AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(9): 17019–17025.
DOI:10.3934/math.2022935
Received: 04 May 2022
Revised: 26 June 2022
Accepted: 30 June 2022
Published: 19 July 2022

Research article

Some aspects in Noetherian modules and rings

Nasr Anwer Zeyada[1,2,*] **and Makkiah S. Makki**[3]

[1] Department of Mathematics, Faculty of Science, Cairo University, Giza 12613, Egypt
[2] Department of Mathematics, College of Science, University of Jeddah, Jeddah 23218, Saudi Arabia
[3] Department of Mathematical Sciences, Faculty of Applied Sciences, Umm Alqura University, Makkah 21955, Saudi Arabia

* **Correspondence:** Email: nzeyada@sci.cu.edu.eg.

Abstract: Many authors have focused on the concept of Noetherian rings. M. Kosan and T. Quynh recently published an article on the Noetherian ring's new properties and their relation to the direct sum of injective hulls of simple right modules and essential extensions. Throughout this article, we extend the results of M. Kosan and T. Quynh to simple singular right modules and strongly singular injectivity.

Keywords: Noetherian ring; singular modules
Mathematics Subject Classification: 16D50, 16L30, 16L60, 16P20

1. Introduction

Researchers have extensively studied Noetherian rings; we list some of the important results. E. Matlis [6] demonstrated that, under direct sum, the class of injective right R-modules is closed if the ring R is right Noetherian. I. Amin, M. Yousif and N. Zeyada [1] showed that the ring R is right Noetherian if and only if every direct sum of strongly soc-injective right R-modules is strongly soc-injective. N. Zeyada [10] proved that the ring R/S_r is Noetherian, where S_r is the right socle of R, if and only if every direct sum of strongly R-injective right R-modules is strongly R-injective. Beidar and Ke [3] obtained that the class of modules that are direct sums of injective R-modules is closed under essential extensions if and only if the ring R is right Noetherian. M. Kosan and T. Quynh [5] extended Beider and Ke's result (mentioned above) and showed that R is right Noetherian if and only if every essential extension of a direct sum of injective hulls of simple right R-modules is a direct sum of either injective right R-modules or projective right R-modules. Our aim in this article is to study the rings such that every essential extension of a direct sum of injective hulls of singular simple right R-modules is a direct sum of either injective right R-modules or projective right R-modules. Also, we investigate how these extensions are close to Noetherian rings. Moreover, we aim to get new characterizations of Noetherian rings.

All rings have an associative ring with identity throughout this article, and all modules are unitary R-modules. The socle of a right R-module M is represented by soc(M). S_r and S_l are used to denote R's right and left socles, respectively. The notations $N \subseteq^{ess} M$ and $N \subseteq^{\oplus} M$ indicate that a submodule N of M is essential and a direct summand, respectively. For all unspecified concepts in this article, we refer to [2, 4, 7, 9].

2. Soc-injectivity and Noetherian rings

Soc-injectivity was first proposed by I. Amin, M. Yousif and N. Zeyada [1].

Definition 2.1. *[1] M is soc-N-injective for any modules M and N if every R-homomorphism $f : soc(N) \to M$ extends to N. If M is soc-N-injective for all modules N, it is strongly soc-injective.*

Theorem 2.1. *[3] Consider M_R to be a module. The following are equivalent:*

(1) M is locally Noetherian.
(2) The essential extension in $\sigma[M]$ of a direct sum of any family of injective modules in $\sigma[M]$ is a direct sum of injective modules in $\sigma[M]$.
(3) If $\{S_i | i \in \mathbb{N}\}$ is a family of simple modules in $\sigma[M]$, any essential extension of $\bigoplus_{i\in\mathbb{N}} E(S_i)$ is a direct sum of injective modules in $\sigma[M]$.

M is locally Noetherian if and only if every direct sum of strongly soc-injective modules in $\sigma[M]$ is strongly soc-injective in $\sigma[M]$, according to Özcan, D. Tütüncü and M. Yousif [8]. Theorem 2.1 is extended to strongly soc-injectivity in the following theorem.

