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Research article

Bonferroni mean operators based on bipolar complex fuzzy setting and their applications in multi-attribute decision making

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Abstract: In our daily life we have to make many decisions and sometimes in a single day we met the situations when correct decision is very compulsory to handle some complicated situations. However, in a professional environment, we need decision-making (DM) techniques to determine the finest alternative from the given alternatives. In this manuscript, we develop one of the finest DM techniques by employing interpreted aggregation operators (AOs). Furthermore, to aggregate the collection of a finite number of information into a singleton set, the Bonferroni mean (BM) operator plays a very beneficial and dominant role. The BM operator is massively powerful than the averaging/geometric operators because they are the specific cases of the BM operator. Based on the above advantages-we initiate the notion of bipolar complex fuzzy BM (BCFBM) operator, bipolar complex fuzzy normalized weighted BM (BCFNWBM) operator and bipolar complex fuzzy ordered weighted BM (BCFOWBM) operator. Furthermore, some well-known and useful properties and results of the initiated operators will be established. We will also apply the described AOs, and evaluate a DM technique, called multi-attribute DM (MADM) to prove the trustworthiness and practicality of the evaluated theory. Finally, to compare the presented work with some prevailing operators, we illustrate some examples and try to evaluate the graphical interpretation of the established work to improve the worth of the proposed theory.

Keywords: bipolar complex fuzzy sets; Bonferroni/weighted Bonferroni mean operators; decision-making strategy
Mathematics Subject Classification: 0352, 90B50

Abbreviations: FST: Fuzzy set theory; BFST: Bipolar fuzzy set theory; CFST: Complex fuzzy set theory; BCFST: Bipolar complex fuzzy set theory; BM: Bonferroni mean; BCFBM: Bipolar complex fuzzy Bonferroni mean; BCFNWBM: Bipolar complex fuzzy normalized weighted Bonferroni mean; BCFOWBM: Bipolar complex fuzzy ordered weighted Bonferroni mean; MADM: Multiattribute decision-making; SG: Supporting grade

1. Introduction

In mathematics, the decision-making technique is the procedure of expressing real life problems and events in a mathematical and statistical format or language. Many techniques exist in the field of mathematics and are used for evaluating or carrying out mathematical and statistical problems and one of these techniques is fuzzy set theory (FST). Zadeh [1] enhanced the classical set theory and initiated the idea of FST in 1965. The information of FST is the group of supportive grade (SG) that gives values belonging to [0, 1]. After successful presentation of the idea of FST, a lot of researchers have enhanced and employed this idea in many fields worldwide, for instance, the qualitative comparative analysis based on FST utilized by Ding and Grundmann [2], Ahamed et al. [3] initiated the layout methodology based on FST and discussed their application, Chen and Tian [4] diagnosed the digital transformation using FST. Furthermore, the fuzzy set measures Python library was deliberated by Sidiropoulos et al. [5] and Kumar et al. [6] diagnosed the fuzzy set qualitative comparative analysis in business and management sciences.

Classical set theory and FST have attained a lot of attention and some researchers have done a lot of work in this direction, but as there are some situations where FST fails to work, for instance, if an expert considers economy of a country then along with exports of the country the expert will have also to consider imports of the country. Then in this scenario the notion of FST fails, because here the expert needs some tool which can handle such type of situation that is there is a need of a tool which can handle not only the positive SG but also the negative SG. To overcome this problem Zhang [7] initiated the notion of bipolar FST (BFST). The main structure of BFST includes two functions, called positive and negative SG having values in [0, 1] and [–1, 0] respectively. After successful presentation of the idea of BFST, many researchers enhanced and employed this notion further, for instance, Mahmood [8] initiated the notion of bipolar soft sets, Jana et al. [9] initiated the Dombi operators for BFST, Wei et al. [10] initiated the Hamacher operators for BFST, Jana et al. [11] presented the Dombi prioritized operators for BFST, Zadrozny and Kacprzyk [12] introduced the bipolar queries, Jana and Pal [13] established the extended BFST with EDAS technique, Lu et al. [14] explored the bipolar 2-tuple linguistic information, Jana [15] established the MABAC technique using BFST and worked on their applications, Zhang et al. [16] exposed supply chain management using BFST and Tchangani [17] initiated the theory of normal classification based on weighted cardinal fuzzy measures for BFST and discussed their applications in DM. Akram et al. [18] presented a BF complex linear system. Haque [19] initiated assessing infrastructural encroachment and fragmentation in east Kolkata. Akram and Arshad [20] presented BF TOPSIS and BF ELECTRE-I methods. MCDM technique in the setting of BF was initiated by Alghamdi et al. [21]. The extended TOPSIS technique in the setting of BF was diagnosed by Sarwar et al. [22]. Singh and Kumar [23] presented BF graph. Akram and Arshad [24] established a novel trapezoidal BF TOPSIS technique for DM. The graphs for the analysis of BF data were interpreted by Akram et al. [25].

For Mathematicians it is of great interest to discuss FST of the type in which the membership values are not the real numbers but the complex numbers. To address this issue, Ramot et al. [26] generalized the notion of FS by enhancing the SG defining from a universal set to the unit disc $\{z \in \mathbb{C}: |z| \leq 1\}$, called complex FS (CFS). After the introduction of CFS theory (CFST), many researchers worked on it, for instance, Liu et al. [27] worked on complex fuzzy cross-entropy measures, Mahmood and Ali [28] worked on complex fuzzy neighborhood operators, Zeeshan et al. [29] initiated the notion of complex fuzzy soft sets, Qudah and Hassan [30] initiated the notion of complex multifuzzy sets, Luqman et al. [31] worked on analysis of hypergraph structure by using CFST, Thirunavukarasu et al. [32] discussed some applications of complex fuzzy soft set theory, Mahmood et al. [33] initiated the notion of complex fuzzy N-soft sets and Alkouri [34] initiated the notion of complex generalised fuzzy soft set.

To generalize the notions of BFST and CFST, Mahmood and Ur Rehman [35] initiated the notion bipolar complex fuzzy set theory (BCFST). Multi-attribute decision making based BCFST is discussed in [36–38].

The main theme of this analysis is stated below:

a) We will show how to utilize this new idea.

b) We will show how to aggregate the information into a singleton set.

c) and how to find the best optimal.

For handling the above problems, we noticed that the theory of BM operators based on BCF information is much more suitable. Because some people have evaluated BM operators based on fuzzy sets and their extensions. The major theme of the BM operator was initiated by Bonferroni [38] in 1950, which proved to be a very effective tool for combining a collection of attributes. Furthermore, Yager [39] diagnosed the generalized BM operators. The geometric BM operator was developed by Xia et al. [40]. The generalized BM operators were also diagnosed by Beliakov et al. [41]. To aggregate or accumulate the collection of a finite number of information into a singleton set, the BM operator plays a very beneficial and dominant role in accurately evaluating the collection of information. The BM operator is massive powerful than the averaging/geometric operators because they are the specific case of the initiated operators. However, in some real life problems, it is a very problematic situation for an expert to capture the relationship between any terms of attributes. For instance, if in some situation we need to find the quality of the laptop, its efficiency, and working capability. Therefore, one thing that is essential for decision-makers is how to find the relation among attributes to make a massive beneficial decision. Additionally, due to the ambiguity and uncertainty of decision-making problems, it is essential to compute a new structure based on BCFST that is helpful to evaluate difficult and unreliable information in real life problems. The main analyses of the introduced approaches are explained below:

a) To deliberate the idea of BCFBM, BCFNWBM, and BCFOWBM operators. Furthermore, some properties and results of the deliberated operators are established.

b) To compute the required decision from the group of opinions, we computed a MADM problem based on the initiated operators for BCF information to evaluate the difficult and unachievable problems.

c) To compare the presented operators with some prevailing operators, we illustrate some examples and try to evaluate the graphical interpretation of the established work to boost the worth of the proposed theory. The influences of the BCFS and their restrictions are available in Table 1.

Table 1. Represented the quality and features of fuzzy sets and their generalizations.

Methods	Positive grade	Positive and negative grade	Contained imaginary part	Deal with two-dimension information
Fuzzy sets	√	×	×	×
Bipolar fuzzy sets	√	√	×	×
Complex fuzzy sets	√	×	√	√
Bipolar complex fuzzy sets	√	√	√	√

Main structure of the current study is organized as: In Section 2, we revise the idea of BCFS and their operational laws with the theory of BM operators. In Section 3, we deliberate the idea of BCFBM, BCFNWBM, and BCFOWBM operators. Furthermore, some properties and results of the deliberated operators are established. In Section 4, to compute the required decision from the group of opinions, we computed a MADM problem based on the initiated operators for BCF information to evaluate the difficult and unachievable problems. Finally, by comparing the presented operators with some existing operators, we illustrate some examples and try to evaluate the graphical interpretation of the diagnosed work to boost the worth of the proposed theory. The final concluding remarks are explained in Section 5.

2. Preliminaries

BM operator works as a tool for aggregating the collection of alternatives into a singleton set. The comparison of BM operators with averaging/geometric operators is massive powerful because they are the particular cases of the initiated operators. The main theme of this review section is to revise the conception of BCFST and their elementary operational laws with BM operators.

Definition 1. [35] A mathematical structure of BCFS is of the form:

$$\mathfrak{I} = \left\{\left(\tau, \Upsilon_{\mathfrak{I}}^{+}(\tau), \Upsilon_{\mathfrak{I}}^{-}(\tau)\right) | \tau \in \mathrm{T}\right\} \tag{1}$$

where $\Upsilon_{\mathfrak{I}}^{+}(\tau): T \rightarrow [0,1] + \imath[0,1]$ and $\Upsilon_{\mathfrak{I}}^{-}(\tau): T \rightarrow [-1,0] + \imath[-1,0]$, as known as the positive and negative SG: $\Upsilon_{\mathfrak{I}}^{+}(\tau) = \lambda_{\mathfrak{I}}^{+}(\tau) + \imath\delta_{\mathfrak{I}}^{+}(\tau)$ and $\Upsilon_{\mathfrak{I}}^{-}(\tau) = \lambda_{\mathfrak{I}}^{-}(\tau) + \imath\delta_{\mathfrak{I}}^{-}(\tau)$, with $\lambda_{\mathfrak{I}}^{+}(\tau),\ \delta_{\mathfrak{I}}^{+}(\tau) \in [0,1]$ and $\lambda_{\mathfrak{I}}^{-}(\tau),\ \delta_{\mathfrak{I}}^{-}(\tau) \in [-1,0]$. In simple words, we named the BCF number (BCFN)

$$\mathfrak{I} = \left(\tau, \Upsilon_{\mathfrak{I}}^{+}(\tau), \Upsilon_{\mathfrak{I}}^{-}(\tau)\right) = \left(\tau, \lambda_{\mathfrak{I}}^{+}(\tau) + \imath\delta_{\mathfrak{I}}^{+}(\tau), \lambda_{\mathfrak{I}}^{-}(\tau) + \imath\delta_{\mathfrak{I}}^{-}(\tau)\right).$$

Definition 2. [36] The score value $\mathcal{S}_B$, explained by using the BCFN

$$\mathfrak{I} = \left(\tau, \Upsilon_{\mathfrak{I}}^{+}(\tau), \Upsilon_{\mathfrak{I}}^{-}(\tau)\right) = \left(\tau, \lambda_{\mathfrak{I}}^{+}(\tau) + \imath\ \delta_{\mathfrak{I}}^{+}(\tau), \lambda_{\mathfrak{I}}^{-}(\tau) + \imath\delta_{\mathfrak{I}}^{-}(\tau)\right),$$

such that

$$\mathcal{S}_B(\mathfrak{I}) = \frac{1}{4}\left(2 + \lambda_{\mathfrak{I}}^{+}(\tau) + \delta_{\mathfrak{I}}^{+}(\tau) + \lambda_{\mathfrak{I}}^{-}(\tau) + \delta_{\mathfrak{I}}^{-}(\tau)\right),\ \mathcal{S}_B \in [0,1]. \tag{2}$$

Definition 3. [36] The accuracy value $\mathcal{S}_B$, explained by using the BCFN

$$\mathfrak{I} = \left(\tau, \Upsilon_{\mathfrak{I}}^{+}(\tau), \Upsilon_{\mathfrak{I}}^{-}(\tau)\right) = \left(\tau, \lambda_{\mathfrak{I}}^{+}(\tau) + \imath\ \delta_{\mathfrak{I}}^{+}(\tau), \lambda_{\mathfrak{I}}^{-}(\tau) + \imath\delta_{\mathfrak{I}}^{-}(\tau)\right),$$

such that

$$\mathcal{H}_B(\mathfrak{J}) = \frac{\lambda_{\mathfrak{J}}^{+}(\tau)+\delta_{\mathfrak{J}}^{+}(\tau)-\lambda_{\mathfrak{J}}^{-}(\tau)-\delta_{\mathfrak{J}}^{-}(\tau)}{4},\ \mathcal{H}_B \in [0,1]. \tag{3}$$

Definition 4. [36] For any $\mathfrak{J} = \left(\tau, \Upsilon_{\mathfrak{J}}^{+}(\tau), \Upsilon_{\mathfrak{J}}^{-}(\tau)\right)$ and $\mathfrak{K} = \left(\tau, \Upsilon_{\mathfrak{K}}^{+}(\tau), \Upsilon_{\mathfrak{K}}^{-}(\tau)\right)$, we computed

1) If $\mathcal{S}_B(\mathfrak{J}) < \mathcal{S}_B(\mathfrak{K})$, then $\mathfrak{J} < \mathfrak{K}$.
2) If $\mathcal{S}_B(\mathfrak{J}) > \mathcal{S}_B(\mathfrak{K})$, then $\mathfrak{J} > \mathfrak{K}$.
3) If $\mathcal{S}_B(\mathfrak{J}) = (\mathfrak{K})$, then
   i. If $\mathcal{H}_B(\mathfrak{J}) < \mathcal{H}_B(\mathfrak{K})$, then $\mathfrak{J} < \mathfrak{K}$.
   ii. If $\mathcal{H}_B(\mathfrak{J}) > \mathcal{H}_B(\mathfrak{K})$, then $\mathfrak{J} > \mathfrak{K}$.
   iii. If $\mathcal{H}_B(\mathfrak{J}) = \mathcal{H}_B(\mathfrak{K})$, then $\mathfrak{J} = \mathfrak{K}$.

