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Research article

The comparative study of resolving parameters for a family of ladder networks

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Abstract: For a simple connected graph $G = (V, E)$, a vertex $x \in V$ distinguishes two elements (vertices or edges) $x_1 \in V, y_1 \in E$ if $d(x, x_1) \neq d(x, y_1)$. A subset $Q_m \subset V$ is a mixed metric generator for G, if every two distinct elements (vertices or edges) of G are distinguished by some vertex of Q_m. The minimum cardinality of a mixed metric generator for G is called the mixed metric dimension and denoted by $dim_m(G)$. In this paper, we investigate the mixed metric dimension for different families of ladder networks. Among these families, we consider Möbius ladder, hexagonal Möbius ladder, triangular Möbius ladder network and conclude that all these families have constant-metric, edge metric and mixed metric dimension.

Keywords: Möbius ladder network; hexagonal Möbius ladder network; triangular Möbius ladder network; mixed metric dimension; mixed metric generator
Mathematics Subject Classification: 05C09, 05C12, 05C92

1. Introduction

Let $G = (V, E)$ be a simple, connected graph. For $x_1, x_2 \in V$, the distance $d(x_1, x_2)$ between vertices x_1 and x_2 is the count of edges between x_1 and x_2. A vertex $v \in V$ is said to distinguish two vertices x_1 and x_2, if $d(v, x_1) \neq d(v, x_2)$. A set $Q \subset V$ is called a metric generator for G, if any pair of distinct vertices of G is distinguished by some element of Q. A metric generator of minimum cardinality is named as metric basis, and its cardinality is the metric dimension of G, denoted by $dim(G)$. For a vertex v from the vertex set and an edge $e = x_1x_2$ from its corresponding edge set, the distance betwixt v and e is measured as $d(e, v) = min\{d(x_1, v), d(x_2, v)\}$. A vertex $x \in V$ categorizes two edges $e_1, e_2 \in E$, if $d(x, e_1) \neq d(x, e_2)$. A subset Q_e having minimum vertices from a graph G is an edge metric generator for G, if each pair of two distinct edges of G are distinguished by some vertex of Q_e. The minimum cardinality of an edge metric generator for a graph G is called the edge metric dimension and is denoted by $dim_e(G)$. A vertex x of a graph G distinguishes two elements (vertices or edges) u, v of G, if $d(u, x) \neq d(v, x)$. A subset Q_m is a mixed metric generator, if there exist any two distinct elements (vertices or edges) of G are distinguished by some vertex of Q_m. The minimum numbers of elements in a mixed metric generator for a graph, is called the mixed metric dimension and is denoted by $dim_m(G)$. Moreover, mixed metric dimension is a blended version of metric and edge metric dimension.

The authors in [1] introduced the concept of metric dimension, where the metric generators were referred to as locating sets due to some connections with the problem of uniquely identifying the position of intruders in networks. Furthermore, it was studied separately in [2], where metric generators were named resolving sets. Aligning with the same pattern of recognizing the vertices of a graph by subset (resolving set) of vertex set, in [3] introduced the edge metric dimension and later analogous to this definition [4] proposed mixed metric dimension in which both concepts are together (recognizing vertices and edges), adding one more condition that no two elements (vertices and edges) have identical representation with respect to the chosen subset.

The standard metric generator is based on the concept of uniquely recognizing the entire vertex set of a graph, such as how to represent different vertices of a graph with respect to the metric generator. This concept might fail when an unknown problem arises in edges instead of vertices, then edge metric generator parameter take attention of entire working, but still, a question can arise, what happened if both of the behaviors came across with an issue? Then recognizing vertices and edges simultaneously to overcome the failure and both elements (vertices and edges) can be correctly recognized the entire structure of graphs with respect to some chosen vertices, which is named mixed metric generator.

The idea of metric dimension has numerous applications in different fields of science, regarding chemical structure [5] robot navigation [6], connections of a metric dimension of Hamming graphs with understanding and analysis of the mastermind of games [7] and variety of coin weighing problems linked with this concept in [8, 9]. Resolving sets have proven worthwhile in a variety of other applications, and they also served as a motivation for the theoretical study of metric dimension.

The metric dimension of different classes of graphs has been extensively studied for the last few decades. It is proved that the metric and one of its generalization named as edge metric, of ladder network is two [10, 11], metric dimension of Möbius ladder network is three [12], while edge metric is increased by one and proved four in [11], it is proved that both parameters of dimension for the hexagonal Möbius ladder network is three [11, 13], similarly, metric dimension of triangular ladder network is three and edge metric dimension increased by one proved in [11, 14]. In [15], discussed vertex metric-based dimension of generalized perimantanes diamondoid structure, some extended topics of metric dimension are discussed in [16–18]. In [19], measured the metric-based resolvability of polycyclic aromatic hydrocarbons.

Moreover, metric dimension of some cycle related graphs studied in [20], detailed analysis of the exact metric dimension of newly designed hexagonal Möbius ladder network can be seen in [13]. The local edge metric dimension is discussed for some graphs, including the ladder network in [21]. Convex polytopes graph discussed in [22] with the concept of edge metric dimension. Mixed metric dimension of generalized Peterson graph discussed in [23], some families of rotationally symmetric graph discussed in [24] with the concept of mixed metric dimension, some lower and upper bounds are discussed in [4] with respect to the girth of graph. Limited to the topic, one can find interesting research on metric dimension [25–27], on edge metric dimension [28, 29].

Theorem 1.1. *[4] Let G be a simple connected graph G. Then* $dim_m(G) \geq max\{dim(G), dim_e(G)\}$.

Definition 1.1. *A size h-gap between two vertices* x_1 *and* x_2 *is a path of length h between them.*

In [4], researcher proved that the mixed metric dimension of a graph is an NP-hard problem. Our emphasis is to provide the exact mixed metric dimension of some families of ladder networks. We investigate the mixed metric dimension of hexagonal Möbius ladder, triangular ladder, triangular Möbius ladder and simple ladder network. We prove that these ladder network families lie in the constant mixed metric dimension category. A comparative analysis between metric, edge metric, and mixed metric dimension is provided in the end to decide which parameter is better to choose.

2. Mixed metric dimension results

As we discuss earlier, edge metric dimension of Möbius ladder network is three. The mixed metric dimension can be three or more. In this part of section we will prove that the mixed metric dimension of Möbius ladder network is four.

Theorem 2.1. *Let* ML_m *be a Möbius ladder network with* $m \geq 3$. *Then*

$$dim_m(ML_m) = 4.$$

Proof. Let the mixed metric generator $Q_m = \{v_1, v_2, v_{\frac{m+4}{2}}, v_{\frac{3m+2}{2}}\}$, when m is even and $Q_m = \{v_1, v_2, v_{\frac{m+3}{2}}, v_{m+1}\}$, when m is odd. The labeling of vertices are defined in Figure 1.

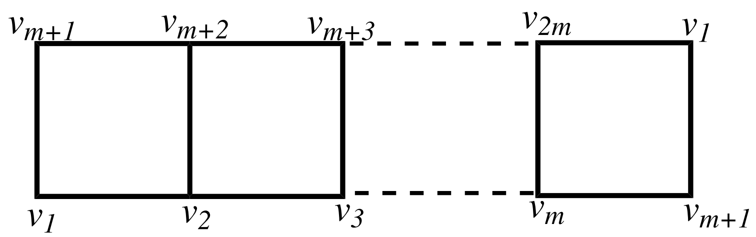

Figure 1. Möbius Ladder Graph ML_m.

To prove that $dim_m(ML_m) = 4$, we use the method of double inequality, for $dim_m(ML_m) \leq 4$, given below are the shortest distance between all edges and the mixed metric generator's vertices:

$$d(v_\epsilon v_{m+\epsilon}, v_1) = \begin{cases} \epsilon - 1 & \text{if } \epsilon = 1, 2, \ldots, \left\lfloor \frac{m+2}{2} \right\rfloor; \\ \left\lfloor \frac{m+3}{2} \right\rfloor - \epsilon + \left\lfloor \frac{m-1}{2} \right\rfloor & \text{if } \epsilon = \left\lfloor \frac{m+3}{2} \right\rfloor, \ldots, m. \end{cases}$$

$$d(v_\epsilon v_{\epsilon+1}, v_1) = \begin{cases} \epsilon - 1 & \text{if } \epsilon = 1, 2, \ldots, \left\lfloor \frac{m+2}{2} \right\rfloor; \\ \left\lfloor \frac{m+4}{2} \right\rfloor - \epsilon + \left\lfloor \frac{m-1}{2} \right\rfloor & \text{if } \epsilon = \left\lfloor \frac{m+4}{2} \right\rfloor, \ldots, m. \end{cases}$$

$$d\left(v_{m+\epsilon}v_{m+\epsilon+1}, v_1\right) = \begin{cases} \epsilon & \text{if } \epsilon = 1, 2, \ldots, \left\lfloor \frac{m}{2} \right\rfloor; \\ \left\lfloor \frac{m-1}{2} \right\rfloor - \epsilon + \left\lfloor \frac{m+2}{2} \right\rfloor & \text{if } \epsilon = \left\lfloor \frac{m+2}{2} \right\rfloor, \ldots, m-1. \end{cases}$$

