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Well-posedness for the Chern-Simons-Schrödinger equations

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Abstract: First, we prove uniform-in-ϵ regularity estimates of local strong solutions to the Chern-Simons-Schrödinger equations in $\mathbb{R}^2$. Here ϵ is the dispersion coefficient. Then we prove the global well-posedness of strong solutions to the limit problem ($\epsilon = 0$).

Keywords: Chern-Simons-Schrödinger; Coulomb gauge; well-posedness
Mathematics Subject Classification: 35B30, 35B65, 35Q55

1. Introduction

In this paper, we consider the following Chern-Simons-Schrödinger equations [1,2]:

$$i\partial_t\phi + (\epsilon\nabla + iA)^2\phi = A_0\phi - \lambda|\phi|^2\phi + if(x,t)\phi, \tag{1.1}$$

$$\partial_t A - \nabla A_0 = \mathrm{Im}(\epsilon\bar{\phi}\nabla^\perp\phi) + A^\perp|\phi|^2, \tag{1.2}$$

$$\mathrm{rot}\, A = -\frac{1}{2}|\phi|^2 \ \text{ in } \ \mathbb{R}^2 \times (0,\infty), \tag{1.3}$$

$$\phi(\cdot,0) = \phi_0(\cdot) \ \text{ in } \ \mathbb{R}^2, \tag{1.4}$$

where ϕ is the complex scalar field, A_0 and $A := \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ are the real gauge fields. $\lambda > 0$ is a coupling constant representing the strength of interaction potential. $\epsilon \geq 0$ is the dispersion coefficient. $i := \sqrt{-1}, \nabla^\perp := \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix}, A^\perp := \begin{pmatrix} -A_2 \\ A_1 \end{pmatrix}, \mathrm{rot}\,\phi := \begin{pmatrix} \partial_2\phi \\ -\partial_1\phi \end{pmatrix} = -\nabla^\perp\phi$ and $\mathrm{rot}\, A := \partial_1 A_2 - \partial_2 A_1$. f is a complex smooth function.

The system (1.1)–(1.3) was proposed in [1,2] to deal with the electromagnetic phenomena in planar domains, such as the fractional quantum Hall effect or high-temperature superconductivity.

The system (1.1)–(1.3) is invariant under the following gauge transformations:

$$\phi \to \phi e^{i\chi}, A_0 \to A_0 - \partial_t\chi, A \to A - \nabla\chi, \tag{1.5}$$

where $\chi : \mathbb{R}^{2+1} \to \mathbb{R}$ is a smooth function. In this work, we fix the Coulomb gauge:

$$\operatorname{div} A = 0 \text{ in } \mathbb{R}^2 \times (0, \infty). \tag{1.6}$$

Taking div to (1.2), rot to (1.3), using $\operatorname{rot}^2 A = -\Delta A + \nabla \operatorname{div} A$ and (1.6), we can reformulate (1.1)–(1.3) as follows.

$$i\partial_t \phi + \epsilon^2 \Delta\phi + 2i\epsilon A \cdot \nabla\phi - |A|^2\phi = A_0\phi - \lambda|\phi|^2\phi + if\phi, \tag{1.7}$$

$$-\Delta A_0 = \epsilon \operatorname{Im}(\partial_1\phi\partial_2\overline{\phi} - \partial_2\phi\partial_1\overline{\phi}) - \operatorname{rot}(|\phi|^2 A), \tag{1.8}$$

$$\Delta A = \frac{1}{2}\operatorname{rot}|\phi|^2 \text{ in } \mathbb{R}^2 \times (0, \infty), \tag{1.9}$$

$$\phi(\cdot, 0) = \phi_0(\cdot) \text{ in } \mathbb{R}^2. \tag{1.10}$$

Bergé-de Bouard-Saut [3] proved that the Cauchy problem is locally well-posed when $\phi_0 \in H^2$. Huh [4] improved it to the case $\phi_0 \in H^1$. Lim [5] refined it to the case $\phi_0 \in H^s$ for $s \geq 1$. In [3], the authors also showed, by deriving a virial identity, that solutions blow up in finite time under certain conditions. The existence of a standing wave solution has also been proved in [6, 7]. Liu-Smith-Tataru [8] proved the local well-posedness of (1.1)–(1.3) for small data $\phi_0 \in H^\sigma$ with any $\sigma > 0$ under the Lorentz (heat) gauge:

$$A_0 = \operatorname{div} A. \tag{1.11}$$

Please see [9–13] for other studies of the problem (1.1)–(1.3).