Definition 5. [36] For any

$$\mathfrak{J} = \left(\tau, \Upsilon_{\mathfrak{J}}^{+}(\tau), \Upsilon_{\mathfrak{J}}^{-}(\tau)\right) = \left(\tau, \lambda_{\mathfrak{J}}^{+}(\tau) + \imath\, \delta_{\mathfrak{J}}^{+}(\tau), \lambda_{\mathfrak{J}}^{-}(\tau) + \imath \delta_{\mathfrak{J}}^{-}(\tau)\right)$$

and

$$\mathfrak{K} = \left(\Upsilon_{\mathfrak{K}}^{+}(\tau), \Upsilon_{\mathfrak{K}}^{-}(\tau)\right) = \left(\lambda_{\mathfrak{K}}^{+}(\tau) + \imath\, \delta_{\mathfrak{K}}^{+}(\tau), \lambda_{\mathfrak{K}}^{-}(\tau) + \imath \delta_{\mathfrak{K}}^{-}(\tau)\right),$$

with $\beta > 0$, then

$$\mathfrak{J} \oplus \mathfrak{K} = \left(\tau, \begin{pmatrix} \lambda_{\mathfrak{J}}^{+}(\tau) + \lambda_{\mathfrak{K}}^{+}(\tau) - \lambda_{\mathfrak{J}}^{+}(\tau)\lambda_{\mathfrak{K}}^{+}(\tau) + \imath\left(\delta_{\mathfrak{J}}^{+}(\tau) + \delta_{\mathfrak{K}}^{+}(\tau) - \delta_{\mathfrak{J}}^{+}(\tau)\delta_{\mathfrak{K}}^{+}(\tau)\right), \\ -\left(\lambda_{\mathfrak{J}}^{-}(\tau)\lambda_{\mathfrak{K}}^{-}(\tau)\right) + \imath\left(-\left(\delta_{\mathfrak{J}}^{-}(\tau)\delta_{\mathfrak{K}}^{-}(\tau)\right)\right) \end{pmatrix}\right). \tag{4}$$

$$\mathfrak{J} \otimes \mathfrak{K} = \left(\tau, \begin{pmatrix} \lambda_{\mathfrak{J}}^{+}(\tau)\lambda_{\mathfrak{K}}^{+}(\tau) + \imath \delta_{\mathfrak{J}}^{+}(\tau)\delta_{\mathfrak{K}}^{+}(\tau), \\ \lambda_{\mathfrak{J}}^{-}(\tau) + \lambda_{\mathfrak{K}}^{-}(\tau) + \lambda_{\mathfrak{J}}^{-}(\tau)\lambda_{\mathfrak{K}}^{-}(\tau) + \imath\left(\delta_{\mathfrak{J}}^{-}(\tau) + \delta_{\mathfrak{K}}^{-}(\tau) + \delta_{\mathfrak{J}}^{-}(\tau)\delta_{\mathfrak{K}}^{-}(\tau)\right) \end{pmatrix}\right). \tag{5}$$

$$\beta\mathfrak{J} = \left(\tau, \left(1 - \left(1 - \lambda_{\mathfrak{J}}^{+}(\tau)\right)^{\beta} + \imath\left(1 - \left(1 - \delta_{\mathfrak{J}}^{+}(\tau)\right)^{\beta}\right), -\left|\lambda_{\mathfrak{J}}^{-}(\tau)\right|^{\beta} + \imath\left(-\left|\delta_{\mathfrak{J}}^{-}(\tau)\right|^{\beta}\right)\right)\right). \tag{6}$$

$$\mathfrak{J}^{\beta} = \left(\tau, \left(\lambda_{\mathfrak{J}}^{+}(\tau)^{\beta} + \imath \delta_{\mathfrak{J}}^{+}(\tau)^{\beta}, -1 + \left(1 + \lambda_{\mathfrak{J}}^{-}(\tau)\right)^{\beta} + \imath\left(-1 + \left(1 + \delta_{\mathfrak{J}}^{-}(\tau)\right)^{\beta}\right)\right)\right). \tag{7}$$

Theorem 1. [37] Under the availability of any BCFNs

$$\mathfrak{J} = \left(\tau, \Upsilon_{\mathfrak{J}}^{+}(\tau), \Upsilon_{\mathfrak{J}}^{-}(\tau)\right) = \left(\tau, \lambda_{\mathfrak{J}}^{+}(\tau) + \imath \delta_{\mathfrak{J}}^{+}(\tau), \lambda_{\mathfrak{J}}^{-}(\tau) + \imath \delta_{\mathfrak{J}}^{-}(\tau)\right)$$

and

$$\mathfrak{K} = \left(\Upsilon_{\mathfrak{K}}^{+}(\tau), \Upsilon_{\mathfrak{K}}^{-}(\tau)\right) = \left(\lambda_{\mathfrak{K}}^{+}(\tau) + \imath \delta_{\mathfrak{K}}^{+}(\tau), \lambda_{\mathfrak{K}}^{-}(\tau) + \imath \delta_{\mathfrak{K}}^{-}(\tau)\right),$$

with $\beta, \beta_1, \beta_2 > 0$, then

1) $\mathfrak{J} \oplus \mathfrak{K} = \mathfrak{K} \oplus \mathfrak{J}$.

2) $\mathfrak{J} \otimes \mathfrak{K} = \mathfrak{K} \otimes \mathfrak{J}$.
3) $\beta(\mathfrak{J} \oplus \mathfrak{K}) = \beta\mathfrak{J} \oplus \beta\mathfrak{K}$.
4) $(\mathfrak{J} \otimes \mathfrak{K})^{\beta} = \mathfrak{J}^{\beta} \otimes \mathfrak{K}^{\beta}$.
5) $\beta_1\mathfrak{J} \oplus \beta_2\mathfrak{J} = (\beta_1 + \beta_2)\mathfrak{J}$.
6) $\mathfrak{J}^{\beta_1} \otimes \mathfrak{J}^{\beta_2} = \mathfrak{J}^{\beta_1+\beta_2}$.
7) $\left(\mathfrak{J}^{\beta_1}\right)^{\beta_2} = \mathfrak{J}^{\beta_1\beta_2}$.

Proof. Trivial.

Definition 6. [38] Suppose that $\hbar_{ķ}$ ($ķ = 1, 2, .., ň$) be a group of positive real numbers, Then

$$B^{\mathrm{þ},\mathrm{q}}(\hbar_1, \hbar_2, \hbar_3, \dots, \hbar_{ň}) = \left(\frac{1}{ň(ň-1)} \sum_{\substack{ķ,ĺ=1 \\ ķ\neq ĺ}}^{ň} \hbar_{ķ}^{\mathrm{þ}} \hbar_{ĺ}^{\mathrm{q}}\right)^{\frac{1}{\mathrm{þ}+\mathrm{q}}} \tag{8}$$

is known as BM, where $\mathrm{þ}, \mathrm{q} \geq 0$.

3. BM operators for BCFSs

The major investigation of this analysis is to deliberate the idea of BCFBM, BCFNWBM, and BCFOWBM operators. Furthermore, some properties and results of the deliberated operators are diagnosed. The terms

$$\hbar_{ķ} = \left(\Upsilon^{+}_{\hbar_{ķ}}, \Upsilon^{-}_{\hbar_{ķ}}\right) = \left(\lambda^{+}_{\hbar_{ķ}} + i\delta^{+}_{\hbar_{ķ}}, \lambda^{-}_{\hbar_{ķ}} + i\delta^{-}_{\hbar_{ķ}}\right) \ (ķ = 1, 2, 3, \dots, ň),$$

stated the family of BCFNs.

3.1. The BCFBM operator

Here, we state the BCFBM operator as follows:

Definition 7. The BCFBM operator $BCFB^{\mathrm{þ},\mathrm{q}}$ with $\mathrm{þ}, \mathrm{q} \geq 0$, simplified by:

$$BCFB^{\mathrm{þ},\mathrm{q}}(\hbar_1, \hbar_2, \hbar_3, \dots, \hbar_{ň}) = \left(\frac{1}{ň(ň-1)}\left(\bigoplus_{\substack{ķ,ĺ=1 \\ ķ\neq ĺ}}^{ň} \left(\hbar_{ķ}^{\mathrm{þ}} \otimes \hbar_{ĺ}^{\mathrm{q}}\right)\right)\right)^{\frac{1}{\mathrm{þ}+\mathrm{q}}}. \tag{9}$$

Theorem 2. For Eq (9) with $\mathrm{þ}, \mathrm{q} \geq 0$, we diagnose

$$BCFB^{\mathfrak{p},\mathfrak{q}}(\hbar_1,\hbar_2,\hbar_3,\dots,\hbar_{\check{n}})=\begin{pmatrix}\left(1-\prod_{\substack{\mathfrak{k},\mathfrak{l}=1\\ \mathfrak{k}\neq\mathfrak{l}}}^{\check{n}}\left(1-\lambda_{\hbar_{\mathfrak{k}}}^{+\mathfrak{p}}\lambda_{\hbar_{\mathfrak{l}}}^{+\mathfrak{q}}\right)^{\frac{1}{\check{n}(\check{n}-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}+\\ i\left(1-\prod_{\substack{\mathfrak{k},\mathfrak{l}=1\\ \mathfrak{k}\neq\mathfrak{l}}}^{\check{n}}\left(1-\delta_{\hbar_{\mathfrak{k}}}^{+\mathfrak{p}}\delta_{\hbar_{\mathfrak{l}}}^{+\mathfrak{q}}\right)^{\frac{1}{\check{n}(\check{n}-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}},\\ -1+\left(1-\left|-\prod_{\substack{\mathfrak{k},\mathfrak{l}=1\\ \mathfrak{k}\neq\mathfrak{l}}}^{\check{n}}\left(-1+\left(1+\lambda_{\hbar_{\mathfrak{k}}}^{-}\right)^{\mathfrak{p}}\left(1+\lambda_{\hbar_{\mathfrak{l}}}^{-}\right)^{\mathfrak{q}}\right)\right|^{\frac{1}{\check{n}(\check{n}-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}+\\ i\left(-1+\left(1-\left|-\prod_{\substack{\mathfrak{k},\mathfrak{l}=1\\ \mathfrak{k}\neq\mathfrak{l}}}^{\check{n}}\left(-1+\left(1+\delta_{\hbar_{\mathfrak{k}}}^{-}\right)^{\mathfrak{p}}\left(1+\delta_{\hbar_{\mathfrak{l}}}^{-}\right)^{\mathfrak{q}}\right)\right|^{\frac{1}{\check{n}(\check{n}-1)}}\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}\right)\end{pmatrix}. \quad (10)$$

Proof. The proof of this theorem is given in Appendix A.

Furthermore, the BCFBM has the following properties:

1) **Idempotency:** If all $\hbar_{\mathfrak{k}}(\mathfrak{k}=1,2,3,..,\check{n})$ are same, that is, $\hbar_{\mathfrak{k}}=\hbar\forall\mathfrak{k}$, then

$$BCFB^{\mathfrak{p},\mathfrak{q}}(\hbar_1,\hbar_2,\hbar_3,\dots,\hbar_{\check{n}})=BCFB^{\mathfrak{p},\mathfrak{q}}(\hbar,\hbar,\hbar,\dots,\hbar)=\hbar. \quad (11)$$

2) **Monotonicity:** Suppose $\hbar_{\mathfrak{k}}(\mathfrak{k}=1,2,3,..,\check{n})$ and $g_{\mathfrak{k}}(\mathfrak{k}=1,2,3,..,\check{n})$ are two collections of BCFNs, if

$$\hbar_{\mathfrak{k}}\leq g_{\mathfrak{k}}\forall\mathfrak{k}(\text{i.e., }\lambda_{\hbar_{\mathfrak{k}}}^{+}\leq\lambda_{g_{\mathfrak{k}}}^{+},\ \delta_{\hbar_{\mathfrak{k}}}^{+}\leq\delta_{g_{\mathfrak{k}}}^{+},\ \lambda_{\hbar_{\mathfrak{k}}}^{-}\leq\lambda_{g_{\mathfrak{k}}}^{-},\text{ and }\delta_{\hbar_{\mathfrak{k}}}^{-}\leq\delta_{g_{\mathfrak{k}}}^{-}),$$

then

$$BCFB^{\mathfrak{p},\mathfrak{q}}(\hbar_1,\hbar_2,\hbar_3,\dots,\hbar_{\check{n}})\leq BCFB^{\mathfrak{p},\mathfrak{q}}(g_1,g_2,g_3,\dots,g_{\check{n}}). \quad (12)$$

3) **Boundedness:** Suppose $\hbar_{\mathfrak{k}}(\mathfrak{k}=1,2,3,..,\check{n})$ is a group of BCFNs, and suppose

$$\hbar^{-}=(\min(\hbar_1,\hbar_2,\hbar_3,\dots,\hbar_{\check{n}}))=\left(\min_{\mathfrak{k}}\left(\lambda_{\hbar_{\mathfrak{k}}}^{+}\right)+i\min_{\mathfrak{k}}\left(\delta_{\hbar_{\mathfrak{k}}}^{+}\right),\max_{\mathfrak{k}}\left(\lambda_{\hbar_{\mathfrak{k}}}^{-}\right)+i\max_{\mathfrak{k}}\left(\delta_{\hbar_{\mathfrak{k}}}^{-}\right)\right),$$

$$\hbar^{+}=(\max(\hbar_1,\hbar_2,\hbar_3,\dots,\hbar_{\check{n}}))=\left(\max_{\mathfrak{k}}\left(\lambda_{\hbar_{\mathfrak{k}}}^{+}\right)+i\max_{\mathfrak{k}}\left(\delta_{\hbar_{\mathfrak{k}}}^{+}\right),\min_{\mathfrak{k}}\left(\lambda_{\hbar_{\mathfrak{k}}}^{-}\right)+i\min_{\mathfrak{k}}\left(\delta_{\hbar_{\mathfrak{k}}}^{-}\right)\right),$$

then,

$$\hbar^{-}\leq BCFB^{\mathfrak{p},\mathfrak{q}}(\hbar_1,\hbar_2,\hbar_3,\dots,\hbar_{\check{n}})\leq\hbar^{+}. \quad (13)$$

4) **Commutativity:** Suppose $\hbar_{\mathfrak{k}}$ $(\mathfrak{k}=1,2,3,..,\check{n})$ is a collection of BCFNs, then

$$BCFB^{\mathfrak{p},\mathfrak{q}}(\hbar_1,\hbar_2,\hbar_3,\dots,\hbar_{\check{n}})=BCFB^{\mathfrak{p},\mathfrak{q}}(\hbar_1',\hbar_2',\hbar_3',\dots,\hbar_{\check{n}}') \quad (14)$$

where $(\hbar_1',\hbar_2',\hbar_3',\dots,\hbar_{\check{n}}')$ is any permutation of $(\hbar_1,\hbar_2,\hbar_3,\dots,\hbar_{\check{n}})$.

By using distinct values of the parameters $\mathfrak{p}$ and $\mathfrak{q}$, we have the following particular cases of BCFBM.