Theorem 2.2. *Assume M_R is a module. The following are equivalent:*

(1) M is locally Noetherian.
(2) The essential extension in $\sigma[M]$ of a direct sum of any family of strongly soc-injective modules in $\sigma[M]$ is a direct sum of strongly soc-injective modules in $\sigma[M]$.
(3) If $\{S_i | i \in \mathbb{N}\}$ is a family of simple modules in $\sigma[M]$, any essential extension of $\bigoplus_{i\in\mathbb{N}} E(S_i)$ is a direct sum of strongly soc-injective modules in $\sigma[M]$.

Proof. (1) ⇒ (2). Let M be locally Noetherian and $\{N_i | i \in I\}$ be a family of strongly soc-injective modules. For all $i \in I$, $N_i = L_i \bigoplus K_i$, where L_i is injective with essential socle and $soc(K_i) = 0$ [1, Theorem 3.1]. Furthermore, $\bigoplus_{i\in I} N_i$ is strongly soc-injective, and $\bigoplus_{i\in I} N_i = L \bigoplus K$, where L is injective with essential socle, and $soc(K) = 0$. As a result, any essential extension of $\bigoplus_{i\in I} N_i$ is strongly soc-injective.
(2) ⇒ (3) Clear.
(3) ⇒ (1) Assume $\{S_i | i \in \mathbb{N}\}$ is a family of simple modules in $\sigma[M]$. Any essential extension E of $\bigoplus_{i\in\mathbb{N}} E(S_i)$ is then strongly soc-injective. Because $E(\bigoplus_{i\in\mathbb{N}} E(S_i))$ has an essential socle, $E(\bigoplus_{i\in\mathbb{N}} E(S_i))$ is injective. As a result, M is locally Noetherian. □

The following Corollary is a direct result of the preceding Theorem.

Corollary 2.1. *Assume* R *is a ring. The following are equivalent:*

(1) R *is Noetherian.*
(2) The essential extension of a direct sum of any family of strongly soc*-injective modules is a direct sum of strongly* soc*-injective modules.*
(3) If $\{S_i | i \in \mathbb{N}\}$ *is a family of simple modules, any essential extension of* $\bigoplus_{i\in\mathbb{N}} E(S_i)$ *is a direct sum of strongly* soc*-injective modules.*

Theorem 2.3. *[5] Assume* M_R *is a module. The following are equivalent:*

(1) M *is locally Noetherian.*
(2) If $\{S_i | i \in \mathbb{N}\}(\subseteq \sigma[M])$ *is a family of simple modules, any essential extension of* $\bigoplus_{i\in\mathbb{N}} S_i$ *in* $\sigma[M]$ *is a direct sum of modules that are either* M*-injective or projective.*
(3) If $\{S_i | i \in \mathbb{N}\}(\subseteq \sigma[M])$ *is a family of simple modules, any essential extension of* $\bigoplus_{i\in\mathbb{N}} E(S_i)$ *in* $\sigma[M]$ *is a direct sum of modules that are either* M*-injective or projective.*

Theorem 2.3 can be extended to include soc-injectivity.

Theorem 2.4. *Assume* M_R *is a module. The following are equivalent:*

(1) M *is locally Noetherian.*
(2) If $\{S_i | i \in \mathbb{N}\}(\subseteq \sigma[M])$ *is a family of simple modules, the essential extension of* $\bigoplus_{i\in\mathbb{N}} S_i$ *in* $\sigma[M]$ *is a direct sum of modules that are either strongly* soc*-injective in* $\sigma[M]$ *or projective.*
(3) If $\{S_i | i \in \mathbb{N}\}(\subseteq \sigma[M])$ *is a family of simple modules, any essential extension of* $\bigoplus_{i\in\mathbb{N}} E(S_i)$ *in* $\sigma[M]$ *is a direct sum of modules that are either strongly* soc*-injective in* $\sigma[M]$ *or projective.*