All the above results dealt with the case $\epsilon = 1$. The aim of this paper is to prove the uniform regularity estimates independent of ϵ and prove the global well-posedness of strong solutions to the limit problem ($\epsilon = 0$). We will prove

Theorem 1.1. *Let $\phi_0 \in H^s$ with $s > 1$. Then there exists $T_0 > 0$ such that the problem* (1.7)–(1.10) *has a unique local strong solution (ϕ, A_0, A) on $[0, T_0]$ satisfying*

$$\sup_{\epsilon\in[0,1]} \sup_{t\in[0,T_0]} \|(\phi, \nabla A_0, \nabla A)(\cdot, t)\|_{H^s} \leq C. \tag{1.12}$$

Remark 1.1. *Our approach is very much technically simpler than that in [5] when $s > 1$.*

Remark 1.2. *We are unable to prove a similar result for the Maxwell-Schrödinger system.*

When $\epsilon = 0$, the problem (1.7)–(1.10) reads as follows.

$$\partial_t\phi = -iA_0\phi - i|A|^2\phi + \lambda i|\phi|^2\phi + f\phi, \tag{1.13}$$

$$\Delta A_0 = \operatorname{rot}(|\phi|^2 A), \tag{1.14}$$

$$\Delta A = \frac{1}{2}\operatorname{rot}|\phi|^2, \tag{1.15}$$

$$\operatorname{div} A = 0 \text{ in } \mathbb{R}^2 \times (0, \infty), \tag{1.16}$$

$$\phi(\cdot, 0) = \phi_0(\cdot) \text{ in } \mathbb{R}^2. \tag{1.17}$$

We have

Theorem 1.2. *Let $\phi_0 \in H^s \cap L^\infty$ with $s \geq 1$. Then the problem* (1.13)–(1.17) *has a unique global strong solution (ϕ, A_0, A) satisfying* (1.12).

Remark 1.3. *For recent literature on the concept of "well-posedness," we refer the reader to [14–16]. Regarding vanishing dispersion limit for related equations, see [17–19].*

In the following proofs, we will use the bilinear commutator and product estimates due to Kato-Ponce [20]:

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leq C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{q_1}} + \|g\|_{L^{p_2}}\|\Lambda^s f\|_{L^{q_2}}), \tag{1.18}$$

$$\|\Lambda^s(fg)\|_{L^p} \leq C(\|f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}) \tag{1.19}$$

with $s > 0, \Lambda := (-\Delta)^{\frac{1}{2}}$ and $\dfrac{1}{p} = \dfrac{1}{p_1} + \dfrac{1}{q_1} = \dfrac{1}{p_2} + \dfrac{1}{q_2}$.

2. Proof of Theorem 1.1

Since the local well-posedness of smooth solution is well-known [5], we only need to show (1.12). (1.7) can be written as

$$\partial_t\phi = i\epsilon^2\Delta\phi - 2\epsilon A\cdot\nabla\phi - iA_0\phi - i|A|^2\phi + i\lambda|\phi|^2\phi + f\phi. \tag{2.1}$$

Testing (2.1) by $\overline{\phi}$, taking the real parts and using (1.5), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\phi|^2\mathrm{d}x = \mathrm{Re}\int f|\phi|^2\mathrm{d}x \leq \|f\|_{L^\infty}\|\phi\|_{L^2}^2,$$