Case 1: If $\mathfrak{q}\to 0$, then by Eq (10) we obtain

$$\lim_{\text{ɋ}\to 0} BCFB^{\text{ꞵ},\text{ɋ}}(\hbar_1, \hbar_2, \hbar_3, \ldots, \hbar_{\text{ň}}) = \lim_{\text{ɋ}\to 0}\left(\frac{1}{\text{ň}(\text{ň}-1)}\left(\bigoplus_{\substack{\text{ķ},\text{ĺ}=1\\ \text{ķ}\neq\text{ĺ}}}^{\text{ň}}\left(\hbar_{\text{ķ}}^{\text{ꞵ}} \otimes \hbar_{\text{ĺ}}^{\text{ɋ}}\right)\right)\right)^{\frac{1}{\text{ꞵ}+\text{ɋ}}}$$

$$= \lim_{\text{ɋ}\to 0}\left(\begin{array}{c} \left(1-\prod_{\substack{\text{ķ},\text{ĺ}=1\\ \text{ķ}\neq\text{ĺ}}}^{\text{ň}}\left(1-\lambda_{\hbar_{\text{ķ}}}^{+\text{ꞵ}}\lambda_{\hbar_{\text{ĺ}}}^{+\text{ɋ}}\right)^{\frac{1}{\text{ň}(\text{ň}-1)}}\right)^{\frac{1}{\text{ꞵ}+\text{ɋ}}} + \\ i\left(1-\prod_{\substack{\text{ķ},\text{ĺ}=1\\ \text{ķ}\neq\text{ĺ}}}^{\text{ň}}\left(1-\delta_{\hbar_{\text{ķ}}}^{+\text{ꞵ}}\delta_{\hbar_{\text{ĺ}}}^{+\text{ɋ}}\right)^{\frac{1}{\text{ň}(\text{ň}-1)}}\right)^{\frac{1}{\text{ꞵ}+\text{ɋ}}}, \\ -1+\left(1-\left|-\prod_{\substack{\text{ķ},\text{ĺ}=1\\ \text{ķ}\neq\text{ĺ}}}^{\text{ň}}\left(-1+\left(1+\lambda_{\hbar_{\text{ķ}}}^{-}\right)^{\text{ꞵ}}\left(1+\lambda_{\hbar_{\text{ĺ}}}^{-}\right)^{\text{ɋ}}\right)\right|^{\frac{1}{\text{ň}(\text{ň}-1)}}\right)^{\frac{1}{\text{ꞵ}+\text{ɋ}}} + \\ i\left(-1+\left(1-\left|-\prod_{\substack{\text{ķ},\text{ĺ}=1\\ \text{ķ}\neq\text{ĺ}}}^{\text{ň}}\left(-1+\left(1+\delta_{\hbar_{\text{ķ}}}^{-}\right)^{\text{ꞵ}}\left(1+\delta_{\hbar_{\text{ĺ}}}^{-}\right)^{\text{ɋ}}\right)\right|^{\frac{1}{\text{ň}(\text{ň}-1)}}\right)^{\frac{1}{\text{ꞵ}+\text{ɋ}}}\right) \end{array}\right)$$

$$=\left(\begin{array}{c}\left(1-\prod_{ķ=1}^{ň}\left(1-\lambda_{ℏ_{ķ}}^{+\mathrm{þ}}\right)^{\frac{ň-1}{ň(ň-1)}}\right)^{\frac{1}{\mathrm{þ}}}+\\ i\left(1-\prod_{ķ=1}^{ň}\left(1-\delta_{ℏ_{ķ}}^{+\mathrm{þ}}\right)^{\frac{ň-1}{ň(ň-1)}}\right)^{\frac{1}{\mathrm{þ}}},\\ -1+\left(1-\left|-\prod_{ķ=1}^{ň}\left(-1+\left(1+\lambda_{ℏ_{ķ}}^{-}\right)^{\mathrm{þ}}\right)\right|^{\frac{ň-1}{ň(ň-1)}}\right)^{\frac{1}{\mathrm{þ}}}+\\ i\left(-1+\left(1-\left|-\prod_{ķ=1}^{ň}\left(-1+\left(1+\delta_{ℏ_{ķ}}^{-}\right)^{\mathrm{þ}}\right)\right|^{\frac{ň-1}{ň(ň-1)}}\right)^{\frac{1}{\mathrm{þ}}}\right)\end{array}\right)$$

$$=\left(\begin{array}{c}\left(1-\prod_{ķ=1}^{ň}\left(1-\lambda_{ℏ_{ķ}}^{+\mathrm{þ}}\right)^{\frac{1}{ň}}\right)^{\frac{1}{\mathrm{þ}}}+\\ i\left(1-\prod_{ķ=1}^{ň}\left(1-\delta_{ℏ_{ķ}}^{+\mathrm{þ}}\right)^{\frac{1}{ň}}\right)^{\frac{1}{\mathrm{þ}}},\\ -1+\left(1-\left|-\prod_{ķ=1}^{ň}\left(-1+\left(1+\lambda_{ℏ_{ķ}}^{-}\right)^{\mathrm{þ}}\right)\right|^{\frac{1}{ň}}\right)^{\frac{1}{\mathrm{þ}}}+\\ i\left(-1+\left(1-\left|-\prod_{ķ=1}^{ň}\left(-1+\left(1+\delta_{ℏ_{ķ}}^{-}\right)^{\mathrm{þ}}\right)\right|^{\frac{1}{ň}}\right)^{\frac{1}{\mathrm{þ}}}\right)\end{array}\right)$$

$$=\left(\frac{1}{ň}\left(\bigoplus_{ķ=1}^{ň}\left(ℏ_{ķ}^{\mathrm{þ}}\right)\right)\right)^{\frac{1}{\mathrm{þ}}}=BCFB^{\mathrm{þ},0}(ℏ_1,ℏ_2,ℏ_3,\dots,ℏ_{ň}). \quad (15)$$

We call it generalized bipolar complex fuzzy mean (GBCFM).

Case 2. If $\mathrm{þ}=2$ and $\mathrm{q}\to 0$, then Eq (10) is converted as

$$BCFB^{2,0}(\hbar_1,\hbar_2,\hbar_3,\dots,\hbar_{ň}) = \left(\frac{1}{ň}\left(\underset{ķ=1}{\overset{ň}{\oplus}}\left(\hbar_{ķ}^2\right)\right)\right)^{\frac{1}{2}}$$

$$= \begin{pmatrix} \left(1-\prod_{ķ=1}^{ň}\left(1-\lambda_{\hbar_{ķ}}^{+2}\right)^{\frac{1}{ň}}\right)^{\frac{1}{2}} + \\ i\left(1-\prod_{ķ=1}^{ň}\left(1-\delta_{\hbar_{ķ}}^{+2}\right)^{\frac{1}{ň}}\right)^{\frac{1}{2}}, \\ -1+\left(1-\left|-\prod_{ķ=1}^{ň}\left(-1+\left(1+\lambda_{\hbar_{ķ}}^{-}\right)^2\right)\right|^{\frac{1}{ň}}\right)^{\frac{1}{2}} + \\ i\left(-1+\left(1-\left|-\prod_{ķ=1}^{ň}\left(-1+\left(1+\delta_{\hbar_{ķ}}^{-}\right)^2\right)\right|^{\frac{1}{ň}}\right)^{\frac{1}{2}}\right) \end{pmatrix}. \quad (16)$$

We call it bipolar complex fuzzy square mean (BCFSM).

Case 3. If $ƥ = 1$ and $ɋ \to 0$, then Eq (10) is converted as

$$BCFB^{1,0}(\hbar_1,\hbar_2,\hbar_3,\dots,\hbar_{ň}) = \begin{pmatrix} \left(1-\prod_{ķ=1}^{ň}\left(1-\lambda_{\hbar_{ķ}}^{+}\right)^{\frac{1}{ň}}\right) + \\ i\left(1-\prod_{ķ=1}^{ň}\left(1-\delta_{\hbar_{ķ}}^{+}\right)^{\frac{1}{ň}}\right), \\ -1+\left(1-\left|-\prod_{ķ=1}^{ň}\left(-1+\left(1+\lambda_{\hbar_{ķ}}^{-}\right)\right)\right|^{\frac{1}{ň}}\right) + \\ i\left(-1+\left(1-\left|-\prod_{ķ=1}^{ň}\left(-1+\left(1+\delta_{\hbar_{ķ}}^{-}\right)\right)\right|^{\frac{1}{ň}}\right)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(1-\prod_{ķ=1}^{ň}\left(1-\lambda_{\hbar_{ķ}}^{+}\right)^{\frac{1}{ň}}\right) + \\ i\left(1-\prod_{ķ=1}^{ň}\left(1-\delta_{\hbar_{ķ}}^{+}\right)^{\frac{1}{ň}}\right), \\ -\left|-\prod_{ķ=1}^{ň}\lambda_{\hbar_{ķ}}^{-}\right|^{\frac{1}{ň}} + \\ i\left(-\left|-\prod_{ķ=1}^{ň}\delta_{\hbar_{ķ}}^{-}\right|^{\frac{1}{ň}}\right) \end{pmatrix} = \frac{1}{ň}\left(\underset{ķ=1}{\overset{ň}{\oplus}}\left(\hbar_{ķ}\right)\right) \quad (17)$$

we call it bipolar complex fuzzy average (BCFA).

Case 4. If $\text{þ} = \text{q} = 1$, then Eq (10) is converted as

$$BCFB^{1,1}(\hbar_1, \hbar_2, \hbar_3, \dots, \hbar_{\check{n}}) = \left(\frac{1}{\check{n}(\check{n}-1)} \left(\bigoplus_{\substack{ķ, ĺ = 1 \\ ķ \neq ĺ}}^{\check{n}} \left(\hbar_{ķ} \otimes \hbar_{ĺ} \right) \right) \right)^{\frac{1}{2}}$$

$$= \left(\begin{array}{c} \left(1 - \prod_{\substack{ķ,ĺ=1 \\ ķ\neq ĺ}}^{\check{n}} \left(1 - \lambda^{+}_{\hbar_{ķ}} \lambda^{+}_{\hbar_{ĺ}}\right)^{\frac{1}{\check{n}(\check{n}-1)}}\right)^{\frac{1}{2}} + \\ \imath \left(1 - \prod_{\substack{ķ,ĺ=1 \\ ķ\neq ĺ}}^{\check{n}} \left(1 - \delta^{+}_{\hbar_{ķ}} \delta^{+}_{\hbar_{ĺ}}\right)^{\frac{1}{\check{n}(\check{n}-1)}}\right)^{\frac{1}{2}}, \\ -1 + \left(1 - \left| - \prod_{\substack{ķ,ĺ=1 \\ ķ\neq ĺ}}^{\check{n}} \left(-1 + \left(1 + \lambda^{-}_{\hbar_{ķ}}\right)\left(1 + \lambda^{-}_{\hbar_{ĺ}}\right)\right) \right|^{\frac{1}{\check{n}(\check{n}-1)}}\right)^{\frac{1}{2}} + \\ \imath \left(-1 + \left(1 - \left| - \prod_{\substack{ķ,ĺ=1 \\ ķ\neq ĺ}}^{\check{n}} \left(-1 + \left(1 + \delta^{-}_{\hbar_{ķ}}\right)\left(1 + \delta^{-}_{\hbar_{ĺ}}\right)\right) \right|^{\frac{1}{\check{n}(\check{n}-1)}}\right)^{\frac{1}{2}} \right) \end{array} \right) \tag{18}$$

we call it bipolar complex fuzzy interrelated square mean (BCFISM).

3.2. The BCFNWBM operator

In MADM, the associated attributes generally have distinct significance and are necessary to be given distinct weights. Thus, AOs should consider the weights of attributes.

Definition 8. The BCFNWBM operator $BCFNWB^{\text{þ},\text{q}}$ with $\text{þ}, \text{q} \geq 0$, simplified by:

$$BCFNWB^{\text{þ},\text{q}}_{\tilde{\omega}}(\hbar_1, \hbar_2, \hbar_3, \dots, \hbar_{\check{n}}) = \left(\bigoplus_{\substack{ķ, ĺ = 1 \\ ķ \neq ĺ}}^{\check{n}} \frac{\tilde{\omega}_{ķ}\tilde{\omega}_{ĺ}}{1-\tilde{\omega}_{ķ}} \left(\hbar_{ķ}^{\text{þ}} \otimes \hbar_{ĺ}^{\text{q}} \right) \right)^{\frac{1}{\text{þ}+\text{q}}} \tag{19}$$

where, $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3, \dots, \tilde{\omega}_{\check{n}})$ is the weight vector (WV) of $\hbar_{ķ}(ķ = 1, 2, 3, \dots, \check{n})$, where $\tilde{\omega}_{ķ}$ denotes the significance degree of $\hbar_{ķ}$ such that $\tilde{\omega}_{ķ} \in [0, 1]$ and $\sum_{ķ=1}^{\check{n}} \tilde{\omega}_{ķ} = 1$, if

Theorem 3. For Eq (19) with $\text{þ}, \text{q} \geq 0$, we diagnose

$$BCFNWB_{\tilde{\omega}}^{p,q}(\hbar_1,\hbar_2,\hbar_3,\ldots,\hbar_{\check{n}})=\left(\begin{array}{c}\left(1-\prod_{\substack{k,l=1\\k\neq l}}^{\check{n}}\left(1-\lambda_{\hbar_k}^{+p}\lambda_{\hbar_l}^{+q}\right)^{\frac{\tilde{\omega}_k\tilde{\omega}_l}{1-\tilde{\omega}_k}}\right)^{\frac{1}{p+q}}+\\ i\left(1-\prod_{\substack{k,l=1\\k\neq l}}^{\check{n}}\left(1-\delta_{\hbar_k}^{+p}\delta_{\hbar_l}^{+q}\right)^{\frac{\tilde{\omega}_k\tilde{\omega}_l}{1-\tilde{\omega}_k}}\right)^{\frac{1}{p+q}},\\ -1+\left(1-\prod_{\substack{k,l=1\\k\neq l}}^{\check{n}}\left|-1+\left(1+\lambda_{\hbar_k}^{-}\right)^{p}\left(1+\lambda_{\hbar_l}^{-}\right)^{q}\right|^{\frac{\tilde{\omega}_k\tilde{\omega}_l}{1-\tilde{\omega}_k}}\right)^{\frac{1}{p+q}}+\\ i\left(-1+\left(1-\prod_{\substack{k,l=1\\k\neq l}}^{\check{n}}\left|-1+\left(1+\delta_{\hbar_k}^{-}\right)^{p}\left(1+\delta_{\hbar_l}^{-}\right)^{q}\right|^{\frac{\tilde{\omega}_k\tilde{\omega}_l}{1-\tilde{\omega}_k}}\right)^{\frac{1}{p+q}}\right)\end{array}\right). \quad (20)$$

Proof. The proof of this theorem is given in Appendix B.

Additionally, the BCFNWBM has the following properties

1) **Idempotency:** If all $\hbar_k$ $(k=1,2,3,..,\check{n})$ are same, that is, $\hbar_k=\hbar\forall k$, then

$$BCFNWB_{\tilde{\omega}}^{p,q}(\hbar_1,\hbar_2,\hbar_3,\ldots,\hbar_{\check{n}})=BCFNWB_{\tilde{\omega}}^{p,q}(\hbar,\hbar,\hbar,\ldots,\hbar)=\hbar. \quad (21)$$

2) **Monotonicity:** Suppose $\hbar_k(k=1,2,3,..,\check{n})$ and $g_k(k=1,2,3,..,\check{n})$ are two collections of BCFNs, if $\hbar_k\le g_k\forall k$ (i.e., $\lambda_{\hbar_k}^{+}\le\lambda_{g_k}^{+}$, $\delta_{\hbar_k}^{+}\le\delta_{g_k}^{+}$, $\lambda_{\hbar_k}^{-}\le\lambda_{g_k}^{-}$, and $\delta_{\hbar_k}^{-}\le\delta_{g_k}^{-}$), then

$$BCFNWB_{\tilde{\omega}}^{p,q}(\hbar_1,\hbar_2,\hbar_3,\ldots,\hbar_{\check{n}})\le BCFNWB_{\tilde{\omega}}^{p,q}(g_1,g_2,g_3,\ldots,g_{\check{n}}). \quad (22)$$

3) **Boundedness:** Suppose $\hbar_k(k=1,2,3,..,\check{n})$ is a group of BCFNs, and suppose

$$\hbar^{-}=(\min(\hbar_1,\hbar_2,\hbar_3,\ldots,\hbar_{\check{n}}))=\left(\min_k\left(\lambda_{\hbar_k}^{+}\right)+i\min_k\left(\delta_{\hbar_k}^{+}\right),\ \max_k\left(\lambda_{\hbar_k}^{-}\right)+i\max_k\left(\delta_{\hbar_k}^{-}\right)\right),$$

$$\hbar^{+}=(\max(\hbar_1,\hbar_2,\hbar_3,\ldots,\hbar_{\check{n}}))=\left(\max_k\left(\lambda_{\hbar_k}^{+}\right)+i\max_k\left(\delta_{\hbar_k}^{+}\right),\ \min_k\left(\lambda_{\hbar_k}^{-}\right)+i\min_k\left(\delta_{\hbar_k}^{-}\right)\right),$$

then,

$$\hbar^{-}\le BCFNWB_{\tilde{\omega}}^{p,q}(\hbar_1,\hbar_2,\hbar_3,\ldots,\hbar_{\check{n}})\le\hbar^{+}. \quad (23)$$

4) **Commutativity:** Suppose $\hbar_k$ $(k=1,2,3,..,\check{n})$ is a collection of BCFNs, then

$$BCFNWB_{\tilde{\omega}}^{p,q}(\hbar_1,\hbar_2,\hbar_3,\ldots,\hbar_{\check{n}})=BCFNWB_{\tilde{\omega}}^{p,q}(\hbar_1',\hbar_2',\hbar_3',\ldots,\hbar_{\check{n}}'). \quad (24)$$

3.3. The BCFOWBM operator

In this subsection, we present the BCFOWBM operator.