$$d\left(v_{2m}v_1, v_1\right) = 0.$$

$$d\left(v_\epsilon v_{m+\epsilon}, v_2\right) = \begin{cases} 1 & \text{if } \epsilon = 1; \\ \epsilon - 2 & \text{if } \epsilon = 2, 3, \ldots, \left\lfloor \frac{m+5}{2} \right\rfloor; \\ \left\lfloor \frac{m}{2} \right\rfloor - \epsilon + \left\lfloor \frac{m+7}{2} \right\rfloor & \text{if } \epsilon = \left\lfloor \frac{m+7}{2} \right\rfloor, \ldots, m. \end{cases}$$

$$d\left(v_\epsilon v_{\epsilon+1}, v_2\right) = \begin{cases} 0 & \text{if } \epsilon = 1; \\ \epsilon - 2 & \text{if } \epsilon = 2, \ldots, \left\lfloor \frac{m+4}{2} \right\rfloor; \\ \left\lfloor \frac{m+6}{2} \right\rfloor - \epsilon + \left\lfloor \frac{m-1}{2} \right\rfloor & \text{if } \epsilon = \left\lfloor \frac{m+6}{2} \right\rfloor, \ldots, m. \end{cases}$$

$$d\left(v_{m+\epsilon}v_{m+\epsilon+1}, v_2\right) = \begin{cases} 1 & \text{if } \epsilon = 1; \\ \epsilon - 1 & \text{if } \epsilon = 2, \ldots, \left\lfloor \frac{m+3}{2} \right\rfloor; \\ \left\lfloor \frac{m+5}{2} \right\rfloor - \epsilon + \left\lfloor \frac{m-1}{2} \right\rfloor & \text{if } \epsilon = \left\lfloor \frac{m+5}{2} \right\rfloor, \ldots, m-1. \end{cases}$$

$$d\left(v_{2m}v_1, v_2\right) = 2.$$

$$d\left(v_\epsilon v_{m+\epsilon}, v_{m+1}\right) = \begin{cases} \epsilon - 1 & \text{if } \epsilon = 1, 2, \ldots, \left\lfloor \frac{m+2}{2} \right\rfloor; \\ \left\lfloor \frac{m+4}{2} \right\rfloor - \epsilon + \left\lfloor \frac{m-1}{2} \right\rfloor & \text{if } \epsilon = \left\lfloor \frac{m+4}{2} \right\rfloor, \ldots, m. \end{cases}$$

$$d\left(v_\epsilon v_{\epsilon+1}, v_{m+1}\right) = \begin{cases} \epsilon & \text{if } \epsilon = 1, 2, \ldots, \left\lfloor \frac{m}{2} \right\rfloor; \\ \left\lfloor \frac{m-1}{2} \right\rfloor - \epsilon + \left\lfloor \frac{m+2}{2} \right\rfloor & \text{if } \epsilon = \left\lfloor \frac{m+2}{2} \right\rfloor, \ldots, m. \end{cases}$$

$$d\left(v_{m+\epsilon}v_{m+\epsilon+1}, v_{m+1}\right) = \begin{cases} \epsilon - 1 & \text{if } \epsilon = 1, 2, \ldots, \left\lfloor \frac{m+2}{2} \right\rfloor; \\ \left\lfloor \frac{m-1}{2} \right\rfloor - \epsilon + \left\lfloor \frac{m+4}{2} \right\rfloor & \text{if } \epsilon = \left\lfloor \frac{m+4}{2} \right\rfloor, \ldots, m-1. \end{cases}$$

$$d\left(v_{2m}v_1, v_{m+1}\right) = 1.$$

$$d\left(v_\epsilon v_{m+\epsilon}, v_b\right) = \begin{cases} \left\lfloor \frac{m-1}{2} \right\rfloor & \text{if } \epsilon = 1; \\ \left|2 - \epsilon + \left\lfloor \frac{m}{2} \right\rfloor\right| & \text{if } \epsilon = 2, 3, \ldots, m. \end{cases}$$

where $b = \left\lfloor \frac{m+4}{2} \right\rfloor = \frac{m+4}{2}$ (m = even) $= \frac{m+3}{2}$ (m = odd),

$$d\left(v_\epsilon v_{\epsilon+1}, v_b\right) = \begin{cases} \left\lfloor \frac{m}{2} \right\rfloor - \epsilon + 1 & \text{if } \epsilon = 1, 2, \ldots, \left\lfloor \frac{m+2}{2} \right\rfloor; \\ \epsilon - \left\lfloor \frac{m+4}{2} \right\rfloor & \text{if } \epsilon = \left\lfloor \frac{m+4}{2} \right\rfloor, \ldots, m. \end{cases}$$

$$d\left(v_{m+\epsilon}v_{m+\epsilon+1}, v_b\right) = \begin{cases} \left\lfloor \frac{m+1}{2} \right\rfloor & \text{if } \epsilon = 1; \\ 2 - \epsilon + \left\lfloor \frac{m-1}{2} \right\rfloor & \text{if } \epsilon = 2, \ldots, \left\lfloor \frac{m+2}{2} \right\rfloor; \\ \epsilon - \left\lfloor \frac{m+4}{2} \right\rfloor + 1 & \text{if } \epsilon = \left\lfloor \frac{m+4}{2} \right\rfloor, \ldots, m-1. \end{cases}$$

$$d\left(v_{2m}v_1, v_b\right) = \left\lfloor \frac{m-1}{2} \right\rfloor.$$

$$d\left(v_\epsilon v_{m+\epsilon}, v_c\right) = \left|\frac{m}{2} - \epsilon + 1\right|,$$

where $c = \frac{3m+2}{2}$,

$$d\left(v_\epsilon v_{\epsilon+1}, v_c\right) = \begin{cases} \frac{m}{2} - \epsilon + 1 & \text{if } \epsilon = 1, 2, \ldots, \frac{m}{2}; \\ \epsilon - \frac{m+2}{2} + 1 & \text{if } \epsilon = \frac{m+2}{2}, \ldots, m. \end{cases}$$

$$d\left(v_{m+\epsilon} v_{m+\epsilon+1}, v_c\right) = \begin{cases} \frac{m-2}{2} - \epsilon + 1 & \text{if } \epsilon = 1, 2, \ldots, \frac{m}{2}; \\ \epsilon - \frac{m+2}{2} & \text{if } \epsilon = \frac{m+2}{2}, \ldots, m. \end{cases}$$

$$d\left(v_{2m} v_1, v_c\right) = \frac{m-2}{2}.$$

It is easy to see the distances between all edges in relation to the chosen mixed metric generator Q_m. Now, the vertices distances according to Q_m are given as follows:

Distance of all vertices with respect to v_1;

$$d(v_\epsilon, v_1) = \begin{cases} \epsilon - 1, & \text{if } \epsilon = 1, 2 \ldots, \left\lfloor \frac{m+2}{2} \right\rfloor; \\ m - \epsilon + 2, & \text{if } \epsilon = \left\lfloor \frac{m+2}{2} \right\rfloor + 1, \ldots, m+1; \\ \epsilon - m, & \text{if } \epsilon = m+2, \ldots, \left\lceil \frac{m+2}{2} \right\rceil; \\ 2m - \epsilon + 1, & \text{if } \epsilon = \left\lceil \frac{m+2}{2} \right\rceil + 1, \ldots, 2m. \end{cases}$$

Distance of all vertices with respect to v_2;

$$d(v_\epsilon, v_2) = \begin{cases} |\epsilon - 2|, & \text{if } \epsilon = 1, 2 \ldots, \left\lceil \frac{m+4}{2} \right\rceil; \\ m - \epsilon + 3, & \text{if } \epsilon = \left\lceil \frac{m+2}{2} \right\rceil + 1, \ldots, m+1; \\ \epsilon - m - 1, & \text{if } \epsilon = m+2, \ldots, \left\lfloor \frac{3m}{2} \right\rfloor + 1; \\ 2m - \epsilon + 2, & \text{if } \epsilon = \left\lfloor \frac{3m}{2} \right\rfloor + 2, \ldots, 2m. \end{cases}$$

Distance of all vertices with respect to $v_{\frac{m+4}{2}}$;

$$d(v_\epsilon, v_{\frac{m+4}{2}}) = \begin{cases} \frac{m}{2}, & \text{if } \epsilon = 1; \\ \frac{m+4}{2} - \epsilon, & \text{if } \epsilon = 2, 3, \ldots, \frac{m+4}{2}; \\ \epsilon - \frac{m+4}{2}, & \text{if } \epsilon = \frac{m+6}{2}, \ldots, m+1; \\ \frac{m}{2}, & \text{if } \epsilon = m+2; \\ \frac{3m+6}{2} - \epsilon, & \text{if } \epsilon = m+3, \ldots, \frac{3m+4}{2}; \\ 2 + \epsilon - \frac{3m+6}{2}, & \text{if } \epsilon = \frac{3m+6}{2} + 2, \ldots, 2m. \end{cases}$$

Distance of all vertices with respect to $v_{\frac{3m+2}{2}}$;

$$d(v_\epsilon, v_{\frac{3m+2}{2}}) = \begin{cases} \frac{m}{2}, & \text{if } \epsilon = 1; \\ \frac{m+4}{2} - \epsilon, & \text{if } \epsilon = 2, 3, \ldots, \frac{m+2}{2}; \\ \epsilon - \frac{m}{2}, & \text{if } \epsilon = \frac{m+4}{2}, \ldots, m; \\ \frac{m}{2}, & \text{if } \epsilon = m+1; \\ \left|\frac{3m+2}{2} - \epsilon\right|, & \text{if } \epsilon = m+2, \ldots, 2m. \end{cases}$$