which gives

$$\|\phi\|_{L^2} \leq C. \tag{2.2}$$

Applying Λ^s to (2.1), testing by $\Lambda^s\overline{\phi}$, taking the real parts, and using (1.5), (1.17) and (1.18), we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\Lambda^s\phi|^2\mathrm{d}x = -2\epsilon\mathrm{Re}\int(\Lambda^s(A\cdot\nabla\phi) - A\cdot\nabla\Lambda^s\phi)\Lambda^s\overline{\phi}\mathrm{d}x\\
&-\mathrm{Re}\, i\int\Lambda^s(A_0\phi)\Lambda^s\overline{\phi}\mathrm{d}x - \mathrm{Re}\, i\int\Lambda^s(|A|^2\phi)\Lambda^s\overline{\phi}\mathrm{d}x\\
&+\mathrm{Re}\lambda i\int\Lambda^s(|\phi|^2\phi)\Lambda^s\overline{\phi}\mathrm{d}x + \mathrm{Re}\int\Lambda^s(f\phi)\Lambda^s\overline{\phi}\mathrm{d}x\\
\leq\ & C\|\nabla A\|_{L^\infty}\|\Lambda^s\phi\|_{L^2}^2 + C\|\nabla\phi\|_{L^p}\|\Lambda^s A\|_{L^{\frac{2p}{p-2}}}\|\Lambda^s\phi\|_{L^2}\\
&\qquad\left(p := \frac{2}{(2-s)^+}\ \text{ if }\ s < 2\ \text{ and }\ p = 4\ \text{ if }\ s \geq 2\right)\\
&+C\|A_0\|_{L^\infty}\|\Lambda^s\phi\|_{L^2}^2 + C\|\phi\|_{L^\infty}\|\Lambda^s A_0\|_{L^2}\|\Lambda^s\phi\|_{L^2}\\
&+C\|A\|_{L^\infty}^2\|\Lambda^s\phi\|_{L^2}^2 + C\|\phi\|_{L^\infty}\|A\|_{L^\infty}\|\Lambda^s A\|_{L^2}\|\Lambda^s\phi\|_{L^2}\\
&+C\|\phi\|_{L^\infty}^2\|\Lambda^s\phi\|_{L^2}^2 + C\|f\|_{L^\infty}\|\Lambda^s\phi\|_{L^2}^2 + C\|\phi\|_{L^\infty}\|\Lambda^s f\|_{L^2}\|\Lambda^s\phi\|_{L^2}.
\end{aligned} \tag{2.3}$$

Noting

$$\begin{aligned}
\|\nabla A\|_{L^\infty} + \|\Lambda^s A\|_{L^{\frac{2p}{p-2}}} + \|\Lambda^s A\|_{L^2} &\leq C\|\nabla A\|_{L^2} + C\|\Lambda^{s+1}A\|_{L^2}\\
&\leq C\|\phi\|_{L^4}^2 + C\|\Lambda^s(|\phi|^2)\|_{L^2}
\end{aligned}$$

$$\leq \quad C\|\phi\|_{L^4}^2 + C\|\phi\|_{L^\infty}\|\Lambda^s\phi\|_{L^2} \leq \|\phi\|_{H^s}^2 \tag{2.4}$$

and

$$\begin{aligned}
\|\Lambda^s A_0\|_{L^2} &\leq C\|\Lambda^{s-2}\mathrm{div}\,(\mathrm{Im}(\epsilon\overline{\phi}\nabla^\perp\phi) + A^\perp|\phi|^2)\|_{L^2} \\
&\leq C\|\phi\|_{H^s}^2 + C\|\phi\|_{L^\infty}^2\|\Lambda^{s-1}A\|_{L^2} + C\|A\|_{L^\infty}\|\phi\|_{L^\infty}\|\Lambda^{s-1}\phi\|_{L^2} \\
&\leq C\|\phi\|_{H^s}^4 + C,
\end{aligned} \tag{2.5}$$

we obtain

$$\begin{aligned}
\|A_0\|_{L^\infty} &\leq C\|A_0\|_{L^4} + C\|\Lambda^s A_0\|_{L^2} \\
&\leq C\|\nabla A_0\|_{L^{\frac{4}{3}}} + C\|\Lambda^s A_0\|_{L^2} \\
&\leq C\|\overline{\phi}\nabla^\perp\phi\|_{L^{\frac{4}{3}}} + C\|A^\perp|\phi|^2\|_{L^{\frac{4}{3}}} + C\|\Lambda^s A_0\|_{L^2} \\
&\leq C\|\phi\|_{L^4}\|\nabla\phi\|_{L^2} + C\|A\|_{L^\infty}\|\phi\|_{L^{\frac{8}{3}}}^2 + C\|\Lambda^s A_0\|_{L^2} \\
&\leq C\|\phi\|_{H^s}^4 + C
\end{aligned} \tag{2.6}$$

due to (1.2), and

$$\begin{aligned}
\|A\|_{L^\infty} &\leq C\|A\|_{L^4} + C\|\Lambda^{s+1}A\|_{L^2} \\
&\leq C\|\nabla A\|_{L^{\frac{4}{3}}} + C\|\Lambda^{s+1}A\|_{L^2} \\
&\leq C\|\phi\|_{L^{\frac{8}{3}}}^2 + C\|\Lambda^s(|\phi|^2)\|_{L^2} \\
&\leq C\|\phi\|_{L^{\frac{8}{3}}}^2 + C\|\phi\|_{L^\infty}\|\Lambda^s\phi\|_{L^2} \\
&\leq C\|\phi\|_{H^s}^2.
\end{aligned} \tag{2.7}$$

Inserting (2.4), (2.5), (2.6), and (2.7) into (2.3), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |\Lambda^s\phi|^2\mathrm{d}x \leq C\|\phi\|_{H^s}^6 + C, \tag{2.8}$$

which gives (1.12).