Definition 9. The BCFOWBM operator $BCFOWB^{\mathrm{p},\mathrm{q}}$ with $\mathrm{p},\mathrm{q} \geq 0$, simplified by:

$$BCFOWB_{\tilde{\varpi}}^{\mathrm{p},\mathrm{q}}(\hbar_1, \hbar_2, \hbar_3, \dots, \hbar_{\check{n}}) = \left(\bigoplus_{\substack{\text{ķ},\text{ĺ}=1 \\ \text{ķ} \neq \text{ĺ}}}^{\check{n}} \frac{\tilde{\varpi}_{\text{ķ}} \tilde{\varpi}_{\text{ĺ}}}{1-\tilde{\varpi}_{\text{ķ}}} \left(\hbar_{\varepsilon(\text{ķ})}^{\mathrm{p}} \otimes \hbar_{\varepsilon(\text{ĺ})}^{\mathrm{q}} \right) \right)^{\frac{1}{\mathrm{p}+\mathrm{q}}} \tag{25}$$

where, $\tilde{\varpi} = (\tilde{\varpi}_1, \tilde{\varpi}_2, \tilde{\varpi}_3, \dots, \tilde{\varpi}_{\check{n}})$ is the WV such that $\tilde{\varpi}_{\text{ķ}} \in [0,1]$ and $\sum_{\text{ķ}=1}^{\check{n}} \tilde{\varpi}_{\text{ķ}} = 1$, and $\varepsilon(1), \varepsilon(2), \dots, \varepsilon(\check{n})$ are the permutation of $(\text{ķ} = 1, 2, 3, \dots, \check{n})$ such that $\hbar_{\varepsilon(\text{ķ}-1)} \geq \hbar_{\varepsilon(\text{ķ})} \forall \text{ķ} = 1, 2, 3, \dots, \check{n}$.

Theorem 4. For Eq (29) with $\mathrm{p},\mathrm{q} \geq 0$, we diagnose

$$BCFOWB_{\tilde{\varpi}}^{\mathrm{p},\mathrm{q}}(\hbar_1, \hbar_2, \hbar_3, \dots, \hbar_{\check{n}}) = \left(\begin{array}{c} \left(1 - \prod_{\substack{\text{ķ},\text{ĺ}=1 \\ \text{ķ}\neq\text{ĺ}}}^{\check{n}} \left(1 - \lambda_{\hbar_{\varepsilon(\text{ķ})}}^{+\mathrm{p}} \lambda_{\hbar_{\varepsilon(\text{ķ})}}^{+\mathrm{q}}\right)^{\frac{\tilde{\varpi}_{\text{ķ}}\tilde{\varpi}_{\text{ĺ}}}{1-\tilde{\varpi}_{\text{ķ}}}}\right)^{\frac{1}{\mathrm{p}+\mathrm{q}}} + \\ i \left(1 - \prod_{\substack{\text{ķ},\text{ĺ}=1 \\ \text{ķ}\neq\text{ĺ}}}^{\check{n}} \left(1 - \delta_{\hbar_{\varepsilon(\text{ķ})}}^{+\mathrm{p}} \delta_{\hbar_{\varepsilon(\text{ķ})}}^{+\mathrm{q}}\right)^{\frac{\tilde{\varpi}_{\text{ķ}}\tilde{\varpi}_{\text{ĺ}}}{1-\tilde{\varpi}_{\text{ķ}}}}\right)^{\frac{1}{\mathrm{p}+\mathrm{q}}}, \\ -1 + \left(1 - \prod_{\substack{\text{ķ},\text{ĺ}=1 \\ \text{ķ}\neq\text{ĺ}}}^{\check{n}} \left|-1 + \left(1 + \lambda_{\hbar_{\varepsilon(\text{ķ})}}^{-}\right)^{\mathrm{p}} \left(1 + \lambda_{\hbar_{\varepsilon(\text{ķ})}}^{-}\right)^{\mathrm{q}}\right|^{\frac{\tilde{\varpi}_{\text{ķ}}\tilde{\varpi}_{\text{ĺ}}}{1-\tilde{\varpi}_{\text{ķ}}}}\right)^{\frac{1}{\mathrm{p}+\mathrm{q}}} + \\ i \left(-1 + \left(1 - \prod_{\substack{\text{ķ},\text{ĺ}=1 \\ \text{ķ}\neq\text{ĺ}}}^{\check{n}} \left|-1 + \left(1 + \delta_{\hbar_{\varepsilon(\text{ķ})}}^{-}\right)^{\mathrm{p}} \left(1 + \delta_{\hbar_{\varepsilon(\text{ķ})}}^{-}\right)^{\mathrm{q}}\right|^{\frac{\tilde{\varpi}_{\text{ķ}}\tilde{\varpi}_{\text{ĺ}}}{1-\tilde{\varpi}_{\text{ķ}}}}\right)^{\frac{1}{\mathrm{p}+\mathrm{q}}}\right) \end{array} \right) \tag{26}$$

where $\varepsilon(1), \varepsilon(2), \dots, \varepsilon(\check{n})$ are the permutation of $(\text{ķ} = 1, 2, 3, \dots, \check{n})$ such that $\hbar_{\varepsilon(\text{ķ}-1)} \geq \hbar_{\varepsilon(\text{ķ})} \forall \text{ķ} = 2, 3, \dots, \check{n}$.

Additionally, the BCFOWBM has the following properties.

1) **Idempotency:** If all $\hbar_{\text{ķ}}$ $(\text{ķ} = 1, 2, 3, \dots, \check{n})$ are same, that is, $\hbar_{\text{ķ}} = \hbar \ \forall \ \text{ķ}$, then

$$BCFOWB_{\tilde{\varpi}}^{\mathrm{p},\mathrm{q}}(\hbar_1, \hbar_2, \hbar_3, \dots, \hbar_{\check{n}}) = BCFNWB_{\tilde{\varpi}}^{\mathrm{p},\mathrm{q}}(\hbar, \hbar, \hbar, \dots, \hbar) = \hbar. \tag{27}$$

2) **Monotonicity:** Suppose $\hbar_{\text{ķ}}(\text{ķ} = 1, 2, 3, \dots, \check{n})$ and $g_{\text{ķ}}(\text{ķ} = 1, 2, 3, \dots, \check{n})$ are two collections of BCFNs, if $\hbar_{\text{ķ}} \leq g_{\text{ķ}} \ \forall \ \text{ķ}$ (i.e., $\lambda_{\hbar_{\text{ķ}}}^{+} \leq \lambda_{g_{\text{ķ}}}^{+}$, $\delta_{\hbar_{\text{ķ}}}^{+} \leq \delta_{g_{\text{ķ}}}^{+}$, $\lambda_{\hbar_{\text{ķ}}}^{-} \leq \lambda_{g_{\text{ķ}}}^{-}$, and $\delta_{\hbar_{\text{ķ}}}^{-} \leq \delta_{g_{\text{ķ}}}^{-}$), then

$$BCFOWB_{\tilde{\varpi}}^{\mathrm{p},\mathrm{q}}(\hbar_1, \hbar_2, \hbar_3, \dots, \hbar_{\check{n}}) \leq BCFOWB_{\tilde{\varpi}}^{\mathrm{p},\mathrm{q}}(g_1, g_2, g_3, \dots, g_{\check{n}}). \tag{28}$$

3) **Boundedness:** Suppose $\hbar_{\text{ķ}}$ $(\text{ķ} = 1, 2, 3, \dots, \check{n})$ is a group of BCFNs, and suppose

$$\hbar^{-} = (\min(\hbar_1, \hbar_2, \hbar_3, \dots, \hbar_{\check{n}})) = \left(\min_{ķ}\left(\lambda^{+}_{\hbar_{ķ}}\right) + i\ \min_{ķ}\left(\delta^{+}_{\hbar_{ķ}}\right), \max_{ķ}\left(\lambda^{-}_{\hbar_{ķ}}\right) + i \max_{ķ}\left(\delta^{-}_{\hbar_{ķ}}\right)\right),$$

$$\hbar^{+} = (\max(\hbar_1, \hbar_2, \hbar_3, \dots, \hbar_{\check{n}})) = \left(\max_{ķ}\left(\lambda^{+}_{\hbar_{ķ}}\right) + i\ \max_{ķ}\left(\delta^{+}_{\hbar_{ķ}}\right), \min_{ķ}\left(\lambda^{-}_{\hbar_{ķ}}\right) + i \min_{ķ}\left(\delta^{-}_{\hbar_{ķ}}\right)\right),$$

then,

$$\hbar^{-} \le BCFOWB^{þ,q}_{\tilde{\omega}}(\hbar_1, \hbar_2, \hbar_3, \dots, \hbar_{\check{n}}) \le \hbar^{+}. \tag{29}$$

4. Application "MADM method"

The decision-making technique is used especially for evaluating the beneficial decision from the family of alternatives. The main theme of this analysis is to compute the required decision from the group of opinions, we computed a MADM problem based on the initiated operators for BCF information to evaluate the difficult and unachievable problems.

4.1. Decision-making process

Suppose $\mathcal{V} = \{v_1, v_2, v_3, \dots, v_{\check{n}}\}$ is a set of $\check{n}$ alternatives, $\mathfrak{D} = \{\mathfrak{o}_1, \mathfrak{o}_2, \mathfrak{o}_3, \dots, \mathfrak{o}_{\mathfrak{m}}\}$ is a set of $\mathfrak{m}$ attributes. The performance of the alternative $v_{ķ}$ concerning the criteria $\mathfrak{o}_{ĺ}$ is measured by the BCFN

$$\hbar_{ķĺ} = \left(\Upsilon^{+}_{\hbar_{ķĺ}}, \Upsilon^{-}_{\hbar_{ķĺ}}\right) = \left(\lambda^{+}_{\hbar_{ķĺ}} + i\delta^{+}_{\hbar_{ķĺ}}, \lambda^{-}_{\hbar_{ķĺ}} + i\delta^{-}_{\hbar_{ķĺ}}\right).$$

Suppose that $\mathfrak{A} = \left(\mathfrak{a}_{ķĺ}\right)_{\mathfrak{m}\times\check{n}} = \left(\Upsilon^{+}_{\hbar_{ķĺ}}, \Upsilon^{-}_{\hbar_{ķĺ}}\right)_{\mathfrak{m}\times\check{n}}$ is a BCF decision matrix, where $\Upsilon^{+}_{\hbar_{ķĺ}}$ is the positive truth grade for which the alternative $v_{ķ}$ fulfills the attribute $\mathfrak{o}_{ĺ}$, provided by the decision analyst, and $\Upsilon^{-}_{\hbar_{ķĺ}}$ is the negative truth grade for which the alternative $v_{ķ}$ doesn't fulfill the attribute $\mathfrak{o}_{ĺ}$, provided by the decision analyst. We initiate the algorithm to solve the MCDM problem in the circumstances of BCFSs as follows.

Step 1. All

$$\mathfrak{a}_{ķĺ} = \left(\Upsilon^{+}_{\hbar_{ķĺ}}, \Upsilon^{-}_{\hbar_{ķĺ}}\right) = \left(\lambda^{+}_{\hbar_{ķĺ}} + i\delta^{+}_{\hbar_{ķĺ}}, \lambda^{-}_{\hbar_{ķĺ}} + i\delta^{-}_{\hbar_{ķĺ}}\right)\ (ķ = 1, 2, .., \check{n})\ \left(ĺ = 1, 2, .., \mathfrak{m}\right)$$

are presented in a BCF decision matrix $\mathfrak{A} = \left(\mathfrak{a}_{ķĺ}\right)_{\mathfrak{m}\times\check{n}} = \left(\Upsilon^{+}_{\hbar_{ķĺ}}, \Upsilon^{-}_{\hbar_{ķĺ}}\right)_{\mathfrak{m}\times\check{n}}$, $\hbar_{ķĺ}$ signifies a BCFN, which is on the $ķth$ row and $ĺth$ column in the matrix.

Step 2. Diagnose the best values $\hbar_{ķĺ}\left(ĺ = 1, 2,3, \dots, \mathfrak{m}\right)$ by aggregating the suggested information based on BCFNWBM for $þ = q = 1$, such that

$$\hbar_{ķĺ} = \left(\Upsilon^{+}_{\hbar_{ķĺ}}, \Upsilon^{-}_{\hbar_{ķĺ}}\right) = BCFNWBM^{þ,q}\left(\hbar_{ķ1}, \hbar_{ķ2}, \hbar_{ķ3}, \dots, \hbar_{ķ\mathfrak{m}}\right)$$

Step 3. Diagnose the score values of the evaluated preferences.

Step 4. Diagnose ranking values.

4.2. Numerical example

The Supplier Sustainability Toolkit is expected exclusively to give general direction for matters of interest and does not comprise proficient counsel. You should not follow up on the data contained in this toolkit without acquiring explicit proficient exhortation. JJ (some company) will utilize sensible endeavors to remember up-to-date and exact data for this toolkit, however, makes no portrayal, guarantees, or affirmation concerning the precision, money, or fulfillment of the data.

JJ will not be responsible for any harm or injury coming about because of your admittance to, or failure to get to, this toolkit, or from your dependence on any data given in this toolkit. This Toolkit might give connections or references to different locales, however, JJ have no liability regarding the substance of such different locales and will not be at risk for any harm or injury emerging from that substance. Any connections to other destinations are given as simply an accommodation to the clients of this toolkit.

Supportability incorporates a scope of natural, social, and monetary themes. These themes additionally alluded to as "Individuals, planet, profit" or "the triple primary concern," can be applied to organizations in all areas, from examination to assembling to administration. Corporate Social Responsibility (CSR), Environmental, Social and Governance (ESG) measures, Corporate Sustainability, Practical Business, and Corporate Citizenship are different terms generally utilized broadly to portray comparative projects, drives, and activities. We urge providers to utilize the term that reverberates best with its association. At JJ, we use the terms Citizenship and sustainability to characterize our desire to further develop wellbeing in all that we do. To examine the above problem, for this, we considered Sustainability & Citizenship at Johnson & Johnson in the form of alternatives:

$\mathcal{v}_1$: Defining Sustainability at J&J.
$\mathcal{v}_2$: Sustainability Reporting at J&J.
$\mathcal{v}_3$: Delivering Health for Humanity.
$\mathcal{v}_4$: Engaging Our Suppliers.