Distance of all vertices with respect to $v_{\frac{m+3}{2}}$;

$$d(v_\epsilon, v_{\frac{m+3}{2}}) = \begin{cases} \frac{m+3}{2} - \epsilon, & \text{if } \epsilon = 1, 2, 3, \ldots, \frac{m+3}{2}; \\ \epsilon - \frac{m+3}{2}, & \text{if } \epsilon = \frac{m+5}{2}, \ldots, m+1; \\ \frac{3m+5}{2} - \epsilon, & \text{if } \epsilon = m+2, \ldots, \frac{3m+3}{2}; \\ 2 + \epsilon - \frac{3m+5}{2}, & \text{if } \epsilon = \frac{3m+5}{2} + 2, \ldots, 2m. \end{cases}$$

Distance of all vertices with respect to v_{m+1};

$$d(v_\epsilon, v_{m+1}) = \begin{cases} \epsilon, & \text{if } \epsilon = 1, 2, \ldots, \frac{m+1}{2}; \\ m - \epsilon + 1, & \text{if } \epsilon = \frac{m+3}{2}, \ldots, m+1; \\ \epsilon - m - 1, & \text{if } \epsilon = m+2, \ldots, \frac{3m+2}{2}; \\ 2m - \epsilon + 2, & \text{if } \epsilon = \frac{3m+4}{2}, \ldots, 2m. \end{cases}$$

It is clear to see the representations in term of distances of all edges and vertices with respect to mixed metric generator Q_m are different, it proved that $dim_m(ML_m) \leq 4$. Now, the converse is directly deduced from Theorem 1.1, which implies that

$$dim_m(ML_m) = 4.$$

□

We know that edge metric dimension of hexagonal Möbius ladder network is three. By the relation in Theorem 1.1, mixed metric dimension can be three or more. In the following, we proved that the mixed metric dimension of the hexagonal Möbius ladder network is four.

Theorem 2.2. *Let HML_m be a hexagonal Möbius ladder network with $m \geq 2$. Then*

$$dim_m(HML_m) = 4.$$

Proof. Suppose that the mixed metric generator is $Q_m = \{v_1, v_2, v_{m+2}, v_{3m+2}\}$. The labeling defined in Figure 2.

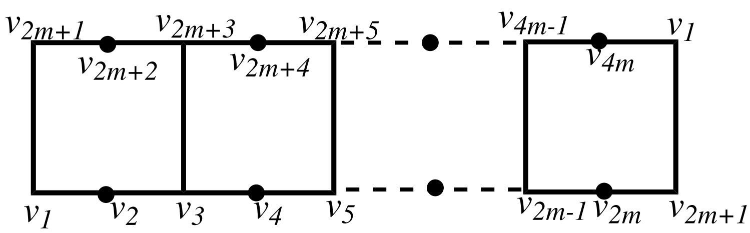

Figure 2. Hexagonal Möbius Ladder Graph HML_m.

To prove that $dim_m(HML_m) = 4$, we will use the method of double inequality, for $dim_m(HML_m) \leq 4$, given below are the distances of all edges with respect to mixed metric generator:

$$d(v_\epsilon v_{m+\epsilon}, v_1) = \begin{cases} \epsilon - 1 & \text{if } \epsilon = 1, 3, \ldots, m+1 \text{ and } m \text{ even}; \\ 2m + 1 - \epsilon & \text{if } \epsilon = m+3, m+5, \ldots, 2m-1 \text{ and } m \text{ even}; \\ \epsilon - 1 & \text{if } \epsilon = 1, 3, \ldots, m \text{ and } m \text{ odd}; \\ 2m + 1 - \epsilon & \text{if } \epsilon = m+2, m+4 \ldots, 2m-1 \text{ and } m \text{ odd}. \end{cases}$$

$$d(v_\epsilon v_{\epsilon+1}, v_1) = \begin{cases} \epsilon - 1 & \text{if } \epsilon = 1, 2, \ldots, m+1; \\ 2m + 1 - \epsilon & \text{if } \epsilon = m+2, \ldots, 2m. \end{cases}$$

$$d(v_{2m+\epsilon} v_{2m+\epsilon+1}, v_1) = \begin{cases} \epsilon & \text{if } \epsilon = 1, 2, \ldots, m; \\ 2m - \epsilon & \text{if } \epsilon = m+1, \ldots, 2m-1. \end{cases}$$

$$d(v_{4m} v_1, v_1) = 0.$$

$$d(v_\epsilon v_{m+\epsilon}, v_2) = \begin{cases} 1 & \text{if } \epsilon = 1; \\ \epsilon - 2 & \text{if } \epsilon = 3, 5, \ldots, m+1 \text{ and } m \text{ even}; \\ 2m + 3 - \epsilon & \text{if } \epsilon = m+3, m+5, \ldots, 2m-1 \text{ and } m \text{ even}; \\ \epsilon - 2 & \text{if } \epsilon = 3, 5, \ldots, m+2 \text{ and } m \text{ odd}; \\ 2m + 3 - \epsilon & \text{if } \epsilon = m+4, m+6, \ldots, 2m-1 \text{ and } m \text{ odd}. \end{cases}$$

$$d(v_\epsilon v_{\epsilon+1}, v_2) = \begin{cases} 0 & \text{if } \epsilon = 1; \\ \epsilon - 2 & \text{if } \epsilon = 2, \ldots, m+2; \\ 2m + 2 - \epsilon & \text{if } \epsilon = m+3, \ldots, 2m. \end{cases}$$

$$d(v_{2m+\epsilon} v_{2m+\epsilon+1}, v_2) = \begin{cases} 2 & \text{if } \epsilon = 1, 2; \\ \epsilon - 1 & \text{if } \epsilon = 3, \ldots, m+1; \\ 2m + 2 - \epsilon & \text{if } \epsilon = m+2, \ldots, 2m-1. \end{cases}$$

$$d(v_{4m} v_1, v_2) = 2.$$

$$d(v_\epsilon v_{m+\epsilon}, v_{m+2}) = \begin{cases} m - 1 & \text{if } \epsilon = 1; \\ m + 2 - \epsilon & \text{if } \epsilon = 3, 5, \ldots, m+1 \text{ and } m \text{ even}; \\ \epsilon - m - 2 & \text{if } \epsilon = m+3, m+5, \ldots, 2m-1 \text{ and } m \text{ even}; \\ m + 2 - \epsilon & \text{if } \epsilon = 3, 5, \ldots, m+2 \text{ and } m \text{ odd}; \\ \epsilon - m - 2 & \text{if } \epsilon = m+4, m+6 \ldots, 2m-1 \text{ and } m \text{ odd}. \end{cases}$$

$$d(v_\epsilon v_{\epsilon+1}, v_{m+2}) = \begin{cases} m - \epsilon + 1 & \text{if } \epsilon = 1, 2, \ldots, m+1; \\ \epsilon - m - 2 & \text{if } \epsilon = m+2, \ldots, 2m. \end{cases}$$

$$d(v_{2m+\epsilon} v_{2m+\epsilon+1}, v_{m+2}) = \begin{cases} m & \text{if } \epsilon = 1; \\ m + 2 - \epsilon & \text{if } \epsilon = 2, \ldots, m \text{ and } m \text{ even}; \\ 2 & \text{if } \epsilon = m+1, m+2 \text{ and } m \text{ even}; \\ \epsilon - m - 1 & \text{if } \epsilon = m+3, \ldots, 2m-1 \text{ and } m \text{ even}; \\ m + 2 - \epsilon & \text{if } \epsilon = 2, \ldots, m+1 \text{ and } m \text{ odd}; \\ \epsilon - m - 1 & \text{if } \epsilon = m+2, \ldots, 2m-1 \text{ and } m \text{ odd}. \end{cases}$$

$$d(v_{4m} v_1, v_{m+2}) = \begin{cases} m & \text{when } m = 2; \\ m - 1 & \text{when } m \geq 3. \end{cases}$$

$$d\left(v_\epsilon v_{m+\epsilon}, v_{3m+2}\right) = \begin{cases} m-1 & \text{if } \epsilon = 1; \\ m+2-\epsilon & \text{if } \epsilon = 3, 5, \ldots, m+1 \text{ and } m \text{ even}; \\ 2 & \text{if } \epsilon = m+1, m+2, \text{ and } m \text{ even}; \\ \epsilon-m-2 & \text{if } \epsilon = m+3, m+5, \ldots, 2m-1 \text{ and } m \text{ even}; \\ m+2-\epsilon & \text{if } \epsilon = 3, 5, \ldots, m+2 \text{ and } m \text{ odd}; \\ \epsilon-m-2 & \text{if } \epsilon = m+4, m+6, \ldots, 2m-1 \text{ and } m \text{ odd}. \end{cases}$$

$$d\left(v_\epsilon v_{\epsilon+1}, v_{3m+2}\right) = \begin{cases} m & \text{if } \epsilon = 1; \\ m+2-\epsilon & \text{if } \epsilon = 2, \ldots, m \text{ and } m \text{ even}; \\ 2 & \text{if } \epsilon = m+1, m+2 \text{ and } m \text{ even}; \\ \epsilon-m-1 & \text{if } \epsilon = m+3, \ldots, 2m \text{ and } m \text{ even}; \\ m+2-\epsilon & \text{if } \epsilon = 2, \ldots, m+1 \text{ and } m \text{ odd}; \\ \epsilon-m-1 & \text{if } \epsilon = m+2, \ldots, 2m \text{ and } m \text{ odd}. \end{cases}$$

$$d\left(v_{2m+\epsilon} v_{2m+\epsilon+1}, v_{3m+2}\right) = \begin{cases} m+1-\epsilon & \text{if } \epsilon = 1, \ldots, m+1; \\ \epsilon-m-2 & \text{if } \epsilon = m+2, \ldots, 2m-1. \end{cases}$$

$$d\left(v_{4m} v_1, v_{3m+2}\right) = m-2.$$

It is easy to verify that no two edges have the same representation with respect to the mixed metric generator. In order to complete the definition of mixed metric, we will check the vertices distance as well.

