligned}
&= S_{2,\lambda}(k,l)+(l+1-k\lambda)S_{2,\lambda}(k-1,l)\\
&\qquad +\langle l+1-k\lambda\rangle_{2,\lambda}S_{2,\lambda}(k-1,l+1)\\
&= S_{2,\lambda}(k,l)+(l+1-k\lambda)S_{2,\lambda}(k-1,l)+\langle l+1-k\lambda\rangle_{2,\lambda}S_{2,\lambda}(k-2,l)\\
&\quad +\cdots+\langle l+1-k\lambda\rangle_{k-l,\lambda}S_{2,\lambda}(k,l)\\
&= \sum_{m=l}^{k}\langle l+1-k\lambda\rangle_{k-m,\lambda}S_{2,\lambda}(l,l).
\end{aligned}
$$

Therefore, by (2.18), we obtain the following theorem.

Theorem 7. *For $l,k \in \mathbb{Z}$ with $0 \le l \le k$, we have*

$$
S_{2,\lambda}(k+1,l+1)=\sum_{m=l}^{k}\langle l+1-k\lambda\rangle_{k-m,\lambda}S_{2,\lambda}(m,l).
$$

From (1.8), we note that

$$
\begin{aligned}
&S_{1,\lambda}(k+1,m+1)=S_{1,\lambda}(k,m)-\big(k-(m+1)\lambda\big)S_{1,\lambda}(k,m+1)\\
&= S_{1,\lambda}(k,m)-\big(k-(m+1)\lambda\big)\\
&\qquad \times\Big(S_{1,\lambda}(k-1,m)-((k-1)-(m+1)\lambda)S_{1,\lambda}(k-1,m+1)\Big)\\
&= S_{1,\lambda}(k,m)-(k-(m+1)\lambda)S_{1,\lambda}(k-1,m)\\
&\qquad +(-1)^2(k-(m+1)\lambda)_2S_{1,\lambda}(k-1,m+1)\\
&= S_{1,\lambda}(k,m)-(k-(m+1)\lambda)S_{1,\lambda}(k-1,m)\\
&\qquad +(-1)^2(k-(m+1)\lambda)_2S_{1,\lambda}(k-1,m+1)\\
&\qquad +\cdots+(-1)^{k-m}(k-(m+1)\lambda)_{k-m}S_{1,\lambda}(m,m)\\
&= \sum_{l=m}^{k}(-1)^{k-l}\big(k-(m+1)\lambda\big)_{k-l}S_{1,\lambda}(l,m).
\end{aligned}
\tag{2.19}
$$

Therefore, by (2.19), we obtain the following theorem.

Theorem 8. *For $m,k \in \mathbb{Z}$ with $k \ge m \ge 0$, we have*

$$
S_{1,\lambda}(k+1,m+1)=\sum_{l=m}^{k}(-1)^{k-l}\big(k-(m+1)\lambda\big)_{k-l}S_{1,\lambda}(l,m).
$$

By (2.3), we get

$$
\begin{aligned}
&S_{2,\lambda}(m+k+1,m)=S_{2,\lambda}(m+k,m-1)+\big(m-(m+k)\lambda\big)S_{2,\lambda}(m+k,m)\\
&= S_{2,\lambda}(m+k-1,m-2)+\big((m-1)-(m+k-1)\lambda\big)S_{2,\lambda}(m+k-1,m-1)\\
&\qquad +\big(m-(m+k)\lambda\big)S_{2,\lambda}(m+k,m)\\
&= \big(m-(m+k)\lambda\big)S_{2,\lambda}(m+k,m)+\big((m-1)-(m+k-1)\lambda\big)\\
&\qquad \times S_{2,\lambda}(m+k-1,m-1)+\cdots+(0-k\lambda)S_{2,\lambda}(k,0)\\
&= \sum_{l=0}^{m}\big(l-(k+l)\lambda\big)S_{2,\lambda}(k+l,l).
\end{aligned}
\tag{2.20}
$$

Therefore, by (2.20), we obtain the following theorem.