To deeply evaluate the above information, we use some attributes in the form: Cost savings through efficiency, improving risk management, driving innovation, and growing customer loyalty and brand position. This section includes a real life example to exhibit the efficiency and advantages of the initiated methods.

4.3. Using BCFNWBM operator

Step 1. As all attributes are of the same sort, thus the data specified in Table 2 don't need to normalize.

Step 2. Aggregate all BCFNs presented in Table 2 by employing the BCFNWBM operator to get the overall BCFNs $\hbar_{\mathfrak{l}}\left(\mathfrak{l} = 1,2,3,\dots,\mathfrak{m}\right)$ of the alternatives $v_{\mathfrak{k}}(\mathfrak{k} = 1,2,3,4)$. The aggregating values are displayed in Table 3.

Step 3. The score function of the alternatives, as per the aggregated values displayed in Table 3, is established in Table 4.

Step 4. The ranking of the alternatives is $\mathcal{v}_1 > \mathcal{v}_2 > \mathcal{v}_3 > \mathcal{v}_4$ as per the score values given in Table 4 and the $\mathcal{v}_1$ is the finest alternative.

Table 2. Decision theory in the form of bipolar complex fuzzy numbers.

	$\mathfrak{o}_1$	$\mathfrak{o}_2$	$\mathfrak{o}_3$	$\mathfrak{o}_4$
$\mathscr{v}_1$	$\begin{pmatrix}0.78+\iota\,0.9,\\ -0.6-\iota\,0.5,\end{pmatrix}$	$\begin{pmatrix}0.36+\iota\,0.65,\\ -0.5-\iota\,0.8\end{pmatrix}$	$\begin{pmatrix}0.45+\iota\,0.7,\\ -0.34-\iota\,0.8\end{pmatrix}$	$\begin{pmatrix}0.9+\iota\,0.5,\\ -0.2-\iota\,0.4\end{pmatrix}$
$\mathscr{v}_2$	$\begin{pmatrix}0.4+\iota\,0.36,\\ -0.39-\iota\,0.4\end{pmatrix}$	$\begin{pmatrix}0.76+\iota\,0.19,\\ -0.28-\iota\,0.5\end{pmatrix}$	$\begin{pmatrix}0.6+\iota\,0.38,\\ -0.5-\iota\,0.87\end{pmatrix}$	$\begin{pmatrix}0.15+\iota\,0.25,\\ -0.43-\iota\,0.34,\end{pmatrix}$
$\mathscr{v}_3$	$\begin{pmatrix}0.5+\iota\,0.46,\\ -0.49-\iota\,0.5\end{pmatrix}$	$\begin{pmatrix}0.67+\iota\,0.29,\\ -0.38-\iota\,0.6\end{pmatrix}$	$\begin{pmatrix}0.5+\iota\,0.48,\\ -0.4-\iota\,0.78\end{pmatrix}$	$\begin{pmatrix}0.25+\iota\,0.35,\\ -0.44-\iota\,0.43\end{pmatrix}$
$\mathscr{v}_4$	$\begin{pmatrix}0.47+\iota\,0.2,\\ -0.7-\iota\,0.8\end{pmatrix}$	$\begin{pmatrix}0.19+\iota\,0.5,\\ -0.7-\iota\,0.8\end{pmatrix}$	$\begin{pmatrix}0.2+\iota\,0.4,\\ -0.3-\iota\,0.4\end{pmatrix}$	$\begin{pmatrix}+\iota\,0.9,\\ -0.8-\iota\,0.1\end{pmatrix}$

Table 3. The aggregating values of the alternatives.

	BCFNWBM
$\mathscr{v}_1$	$(0.5449+\iota\,0.7078, -0.4364-\iota\,0.6953)$
$\mathscr{v}_2$	$(0.5613+\iota\,0.2862, -0.3909-\iota\,0.5777)$
$\mathscr{v}_3$	$(0.53+\iota\,0.3882, -0.4174-\iota\,0.402)$
$\mathscr{v}_4$	$(0.2586+\iota\,0.4539, -0.606-\iota\,0.6066)$

Table 4. The score values of the alternatives.

	Score value
$\mathscr{v}_1$	0.5303
$\mathscr{v}_2$	0.4697
$\mathscr{v}_3$	0.5247

4.4. Using BCFOWBM operator

Step 1. As all attributes are of the same sort, thus the data specified in Table 2 don't need to normalize.

Step 2. Aggregate all BCFNs presented in Table 2 by employing the BCFOWBM operator to get the overall BCFNs $\hbar_{\mathrm{Í}}(\mathrm{Í}=1,2,3,\dots,\mathfrak{m})$ of the alternatives $\mathscr{v}_{\mathrm{k}}(\mathrm{k}=1,2,3,4)$. The aggregating values are displayed in Table 5.

Step 3. The score function of the alternatives, as per the aggregated values displayed in Table 5, is established in Table 6.

Step 4. The ranking of the alternatives is $\mathscr{v}_1 > \mathscr{v}_3 > \mathscr{v}_2 > \mathscr{v}_4$ as per the score values given in Table 6 and the $\mathscr{v}_1$ is the finest alternative.

Table 5. Representation of the aggregated values.

	BCFNWBM
$\mathscr{v}_1$	$(0.6697+\iota\,0.7233, -0.4235-\iota\,0.6113)$
$\mathscr{v}_2$	$(0.4194+\iota\,0.2885, -0.3899-\iota\,0.4541)$
$\mathscr{v}_3$	$(0.53+\iota\,0.3882, -0.4174-\iota\,0.402)$
$\mathscr{v}_4$	$(0.244+\iota\,0.5161, -0.6057-\iota\,0.5003)$

Table 6. The score values of the alternatives.

	Score value
$\boldsymbol{v}_1$	0.5895
$\boldsymbol{v}_2$	0.466
$\boldsymbol{v}_3$	0.5247
$\boldsymbol{v}_4$	0.4135

4.5. Case study

To verify the worth of the diagnosed operators, we discussed different aspects of the proposed theory by considering their different values. If we use the value of the imaginary part as zero, then what happened for this, we use the information in Table 7.

Table 7. Represented bipolar fuzzy information.

	$\mathfrak{o}_1$	$\mathfrak{o}_2$	$\mathfrak{o}_3$	$\mathfrak{o}_4$
$\boldsymbol{v}_1$	$\begin{pmatrix}0.78+\iota\,0.0,\\-0.6-\iota\,0.0,\end{pmatrix}$	$\begin{pmatrix}0.36+\iota\,0.0,\\-0.5-\iota\,0.0\end{pmatrix}$	$\begin{pmatrix}0.45+\iota\,0.0,\\-0.34-\iota\,0.0\end{pmatrix}$	$\begin{pmatrix}0.9+\iota\,0.0,\\-0.2-\iota\,0.0\end{pmatrix}$
$\boldsymbol{v}_2$	$\begin{pmatrix}0.4+\iota\,0.0,\\-0.39-\iota\,0.0\end{pmatrix}$	$\begin{pmatrix}0.76+\iota\,0.0,\\-0.28-\iota\,0.0\end{pmatrix}$	$\begin{pmatrix}0.6+\iota\,0.0,\\-0.5-\iota\,0.0\end{pmatrix}$	$\begin{pmatrix}0.15+\iota\,0.0,\\-0.43-\iota\,0.0,\end{pmatrix}$
$\boldsymbol{v}_3$	$\begin{pmatrix}0.5+\iota\,0.0,\\-0.49-\iota\,0.0\end{pmatrix}$	$\begin{pmatrix}0.67+\iota\,0.0,\\-0.38-\iota\,0.0\end{pmatrix}$	$\begin{pmatrix}0.5+\iota\,0.0,\\-0.4-\iota\,0.0\end{pmatrix}$	$\begin{pmatrix}0.25+\iota\,0.0,\\-0.44-\iota\,0.0\end{pmatrix}$
$\boldsymbol{v}_4$	$\begin{pmatrix}0.47+\iota\,0.0,\\-0.7-\iota\,0.0\end{pmatrix}$	$\begin{pmatrix}0.19+\iota\,0.0,\\-0.7-\iota\,0.0\end{pmatrix}$	$\begin{pmatrix}0.2+\iota\,0.0,\\-0.3-\iota\,0.0\end{pmatrix}$	$\begin{pmatrix}0.29+\iota\,0.0,\\-0.8-\iota\,0.0\end{pmatrix}$

Aggregate all BCFNs presented in Table 2 by employing the BCFNWBM operator and BCFOWBM operator to get the overall BCFNs $\hbar_{\mathfrak{l}}(\mathfrak{l}=1,2,3,\dots,\mathfrak{m})$ of the alternatives $v_{\mathfrak{k}}(\mathfrak{k}=1,2,3,4)$. The aggregating values are displayed in Table 8.

Table 8. The aggregating values of the alternatives.

	BCFNWBM	BCFOWBM
$\boldsymbol{v}_1$	$(0.5449+\iota\,0.0,-0.4364-\iota\,0.0)$	$(0.6697+\iota\,0.0,-0.4235-\iota\,0.0)$
$\boldsymbol{v}_2$	$(0.5613+\iota\,0.0,-0.3909-\iota\,0.0)$	$(0.4194+\iota\,0.0,-0.3899-\iota\,0.0)$
$\boldsymbol{v}_3$	$(0.53+\iota\,0.0,-0.4174-\iota\,0.0)$	$(0.53+\iota\,0.0,-0.4174-\iota\,0.0)$
$\boldsymbol{v}_4$	$(0.2586+\iota\,0.0,-0.606-\iota\,0.0)$	$(0.244+\iota\,0.0,-0.6057-\iota\,0.0)$

The score function of the alternatives, as per the aggregated values displayed in Table 8, is established in Table 9.

Table 9. The score values of the alternatives.

	BCFNWBM	BCFOWBM
$\boldsymbol{v}_1$	0.5271	0.6515
$\boldsymbol{v}_2$	0.5426	0.5074
$\boldsymbol{v}_3$	0.5282	0.5282
$\boldsymbol{v}_4$	0.4132	0.4096

The ranking of the alternatives is $\mathcal{v}_2 > \mathcal{v}_3 > \mathcal{v}_1 > \mathcal{v}_4$ and $\mathcal{v}_1 > \mathcal{v}_3 > \mathcal{v}_2 > \mathcal{v}_4$ as per the score values given in Table 9 and the $\mathcal{v}_2$ and $\mathcal{v}_1$ is the finest alternative.

4.6. Comparison

Here, we compare this analysis with some prevailing algorithms and DM techniques such as [20,36,37,42–45]. The outcome of this comparison is portrayed in Table 10 and Figure 1.

From Table 10, we noticed that the TOPSIS and ELECTRIC-I methods initiated by Akram and Arshad [20] failed to provide any sort of result because this method can't deal with the imaginary part of the information. Likewise, the method ELECTRIC-II diagnosed by Akram and Al-Kenani [43] also failed to provide the result as it can't overcome the imaginary part of both positive and negative SGs. Furthermore, Jana et al. [44] diagnosed Dombi AOs but these operators are not able to provide a result in any sort of DM where two dimensions are involved. The Hamacher AOs for BFS [45] failed as these operators are also not able to provide a result in any sort of DM where two dimensions are involved. The operators discussed in [36,37,42] and proposed operators for weighted averaging are given the same ranking results in the form of $\mathcal{v}_1$, but the information is given in Ref. [36] and the proposed operator for weighted geometric gives their results in the form of $\mathcal{v}_2$ and the weighted geometric operator in [37] give their result in the form of $\mathcal{v}_4$. The operators of [42] give that $\mathcal{v}_1$ is the finest one.

Below we will display that the diagnosed operators and DM technique are more generalized and modified than the prevailing work. For this, we take an example from Akram and Arshad [20] and solve it by using the DM technique given in this analysis. The result of this example is portrayed in Table 11.

Table 10. Interpretation of the comparative analysis.

Operators	$\mathcal{S}_{\mathcal{B}}(\mathcal{v}_1)$	$\mathcal{S}_{\mathcal{B}}(\mathcal{v}_2)$	$\mathcal{S}_{\mathcal{B}}(\mathcal{v}_3)$	$\mathcal{S}_{\mathcal{B}}(\mathcal{v}_4)$
Akram and Arshad [20]	Failed	Failed	Failed	Failed
Akram and Al-Kenani [43]	Failed	Failed	Failed	Failed
Jana et al. [44]	Failed	Failed	Failed	Failed
Wei et al. [45]	Failed	Failed	Failed	Failed
BCFDWA [36]	0.671	0.508	0.521	0.552
BCFDWG [36]	0.2177	0.4041	0.3892	0.3663
BCFHWA [37]	0.6329	0.4876	0.4837	0.4812
BCFHWG [37]	0.3671	0.5124	0.5163	0.5188
BCFWAA [42]	0.555	0.499	0.494	0.424
BCFOWAA [42]	0.6329	0.4876	0.4837	0.4812
BCFWGA [42]	0.499	0.422	0.484	0.337
BCFOWGA [42]	0.563	0.439	0.484	0.377
BCFNWBM	0.5303	0.5697	0.5247	0.375
BCFOWBM	0.5895	0.466	0.5247	0.4135

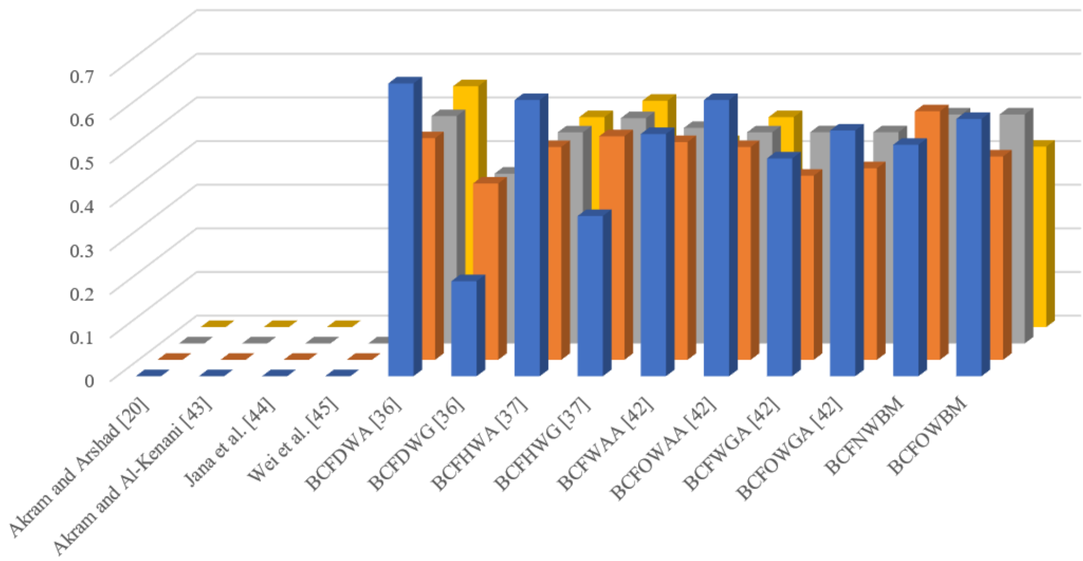

Figure 1. The graphical display of the comparison.

In Table 11, the outcome of the example is taken from Akram and Arshad [20]. Akram and Arshad imitated the TOPSIS technique and found outcome are shown in Table 11, while we employed the proposed DM technique on the same example and the obtained results are also shown in Table 11. From the above discussion, it is evident that the diagnosed approach is better and is more generalized than the prevailing ones, as the prevailing ones can't deal with the BCF information, but the adopted approach can handle the fuzzy information, BF information, and complex fuzzy information.