The proof is complete. □

3. Proof of Theorem 1.2

First, we still have (2.2).

It is clear that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\phi|^2 = (f + \overline{f})|\phi|^2 \leq 2\|f\|_{L^\infty}\|\phi\|_{L^\infty}^2,$$

and thus

$$\|\phi\|_{L^\infty}^2 \leq \|\phi_0\|_{L^\infty}^2 + 2\int_0^t \|f\|_{L^\infty}\|\phi\|_{L^\infty}^2\mathrm{d}\tau,$$

which yields

$$\|\phi\|_{L^\infty} \leq C. \tag{3.1}$$

Applying ∇ to (1.12), testing by $\nabla\overline{\phi}$, taking the real parts and using (1.15), (2.2) and (3.1), we see that

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla\phi|^2\mathrm{d}x &= -\mathrm{Re}\,i\int\phi\nabla A_0\nabla\overline{\phi}\mathrm{d}x-\mathrm{Re}\,i\int\phi\nabla|A|^2\cdot\nabla\overline{\phi}\mathrm{d}x\\
&\quad+\lambda\mathrm{Re}\,i\int\phi\nabla\overline{\phi}\nabla|\phi|^2\mathrm{d}x+\mathrm{Re}\int\nabla(f\phi)\nabla\overline{\phi}\mathrm{d}x\\
&\leq \|\phi\|_{L^\infty}\|\nabla A_0\|_{L^2}\|\nabla\phi\|_{L^2}+2\|\phi\|_{L^\infty}\|A\|_{L^4}\|\nabla A\|_{L^4}\|\nabla\phi\|_{L^2}\\
&\quad+2\lambda\|\phi\|_{L^\infty}^2\|\nabla\phi\|_{L^2}^2+\|f\|_{L^\infty}\|\nabla\phi\|_{L^2}^2+\|\phi\|_{L^\infty}\|\nabla f\|_{L^2}\|\nabla\phi\|_{L^2}\\
&\leq C\||\phi|^2A\|_{L^2}\|\nabla\phi\|_{L^2}+C\|A\|_{L^4}\|\nabla A\|_{L^4}\|\nabla\phi\|_{L^2}+C\|\nabla\phi\|_{L^2}^2+C\\
&\leq C\|\phi\|_{L^8}^2\|A\|_{L^4}\|\nabla\phi\|_{L^2}+C\|\nabla A\|_{L^{\frac{4}{3}}}\|\nabla A\|_{L^4}\|\nabla\phi\|_{L^2}+C\|\nabla\phi\|_{L^2}^2+C\\
&\leq C\|\nabla A\|_{L^{\frac{4}{3}}}\|\nabla\phi\|_{L^2}+C\|\nabla A\|_{L^{\frac{4}{3}}}\|\nabla A\|_{L^4}\|\nabla\phi\|_{L^2}+C\|\nabla\phi\|_{L^2}^2+C\\
&\leq C\||\phi|^2\|_{L^{\frac{4}{3}}}\|\nabla\phi\|_{L^2}+C\||\phi|^2\|_{L^{\frac{4}{3}}}\||\phi|^2\|_{L^4}\|\nabla\phi\|_{L^2}+C\|\nabla\phi\|_{L^2}^2+C\\
&\leq C\|\nabla\phi\|_{L^2}^2+C,
\end{aligned}$$

which implies

$$\|\phi(\cdot,t)\|_{H^1}\leq C. \tag{3.2}$$

Here we have used the estimates

$$\|\nabla A\|_{L^q}\leq C\||\phi|^2\|_{L^q}\leq C \text{ for } 1<q<\infty, \tag{3.3}$$

and

$$\|\nabla A_0\|_{L^2}\leq C\|A^\perp|\phi|^2\|_{L^2}\leq C\|A\|_{L^4}\leq C\|\nabla A\|_{L^{\frac{4}{3}}}\leq C, \tag{3.4}$$