Proof. (1) ⇒ (2) and (2) ⇒ (3) are clear.
(3) ⇒ (1) Assume $\{S_i | i \in \mathbb{N}\}$ is a family of simple modules in $\sigma[M]$. Then, any essential extension E of $\bigoplus_{i\in\mathbb{N}} E(S_i)$ is strongly soc-injective in $\sigma[M]$ or projective. Because $E(\bigoplus_{i\in\mathbb{N}} E(S_i))$ has an essential socle, any strongly soc-injective summand of $E(\bigoplus_{i\in\mathbb{N}} E(S_i))$ is M-injective. As a result, M is locally Noetherian. □

Corollary 2.2. *Let* R *represent a ring. The following are equivalent:*

(1) R *is right Noetherian.*
(2) If $\{S_i | i \in \mathbb{N}\}$ *is a family of simple modules, any essential extension of* $\bigoplus_{i\in\mathbb{N}} S_i$ *is a direct sum of modules that are either strongly* soc*-injective or projective.*
(3) If $\{S_i | i \in \mathbb{N}\}$ *is a family of simple modules, any essential extension of* $\bigoplus_{i\in\mathbb{N}} E(S_i)$ *is a direct sum of modules that are either strongly* soc*-injective or projective.*

Proof. The preceding Theorem makes this clear. □

If a ring R is right (or left) Artinian and right (or left) self-injective, it is referred to as a quasi-Frobenius (QF-ring). Equivalently, R is QF if and only if every injective right R-module is projective if and only if every projective right R-module is injective.

Corollary 2.3. *Let* R *represents a ring. The following are equivalent:*

(1) R *is a* QF *ring.*
(2) Every essential extension of the direct sum of strongly soc*-injective right* R*-modules is projective.*

Proof. If R is QF, then every strongly soc-injective module is injective [1]. As a result, [5, Corollary 2.8] completes the proof. □

3. S-injectivity and Noetherian rings

Let M be a module and N and L be members of σ[M]. If any homomorphism f : K → N extends to L for any singular submodule K of L, N is said to be s-L-injective. If N is s-N-injective, a module N in σ[M] is called s-quasi-injective in σ[M]. If N is s-M-injective, it is called s-injective in σ[M]. If N is s-L-injective for all T in σ[M], it is said to be strongly s-injective in σ[M].

Theorem 3.1. *Let* N ∈ σ[M]. *The following are equivalent:*

(1) In σ[M], N *is strongly* s*-injective.*
(2) N *is* s-E_M(N)*-injective, where* E_M(N) *is the injective hull of* N *in* σ[M].
(3) N = E ⊕ L*, where* E *is injective in* σ[M]*, and* L *is nonsingular.*

Furthermore, if Z(N) ≠ 0*, then* E *has an essential singular submodule.*

Proof. (1) ⇒ (2) Clear.
(2) ⇒ (3) We're done if Z(N) = 0. Assume Z(N) ≠ 0, and consider the diagram below.

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ 0 & \longrightarrow & Z(N) & \xrightarrow{i_1} & E_M(Z(N)) \\ & & i_2 \downarrow & & \\ & & N & & \end{array}$$

Here, i_1 and i_2 are inclusion maps. N is s-E_M(Z(N))- injective because N is s-E_M(N)-injective. As a result, there exists a homomorphism σ : E_M(Z(N)) → N, which extends i_2. Since Z(N) is an embedding of E_M(Z(N)) in N, Z(N) $\subseteq^e$ E_M(Z(N)). If E = E_M(Z(N)), then N = E ⊕ L for some submodule L of N. E is injective, and L is nonsingular.
(3) ⇒ (1) This is obvious because nonsingular modules are strongly s-injective in σ[M], and the finite direct sum of strongly s-injective modules is also strongly s-injective in σ[M]. The final statement of the theorem is Z(E) $\subseteq^e$ E. On the other hand, Z(E) = Z(N) = σ(Z(N)) $\subseteq^e$ E implies that Z(E) $\subseteq^e$ E. □

Recall that a module *M* satisfies ACC on essential submodules if *M* satisfies the ascending chain condition on essential submodules.