Table 11. The outcomes of the diagnosed DM and TOPSIS technique are presented in [20].

Methods	$\mathcal{S}_B(\boldsymbol{v}_1)$	$\mathcal{S}_B(\boldsymbol{v}_2)$	$\mathcal{S}_B(\boldsymbol{v}_3)$	$\mathcal{S}_B(\boldsymbol{v}_4)$
Akram and Arshad [20]	0.2639	0.7316	0.7292	0.4045
BCFNWBM	0.4516	0.5468	0.5861	0.4836
BCFOWBM	0.5007	0.5458	0.5914	0.5314

4.7. Influence of parameters

We discussed the influence of parameters by using their different values. Using the information in Table 2 and the proposed AOs, the stability of the parameters is discussed in the form of Table 12 and Figure 2.

Hence, for every value of the parameter, we get the same ranking result as $\boldsymbol{v}_1$. Furthermore, using the information in Table 2 and the proposed AOs, the stability of the parameters is discussed in the form of Table 13 and Figure 3.

Similarly, for every value of the parameter, we again get the same ranking result as $\boldsymbol{v}_1$. Therefore, the presented operators are not yet diagnosed by any researcher and these are more generalized than the information in [36,37,42]. Hence, the diagnosed operator is more beneficial and dominant to handle difficult and unreliable information.

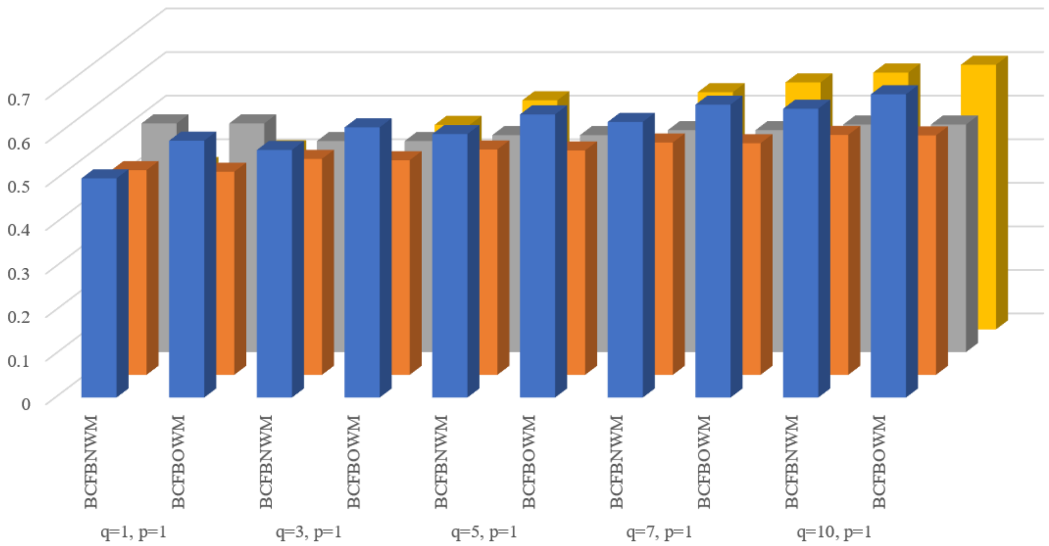

Figure 2. The graphical representation of the stability of parameters for different values of q.

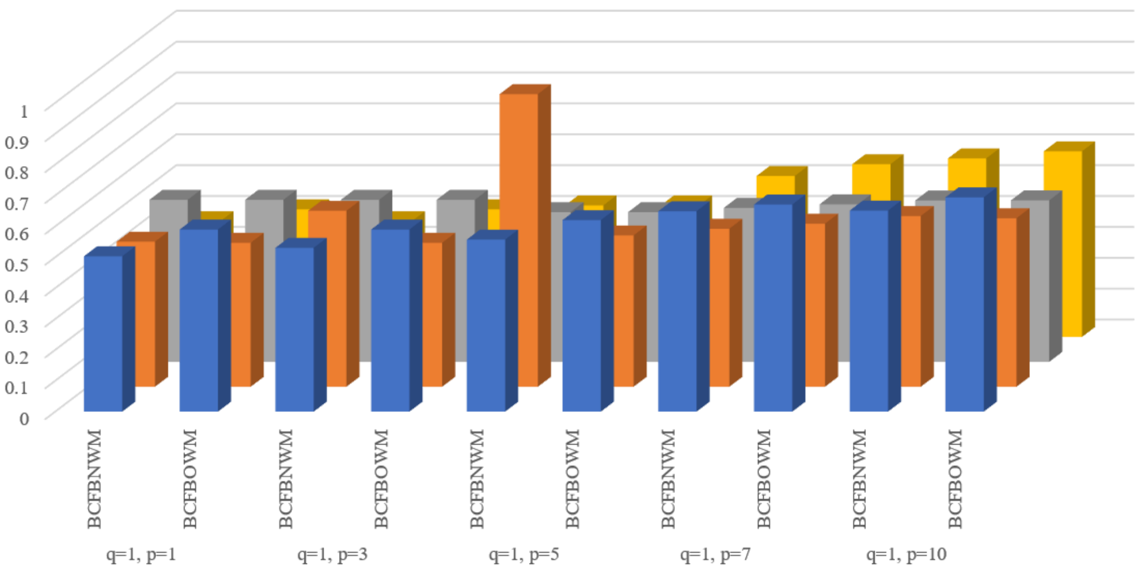

Figure 3. The graphical representation of the stability of parameters for different values of q.

Table 12. represented the stability of parameters for different values of ɋ.

$\text{ƥ} = \mathbf{1}$	Operator	$\mathcal{S}_{B}(\boldsymbol{v}_1)$	$\mathcal{S}_{B}(\boldsymbol{v}_2)$	$\mathcal{S}_{B}(\boldsymbol{v}_3)$	$\mathcal{S}_{B}(\boldsymbol{v}_4)$	Ranking value
$\text{ɋ} = \mathbf{1}$	BCFBNWM	0.5303	0.4697	0.5247	0.375	$\boldsymbol{v}_1 > \boldsymbol{v}_3 > \boldsymbol{v}_2 > \boldsymbol{v}_4$
	BCFBOWM	0.5895	0.466	0.5247	0.4135	$\boldsymbol{v}_1 > \boldsymbol{v}_3 > \boldsymbol{v}_2 > \boldsymbol{v}_4$
$\text{ɋ} = \mathbf{3}$	BCFBNWM	0.5681	0.4954	0.4843	0.4369	$\boldsymbol{v}_1 > \boldsymbol{v}_2 > \boldsymbol{v}_3 > \boldsymbol{v}_4$
	BCFBOWM	0.62	0.4924	0.4843	0.4694	$\boldsymbol{v}_1 > \boldsymbol{v}_2 > \boldsymbol{v}_3 > \boldsymbol{v}_4$
$\text{ɋ} = \mathbf{5}$	BCFBNWM	0.6046	0.5172	0.498	0.4988	$\boldsymbol{v}_1 > \boldsymbol{v}_2 > \boldsymbol{v}_4 > \boldsymbol{v}_3$
	BCFBOWM	0.6497	0.5149	0.498	0.5257	$\boldsymbol{v}_1 > \boldsymbol{v}_4 > \boldsymbol{v}_2 > \boldsymbol{v}_3$
$\text{ɋ} = \mathbf{7}$	BCFBNWM	0.6325	0.5332	0.5093	0.5443	$\boldsymbol{v}_1 > \boldsymbol{v}_4 > \boldsymbol{v}_2 > \boldsymbol{v}_3$
	BCFBOWM	0.6721	0.5315	0.5093	0.5667	$\boldsymbol{v}_1 > \boldsymbol{v}_4 > \boldsymbol{v}_2 > \boldsymbol{v}_3$
$\text{ɋ} = \mathbf{10}$	BCFBNWM	0.6628	0.5506	0.5222	0.5894	$\boldsymbol{v}_1 > \boldsymbol{v}_4 > \boldsymbol{v}_2 > \boldsymbol{v}_3$
	BCFBOWM	0.696	0.5493	0.5222	0.6073	$\boldsymbol{v}_1 > \boldsymbol{v}_4 > \boldsymbol{v}_2 > \boldsymbol{v}_3$

Table 13. Represented the stability of parameters for different values of ƥ.

$\text{ɋ} = \mathbf{1}$	Operator	$\mathcal{S}_{B}(\boldsymbol{v}_1)$	$\mathcal{S}_{B}(\boldsymbol{v}_2)$	$\mathcal{S}_{B}(\boldsymbol{v}_3)$	$\mathcal{S}_{B}(\boldsymbol{v}_4)$	Ranking value
$\text{ƥ} = \mathbf{1}$	BCFBNWM	0.5303	0.4697	0.5247	0.375	$\boldsymbol{v}_1 > \boldsymbol{v}_3 > \boldsymbol{v}_2 > \boldsymbol{v}_4$
	BCFBOWM	0.5895	0.466	0.5247	0.4135	$\boldsymbol{v}_1 > \boldsymbol{v}_3 > \boldsymbol{v}_2 > \boldsymbol{v}_4$
$\text{ƥ} = \mathbf{3}$	BCFBNWM	0.557	0.4974	0.4847	0.4261	$\boldsymbol{v}_1 > \boldsymbol{v}_2 > \boldsymbol{v}_3 > \boldsymbol{v}_4$
	BCFBOWM	0.6201	0.4899	0.4847	0.4647	$\boldsymbol{v}_1 > \boldsymbol{v}_2 > \boldsymbol{v}_3 > \boldsymbol{v}_4$
$\text{ƥ} = \mathbf{5}$	BCFBNWM	0.5915	0.5195	0.4985	0.4853	$\boldsymbol{v}_1 > \boldsymbol{v}_2 > \boldsymbol{v}_4 > \boldsymbol{v}_3$
	BCFBOWM	0.6489	0.5115	0.4985	0.5187	$\boldsymbol{v}_1 > \boldsymbol{v}_4 > \boldsymbol{v}_2 > \boldsymbol{v}_3$
$\text{ƥ} = \mathbf{7}$	BCFBNWM	0.6195	0.5356	0.5099	0.5312	$\boldsymbol{v}_1 > \boldsymbol{v}_4 > \boldsymbol{v}_2 > \boldsymbol{v}_3$
	BCFBOWM	0.6706	0.5279	0.5099	0.5596	$\boldsymbol{v}_1 > \boldsymbol{v}_4 > \boldsymbol{v}_2 > \boldsymbol{v}_3$
$\text{ƥ} = \mathbf{10}$	BCFBNWM	0.651	0.5528	0.5225	0.5782	$\boldsymbol{v}_1 > \boldsymbol{v}_4 > \boldsymbol{v}_2 > \boldsymbol{v}_3$
	BCFBOWM	0.6939	0.5457	0.5229	0.6011	$\boldsymbol{v}_1 > \boldsymbol{v}_4 > \boldsymbol{v}_2 > \boldsymbol{v}_3$

5. Conclusions

The Decision-making technique is the procedure of expressing real life problems and events in a mathematical and statistical format or language. Many kinds of techniques and methods are present in various theories such as FST, BFST, CFST, etc. However, in regard to evaluating the difficult and unreliable information, for example, the information in two dimensions with positive and negative grades or opinion of human beings, then the decision-maker has no such kind of tool or DM technique that can handle such sort of information. The only concept to handle such sort of information is BCFST. The BCFST contains both positive and negative opinions of human beings with both real and unreal parts. The major investigation of this analysis is evaluated below:

1) We employed the BM operators in the setting of BCFST with the described idea of BCFBM, BCFNWBM, and BCFOWBM operators.
2) Furthermore, some properties and results of the deliberated operators are diagnosed.
3) We computed the required decision from the group of opinions, we computed a MADM problem based on the initiated operators for BCF information.
4) For comparing the presented work with some prevailing operators, we illustrated some examples and tried to evaluate the graphical interpretation of the diagnosed work to prove the authenticity of the proposed work.

6. Future work

In future, we try to review the theory of similarity measures for Fermatean fuzzy sets [46], new score values based on Fermatean fuzzy sets [47], TOPSIS technique based on Fermatean fuzzy sets [48], complex spherical fuzzy sets [49], picture fuzzy aggregation operators [50], Aczel-Alsina operators for T-spherical fuzzy sets [51], complex Fermatean fuzzy N-soft set [52] and try to utilize it in the environment of bipolar complex fuzzy sets as in the current analysis these areas are not covered. These areas have a significant role in the generalization of FST, for example, Fermatean FS theory (FFST) handles the information that can't be handled by intuitionistic FST.

Appendix

Appendix A.

Proof. From Definition 5, we have

$$\hbar^{\text{þ}} = \left((\lambda_{\hbar}^{+})^{\text{þ}} + i\,(\delta_{\hbar}^{+})^{\text{þ}}, -1 + (1+\lambda_{\hbar}^{-})^{\text{þ}} + i\left(-1 + (1+\delta_{\hbar}^{-})^{\text{þ}}\right)\right),$$

$$\hbar^{\text{q}} = \left((\lambda_{\hbar}^{+})^{\text{q}} + i\,(\delta_{\hbar}^{+})^{\text{q}}, -1 + (1+\lambda_{\hbar}^{-})^{\text{q}} + i\left(-1 + (1+\delta_{\hbar}^{-})^{\text{q}}\right)\right),$$

Then we have

$$\hbar_{\text{ķ}}^{\text{þ}} \otimes \hbar_{\text{ĺ}}^{\text{q}} = \begin{pmatrix} \left(\lambda_{\hbar_{\text{ķ}}}^{+}\right)^{\text{þ}}\left(\lambda_{\hbar_{\text{ĺ}}}^{+}\right)^{\text{q}} + i\left(\delta_{\hbar_{\text{ķ}}}^{+}\right)^{\text{þ}}\left(\delta_{\hbar_{\text{ĺ}}}^{+}\right)^{\text{q}}, \\ -1 + \left(1+\lambda_{\hbar_{\text{ķ}}}^{-}\right)^{\text{þ}}\left(1+\lambda_{\hbar_{\text{ĺ}}}^{-}\right)^{q} + i\left(-1 + \left(1+\delta_{\hbar_{\text{ķ}}}^{-}\right)^{\text{þ}}\left(1+\delta_{\hbar_{\text{ĺ}}}^{-}\right)^{q}\right) \end{pmatrix}. \quad (30)$$

Now by mathematical induction we prove the following:

$$\left(\bigoplus_{\substack{\text{ķ},\text{ĺ}=1 \\ \text{ķ}\neq\text{ĺ}}}^{\text{ň}} \left(\hbar_{\text{ķ}}^{\text{þ}} \otimes \hbar_{\text{ĺ}}^{\text{q}}\right)\right) = \begin{pmatrix} 1 - \prod_{\substack{\text{ķ},\text{ĺ}=1 \\ \text{ķ}\neq\text{ĺ}}}^{\text{ň}}\left(1 - \lambda_{\hbar_{\text{ķ}}}^{+\text{þ}}\lambda_{\hbar_{\text{ĺ}}}^{+\text{q}}\right) \\ +i\left(1 - \prod_{\substack{\text{ķ},\text{ĺ}=1 \\ \text{ķ}\neq\text{ĺ}}}^{\text{ň}}\left(1 - \delta_{\hbar_{\text{ķ}}}^{+\text{þ}}\delta_{\hbar_{\text{ĺ}}}^{+\text{q}}\right)\right), \\ -\prod_{\substack{\text{ķ},\text{ĺ}=1 \\ \text{ķ}\neq\text{ĺ}}}^{\text{ň}}\left(-1 + \left(1+\lambda_{\hbar_{\text{ķ}}}^{-}\right)^{\text{þ}}\left(1+\lambda_{\hbar_{\text{ĺ}}}^{-}\right)^{\text{q}}\right) \\ +i\left(-\prod_{\substack{\text{ķ},\text{ĺ}=1 \\ \text{ķ}\neq\text{ĺ}}}^{\text{ň}}\left(-1 + \left(1+\delta_{\hbar_{\text{ķ}}}^{-}\right)^{\text{þ}}\left(1+\delta_{\hbar_{\text{ĺ}}}^{-}\right)^{\text{q}}\right)\right) \end{pmatrix}. \quad (31)$$