Distance of all vertices with respect to v_1;

$$d(v_\epsilon, v_1) = \begin{cases} \epsilon-1, & \text{if } \epsilon = 1, 2 \ldots, m+1; \\ 2m-\epsilon+2, & \text{if } \epsilon = m+2, \ldots, 2m+1; \\ \epsilon-2m, & \text{if } \epsilon = 2m+2, \ldots, 3m; \\ 4m-\epsilon+1, & \text{if } \epsilon = 3m+1, \ldots, 4m. \end{cases}$$

Distance of all vertices with respect to v_2;

$$d(v_\epsilon, v_2) = \begin{cases} |\epsilon-2|, & \text{if } \epsilon = 1, 2 \ldots, m+2; \\ 2m-\epsilon+3, & \text{if } \epsilon = m+3, \ldots, 2m+1; \\ \epsilon-2m-1, & \text{if } \epsilon = 2m+3, \ldots, 3m+1; \\ \epsilon-2m+1, & \text{if } \epsilon = 2m+2; \\ 4m-\epsilon+2, & \text{if } \epsilon = 3m+2, \ldots, 4m. \end{cases}$$

Distance of all vertices with respect to v_{m+2};

$$d(v_\epsilon, v_{m+2}) = \begin{cases} m, & \text{if } \epsilon = 1; \\ m-\epsilon+2, & \text{if } \epsilon = 2, 3, \ldots, m+2; \\ \epsilon-m-2, & \text{if } \epsilon = m+3, \ldots, 2m+1; \\ 3, & \text{if } \epsilon = 2m+2 \text{ and } m = 2; \\ m, & \text{if } \epsilon = 2m+2 \text{ and } m \geq 3; \\ 3m+3-\epsilon, & \text{if } \epsilon = 2m+3, \ldots, 3m+1. \end{cases}$$

$$d(v_\epsilon, v_{m+2}) = \begin{cases} 3m+2-\epsilon+\left\lfloor\frac{m+6}{4}\right\rfloor, & \text{if } \epsilon = 3m+2,\ldots,3m+2+\left\lfloor\frac{m+6}{4}\right\rfloor, \text{ and } m \text{ even};\\ \epsilon-3m-1-\left\lfloor\frac{m-1}{4}\right\rfloor, & \text{if } \epsilon = 3m+3+\left\lfloor\frac{m-1}{4}\right\rfloor,\ldots,4m \text{ and } m \text{ even};\\ \epsilon-3m-1, & \text{if } \epsilon = 3m+2,\ldots,4m \text{ and } m \text{ odd}.\end{cases}$$

Distance of all vertices with respect to v_{3m+2};

$$d(v_\epsilon, v_{m+2}) = \begin{cases} m-2+\epsilon, & \text{if } \epsilon = 1,2;\\ m-\epsilon+3, & \text{if } \epsilon = 3,4,\ldots,m+1 \text{ and } m \text{ even};\\ 5-\epsilon+m, & \text{if } \epsilon = m+2, m+3 \text{ and } m \text{ even};\\ \epsilon-m-1, & \text{if } \epsilon = m+4,\ldots,2m+1 \text{ and } m \text{ even};\\ 3m+2-\epsilon, & \text{if } \epsilon = 2m+2,\ldots,3m+2 \text{ and } m \text{ even};\\ \epsilon-3m-2, & \text{if } \epsilon = 3m+3,\ldots,4m \text{ and } m \text{ even};\\ m-\epsilon+3, & \text{if } \epsilon = 3,4,\ldots,m+2 \text{ and } m \text{ odd};\\ \epsilon-m-1, & \text{if } \epsilon = m+3,\ldots,2m+1 \text{ and } m \text{ odd};\\ 3m+2-\epsilon, & \text{if } \epsilon = 2m+2,\ldots,3m+2 \text{ and } m \text{ odd};\\ \epsilon-3m-2, & \text{if } \epsilon = 3m+3,\ldots,4m \text{ and } m \text{ odd}.\end{cases}$$

All the edges and vertices with respect to the selected mixed metric generator Q_m have different representation, it is proved that $dim_m(HML_m) \leq 4$. Now, assume on the contrary that $dim_m(HML_m) = 3$, which makes the minimum members in a mixed metric generator $Q_m^{'}$ are three, given below are few deliberation on the behalf of argument.

If all three vertices have a disparity of every arbitrary length, $Q_m^{'} = \{v_i, v_j, v_k\}$ with $1 \leq i, j, k \leq 2m-1$, then it denotes the same representation in the edges that are $d\left(v_1v_{2m+1}|Q_m^{'}\right) = d\left(v_1v_{4m}|Q_m^{'}\right)$ and including $i, j, k = 2m, 2m+1$ then the same representation in a vertex and an edge, $d\left(v_1|Q_m^{'}\right) = d\left(v_1v_{4m}|Q_m^{'}\right)$. When $2m+2 \leq i, j, k \leq 4m-1$, then it implies that the same representation in the edges which are $d\left(v_1v_{2m+1}|Q_m^{'}\right) = d\left(v_{2m}v_{2m+1}|Q_m^{'}\right)$. To generalize this concept, we choose the vertices $v_i, v_j, v_k \in V(HML_m)$, then it follows that the same representations in the edges which are either $d\left(v_pv_{p+1}|Q_m^{'}\right) = d\left(v_{2m+p}v_{2m+p+1}|Q_m^{'}\right)$ where $1 \leq p \leq 2m$ or $d\left(v_1|Q_m^{'}\right) = d\left(v_1v_{2m+1}|Q_m^{'}\right)$ or $d\left(v_2|Q_m^{'}\right) = d\left(v_2v_3|Q_m^{'}\right)$ or $d\left(v_3|Q_m^{'}\right) = d\left(v_3v_4|Q_m^{'}\right)$ or $d\left(v_i|Q_m^{'}\right) = d\left(v_iv_{m+i}|Q_m^{'}\right)$ where $1 \leq i \text{ (odd)} \leq 2m-1$. Thus, it is concluded that we can not take vertices in a mixed metric generator with cardinality three. Therefore, $dim_m(HML_m) = 4$. □

In [11], it is discussed that the edge metric dimension of triangular ladder network is four. By the relation minimum mixed metric dimension can be four and it is proved in this part.

Theorem 2.3. *Let TL_m be a triangular ladder network with $m \geq 3$. Then*

$$dim_m(TL_m) = 4.$$

Proof. Suppose that the mixed metric generator $Q_m = \{v_1, v_2, v_{2m-1}, v_{2m}\}$. In Figure 3 labeling is defined.

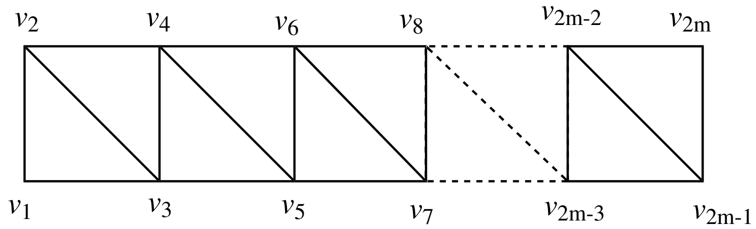

Figure 3. Triangular Ladder Graph TL_m.

