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Research article

Ulam stability of hom-ders in fuzzy Banach algebras

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Abstract: This paper aims to investigate a new type of derivations in a fuzzy Banach algebra. Moreover, by using the fixed point method, we obtain some stability results of the hom-der in fuzzy Banach algebras associated with the functional equation

$$f(x + \mathbf{k}y) = f(x) + \mathbf{k}f(y)$$

where $\mathbf{k}$ is a fixed positive integer greater than 1.

Keywords: Ulam stability result; hom-der; fuzzy Banach algebra; fixed point method
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1. Introduction and preliminaries

In the sequel, $\mathbb{C}, \mathbb{N}$, and $\mathbb{N}_0$ denote the sets of complex numbers, of positive integers, and of non-negative integers, respectively.

The following stability problem of group homomorphisms was raised by Ulam [27]:

> *Suppose that $(G, *)$ is a group and $(H, +, d)$ a metric group (a group with a metric). Given $\varepsilon > 0$, does there always exist a $\delta := \delta(\varepsilon) > 0$ such that if $f : G \to H$ is a mapping that satisfies $d(f(x * y), f(x) + f(y)) < \delta$ for all $x, y \in G$, then there exists a group homomorphism $F : G \to H$ such that $d(F(x), f(x)) < \varepsilon$ for all $x \in G$?*

In 1941, Hyers [10] presented the first affirmative partial answer where the groups are Banach spaces. The method used by Hyers is now called the *direct method*. Using this method, Aoki [1] generalized

Hyers' result in a natural way. Also, Rassias [23] independently investigated a generalization of Hyers' result related to that of Aoki. To compliment the result of Aoki (and also Rassias), Gajda [9] presented a stability result and an interesting counterexample concerning the *unstable* result. Găvruta [8] again presented some generalizations of Rassias' result.

The stability of functional equations is quite related to Ulam's problem. Nowadays, many authors study the stability of various functional equations in different structures (see [12, 18, 25, 26]). A new look for proving the stability results seems to be pioneered by Radu [22] (see also [4, 5]). In fact, the stability results of some functional equations can be regarded as a consequence of a fixed point alternative. Using this notion, a number of stability papers have been rapidly published. It is important to recall the *fixed point result* inverstigated by Diaz and Margolis [7] here. By a *generalized metric* on a non-empty set X, we mean a function $d : X \times X \to [0, +\infty]$ satisfying the following metric space axioms: For every $x, y, z \in X$, $d(x, y) = 0$ if and only if $x = y$; $d(x, y) = d(y, x)$; $d(x, z) \leq d(x, y) + d(y, z)$.

The following result is one of fundamental theorems in fixed point theory.

Theorem 1.1 ([7]). *Let (S, d) be a complete generalized metric space and let $\mathcal{J} : S \to S$ be a contraction with a positive constant $L < 1$. Then, for each $s \in S$, either $d(\mathcal{J}^n s, \mathcal{J}^{n+1} s) = +\infty$ for all $n \in \mathbb{N}_0$ or there exists an $n_0 \in \mathbb{N}_0$ satisfying:*

i) *$d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) < +\infty$ for all $n \geq n_0$;*

ii) *$\{\mathcal{J}^n x\}_n$ is convergent and $\lim_{n\to+\infty} \mathcal{J}^n s = s^*$ where s^* is a fixed point of $\mathcal{J}$;*

iii) *s^* is the unique fixed point of $\mathcal{J}$ in the set $Y := \{y \in X : d(\mathcal{J}^{n_0} s, y) < +\infty\}$;*

iv) *$d(y, s^*) \leq \frac{1}{1-L} d(y, \mathcal{J} y)$ for all $y \in Y$.*

Many papers concerning the stability of functional equations based on Theorem 1.1 can be seen for instance in [4, 5, 13, 19, 20].

Due to the fixed point method, Isar et al. [11] studied Hyers-Ulam stability of homomorphisms in Banach algebras and Banach Lie algebras. Moreover, they also presented some stability results concerning derivations on Banach algebras and Banach Lie algebras associated with the general additive functional equation:

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

In this work, we now introduce the notion of a *hom-der* (see Definition 1.6) in a fuzzy complex Banach algebra and derive some stability results. In fact, we also deal with the functional equation

$$f(x + \mathbf{k} y) = f(x) + \mathbf{k} f(y)$$

where $\mathbf{k}$ is a fixed positive integer greater than 1. It is easily seen that the functional equation is equivalent to the well-known *additive* functional equation. To define our certain type of derivations, we begin with recalling the following basic notions of a *fuzzy normed space* as follows:

Definition 1.2 ([3, 17]). (Fuzzy normed space) Let X be a linear vector space (over $\mathbb{R}$ or $\mathbb{C}$). A function $N : X \times \mathbb{R} \to [0, 1]$ is called a *fuzzy norm* on X if the following conditions are satisfied: for every $x, y \in X$,

(N1) $N(x, t) = 0$ for all $(x, t) \in X \times (-\infty, 0]$;

(N2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;

(N3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;

(N4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$ for all $s, t \in \mathbb{R}$;

(N5) $N(x, \cdot)$ is non-decreasing function on $\mathbb{R}$ and $\lim_{t\to+\infty} N(x, t) = 1$;

(N6) $N(x, \cdot)$ is continuous on $\mathbb{R}$ for each fixed non-zero vector $x \in X$.