For $\text{ň} = 2$, we obtain

$$\bigoplus_{\substack{ꝁ,ĺ=1\\ ꝁ\neq ĺ}}^{2}\left(\hbar_{ꝁ}^{\mathfrak{p}}\otimes\hbar_{ĺ}^{\mathfrak{q}}\right)=\left(\hbar_{1}^{\mathfrak{p}}\otimes\hbar_{2}^{\mathfrak{q}}\right)\oplus\left(\hbar_{2}^{\mathfrak{p}}\otimes\hbar_{1}^{\mathfrak{q}}\right)$$

$$=1-\left(1-\lambda_{\hbar_1}^{+\mathfrak{p}}\lambda_{\hbar_2}^{+\mathfrak{q}}\right)\left(1-\lambda_{\hbar_2}^{+\mathfrak{p}}\lambda_{\hbar_1}^{+\mathfrak{q}}\right)+\imath\left(1-\left(1-\delta_{\hbar_1}^{+\mathfrak{p}}\delta_{\hbar_2}^{+\mathfrak{q}}\right)\left(1-\delta_{\hbar_2}^{+\mathfrak{p}}\delta_{\hbar_1}^{+\mathfrak{q}}\right)\right),-1$$
$$+\left(1+\lambda_{\hbar_1}^{-}\right)^{\mathfrak{p}}\left(1+\lambda_{\hbar_2}^{-}\right)^{\mathfrak{q}}$$
$$\times\left(-1+\left(1+\lambda_{\hbar_2}^{-}\right)^{\mathfrak{p}}\left(1+\lambda_{\hbar_1}^{-}\right)^{\mathfrak{q}}\right)+\imath\left(-\left(\begin{array}{c}-1+\left(1+\delta_{\hbar_1}^{-}\right)^{\mathfrak{p}}\left(1+\delta_{\hbar_2}^{-}\right)^{\mathfrak{q}}\\ \times\left(-1+\left(1+\delta_{\hbar_2}^{-}\right)^{\mathfrak{p}}\left(1+\delta_{\hbar_1}^{-}\right)^{\mathfrak{q}}\right)\end{array}\right)\right). \quad (32)$$

If Eq (32) true for $ň=\mathcal{T}$,

$$\bigoplus_{\substack{ꝁ,ĺ=1\\ ꝁ\neq ĺ}}^{\mathcal{T}}\left(\hbar_{ꝁ}^{\mathfrak{p}}\otimes\hbar_{ĺ}^{\mathfrak{q}}\right)=\left(\begin{array}{c}1-\prod_{\substack{ꝁ,ĺ=1\\ ꝁ\neq ĺ}}^{\mathcal{T}}\left(1-\lambda_{\hbar_ꝁ}^{+\mathfrak{p}}\lambda_{\hbar_ĺ}^{+\mathfrak{q}}\right)+\imath\left(1-\prod_{\substack{ꝁ,ĺ=1\\ ꝁ\neq ĺ}}^{\mathcal{T}}\left(1-\delta_{\hbar_ꝁ}^{+\mathfrak{p}}\delta_{\hbar_ĺ}^{+\mathfrak{q}}\right)\right),\\ -\prod_{\substack{ꝁ,ĺ=1\\ ꝁ\neq ĺ}}^{\mathcal{T}}\left(-1+\left(1+\lambda_{\hbar_ꝁ}^{-}\right)^{\mathfrak{p}}\left(1+\lambda_{\hbar_ĺ}^{-}\right)^{\mathfrak{q}}\right)+\imath\left(-\prod_{\substack{ꝁ,ĺ=1\\ ꝁ\neq ĺ}}^{\mathcal{T}}\left(-1+\left(1+\delta_{\hbar_ꝁ}^{-}\right)^{\mathfrak{p}}\left(1+\delta_{\hbar_ĺ}^{-}\right)^{\mathfrak{q}}\right)\right)\end{array}\right),$$

then, $ň=\mathcal{T}+1$, by Definition 5, we obtain

$$\bigoplus_{\substack{ꝁ,ĺ=1\\ ꝁ\neq ĺ}}^{\mathcal{T}+1}\left(\hbar_{ꝁ}^{\mathfrak{p}}\otimes\hbar_{ĺ}^{\mathfrak{q}}\right)=\left(\bigoplus_{\substack{ꝁ,ĺ=1\\ ꝁ\neq ĺ}}^{\mathcal{T}}\left(\hbar_{ꝁ}^{\mathfrak{p}}\otimes\hbar_{ĺ}^{\mathfrak{q}}\right)\right)\oplus\left(\bigoplus_{ꝁ=1}^{\mathcal{T}}\left(\hbar_{ꝁ}^{\mathfrak{p}}\otimes\hbar_{\mathcal{T}+1}^{\mathfrak{q}}\right)\right)\oplus\left(\bigoplus_{ĺ=1}^{\mathcal{T}}\left(\hbar_{\mathcal{T}+1}^{\mathfrak{p}}\otimes\hbar_{ĺ}^{\mathfrak{q}}\right)\right).$$

Next, we show that

$$\bigoplus_{ꝁ=1}^{\mathcal{T}}\left(\hbar_{ꝁ}^{\mathfrak{p}}\otimes\hbar_{\mathcal{T}+1}^{\mathfrak{q}}\right)$$
$$=\left(\begin{array}{c}1-\prod_{ꝁ=1}^{\mathcal{T}}\left(1-\lambda_{\hbar_ꝁ}^{+\mathfrak{p}}\lambda_{\hbar_{\mathcal{T}+1}}^{+\mathfrak{q}}\right)+\imath\left(1-\prod_{ꝁ=1}^{\mathcal{T}}\left(1-\delta_{\hbar_ꝁ}^{+\mathfrak{p}}\delta_{\hbar_{\mathcal{T}+1}}^{+\mathfrak{q}}\right)\right),\\ -\prod_{ꝁ=1}^{\mathcal{T}}\left(-1+\left(1+\lambda_{\hbar_ꝁ}^{-}\right)^{\mathfrak{p}}\left(1+\lambda_{\hbar_{\mathcal{T}+1}}^{-}\right)^{\mathfrak{q}}\right)+\imath\left(-\prod_{ꝁ=1}^{\mathcal{T}}\left(-1+\left(1+\delta_{\hbar_ꝁ}^{-}\right)^{\mathfrak{p}}\left(1+\delta_{\hbar_{\mathcal{T}+1}}^{-}\right)^{\mathfrak{q}}\right)\right)\end{array}\right) \quad (33)$$

by utilizing mathematical induction on $\mathcal{T}$ as below.

For $\mathcal{T}=2$, then by Eq (31), we have

$$\hbar_{ķ}^{þ} \otimes \hbar_{2+1}^{q}$$

$$= \begin{pmatrix} \left(\lambda_{\hbar_{ķ}}^{+}\right)^{þ} \left(\lambda_{\hbar_{2+1}}^{+}\right)^{q} + i \left(\delta_{\hbar_{ķ}}^{+}\right)^{þ} \left(\delta_{\hbar_{2+1}}^{+}\right)^{q}, \\ -1 + \left(\left(1 + \lambda_{\hbar_{ķ}}^{-}\right)^{þ}\right) \left(\left(1 + \lambda_{\hbar_{2+1}}^{-}\right)^{q}\right) + i \left(-1 + \left(\left(1 + \delta_{\hbar_{ķ}}^{-}\right)^{þ}\right) \left(\left(1 + \delta_{2+1}^{-}\right)^{q}\right)\right) \end{pmatrix}, \; ķ = 1, 2.$$

And consequently,

$$\bigoplus_{ķ=1}^{2} \left(\hbar_{ķ}^{þ} \otimes \hbar_{2+1}^{q}\right) = \left(\hbar_{1}^{þ} \otimes \hbar_{2+1}^{q}\right) \oplus \left(\hbar_{2}^{þ} \otimes \hbar_{2+1}^{q}\right)$$

$$= \begin{pmatrix} 1 - \prod_{ķ=1}^{2} \left(1 - \lambda_{\hbar_{ķ}}^{+þ} \lambda_{\hbar_{3}}^{+q}\right) + i \left(1 - \prod_{ķ=1}^{2} \left(1 - \delta_{\hbar_{ķ}}^{+þ} \delta_{\hbar_{3}}^{+q}\right)\right), \\ -\prod_{ķ=1}^{2} \left(-1 + \left(1 + \lambda_{\hbar_{ķ}}^{-}\right)^{þ} \left(1 + \lambda_{\hbar_{3}}^{-}\right)^{q}\right) + i \left(-\prod_{ķ=1}^{2} \left(-1 + \left(1 + \delta_{\hbar_{ķ}}^{-}\right)^{þ} \left(1 + \delta_{\hbar_{3}}^{-}\right)^{q}\right)\right) \end{pmatrix}.$$

If Eq (33) holds for $\mathcal{T} = \mathcal{T}_0$, that is,

$$\bigoplus_{ķ=1}^{\mathcal{T}_0} \left(\hbar_{ķ}^{þ} \otimes \hbar_{\mathcal{T}_0+1}^{q}\right)$$

$$= \begin{pmatrix} 1 - \prod_{ķ=1}^{\mathcal{T}_0} \left(1 - \lambda_{\hbar_{ķ}}^{+þ} \lambda_{\hbar_{\mathcal{T}_0+1}}^{+q}\right) + i \left(1 - \prod_{ķ=1}^{\mathcal{T}_0} \left(1 - \delta_{\hbar_{ķ}}^{+þ} \delta_{\hbar_{\mathcal{T}_0+1}}^{+q}\right)\right), \\ -\prod_{ķ=1}^{\mathcal{T}_0} \left(-1 + \left(1 + \lambda_{\hbar_{ķ}}^{-}\right)^{þ} \left(1 + \lambda_{\hbar_{\mathcal{T}_0+1}}^{-}\right)^{q}\right) + i \left(-\prod_{ķ=1}^{\mathcal{T}_0} \left(-1 + \left(1 + \delta_{\hbar_{ķ}}^{-}\right)^{þ} \left(1 + \delta_{\hbar_{\mathcal{T}_0+1}}^{-}\right)^{q}\right)\right) \end{pmatrix}$$

then, when $\mathcal{T} = \mathcal{T}_0 + 1$, by Eq (31), and Definition 5, we have

$$\bigoplus_{ķ=1}^{\mathcal{T}_0+1} \left(\hbar_{ķ}^{þ} \otimes \hbar_{\mathcal{T}_0+2}^{q}\right)$$

$$= \bigoplus_{ķ=1}^{\mathcal{T}_0} \left(\hbar_{ķ}^{þ} \otimes \hbar_{\mathcal{T}_0+2}^{q}\right) \oplus \left(\hbar_{\mathcal{T}_0+1}^{þ} \otimes \hbar_{\mathcal{T}_0+2}^{q}\right)$$

$$= \begin{pmatrix} 1 - \prod_{ķ=1}^{\mathcal{T}_0+1} \left(1 - \lambda_{\hbar_{ķ}}^{+þ} \lambda_{\hbar_{\mathcal{T}_0+2}}^{+q}\right) + i \left(1 - \prod_{ķ=1}^{\mathcal{T}_0+1} \left(1 - \delta_{\hbar_{ķ}}^{+þ} \delta_{\hbar_{\mathcal{T}_0+2}}^{+q}\right)\right), \\ -\prod_{ķ=1}^{\mathcal{T}_0+1} \left(-1 + \left(1 + \lambda_{\hbar_{ķ}}^{-}\right)^{þ} \left(1 + \lambda_{\hbar_{\mathcal{T}_0+2}}^{-}\right)^{q}\right) + i \left(-\prod_{ķ=1}^{\mathcal{T}_0+1} \left(-1 + \left(1 + \delta_{\hbar_{ķ}}^{-}\right)^{þ} \left(1 + \delta_{\hbar_{\mathcal{T}_0+2}}^{-}\right)^{q}\right)\right) \end{pmatrix}.$$

This shows that Eq (33) true for $\mathcal{T} = \mathcal{T}_0 + 1$. Thus, Eq (33) holds $\forall \, \mathcal{T}$. In the same manner, one can show that

$$\bigoplus_{\text{ĺ}=1}^{\mathcal{T}}\left(\hbar_{\mathcal{T}+1}^{\text{þ}} \otimes \hbar_{\text{ĺ}}^{\text{q}}\right)=\begin{pmatrix} 1-\prod_{\text{ķ}=1}^{\mathcal{T}}\left(1-\lambda_{\hbar_{\mathcal{T}+1}}^{+\text{þ}}\lambda_{\hbar_{\text{ĺ}}}^{+\text{q}}\right)+i\left(1-\prod_{\text{ķ}=1}^{\mathcal{T}}\left(1-\delta_{\hbar_{\mathcal{T}+1}}^{+\text{þ}}\delta_{\hbar_{\text{ĺ}}}^{+\text{q}}\right)\right), \\ -\prod_{\text{ķ}=1}^{\mathcal{T}}\left(-1+\left(1+\lambda_{\hbar_{\mathcal{T}+1}}^{-}\right)^{\text{þ}}\left(1+\lambda_{\hbar_{\text{ĺ}}}^{-}\right)^{\text{q}}\right)+i\left(-\prod_{\text{ķ}=1}^{\mathcal{T}}\left(-1+\left(1+\delta_{\hbar_{\mathcal{T}+1}}^{-}\right)^{\text{þ}}\left(1+\delta_{\hbar_{\text{ĺ}}}^{-}\right)^{\text{q}}\right)\right) \end{pmatrix}.$$