Lemma 3.1. *[10] For a right* R*-module* M*, the following are equivalent:*

(1) M *satisfies* ACC *on essential submodules.*
(2) M/Soc(M) *is right Noetherian.*

Proposition 3.1. *For a right R-module* M*, the following are equivalent:*

(1) M/soc(M) *is locally Noetherian.*
(2) Every direct sum of strongly s*-injective modules is strongly* s*-injective in* σ[M].

Proof. (1) ⇒ (2) Let $\{M_i\}_{i\in I}$ represent a family of strongly s-injective modules in σ[M]. According to Theorem 3.1, write $M_i = E_i \oplus T_i$ for each $i \in I$, where E_i is injective in σ[M], and $Z(T_i) = 0$. If $E = \oplus_{i\in I}E_i$, and $T = \oplus_{i\in I}T_i$, then $\oplus_{i\in I}M_i = E \oplus T$, and $Z(T) = 0$. E is M/soc(M)-injective because M/soc(M) is locally Noetherian, and E can be considered in σ[M/soc(M)]. As a result, E is injective in $\sigma[M]$, and $\oplus_{i\in I}M_i$ is strongly s-injective. Since M/soc(M) is locally Noetherian, and E may be considered in σ[M/soc(M)], E is M/soc(M)-injective. As a result, E is injective in σ[M], and $\oplus_{i\in I}M_i$ is strongly s-injective.
(2) ⇒ (1). Consider a chain $L_1 \subseteq L_2 \subseteq ......$ of essential submodules of a submodule K of M with $soc(M) \subseteq K$, and K/soc(M) is finitely generated. Let $E(K/L_i)$ be the injective hull of K/L_i, $i \geq 1$, and $f : L \longrightarrow \bigoplus_{1\leq i} E(K/L_i)$ be a map defined by $f(l) = (l + L_i)$ where $L = \bigcup_{1\leq i} L_i$. Since $\bigoplus_{1\leq i} E(K/L_i)$ is strongly s-injective in σ[M], and $\bigoplus_{1\leq i} E(K/L_i)$ has an essential singular submodule, $\bigoplus_{1\leq i} E(K/L_i)$ is injective, and f can be extended to an R-homomorphism $\widehat{f} : K \longrightarrow \bigoplus_{1\leq i} E(K/L_i)$. Since f (Soc(M)) = 0, there exists $h : K/soc(M) \longrightarrow \bigoplus_{1\leq i} E(K/L_i)$ with $h(k + soc(M)) = \widehat{f}(k)$ for all $k \in K$.
Then $\widehat{f}(K) \subseteq \bigoplus_{1\leq i\leq n} E(K/L_i)$ for some n, and $f(K) \subseteq \bigoplus_{1\leq i\leq n} E(K/L_i)$. Thus $L = L_{j+n}$ for every $j \geq 1$, and K has ACC on essential submodules. It follows that K/soc(K) is Noetherian by the above Lemma. Since $soc(K) = soc(M)$, every finitely generated submodule of M/soc(M) is Noetherian. Therefore, M/soc(M) is locally Noetherian. □

Proposition 3.2. *Consider* M_R *to be a module. Then, the following are equivalent:*

(1) M/soc(M) *is locally Noetherian.*
(2) The essential extension in σ[M] *of a direct sum of any family of strongly* s*-injective modules in* σ[M] *is a direct sum of* s*-injective modules in* σ[M].
(3) For a given family $\{S_i | i \in \mathbb{N}\}$ *of singular simple modules in* σ[M]*, any essential extension of* $\bigoplus_{i\in\mathbb{N}} E(S_i)$ *is a direct sum of injective modules in* σ[M].

Proof. (1) ⇒ (2) and (2) ⇒ (3) are clear.
(3) ⇒ (1) Because the simple modules in σ[M/soc(M)] are the singular simple modules in σ[M], Theorem 2.1 completes the proof. □

Theorem 3.2. *Allow* M_R *to be a module. Then, the following are equivalent:*

(1) M/soc(M) *is locally Noetherian.*
(2) If $\{S_i | i \in \mathbb{N}\}(\subseteq \sigma[M])$ *is a family of singular simple modules, then any essential extension of* $\bigoplus_{i\in\mathbb{N}} E(S_i)$ *in* σ[M] *is a direct sum of modules that are either strongly* s*-injective or projective in* σ[M].