If N forms a fuzzy norm on X, then (X, N) is called a *fuzzy normed space*.

It is known that every normed space can be viewed as a fuzzy normed space together with the fuzzy norm N defined by

$$N(x, t) := \begin{cases} 0 & \text{if } x \in X \text{ and } t \leq 0; \\ \frac{t}{t+\|x\|} & \text{if } x \in X \text{ and } t > 0. \end{cases}$$

Definition 1.3 ([2, 3]). (Convergence) A sequence $\{x_n\}$ in a fuzzy normed space (X, N) is said to be *convergent* if there exists $x \in X$ such that

$$\lim_{n\to+\infty} N(x_n - x, t) = 1 \quad \text{for all } t > 0.$$

In this case, x is called the *limit* of the sequence $\{x_n\}$ and is denoted by $N\text{-}\lim_{n\to+\infty} x_n$.

It is natural to define a *fuzzy Banach space* to be a fuzzy normed space which is every Cauchy (see [3, Definition 2.8]) sequence converges.

Definition 1.4 ([3, 16]). Let X be an algebra. A fuzzy normed space (X, N) is called a *fuzzy normed algebra* if

$$N(xy, st) \geq N(x, s)N(y, t)$$

for all $x, y \in X$ and all $s, t \in \mathbb{R}$.

Remark 1.5. It can be seen in [3, Theorem 3.7] that the multiplication is continuous. Also, the addition is trivially continous.

Some interesting papers concerning stability results on fuzzy normed spaces can be seen in [6, 14, 15, 24].

Now, we are ready to investigate the notion of the *hom-der* in a fuzzy complex Banach algebra as in the following definition.

Definition 1.6. Let $\mathcal{A}$ be a complex algebra. A $\mathbb{C}$-linear mapping $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ is called a *hom-der* if it satisfies

$$\mathcal{D}(x)\mathcal{D}(y) = \mathcal{D}(x)y + x\mathcal{D}(y) \quad \text{for all } x, y \in \mathcal{A}.$$

Hence, this paper aims to prove some stability results of the hom-der in a fuzzy complex Banach algebra by using the fixed point theorem (Theorem 1.1).

Throughout this paper, we assume that (X, N) is a fuzzy (complex) Banach algebra if otherwise stated.

2. Stability of hom-ders in fuzzy Banach algebras

Firstly, let us fix $\mathbf{k} \in \mathbb{N} - \{1\}$ and define $\mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

Theorem 2.1. *Let $\varphi : X \times X \to [0, +\infty)$ be a function and suppose that there exists a positive real number $L < 1$ satisfying*

$$\varphi\left(\frac{x}{\mathbf{k}}, \frac{y}{\mathbf{k}}\right) \leq \frac{L}{\mathbf{k}^2}\varphi(x, y) \quad \text{for all } x, y \in X. \tag{2.1}$$

Suppose that $f : X \to X$ is a mapping that satisfies the following inequalities:

$$N\left(f(\lambda x + \mathbf{k}y) - \lambda f(x) - \mathbf{k}f(y), t\right) \geq \frac{t}{t + \varphi(x, y)}, \tag{2.2}$$

$$N\left(f(x)f(y) - f(x)y - xf(y), t\right) \geq \frac{t}{t + \varphi(x, y)}, \tag{2.3}$$

for all $x, y \in X, t > 0$, and all $\lambda \in \mathbb{T}^1$. Then there exists a unique hom-der $\mathcal{H} : X \to X$ such that

$$N\left(\mathcal{H}(x) - f(x), t\right) \geq \frac{(\mathbf{k} - \mathbf{k}L)t}{(\mathbf{k} - \mathbf{k}L)t + L\varphi(0, x)} \quad \text{for all } x \in X, t > 0. \tag{2.4}$$

Proof. We first note that $\varphi(0, 0) \leq L\varphi(0, 0)$. Since $L < 1$, it follows that $\varphi(0, 0) = 0$ and hence

$$N\left(f(0), \frac{t}{\mathbf{k}}\right) = N(-\mathbf{k}f(0), t) \geq \frac{t}{t} = 1 \quad \text{for all } t > 0.$$

By Definition 1.2(N2), we have $\mathbf{k}f(0) = 0$ and then $f(0) = 0$.