Theorem 9. *For $k,m \in \mathbb{Z}$ with $k,m \geq 0$, we have*

$$S_{2,\lambda}(m+k+1,m) = \sum_{l=0}^{m} \big(l-(k+l)\lambda\big) S_{2,\lambda}(k+l,l).$$

From Theorem 3, we note that

$$S_{2,\lambda}(k+1,m+1) = \sum_{l=m}^{k} \binom{k}{l} (1-\lambda)_{k-l,\lambda} S_{2,\lambda}(l,m).$$

Now, by using Proposition 1, we have

$$\begin{aligned}\sum_{l=m}^{k} S_{2,\lambda}(k+1,l+1)S_{1,\lambda}(l,m) &= \sum_{l=m}^{k}\sum_{p=l}^{k} \binom{k}{p}(1-\lambda)_{k-p,\lambda} S_{2,\lambda}(p,l)S_{1,\lambda}(l,m) \\ &= \sum_{p=m}^{k} \binom{k}{p}(1-\lambda)_{k-p,\lambda} \sum_{l=m}^{p} S_{2,\lambda}(p,l)S_{1,\lambda}(l,m) \\ &= \binom{k}{m}(1-\lambda)_{k-m,\lambda}.\end{aligned} \tag{2.21}$$

Therefore, by (2.21), we obtain the following theorem.

Theorem 10. *For $m,k \in \mathbb{Z}$ with $k \geq m \geq 0$, we have*

$$\binom{k}{m} = \frac{1}{(1-\lambda)_{k-m,\lambda}} \sum_{l=m}^{k} S_{2,\lambda}(k+1,l+1)S_{1,\lambda}(l,m).$$

By (1.8), we get

$$\begin{aligned}&S_{1,\lambda}(m+k+1,m) = S_{1,\lambda}(m+k,m-1) - (m+k-m\lambda)S_{1,\lambda}(m+k,m) \\ &= S_{1,\lambda}(m+k-1,m-2) - (m+k-1-(m-1)\lambda))S_{1,\lambda}(m+k-1,m-1) \\ &\qquad - (m+k-m\lambda)S_{1,\lambda}(m+k,m) \\ &= -(m+k-m\lambda)S_{1,\lambda}(m+k,m) - (m+k-1-(m-1)\lambda) \\ &\qquad \times S_{1,\lambda}(m+k-1,m-1) - \cdots - kS_{1,\lambda}(k,0) \\ &= -\sum_{l=0}^{m}(k+l-l\lambda)S_{1,\lambda}(k+l,l).\end{aligned} \tag{2.22}$$

Therefore, by (2.22), we obtain the following theorem.

Theorem 11. *For $m,k \in \mathbb{Z}$ with $m,k \geq 0$, we have*

$$S_{1,\lambda}(m+k+1,m) = -\sum_{l=0}^{m}(k+l-l\lambda)S_{1,\lambda}(k+l,l).$$

3. Conclusions

In recent years, studying degenerate versions of some special numbers and polynomials has drawn the attention of many mathematicians with regained interests not only in their combinatorial and arithmetical properties but also in their applications to differential equations, identities of symmetry and probability theory. These degenerate versions include the degenerate Stirling numbers of the first and second kinds, degenerate Bernoulli numbers of the second kind and degenerate Bell numbers and polynomials.

As a degenerate version of the well known normal ordering of an integral power of the number operator, we considered the normal ordering of a degenerate integral power of the number operator in terms of boson operators and its inversion, as well. We derived some identities and recurrence relations for the degenerate Stirling numbers of the first kind and of the second kind by using the normal ordering in (1.1) and its inversion in (1.2), certain relations of boson operators and the recurrence relations of the Stirling numbers themselves.

It is one of our future projects to continue to explore various degenerate versions of many special polynomials and numbers by using various methods mentioned in the introduction.

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Conflict of interest

The authors declare no conflicts of interest.

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