(3) *Provided a family of singular simple modules* $\{S_i | i \in \mathbb{N}\} (\subseteq \sigma[M])$*, any essential extension of* $\bigoplus_{i\in\mathbb{N}} E(S_i)$ *in* $\sigma[M]$ *would be a direct sum of modules that are either strongly* s*-injective in* $\sigma[M]$ *or projective.*

Proof. (1) ⇒ (2) and (2) ⇒ (3) are clear.
(3) ⇒ (1) Assume $\{S_i | i \in \mathbb{N}\}$ is a family of singular simple modules in $\sigma[M]$. Then, any essential extension E of $E(\bigoplus_{i\in\mathbb{N}}(S_i))$ is strongly s-injective in $\sigma[M]$ or projective. Because $E(\bigoplus_{i\in\mathbb{N}} S_i)$ contains an essential singular submodule, every strongly s-injective summand of $E(\bigoplus_{i\in\mathbb{N}} S_i)$ is $M/soc(M)$-injective. As a result, according to Theorem 2.3, $M/soc(M)$ is locally Noetherian. □

Corollary 3.1. *Let* R *represent a ring. The following are equivalent:*

(1) R/S_r *is right Noetherian.*
(2) *The essential extension of a direct sum of any family of strongly* s*-injective right* R*-modules is a direct sum of* s*-injective modules.*
(3) *The essential extension of a direct sum of any family of injective right* R*-modules is a direct sum of modules that are either strongly* s*-injective or projective.*
(4) *If* $\{S_i | i \in \mathbb{N}\}$ *is a family of singular simple right* R*-modules, any essential extension of* $\bigoplus_{i\in\mathbb{N}} E(S_i)$ *is a direct sum of injective modules.*
(5) *Given a family* $\{S_i | i \in \mathbb{N}\}$ *of singular simple* R*-modules, any essential extension of* $\bigoplus_{i\in\mathbb{N}} E(S_i)$ *is a direct sum of right R-modules that are either injective or projective.*

Conflict of interest

The authors declare that they have no competing interests.

References

1. I. Amin, M. Yousif, N. Zeyada, Soc-injective rings and modules, *Commun. Algebra*, **33** (2005), 4229–4250. https://doi.org/10.1080/00927870500279001
2. F. Anderson, K. Fuller, *Rings and category of modules*, New York: Springer-Verlag, 1992. https://doi.org/10.1007/978-1-4612-4418-9
3. K. Beidar, W. Ke, On essential extensions of direct sums of injective modules, *Arch. Math.*, **78** (2002), 120–123. https://doi.org/10.1007/s00013-002-8224-2
4. P. Guil-Asensio, S. Jain, A. Srivastava, Direct sums of injective and projective modules, *J. Algebra*, **324** (2010), 1429–1434. https://doi.org/10.1016/j.jalgebra.2010.05.020
5. M. Kosan, T. Quynh, On essential extensions of direct sums of either injective or projective modules, *J. Algebra Appl.*, **13** (2014), 1450038. https://doi.org/10.1142/S0219498814500388
6. E. Matlis, Injective modules over Noetherian rings, *Pacific J. Math.*, **8** (1958), 511–528.
7. W. Nicholson, M. Yousif, *Quasi-Frobenius rings*, Cambridge: Cambridge University Press, 2003. https://doi.org/10.1017/CBO9780511546525

8. A. Özcan, D. Tütüncü, M. Yousif, On some injective modules in $\sigma[M]$, In: *Modules and comodules*, Basel: Birkhäuser, 2008, 313–329. https://doi.org/10.1007/978-3-7643-8742-6_21

9. R. Wisbauer, *Foundations of module and ring theory*, Reading: Routledge, 1991. https://doi.org/10.1201/9780203755532

10. N. Zeyada, S-injective modules and rings, *Advances in Pure Mathematics*, **4** (2014), 25–33. https://doi.org/10.4236/apm.2014.41004

AIMS Press