Letting $x = 0$ and $\lambda = 1$ in (2.2), we get that

$$N\left(f(\mathbf{k}y) - \mathbf{k}f(y), t\right) \geq \frac{t}{t + \varphi(0, y)} \quad \text{for all } y \in X \text{ and } t > 0. \tag{2.5}$$

We define $S := \{g \in X^X : g(0) = 0\}$ and a generalized metric $d : S \times S \to [0, +\infty]$ by

$$d(g, h) := \inf\left\{\delta \in \mathbb{R}_+ : N(g(x) - h(x), \delta t) \geq \frac{t}{t + \varphi(0, x)} \text{ for all } x \in X \text{ and all } t > 0\right\},$$

where $\inf \emptyset = +\infty$. Then a couple (S, d) forms a complete generalized metric space.

Now, we consider the mapping $\mathcal{J} : S \to S$ defined by, for each $g \in S$,

$$(\mathcal{J}g)(x) := \mathbf{k}g\left(\frac{x}{\mathbf{k}}\right) \quad \text{for all } x \in X.$$

Let $g, h \in S$ be given and $\theta := d(g, h) < +\infty$. Then we see that

$$N\left(g(x) - h(x), \theta t\right) \geq \frac{t}{t + \varphi(0, x)} \quad \text{for all } x \in X \text{ and } t > 0.$$

So, for any $x \in X$ and $t > 0$ we have

$$
\begin{aligned}
N\left((\mathcal{J}g)(x)-(\mathcal{J}h)(x),(L\theta)t\right) &= N\left(\mathbf{k}g\left(\frac{x}{\mathbf{k}}\right)-\mathbf{k}h\left(\frac{x}{\mathbf{k}}\right),(L\theta)t\right)\\
&= N\left(g\left(\frac{x}{\mathbf{k}}\right)-h\left(\frac{x}{\mathbf{k}}\right),\theta\left(\frac{Lt}{\mathbf{k}}\right)\right)\geq \frac{\frac{Lt}{\mathbf{k}}}{\frac{Lt}{\mathbf{k}}+\varphi\left(0,\frac{x}{\mathbf{k}}\right)}\geq \frac{t}{t+\varphi\left(0,x\right)}.
\end{aligned}
$$

This means that $d(\mathcal{J}g,\mathcal{J}h)\leq Ld(g,h)$ for all $g,h\in S$, that is, $\mathcal{J}$ is a strictly contractive mapping with the Lipschitz constant $L<1$. We also note from (2.5) that

$$
N\left(f(x),(\mathcal{J}f)(x),\frac{L}{\mathbf{k}}t\right)=N\left(f(x)-\mathbf{k}f\left(\frac{x}{\mathbf{k}}\right),\frac{L}{\mathbf{k}}t\right)\geq\frac{t}{t+\varphi(0,x)}
$$

for all $x\in X$ and all $t>0$. So, we can see that $d(f,\mathcal{J}f)\leq\frac{L}{\mathbf{k}}<+\infty$. By Theorem 1.1, there exists a mapping $\mathcal{H}:X\to X$ such that the following three assertions are true:

- $\mathcal{H}$ is a fixed point of the mapping $\mathcal{J}$, that is,

$$
\mathcal{H}(x)=(\mathcal{J}\mathcal{H})(x)=\mathbf{k}\mathcal{H}\left(\frac{x}{\mathbf{k}}\right)\quad\text{for all } x\in X.
$$

- $\lim_{n\to+\infty}d(\mathcal{J}^nf,\mathcal{H})=0$. This means that

$$
\mathcal{H}(x)=N\text{-}\lim_{n\to+\infty}(\mathcal{J}^nf)(x)=N\text{-}\lim_{n\to+\infty}\mathbf{k}^nf\left(\frac{x}{\mathbf{k}^n}\right)\quad\text{for all } x\in X.
$$

- $d(\mathcal{H},f)\leq\frac{1}{1-L}d(f,\mathcal{J}f)\leq\frac{L}{\mathbf{k}-\mathbf{k}L}$. It follows from Definition 1.2(N5) that

$$
\begin{aligned}
N\left(\mathcal{H}(x)-f(x),t\right) &= N\left(\mathcal{H}(x)-f(x),\frac{L}{\mathbf{k}(1-L)}\left(\frac{\mathbf{k}(1-L)}{L}t\right)\right)\\
&\geq N\left(\mathcal{H}(x)-f(x),d(\mathcal{H},f)\left(\frac{\mathbf{k}(1-L)}{L}t\right)\right)\\
&\geq \frac{\frac{\mathbf{k}(1-L)}{L}t}{\frac{\mathbf{k}(1-L)}{L}t+\varphi(0,x)}\\
&= \frac{(\mathbf{k}-\mathbf{k}L)t}{(\mathbf{k}-\mathbf{k}L)t+L\varphi(0,x)}
\end{aligned}
$$

  for all $x\in X$ and all $t>0$. This implies that the inequality (2.4) holds.