To prove that $dim_m(TL_m) = 4$, we first prove that $dim_m(TL_m) \leq 4$. The shortest distances of all the edges of our network according to the chosen mixed metric generator are given as follows:

$$d(v_\epsilon v_{\epsilon+1}, v_1) = \begin{cases} \frac{\epsilon-1}{2}, & \text{if } 1 \leq \epsilon \text{ (odd)} \leq 2m-1; \\ \frac{\epsilon+2}{2}, & \text{if } 2 \leq \epsilon \text{ (even)} \leq 2m-2. \end{cases}$$

$$d(v_\epsilon v_{\epsilon+1}, v_1) = \begin{cases} \frac{\epsilon-1}{2}, & \text{if } 1 \leq \epsilon \text{ (odd)} \leq 2m-3; \\ \frac{\epsilon}{2}, & \text{if } 2 \leq \epsilon \text{ (even)} \leq 2m-2. \end{cases}$$

$$d(v_\epsilon v_{\epsilon+1}, v_2) = \begin{cases} \frac{\epsilon-1}{2}, & \text{if } 1 \leq \epsilon \text{ (odd)} \leq 2m-1; \\ \frac{\epsilon-2}{2}, & \text{if } 2 \leq \epsilon \text{ (even)} \leq 2m-2. \end{cases}$$

$$d(v_\epsilon v_{\epsilon+2}, v_2) = \begin{cases} 1, & \text{if } \epsilon = 1; \\ \frac{\epsilon-1}{2}, & \text{if } 1 \leq \epsilon \text{ (odd)} \leq 2m-3; \\ \frac{\epsilon-2}{2}, & \text{if } 2 \leq \epsilon \text{ (even)} \leq 2m-2. \end{cases}$$

$$d(v_\epsilon v_{\epsilon+1}, v_{2m-1}) = \begin{cases} \frac{2m-\epsilon-1}{2}, & \text{if } 1 \leq \epsilon \text{ (odd)} \leq 2m-1; \\ \frac{2m-\epsilon-2}{2}, & \text{if } 2 \leq \epsilon \text{ (even)} \leq 2m-2. \end{cases}$$

$$d(v_\epsilon v_{\epsilon+2}, v_{2m-1}) = \begin{cases} \frac{2m-\epsilon-1}{2}, & \text{if } 1 \leq \epsilon \text{ (odd)} \leq 2m-3; \\ \frac{2m-\epsilon}{2}, & \text{if } 2 \leq \epsilon \text{ (even)} \leq 2m-2. \end{cases}$$

$$d(v_\epsilon v_{\epsilon+1}, v_{2m}) = \begin{cases} \frac{2m-\epsilon-1}{2}, & \text{if } 1 \leq \epsilon \text{ (odd)} \leq 2m-1; \\ \frac{2m-\epsilon+1}{2}, & \text{if } 2 \leq \epsilon \text{ (even)} \leq 2m-2. \end{cases}$$

$$d(v_\epsilon v_{\epsilon+2}, v_{2m}) = \begin{cases} \frac{2m-\epsilon+1}{2}, & \text{if } 1 \leq \epsilon \text{ (odd)} \leq 2m-3; \\ \frac{2m-\epsilon-2}{2}, & \text{if } 2 \leq \epsilon \text{ (even)} \leq 2m-2. \end{cases}$$

The representations of edges is unique with respect to the mixed metric generator. To complete the proof, we need to check the distances of vertices with respect to chosen mixed metric generator:

$$d(v_\epsilon, v_1) = \left\lfloor \frac{\epsilon}{2} \right\rfloor,$$

$$d(v_\epsilon, v_2) = \begin{cases} \left\lfloor \frac{1}{\epsilon} \right\rfloor, & \text{if } \epsilon = 1, 2; \\ \left\lfloor \frac{\epsilon-1}{2} \right\rfloor, & \text{if } \epsilon = 3, 4, \ldots, 2m. \end{cases}$$

$$d(v_\epsilon, v_{2m-1}) = \begin{cases} m-1-\left\lfloor \frac{\epsilon-1}{2} \right\rfloor, & \text{if } \epsilon = 1, 2, \ldots, 2m-1; \\ 1, & \text{if } \epsilon = 2m. \end{cases}$$

$$d(v_\epsilon, v_{2m}) = m - \left\lfloor \frac{\epsilon}{2} \right\rfloor.$$

The representations of all edges and vertices with respect to the mixed metric generator Q_m are easily visible, it is proved that $dim_m(TL_m) \leq 4$. Now, for $dim_m(TL_m) \geq 4$, the proof can proceed similar to the proof of Theorem 2.1, as a result, we omit it. □

Theorem 2.4. *Let TML_m be a triangular Möbius ladder network with $m \geq 5$. Then*

$$dim_m(TML_m) = 5.$$

Proof. Consider the mixed metric generator $Q_m = \{v_1, v_2, v_4, v_5, v_6\}$, when $m = 5$, $Q_m = \{v_1, v_3, v_4, v_{m+1}, v_{m+2}\}$, when $m \geq 6$ (even) and $Q_m = \{v_1, v_4, v_5, v_{m+3}, v_{m+4}\}$, when $m \geq 7$ (odd). The labeling of vertices is shown in Figure 4.

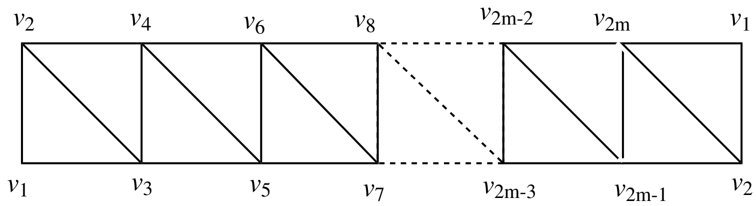

Figure 4. Triangonal Möbius Ladder Graph TML_m.

To prove that $dim_m(TML_m) = 5$, we prove that $dim_m(TML_m) \leq 5$. The distances of all edges according to mixed metric generator are given below:

When $m = 5$, the distance of the three vertices v_1, v_4, v_5 to all edges will be explained in later parts of proof.

$$d(v_\epsilon v_{\epsilon+1}, v_2) = \begin{cases} \frac{\epsilon-1}{2}; & \text{if } \epsilon = 1, 3, 5 \\ 1; & \text{if } \epsilon = 4, 6, 7 \\ 0; & \text{if } \epsilon = 2. \end{cases}$$

$$d(v_{2m-2}v_2, v_2) = d(v_{2m-3}v_2, v_2) = 0.$$

$$d(v_\epsilon v_{\epsilon+2}, v_2) = \begin{cases} 1; & \text{if } \epsilon = 1, 5 \\ 2; & \text{if } \epsilon = 3 \\ \frac{\epsilon-2}{2}, & \text{if } \epsilon = 2, 4, 6. \end{cases}$$

$$d(v_{2m-1}v_1, v_2) = d(v_{2m-2}v_2, v_6) = d(v_{2m-1}v_1, v_6) = 1$$

$$d(v_\epsilon v_{\epsilon+1}, v_6) = \begin{cases} \left|\frac{5-\epsilon}{2}\right|, & \text{if } 1 \leq \epsilon \text{ (odd)} \leq 2m-3; \\ \left|\frac{6-\epsilon}{2}\right|, & \text{if } 2 \leq \epsilon \text{ (even)} \leq 2m-4. \end{cases}$$

$$d(v_\epsilon v_{\epsilon+2}, v_6) = \begin{cases} 2; & \text{if } \epsilon = 1; \\ 1; & \text{if } \epsilon = 2, 3, 5; \\ 0; & \text{if } \epsilon = 4, 6. \end{cases}$$

$$d(v_{2m-3}v_2, v_6) = 2.$$

When $m \geq 6$ (even), we obtain the following:

$$d(v_\epsilon v_{\epsilon+1}, v_1) = \begin{cases} \frac{\epsilon-2}{2}; & \text{if } \epsilon = 1, 3, \ldots, m-1 \\ \frac{2m-\epsilon-3}{2}; & \text{if } \epsilon = m+1, m+3, \ldots, 2m-3 \\ \frac{\epsilon}{2}; & \text{if } \epsilon = 2, 4, \ldots, m \\ \frac{2m-\epsilon}{2}; & \text{if } \epsilon = m+2, m+4, \ldots, 2m-4. \end{cases}$$

$$d(v_{2m-2}v_2, v_1) = d(v_{2m-3}v_2, v_1) = 1.$$

$$d(v_\epsilon v_{\epsilon+2}, v_1) = \begin{cases} \frac{\epsilon-2}{2}; & \text{if } \epsilon = 1, 3, \ldots, m-1 \\ \frac{2m-\epsilon-1}{2}; & \text{if } \epsilon = m+1, m+3, \ldots, 2m-5 \\ \frac{\epsilon}{2}; & \text{if } \epsilon = 2, 4, \ldots, m-2 \\ \frac{2m-\epsilon-2}{2}; & \text{if } \epsilon = m, m+2, \ldots, 2m-4. \end{cases}$$

$$d(v_{2m-1}v_1, v_1) = 0.$$

$$d(v_\epsilon v_{\epsilon+1}, v_3) = \begin{cases} 1; & \text{if } \epsilon = 1 \\ \frac{\epsilon-3}{2}; & \text{if } \epsilon = 3, 5, \ldots, m+3 \\ \frac{2m+3-\epsilon}{2}; & \text{if } \epsilon = m+5, m+7, \ldots, 2m-3 \\ \frac{\epsilon-2}{2}; & \text{if } \epsilon = 2, 4, \ldots, m+2 \\ \frac{2m+2-\epsilon}{2}; & \text{if } \epsilon = m+4, m+6, \ldots, 2m-4. \end{cases}$$

$$d(v_{2m-2}v_2, v_3) = d(v_{2m-3}v_2, v_3) = d(v_{2m-1}v_1, v_3) = 2.$$

$$d(v_\epsilon v_{\epsilon+2}, v_3) = \begin{cases} \epsilon-1; & \text{if } \epsilon = 1, 2 \\ \frac{\epsilon-3}{2}; & \text{if } \epsilon = 3, 5, \ldots, m+1 \\ \frac{2m+1-\epsilon}{2}; & \text{if } \epsilon = m+3, m+5, \ldots, 2m-5 \\ \frac{\epsilon-2}{2}; & \text{if } \epsilon = 4, 6, \ldots, m+2 \\ \frac{2m+2-\epsilon}{2}; & \text{if } \epsilon = m+4, m+6, \ldots, 2m-4. \end{cases}$$