Thus,

$$\bigoplus_{\substack{\text{ķ},\text{ĺ}=1 \\ \text{ķ}\neq\text{ĺ}}}^{\mathcal{T}+1}\left(\hbar_{\text{ķ}}^{\text{þ}} \otimes \hbar_{\text{ĺ}}^{\text{q}}\right)$$

$$=\begin{pmatrix} 1-\prod_{\substack{\text{ķ},\text{ĺ}=1 \\ \text{ķ}\neq\text{ĺ}}}^{\mathcal{T}}\left(1-\lambda_{\hbar_{\text{ķ}}}^{+\text{þ}}\lambda_{\hbar_{\text{ĺ}}}^{+\text{q}}\right)+i\left(1-\prod_{\substack{\text{ķ},\text{ĺ}=1 \\ \text{ķ}\neq\text{ĺ}}}^{\mathcal{T}}\left(1-\delta_{\hbar_{\text{ķ}}}^{+\text{þ}}\delta_{\hbar_{\text{ĺ}}}^{+\text{q}}\right)\right), \\ -\prod_{\substack{\text{ķ},\text{ĺ}=1 \\ \text{ķ}\neq\text{ĺ}}}^{\mathcal{T}}\left(-1+\left(1+\lambda_{\hbar_{\text{ķ}}}^{-}\right)^{\text{þ}}\left(1+\lambda_{\hbar_{\text{ĺ}}}^{-}\right)^{\text{q}}\right)+i\left(-\prod_{\substack{\text{ķ},\text{ĺ}=1 \\ \text{ķ}\neq\text{ĺ}}}^{\mathcal{T}}\left(-1+\left(1+\delta_{\hbar_{\text{ķ}}}^{-}\right)^{\text{þ}}\left(1+\delta_{\hbar_{\text{ĺ}}}^{-}\right)^{\text{q}}\right)\right) \end{pmatrix}$$

$$\oplus$$

$$\begin{pmatrix} 1-\prod_{\text{ķ}=1}^{\mathcal{T}}\left(1-\lambda_{\hbar_{\text{ķ}}}^{+\text{þ}}\lambda_{\hbar_{\mathcal{T}+1}}^{+\text{q}}\right)+i\left(1-\prod_{\text{ķ}=1}^{\mathcal{T}}\left(1-\delta_{\hbar_{\text{ķ}}}^{+\text{þ}}\delta_{\hbar_{\mathcal{T}+1}}^{+\text{q}}\right)\right), \\ -\prod_{\text{ķ}=1}^{\mathcal{T}}\left(-1+\left(1+\lambda_{\hbar_{\text{ķ}}}^{-}\right)^{\text{þ}}\left(1+\lambda_{\hbar_{\mathcal{T}+1}}^{-}\right)^{\text{q}}\right)+i\left(-\prod_{\text{ķ}=1}^{\mathcal{T}}\left(-1+\left(1+\delta_{\hbar_{\text{ķ}}}^{-}\right)^{\text{þ}}\left(1+\delta_{\hbar_{\mathcal{T}+1}}^{-}\right)^{\text{q}}\right)\right) \end{pmatrix}$$

$$\oplus$$

$$\begin{pmatrix} 1-\prod_{\text{ķ}=1}^{\mathcal{T}}\left(1-\lambda_{\hbar_{\mathcal{T}+1}}^{+\text{þ}}\lambda_{\hbar_{\text{ĺ}}}^{+\text{q}}\right)+i\left(1-\prod_{\text{ķ}=1}^{\mathcal{T}}\left(1-\delta_{\hbar_{\mathcal{T}+1}}^{+\text{þ}}\delta_{\hbar_{\text{ĺ}}}^{+\text{q}}\right)\right), \\ -\prod_{\text{ķ}=1}^{\mathcal{T}}\left(-1+\left(1+\lambda_{\hbar_{\mathcal{T}+1}}^{-}\right)^{\text{þ}}\left(1+\lambda_{\hbar_{\text{ĺ}}}^{-}\right)^{\text{q}}\right)+i\left(-\prod_{\text{ķ}=1}^{\mathcal{T}}\left(-1+\left(1+\delta_{\hbar_{\mathcal{T}+1}}^{-}\right)^{\text{þ}}\left(1+\delta_{\hbar_{\text{ĺ}}}^{-}\right)^{\text{q}}\right)\right) \end{pmatrix}$$

$$=\begin{pmatrix} 1-\prod_{\substack{\text{ķ},\text{ĺ}=1 \\ \text{ķ}\neq\text{ĺ}}}^{\mathcal{T}+1}\left(1-\lambda_{\hbar_{\text{ķ}}}^{+\text{þ}}\lambda_{\hbar_{\text{ĺ}}}^{+\text{q}}\right)+i\left(1-\prod_{\substack{\text{ķ},\text{ĺ}=1 \\ \text{ķ}\neq\text{ĺ}}}^{\mathcal{T}+1}\left(1-\delta_{\hbar_{\text{ķ}}}^{+\text{þ}}\delta_{\hbar_{\text{ĺ}}}^{+\text{q}}\right)\right), \\ -\prod_{\substack{\text{ķ},\text{ĺ}=1 \\ \text{ķ}\neq\text{ĺ}}}^{\mathcal{T}+1}\left(-1+\left(1+\lambda_{\hbar_{\text{ķ}}}^{-}\right)^{\text{þ}}\left(1+\lambda_{\hbar_{\text{ĺ}}}^{-}\right)^{\text{q}}\right)+i\left(-\prod_{\substack{\text{ķ},\text{ĺ}=1 \\ \text{ķ}\neq\text{ĺ}}}^{\mathcal{T}+1}\left(-1+\left(1+\delta_{\hbar_{\text{ķ}}}^{-}\right)^{\text{þ}}\left(1+\delta_{\hbar_{\text{ĺ}}}^{-}\right)^{\text{q}}\right)\right) \end{pmatrix}.$$

This implies that Eq (12) true for $\check{n} = \mathcal{T} + 1$. Consequently, Eq (12) true $\forall\ \check{n}$.

Next, by Eq (32), and Definition 5, we have

$$\frac{1}{\check{n}(\check{n}-1)}\left(\bigoplus_{\substack{ķ,ĺ=1\\ ķ\neq ĺ}}^{\check{n}}\left(\hbar_{ķ}^{\flat}\otimes \hbar_{ĺ}^{q}\right)\right)=\left(\begin{array}{c} 1-\prod_{\substack{ķ,ĺ=1\\ ķ\neq ĺ}}^{\check{n}}\left(1-\lambda_{\hbar_{ķ}}^{+\flat}\lambda_{\hbar_{ĺ}}^{+q}\right)^{\frac{1}{\check{n}(\check{n}-1)}}+\\ i\left(1-\prod_{\substack{ķ,ĺ=1\\ ķ\neq ĺ}}^{\check{n}}\left(1-\delta_{\hbar_{ķ}}^{+\flat}\delta_{\hbar_{ĺ}}^{+q}\right)^{\frac{1}{\check{n}(\check{n}-1)}}\right),\\ -\left|-\prod_{\substack{ķ,ĺ=1\\ ķ\neq ĺ}}^{\check{n}}\left(-1+\left(1+\lambda_{\hbar_{ķ}}^{-}\right)^{\flat}\left(1+\lambda_{\hbar_{ĺ}}^{-}\right)^{q}\right)\right|^{\frac{1}{\check{n}(\check{n}-1)}}+\\ i\left(-\left|-\prod_{\substack{ķ,ĺ=1\\ ķ\neq ĺ}}^{\check{n}}\left(-1+\left(1+\delta_{\hbar_{ķ}}^{-}\right)^{\flat}\left(1+\delta_{\hbar_{ĺ}}^{-}\right)^{q}\right)\right|^{\frac{1}{\check{n}(\check{n}-1)}}\right)\end{array}\right) \tag{34}$$

and then, by Eq (34), and Definition 5,

$$BCFB^{\flat,q}(\hbar_1,\hbar_2,\hbar_3,\dots,\hbar_{\check{n}})=\left(\frac{1}{\check{n}(\check{n}-1)}\left(\bigoplus_{\substack{ķ,ĺ=1\\ ķ\neq ĺ}}^{\check{n}}\left(\hbar_{ķ}^{\flat}\otimes \hbar_{ĺ}^{q}\right)\right)\right)^{\frac{1}{\flat+q}}$$

$$=\left(\begin{array}{c} \left(1-\prod_{\substack{ķ,ĺ=1\\ ķ\neq ĺ}}^{\check{n}}\left(1-\lambda_{\hbar_{ķ}}^{+\flat}\lambda_{\hbar_{ĺ}}^{+q}\right)^{\frac{1}{\check{n}(\check{n}-1)}}\right)^{\frac{1}{\flat+q}}+\\ i\left(1-\prod_{\substack{ķ,ĺ=1\\ ķ\neq ĺ}}^{\check{n}}\left(1-\delta_{\hbar_{ķ}}^{+\flat}\delta_{\hbar_{ĺ}}^{+q}\right)^{\frac{1}{\check{n}(\check{n}-1)}}\right)^{\frac{1}{\flat+q}},\\ -1+\left(1-\left|-\prod_{\substack{ķ,ĺ=1\\ ķ\neq ĺ}}^{\check{n}}\left(-1+\left(1+\lambda_{\hbar_{ķ}}^{-}\right)^{\flat}\left(1+\lambda_{\hbar_{ĺ}}^{-}\right)^{q}\right)\right|^{\frac{1}{\check{n}(\check{n}-1)}}\right)^{\frac{1}{\flat+q}}+\\ i\left(-1+\left(1-\left|-\prod_{\substack{ķ,ĺ=1\\ ķ\neq ĺ}}^{\check{n}}\left(-1+\left(1+\delta_{\hbar_{ķ}}^{-}\right)^{\flat}\left(1+\delta_{\hbar_{ĺ}}^{-}\right)^{q}\right)\right|^{\frac{1}{\check{n}(\check{n}-1)}}\right)^{\frac{1}{\flat+q}}\right)\end{array}\right).$$

Completed proof of the results.

Appendix B.

Proof. From Definition 5, we have

$$\hbar^{\flat} = \left((\lambda_{\hbar}^{+})^{\flat} + i(\delta_{\hbar}^{+})^{\flat}, -1 + (1+\lambda_{\hbar}^{-})^{\flat} + i\left(-1 + (1+\delta_{\hbar}^{-})^{\flat}\right)\right),$$

$$\hbar^{q} = \left((\lambda_{\hbar}^{+})^{q} + i(\delta_{\hbar}^{+})^{q}, -1 + (1+\lambda_{\hbar}^{-})^{q} + i(-1 + (1+\delta_{\hbar}^{-})^{q})\right),$$

then we have

$$\hbar_{k}^{\flat} \otimes \hbar_{l}^{q} = \begin{pmatrix} \left(\lambda_{\hbar_k}^{+}\right)^{\flat}\left(\lambda_{\hbar_l}^{+}\right)^{q} + i\left(\delta_{\hbar_k}^{+}\right)^{\flat}\left(\delta_{\hbar_l}^{+}\right)^{q}, \\ -1 + \left(1+\lambda_{\hbar_k}^{-}\right)^{\flat}\left(1+\lambda_{\hbar_l}^{-}\right)^{q} + i\left(-1 + \left(1+\delta_{\hbar_k}^{-}\right)^{\flat}\left(1+\delta_{\hbar_l}^{-}\right)^{q}\right) \end{pmatrix}.$$

$$\frac{\omega_k\omega_l}{1-\omega_k}\left(\hbar_{k}^{\flat} \otimes \hbar_{l}^{q}\right) = \begin{pmatrix} 1 - \left(1 - \left(\lambda_{\hbar_k}^{+}\right)^{\flat}\left(\lambda_{\hbar_l}^{+}\right)^{q}\right)^{\frac{\omega_k\omega_l}{1-\omega_k}} \\ +i\left(1 - \left(1 - \left(\delta_{\hbar_k}^{+}\right)^{\flat}\left(\delta_{\hbar_l}^{+}\right)^{q}\right)^{\frac{\omega_k\omega_l}{1-\omega_k}}\right), \\ -\left(\left|-1 + \left(1+\lambda_{\hbar_k}^{-}\right)^{\flat}\left(1+\lambda_{\hbar_l}^{-}\right)^{q}\right|^{\frac{\omega_k\omega_l}{1-\omega_k}}\right) \\ +i\left(-\left(\left|-1 + \left(1+\delta_{\hbar_k}^{-}\right)^{\flat}\left(1+\delta_{\hbar_l}^{-}\right)^{q}\right|^{\frac{\omega_k\omega_l}{1-\omega_k}}\right)\right) \end{pmatrix}.$$

$$\bigoplus_{\substack{k,l=1\\ k\neq l}}^{\check{n}} \frac{\omega_k\omega_l}{1-\omega_k}\left(\hbar_{k}^{\flat} \otimes \hbar_{l}^{q}\right) = \begin{pmatrix} 1 - \prod_{\substack{k,l=1\\ k\neq l}}^{\check{n}}\left(1 - \lambda_{\hbar_k}^{+\flat}\lambda_{\hbar_l}^{+q}\right)^{\frac{\omega_k\omega_l}{1-\omega_k}} \\ +i\left(1 - \prod_{\substack{k,l=1\\ k\neq l}}^{\check{n}}\left(1 - \delta_{\hbar_k}^{+\flat}\delta_{\hbar_l}^{+q}\right)^{\frac{\omega_k\omega_l}{1-\omega_k}}\right), \\ -\prod_{\substack{k,l=1\\ k\neq l}}^{\check{n}}\left(\left|-1 + \left(1+\lambda_{\hbar_k}^{-}\right)^{\flat}\left(1+\lambda_{\hbar_l}^{-}\right)^{q}\right|^{\frac{\omega_k\omega_l}{1-\omega_k}}\right) \\ +\, i\left(-\prod_{\substack{k,l=1\\ k\neq l}}^{\check{n}}\left(\left|-1 + \left(1+\delta_{\hbar_k}^{-}\right)^{\flat}\left(1+\delta_{\hbar_l}^{-}\right)^{q}\right|^{\frac{\omega_k\omega_l}{1-\omega_k}}\right)\right) \end{pmatrix}.$$

$$\left(\bigoplus_{\substack{k,l=1\\ k\neq l}}^{\check{n}} \frac{\omega_k\omega_l}{1-\omega_k}\left(\hbar_{k}^{\flat} \otimes \hbar_{l}^{q}\right)\right)^{\frac{1}{\flat+q}}$$

$$
= \left(\begin{array}{c}
\left(1 - \prod_{\substack{k,l=1 \\ k \neq l}}^{\check{n}} \left(1 - \lambda_{\hbar_k}^{+p} \lambda_{\hbar_l}^{+q}\right)^{\frac{\tilde{\omega}_k \tilde{\omega}_l}{1-\tilde{\omega}_k}}\right)^{\frac{1}{p+q}} \\
+i \left(1 - \prod_{\substack{k,l=1 \\ k \neq l}}^{\check{n}} \left(1 - \delta_{\hbar_k}^{+p} \delta_{\hbar_l}^{+q}\right)^{\frac{\tilde{\omega}_k \tilde{\omega}_l}{1-\tilde{\omega}_k}}\right)^{\frac{1}{p+q}}, \\
-1 + \left(1 - \prod_{\substack{k,l=1 \\ k \neq l}}^{\check{n}} \left(\left|-1 + \left(1 + \lambda_{\hbar_k}^{-}\right)^{p} \left(1 + \lambda_{\hbar_l}^{-}\right)^{q}\right|^{\frac{\tilde{\omega}_k \tilde{\omega}_l}{1-\tilde{\omega}_k}}\right)\right)^{\frac{1}{p+q}} \\
+i \left(-1 + \left(1 - \prod_{\substack{k,l=1 \\ k \neq l}}^{\check{n}} \left(\left|-1 + \left(1 + \delta_{\hbar_k}^{-}\right)^{p} \left(1 + \delta_{\hbar_l}^{-}\right)^{q}\right|^{\frac{\tilde{\omega}_k \tilde{\omega}_l}{1-\tilde{\omega}_k}}\right)\right)^{\frac{1}{p+q}}\right)
\end{array} \right)
$$

$$
= BCFNWB_{\tilde{\omega}}^{p,q}(\hbar_1, \hbar_2, \hbar_3, \ldots, \hbar_{\check{n}}).
$$

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Data availability

The data employed in this analysis are speculative and artificial. One can employ the data before earlier approval simply by citing this article.

Conflict of interest

The authors state that they have no conflicts of interest.

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