On the other hand, noting that

$$\begin{aligned}
\|A_0\|_{L^\infty} &\leq C\|A_0\|_{L^4}+C\|\nabla A_0\|_{L^4}\\
&\leq C\|\nabla A_0\|_{L^{\frac{4}{3}}}+C\|\nabla A_0\|_{L^4}\\
&\leq C\||\phi|^2A\|_{L^{\frac{4}{3}}}+C\||\phi|^2A\|_{L^4}\\
&\leq C\|A\|_{L^4}\leq C,
\end{aligned} \tag{3.5}$$

and

$$\|A\|_{L^\infty}\leq C\|A\|_{L^4}+C\|\nabla A\|_{L^4}\leq C. \tag{3.6}$$

Then applying Λ^s to (1.12), testing by $\Lambda^s\overline{\phi}$, taking the real parts, using (1.18), (3.1), (3.5) and (3.6), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\Lambda^s\phi|^2\mathrm{d}x=-\mathrm{Re}\,i\int\Lambda^s(A_0\phi)\Lambda^s\overline{\phi}\mathrm{d}x-\mathrm{Re}\,i\int\Lambda^s(|A|^2\phi)\Lambda^s\overline{\phi}\mathrm{d}x\\
&+\lambda\mathrm{Re}\,i\int\Lambda^s(|\phi|^2\phi)\Lambda^s\overline{\phi}\mathrm{d}x+\mathrm{Re}\int\Lambda^s(f\phi)\Lambda^s\overline{\phi}\mathrm{d}x\\
\leq\ & C\|A_0\|_{L^\infty}\|\Lambda^s\phi\|_{L^2}^2+C\|\phi\|_{L^\infty}\|\Lambda^sA_0\|_{L^2}\|\Lambda^s\phi\|_{L^2}\\
&+C\|A\|_{L^\infty}^2\|\Lambda^s\phi\|_{L^2}^2+C\|\phi\|_{L^\infty}\|A\|_{L^\infty}\|\Lambda^sA\|_{L^2}\|\Lambda^s\phi\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&+C\|\phi\|_{L^\infty}^2\|\Lambda^s\phi\|_{L^2}^2 + C\|f\|_{L^\infty}\|\Lambda^s\phi\|_{L^2}^2 + C\|\phi\|_{L^\infty}\|\Lambda^s f\|_{L^2}\|\Lambda^s\phi\|_{L^2}\\
\leq\ & C\|\Lambda^s\phi\|_{L^2}^2 + C\|\Lambda^s A_0\|_{L^2}\|\Lambda^s\phi\|_{L^2} + C\|\Lambda^s A\|_{L^2}\|\Lambda^s\phi\|_{L^2} + C\\
\leq\ & C\|\Lambda^s\phi\|_{L^2}^2 + C\|\Lambda^{s-1}(|\phi|^2A)\|_{L^2}\|\Lambda^s\phi\|_{L^2} + C\|\Lambda^{s-1}(|\phi|^2)\|_{L^2}\|\Lambda^s\phi\|_{L^2} + C\\
\leq\ & C\|\Lambda^s\phi\|_{L^2}^2 + C\|\Lambda^{s-1}A\|_{L^2}\|\Lambda^s\phi\|_{L^2} + C\|\Lambda^{s-1}\phi\|_{L^2}\|\Lambda^s\phi\|_{L^2} + C\\
\leq\ & C\|\Lambda^s\phi\|_{L^2}^2 + C,
\end{aligned}
$$

which leads to (1.12).

Here we have used the estimates

$$
\begin{aligned}
\|\Lambda^s A\|_{L^2} &\leq C\|\Lambda^{s-1}(|\phi|^2)\|_{L^2}\\
&\leq C\|\Lambda^{s-1}\phi\|_{L^2} \leq C\|\Lambda^s\phi\|_{L^2} + C,
\end{aligned} \tag{3.7}
$$

and

$$
\begin{aligned}
\|\Lambda^s A_0\|_{L^2} &\leq C\|\Lambda^{s-1}(|\phi|^2A)\|_{L^2}\\
&\leq C\|\Lambda^{s-1}\phi\|_{L^2} + C\|\Lambda^{s-1}A\|_{L^2}\\
&\leq C\|\Lambda^s\phi\|_{L^2} + C.
\end{aligned} \tag{3.8}
$$

The proof is complete. □

4. Conclusions

We have obtained the Sobolev estimates on local time interval uniformly in the dispersion coefficient $\epsilon \in (0, 1]$. Moreover, we have proved the existence and uniqueness of global solutions to the limit problem $\epsilon = 0$.

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Conflict of interest

The authors declare no conflict of interest.

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