$$d(v_\epsilon v_{\epsilon+1}, v_4) = \begin{cases} 1; & \text{if } \epsilon = 1, 2 \\ \frac{\epsilon-3}{2}; & \text{if } \epsilon = 3, 5, \ldots, m+3 \\ \frac{2m-\epsilon+3}{2}; & \text{if } \epsilon = m+5, m+7 \ldots, 2m-3 \\ \frac{\epsilon-4}{2}; & \text{if } \epsilon = 4, 6, \ldots, m+2 \\ \frac{2m-\epsilon+2}{2}; & \text{if } \epsilon = m+4, m+6, \ldots, 2m-4. \end{cases}$$

$$d(v_{2m-2}v_2, v_4) = d(v_{2m-3}v_2, v_4) = d(v_{2m-1}v_1, v_4) = 2.$$

$$d(v_\epsilon v_{\epsilon+2}, v_4) = \begin{cases} 2-\epsilon; & \text{if } \epsilon = 1, 2 \\ \frac{\epsilon-1}{2}; & \text{if } \epsilon = 3, 5, \ldots, m+1 \\ \frac{2m-\epsilon+1}{2}; & \text{if } \epsilon = m+3, m+5, \ldots, 2m-5 \\ \frac{\epsilon-4}{2}; & \text{if } \epsilon = 4, 6, \ldots, m+2 \\ \frac{2m-\epsilon+2}{2}; & \text{if } \epsilon = m+4, m+6 \ldots, 2m-4. \end{cases}$$

$$d(v_\epsilon v_{\epsilon+1}, v_{m+1}) = \begin{cases} \frac{m-2}{2}; & \text{if } \epsilon = 1 \\ \left|\frac{m-\epsilon+1}{2}\right|; & \text{if } \epsilon = 3, 5, \ldots, 2m-3 \\ \left|\frac{m-\epsilon}{2}\right|; & \text{if } \epsilon = 2, 4, \ldots, 2m-4. \end{cases}$$

$$d(v_{2m-2}v_2, v_{m+1}) = d(v_{2m-1}v_1, v_{m+1}) = \frac{m-2}{2},\ d(v_{2m-3}v_2, v_{m+1}) = \frac{m-4}{2}.$$

$$d(v_\epsilon v_{\epsilon+2}, v_{m+1}) = \begin{cases} \frac{m-\epsilon-1}{2}; & \text{if } \epsilon = 1, 3, \ldots, m-1 \\ \frac{\epsilon-m-1}{2}; & \text{if } \epsilon = m+1, m+3, \ldots, 2m-5 \\ \frac{m-\epsilon-1}{2}; & \text{if } \epsilon = 2, 4, \ldots, m-2 \\ 1; & \text{if } \epsilon = m \\ \frac{\epsilon-m}{2}; & \text{if } \epsilon = m+2, m+4, \ldots, 2m-4. \end{cases}$$

$$d(v_\epsilon v_{\epsilon+1}, v_{m+2}) = \begin{cases} \frac{m-2}{2}; & \text{if } \epsilon = 1 \\ \left|\frac{m-\epsilon+1}{2}\right|; & \text{if } \epsilon = 3, 5, \ldots, 2m-3 \\ \left|\frac{m-\epsilon-2}{2}\right|; & \text{if } \epsilon = 2, 4, \ldots, 2m-4. \end{cases}$$

$$d(v_{2m-2}v_2, v_{m+2}) = d(v_{2m-3}v_2, v_{m+2}) = d(v_{2m-1}v_1, v_{m+2}) = \frac{m-4}{2}.$$

$$d(v_\epsilon v_{\epsilon+2}, v_{m+2}) = \begin{cases} \frac{m-\epsilon+1}{2}; & \text{if } \epsilon = 1, 3, \ldots, m-1 \\ \frac{m-\epsilon+3}{2}; & \text{if } \epsilon = m+1, m+3, \ldots, 2m-5 \\ \frac{m-\epsilon}{2}; & \text{if } \epsilon = 2, 4, \ldots, m \\ \frac{\epsilon-m-2}{2}; & \text{if } \epsilon = m+2, m+4, \ldots, 2m-4. \end{cases}$$

When $m \geq 7$ (odd), we have

$$d(v_\epsilon v_{\epsilon+1}, v_1) = \begin{cases} \frac{\epsilon-1}{2}; & \text{if } \epsilon = 1, 3, \ldots, m \\ \frac{2m-\epsilon-1}{2}; & \text{if } \epsilon = m+2, m+4, \ldots, 2m-3 \\ \frac{\epsilon}{2}; & \text{if } \epsilon = 2, 4, \ldots, m-1 \\ \frac{2m-\epsilon}{2}; & \text{if } \epsilon = m+1, m+3, \ldots, 2m-4. \end{cases}$$

$$d(v_{2m-2}v_2, v_1) = d(v_{2m-3}v_2, v_1) = 1,\ d(v_{2m-1}v_1, v_1) = 0.$$

$$d(v_\epsilon v_{\epsilon+2}, v_1) = \begin{cases} \frac{\epsilon-1}{2}; & \text{if } \epsilon = 1, 3, \ldots, m \\ \frac{2m-\epsilon-1}{2}; & \text{if } \epsilon = m+2, m+4, \ldots, 2m-5 \\ \frac{\epsilon}{2}; & \text{if } \epsilon = 2, 4, \ldots, m-1 \\ \frac{2m-\epsilon-2}{2}; & \text{if } \epsilon = m+1, m+3, \ldots, 2m-4. \end{cases}$$

$$d(v_\epsilon v_{\epsilon+1}, v_4) = \begin{cases} 1; & \text{if } \epsilon = 1, 2 \\ \frac{\epsilon-3}{2}; & \text{if } \epsilon = 3, 5, \ldots, m+2 \\ \frac{2m-\epsilon+3}{2}; & \text{if } \epsilon = m+4, m+6, \ldots, 2m-3 \\ \frac{\epsilon-4}{2}; & \text{if } \epsilon = 4, 6, \ldots, m+3 \\ \frac{2m-\epsilon+4}{2}; & \text{if } \epsilon = m+5, m+6, \ldots, 2m-5. \end{cases}$$

$$d(v_{2m-2}v_2, v_4) = d(v_{2m-3}v_2, v_4) = d(v_{2m-1}v_1, v_4) = 2.$$

$$d(v_\epsilon v_{\epsilon+2}, v_4) = \begin{cases} 2-\epsilon; & \text{if } \epsilon = 1,2 \\ \frac{\epsilon-1}{2}; & \text{if } \epsilon = 3,5,\ldots,m \\ \frac{2m-\epsilon+1}{2}; & \text{if } \epsilon = m+2, m+4, \ldots, 2m-5 \\ \frac{\epsilon-4}{2}; & \text{if } \epsilon = 4,6,\ldots,m+3 \\ \frac{2m-\epsilon+2}{2}; & \text{if } \epsilon = m+5, m+7, \ldots, 2m-4. \end{cases}$$

$$d(v_\epsilon v_{\epsilon+1}, v_5) = \begin{cases} \left|\frac{5-\epsilon}{2}\right|; & \text{if } \epsilon = 1,3,\ldots,m+4 \\ \frac{2m-\epsilon+5}{2}; & \text{if } \epsilon = m+6, m+8, \ldots, 2m-3 \\ \frac{4-\epsilon}{2}; & \text{if } \epsilon = 2,4 \\ \frac{\epsilon-4}{2}; & \text{if } \epsilon = 6,8,\ldots,m+5 \\ \frac{2m-\epsilon+3}{2}; & \text{if } \epsilon = m+7, m+9, \ldots, 2m-4. \end{cases}$$

$$d(v_{2m-2}v_2, v_5) = d(v_{2m-3}v_2, v_5) = d(v_{2m-1}v_1, v_5) = 3.$$

$$d(v_\epsilon v_{\epsilon+2}, v_5) = \begin{cases} \frac{3-\epsilon}{2}; & \text{if } \epsilon = 1,3 \\ \frac{\epsilon-5}{2}; & \text{if } \epsilon = 5,7,\ldots,m+4 \\ \frac{2m-\epsilon+3}{2}; & \text{if } \epsilon = m+6, m+8, \ldots, 2m-5 \\ 1; & \text{if } \epsilon = 2,4 \\ \frac{6-\epsilon}{2}; & \text{if } \epsilon = 6,8,\ldots,m+2 \\ \frac{2m-\epsilon+3}{2}; & \text{if } \epsilon = m+4, m+6, \ldots, 2m-4. \end{cases}$$

$$d(v_\epsilon v_{\epsilon+1}, v_{m+3}) = \begin{cases} \frac{m-3}{2}; & \text{if } \epsilon = 1 \\ \left|\frac{m-\epsilon+2}{2}\right|; & \text{if } \epsilon = 3,5,\ldots,2m-3 \\ \frac{m-1}{2}; & \text{if } \epsilon = 2 \\ \left|\frac{m-\epsilon+3}{2}\right|; & \text{if } \epsilon = 4,6\ldots,2m-4. \end{cases}$$

$$d(v_{2m-2}v_2, v_{m+3}) = d(v_{2m-1}v_1, v_{m+3}) = \frac{m-5}{2},\ d(v_{2m-3}v_2, v_{m+3}) = \frac{m-3}{2}.$$

$$d(v_\epsilon v_{\epsilon+2}, v_{m+3}) = \begin{cases} \frac{m-1}{2}; & \text{if } \epsilon = 1 \\ \frac{m-\epsilon+2}{2}; & \text{if } \epsilon = 3,5,\ldots,m \\ 1; & \text{if } \epsilon = m+2 \\ \frac{\epsilon-m-2}{2}; & \text{if } \epsilon = m+4, m+6, \ldots, 2m-5 \\ \frac{m-\epsilon+1}{2}; & \text{if } \epsilon = 2,4,\ldots,m+1 \\ \frac{\epsilon-m-3}{2}; & \text{if } \epsilon = m+3, m+5, \ldots, 2m-4. \end{cases}$$

$$d(v_\epsilon v_{\epsilon+1}, v_{m+4}) = \begin{cases} \frac{m-5}{2}; & \text{if } \epsilon = 1 \\ \frac{m-1}{2}; & \text{if } \epsilon = 3 \\ \left|\frac{m-\epsilon+4}{2}\right|; & \text{if } \epsilon = 5,7,\ldots,2m-3 \\ \frac{m-3}{2}; & \text{if } \epsilon = 2 \\ \left|\frac{m-\epsilon+3}{2}\right|; & \text{if } \epsilon = 4,6,\ldots,2m-4. \end{cases}$$

$$d(v_{2m-2}v_2, v_{m+4}) = d(v_{2m-1}v_1, v_{m+4}) = \frac{m-5}{2},\ d(v_{2m-3}v_2, v_{m+4}) = \frac{m-7}{2}.$$

$$d(v_\epsilon v_{\epsilon+2}, v_{m+4}) = \begin{cases} \frac{m-\epsilon-2}{2}; & \text{if } \epsilon = 1,3 \\ \frac{m-\epsilon+4}{2}; & \text{if } \epsilon = 5,7,\ldots,m+2 \\ \frac{\epsilon-m-4}{2}; & \text{if } \epsilon = m+4, m+6,\ldots,2m-5 \\ \frac{m-3}{2}; & \text{if } \epsilon = 2 \\ \frac{m-\epsilon+3}{2}; & \text{if } \epsilon = 4,6,\ldots,m+1 \\ 1; & \text{if } \epsilon = m+3 \\ \frac{\epsilon-m-3}{2}; & \text{if } \epsilon = m+5, m+7,\ldots,2m-4. \end{cases}$$

Now, the distances of vertices according to mixed metric generator are classified as follows:

When $m = 5$, the distance of the vertices v_1, v_4, v_5 to all other vertices will be given in later parts of proof.

$$d(v_\epsilon, v_2) = \begin{cases} 1; & \text{if } \epsilon = 1,3,4,7,8 \\ 2; & \text{if } \epsilon = 5,6 \\ 0; & \text{if } \epsilon = 2. \end{cases}$$

$$d(v_\epsilon, v_6) = \begin{cases} 2; & \text{if } \epsilon = 1,2,3 \\ 1; & \text{if } \epsilon = 4,5,7,8 \\ 0; & \text{if } \epsilon = 6. \end{cases}$$

When $m \geq 6$ (even), then

$$d(v_\epsilon, v_1) = \begin{cases} \frac{\epsilon-1}{2}; & \text{if } \epsilon = 1,3,\ldots,m+1 \\ \frac{2m-\epsilon+1}{2}; & \text{if } \epsilon = m+3, m+5,\ldots,2m-3 \\ \frac{\epsilon}{2}; & \text{if } \epsilon = 2,4,\ldots,m \\ \frac{2m-\epsilon}{2}; & \text{if } \epsilon = m+2, m+4,\ldots,2m-2. \end{cases}$$

$$d(v_\epsilon, v_3) = \begin{cases} 1; & \text{if } \epsilon = 1,2 \\ \frac{\epsilon-3}{2}; & \text{if } \epsilon = 3,5,\ldots,m+3 \\ \frac{2m-\epsilon+3}{2}; & \text{if } \epsilon = m+5, m+7,\ldots,2m-3 \\ \frac{\epsilon-2}{2}; & \text{if } \epsilon = 4,6,\ldots,m+2 \\ \frac{2m-\epsilon+2}{2}; & \text{if } \epsilon = m+4, m+6,\ldots,2m-2. \end{cases}$$

$$d(v_\epsilon, v_4) = \begin{cases} \frac{5-\epsilon}{2}; & \text{if } \epsilon = 1,3 \\ \frac{\epsilon-3}{2}; & \text{if } \epsilon = 5,7,\ldots,m+1 \\ \frac{2m-\epsilon+1}{2}; & \text{if } \epsilon = m+3, m+5,\ldots,2m-3 \\ \left|\frac{4-\epsilon}{2}\right|; & \text{if } \epsilon = 2,4,\ldots,m+4 \\ \frac{2m-\epsilon+4}{2}; & \text{if } \epsilon = m+6, m+8,\ldots,2m-2. \end{cases}$$

$$d(v_\epsilon, v_{m+1}) = \begin{cases} \left|\frac{m-\epsilon+1}{2}\right|; & \text{if } \epsilon = 1,3,\ldots,2m-3 \\ \frac{m-\epsilon+2}{2}; & \text{if } \epsilon = 2,4,\ldots,m \\ \frac{\epsilon-m}{2}; & \text{if } \epsilon = m+2, m+4,\ldots,2m-2. \end{cases}$$

$$d(v_\epsilon, v_{m+2}) = \begin{cases} \frac{m-2}{2}; & \text{if } \epsilon = 1 \\ \frac{m-\epsilon+3}{2}; & \text{if } \epsilon = 3,5,\ldots,m+1 \\ \frac{\epsilon-m}{2}; & \text{if } \epsilon = m+2, m+4,\ldots,2m-3 \\ \frac{m}{2}; & \text{if } \epsilon = 2 \\ \left|\frac{m-\epsilon+2}{2}\right|; & \text{if } \epsilon = 4,6,\ldots,2m-2. \end{cases}$$

When $m \geq 7$ (odd)

$$d(v_\epsilon, v_1) = \begin{cases} \frac{\epsilon-1}{2}; & \text{if } \epsilon = 1,3,\ldots,m \\ \frac{2m-\epsilon+1}{2}; & \text{if } \epsilon = m+2, m+4,\ldots,2m-3 \\ \frac{\epsilon}{2}; & \text{if } \epsilon = 2,4,\ldots,m-1 \\ \frac{2m-\epsilon}{2}; & \text{if } \epsilon = m+1, m+3,\ldots,2m-2. \end{cases}$$

$$d(v_\epsilon, v_4) = \begin{cases} \frac{5-\epsilon}{2}; & \text{if } \epsilon = 1,3 \\ \frac{\epsilon-3}{2}; & \text{if } \epsilon = 5,7,\ldots,m+2 \\ \frac{2m-\epsilon+1}{2}; & \text{if } \epsilon = m+4, m+6,\ldots,2m-3 \\ \left|\frac{4-\epsilon}{2}\right|; & \text{if } \epsilon = 2,4,\ldots,m+3 \\ \frac{2m-\epsilon+6}{2}; & \text{if } \epsilon = m+5, m+7,\ldots,2m-2. \end{cases}$$

$$d(v_\epsilon, v_5) = \begin{cases} \left|\frac{5-\epsilon}{2}\right|; & \text{if } \epsilon = 1,3,\ldots,m+4 \\ \frac{2m-\epsilon+7}{2}; & \text{if } \epsilon = m+6, m+8,\ldots,2m-3 \\ \frac{6-\epsilon}{2}; & \text{if } \epsilon = 2,4 \\ \frac{\epsilon-4}{2}; & \text{if } \epsilon = 6,8,\ldots,m+3 \\ \frac{2m-\epsilon+4}{2}; & \text{if } \epsilon = m+5, m+7,\ldots,2m-2. \end{cases}$$

$$d(v_\epsilon, v_{m+3}) = \begin{cases} \frac{m+\epsilon-4}{2}; & \text{if } \epsilon = 1,3 \\ \frac{m-\epsilon+4}{2}; & \text{if } \epsilon = 5,7,\ldots,m+2 \\ \frac{\epsilon-m-2}{2}; & \text{if } \epsilon = m+4, m+6,\ldots,2m-3 \\ \frac{m-3}{2}; & \text{if } \epsilon = 2 \\ \left|\frac{m-\epsilon+3}{2}\right|; & \text{if } \epsilon = 4,6,\ldots,2m-2. \end{cases}$$

$$d(v_\epsilon, v_{m+4}) = \begin{cases} \frac{m-3}{2}; & \text{if } \epsilon = 1,3 \\ \left|\frac{m-\epsilon+4}{2}\right|; & \text{if } \epsilon = 5,7,\ldots,2m-3 \\ \frac{\epsilon+m-7}{2}; & \text{if } \epsilon = 2,4 \\ \frac{m-\epsilon+5}{2}; & \text{if } \epsilon = 6,8,\ldots,m+3 \\ \frac{\epsilon-m-3}{2}; & \text{if } \epsilon = m+5, m+7,\ldots,2m-2. \end{cases}$$

In the previously described distances, there are no two elements (vertices and edges) have same representations under the mixed metric generator Q_m. It is shown that $dim_m(TML_m) \leq 5$. Now, a brief discussion for $dim_m(TML_m) = 4$, which makes the minimum number of elements of the chosen mixed metric generator $Q_m^{'}$ equal to four.

If all four nodes have a gap of every arbitrary length, we can choose like, $Q_m^{'} = \{v_i, v_j, v_k, v_l\}$ with $1 \leq i, j, k, l \leq 2m-2$. Then it concludes to the identical representation in the edges which are $d\left(v_1v_2|Q_m^{'}\right) = d\left(v_1v_{2m-1}|Q_m^{'}\right)$ where indices are odd and if indices are even then the identical representation in edge either $d\left(v_1v_2|Q_m^{'}\right) = d\left(v_2v_3|Q_m^{'}\right)$or $d\left(v_pv_{p+1}|Q_m^{'}\right) = d\left(v_qv_{q+1}|Q_m^{'}\right)$ where $1 \leq p$ (odd) $\leq 2m-3$ and $1 \leq q$ (even) $\leq 2m-4$, to generalize this concept and taking vertices $v_i, v_j, v_k, v_l \in V(TML_m)$ then it implies to the identical representations in the edges which are either $d\left(v_pv_{p+1}|Q_m^{'}\right) = d\left(v_qv_{q+1}|Q_m^{'}\right)$ where $1 \leq p$ (odd) $\leq 2m-3$ and $1 \leq q$ (even) $\leq 2m-4$ or $d\left(v_i|Q_m^{'}\right) = d\left(v_pv_{p+2}|Q_m^{'}\right)$, where i = even and $2 \leq p$ (even) $\leq 2m-4$ or $d\left(v_i|Q_m^{'}\right) = d\left(v_pv_{p+2}|Q_m^{'}\right)$ where i = odd and $1 \leq p$ (odd) $\leq 2m-5$, then it is concluded that we can not take vertices in mixed metric generator with cardinality four. Thus,

$$dim_m(TML_m) = 5.$$

□

The edge metric dimension of a ladder network is two. By the relation mixed metric dimension can be two or more. In the following result, we show that the mixed metric dimension of ladder network is indeed three.

Theorem 2.5. *Let L_m be a ladder network with $m \geq 2$. Then*

$$dim_m(L_m) = 3.$$

Proof. Suppose that the mixed metric generator is $Q_m = \{v_1, v_2, v_{2m-1}\}$. The labeling of vertices is shown in Figure 5.

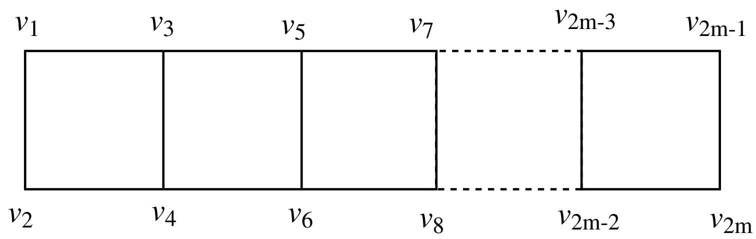

Figure 5. Ladder Graph L_m.

To prove that $dim_m(L_m) = 3$, we use the method of the double inequality. First, we verify that $dim_m(L_m) \leq 3$ by moving towards to show the distances of all edges according to mixed metric generator:

$$d\left(v_\epsilon v_{\epsilon+1}|Q_m\right) = \left(\frac{\epsilon-1}{2}, \frac{\epsilon-1}{2}, m-1-\frac{\epsilon-1}{2}\right); \quad 1 \leq \epsilon \text{ (odd)} \leq 2m-1$$

$$d\left(v_\epsilon v_{\epsilon+2}|Q_m\right) = \left(\frac{\epsilon-1}{2}, \frac{\epsilon+1}{2}, m-2-\frac{\epsilon-1}{2}\right); \quad 1 \leq \epsilon \text{ (odd)} \leq 2m-3$$

$$d\left(v_{\epsilon}v_{\epsilon+1}|Q_m\right)=\left(\frac{\epsilon}{2},\ \frac{\epsilon-2}{2},\ m-\frac{\epsilon}{2}\right);\quad 2\leq\epsilon\ (\text{even})\leq 2m-2$$

Vertices representations according to selected mixed metric generator;

$$d\left(v_{\epsilon}|Q_m\right)=\begin{cases}\left(\frac{\epsilon-1}{2},\ \frac{\epsilon+1}{2},\ m-\frac{\epsilon+1}{2}\right) & \text{if } \epsilon=1,3,\ldots,2m-1;\\ \left(\frac{\epsilon}{2},\ \frac{\epsilon-2}{2},\ m-\frac{\epsilon-2}{2}\right) & \text{if } \epsilon=2,4,\ldots,2m.\end{cases}$$

It is easy to see that the distances between all edges and vertices based on the chosen mixed metric generator Q_m are fulfilling the requirement of definition which means that $dim_m(L_m)\leq 3$. On the contrary, we choose the cardinality of mixed metric generator $Q_m^{'}$ to be two. Then the following discussion verifies the contradiction.

Now, it is possible that by taking $Q_m^{'}$ with cardinality 2, then either in two vertices or two edges have the same representations and it is also possible that an edge have the same distance according to $Q_m^{'}$ with a particular vertex. For example, if we take $Q_m^{'}=\{v_1,v_2\}$, then $d\left(v_1|Q_m^{'}\right)=d\left(v_1v_3|Q_m^{'}\right)$.

Case 2.1. *If we fix the first vertex v_1 and choose another vertex v_i with $3\leq i\ (odd)\leq 2m-3$ which means that $Q_m^{'}=\{v_1,v_i\}$, then the edges v_iv_{i+1} & v_iv_{i+2} have the same representation with respect to $Q_m^{'}$. If $i=2m-1$, then the vertex v_1 and the edge v_1v_2 have the same representation with respect to $Q_m^{'}$. Therefore, it is verified that the cardinality of mixed metric generator set cannot be equal to two.*

Case 2.2. *If two of vertices with moving indices with any kind of arbitrarily chosen gap size, we can choose $Q_m^{'}=\{v_i,v_j\}$ with $1\leq i\ (odd)\leq 2m-1$, then it follows that the identical representation in the edges which are $d\left(v_1v_2|Q_m^{'}\right)=d\left(v_2v_4|Q_m^{'}\right)$. it is verified that we can not take vertices in mixed metric generator with cardinality two.*

Case 2.3. *Now, taking even index vertices with fixed vertex v_2 can choose like, $Q_m^{'}=\{v_2,v_j\}$ with $2\leq i\ (even)\leq 2m-4$ then it implies to the identical representation in the edges which are $d\left(v_{i-1}v_{i+1}|Q_m^{'}\right)=d\left(v_{i+1}v_{i+2}|Q_m^{'}\right)$ and when $j=2m-2,2m$ then the identical representation in a vertex with and edge, $d\left(v_2|Q_m^{'}\right)=d\left(v_1v_3|Q_m^{'}\right)$, it is verified that we can not take vertices in mixed metric generator with cardinality two.*

Case 2.4. *If both of two vertices from even and moving indices with any kind of arbitrarily chosen gap size can choose like, $Q_m^{'}=\{v_i,v_j\}$ with $2\leq i,j\leq 2m$ then it implies to the identical representation in the edges which are $d\left(v_1v_2|Q_m^{'}\right)=d\left(v_1v_3|Q_m^{'}\right)$ and it is verified that we can not take vertices in mixed metric generator with cardinality two.*

Case 2.5. *Take the vertices $v_i,v_j\in V(L_m)$ without choosing the size of gap, then there exist either two edges with distances or an edge with vertex i.e; $d\left(v_1v_2|Q_m^{'}\right)=d\left(v_1v_3|Q_m^{'}\right)$ or $d\left(v_1v_3|Q_m^{'}\right)=d\left(v_3v_4|Q_m^{'}\right)$ or $d\left(v_3v_4|Q_m^{'}\right)=d\left(v_4|Q_m^{'}\right)$ or $d\left(v_1v_2|Q_m^{'}\right)=d\left(v_2v_4|Q_m^{'}\right)$ or $d\left(v_3v_5|Q_m^{'}\right)=d\left(v_5v_6|Q_m^{'}\right)$ or $d\left(v_2|Q_m^{'}\right)=d\left(v_1v_3|Q_m^{'}\right)$ and finally verified that we can not take any type of vertices with any gap-size in the mixed metric generator with cardinality two, which implies to the contradiction that $dim_m(L_m)\neq 2$. Therefore, this proves the double inequality which is*

$$dim_m(L_m)=3.$$

□

3. Conclusions

Metric dimension or metric basis is a concept in which the whole vertex set of a structure is uniquely identified with a chosen subset named as locating set. Edge Metric dimension or edge metric basis is a concept in which the whole edge set of a structure is uniquely identified with a chosen subset from vertex set and it is named as edge locating set. Mixed-metric basis or mixed-metric dimension is a generalized version, in which the whole vertex and edge set of a structure is uniquely identified with a chosen subset from vertex set and it is named as edge mixed-metric locating set. In this comparative study, we discuss the relationships between metric, edge metric, and mixed metric dimension for different families of graphs. Precisely, the obtained results demonstrate the mixed metric dimension of some interesting graphs including Möbius ladder, hexagonal Möbius ladder, triangular Möbius ladder networks, in the given Table 1.

Table 1. Relation of metric parameters between graphs.

G	dim	dim_e	dim_m
L_m	2	2	3
ML_m	3	4	4
HML_m	3	3	4
TL_m	3	4	4
TML_m	3	4	5

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Conflict of interest

The authors have no conflicts of interest to declare.

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