Next, we show that $\mathcal{H}$ is a $\mathbb{C}$-linear mapping. To verify this, let $x,y\in X,\lambda\in\mathbb{T}^1$, and $n\in\mathbb{N}_0$ be given. We see from (2.2) and (2.1) that

$$
\begin{aligned}
&N\left(\mathbf{k}^nf\left(\frac{\lambda x+\mathbf{k}y}{\mathbf{k}^n}\right)-\lambda\mathbf{k}^nf\left(\frac{x}{\mathbf{k}^n}\right)-\mathbf{k}^{n+1}f\left(\frac{y}{\mathbf{k}^n}\right),t\right)\\
=&N\left(f\left(\frac{\lambda x}{\mathbf{k}^n}+\mathbf{k}\left(\frac{y}{\mathbf{k}^n}\right)\right)-\lambda f\left(\frac{x}{\mathbf{k}^n}\right)-\mathbf{k}f\left(\frac{y}{\mathbf{k}^n}\right),\frac{t}{\mathbf{k}^n}\right)\\
\geq&\frac{\frac{t}{\mathbf{k}^n}}{\frac{t}{\mathbf{k}^n}+\varphi\left(\frac{x}{\mathbf{k}^n},\frac{y}{\mathbf{k}^n}\right)}\geq\frac{t}{t+L^n\varphi\left(x,y\right)}.
\end{aligned}
$$

Since $\lim_{n\to+\infty} L^n\varphi(x,y) = 0$ for all $x, y \in X$, one gets from Definition 1.2(N5) that

$$N(\mathcal{H}(\lambda x + \mathbf{k}y) - \lambda\mathcal{H}(x) - \mathbf{k}\mathcal{H}(y), t) = 1 \quad \text{for all } x, y \in X \text{ and all } t > 0.$$

Thus, $\mathcal{H}(\lambda x + \mathbf{k}y) = \lambda\mathcal{H}(x) + \mathbf{k}\mathcal{H}(y)$ for all $x, y \in X, \lambda \in \mathbb{T}^1$. It is easy to see that $\mathcal{H}$ is an additive mapping that satisfies $\mathcal{H}(\lambda x) = \lambda\mathcal{H}(x)$ for all $x \in X$ and all $\lambda \in \mathbb{T}^1$. Under the same reason as in the proof of [21, Theorem 2.1], one can see that $\mathcal{H}$ is $\mathbb{C}$-linear, as desired.

To show that $\mathcal{H}$ is a hom-der, we see from (2.3) that

$$\begin{aligned}
&N\left(\mathbf{k}^{2n} f\left(\frac{x}{\mathbf{k}^n}\right) f\left(\frac{y}{\mathbf{k}^n}\right) - \mathbf{k}^n f\left(\frac{x}{\mathbf{k}^n}\right)\cdot\frac{\mathbf{k}^n y}{\mathbf{k}^n} - \frac{\mathbf{k}^n x}{\mathbf{k}^n}\cdot\mathbf{k}^n f\left(\frac{y}{\mathbf{k}^n}\right), t\right)\\
=&N\left(f\left(\frac{x}{\mathbf{k}^n}\right) f\left(\frac{y}{\mathbf{k}^n}\right) - f\left(\frac{x}{\mathbf{k}^n}\right)\cdot\frac{y}{\mathbf{k}^n} - \frac{x}{\mathbf{k}^n}\cdot f\left(\frac{y}{\mathbf{k}^n}\right), \frac{t}{\mathbf{k}^{2n}}\right)\\
\geq&\frac{\frac{t}{\mathbf{k}^{2n}}}{\frac{t}{\mathbf{k}^{2n}} + \varphi\left(\frac{x}{\mathbf{k}^n}, \frac{y}{\mathbf{k}^n}\right)}\\
\geq&\frac{t}{t + L^n\varphi(x,y)}
\end{aligned} \tag{2.6}$$

for all $x, y \in X, t > 0$, and $n \in \mathbb{N}_0$. Since $\lim_{n\to+\infty} L^n\varphi(x,y) = 0$ for all $x, y \in X$ and all $t > 0$, Definition 1.2(N5) and Definition 1.4 imply that

$$N(\mathcal{H}(x)\mathcal{H}(y) - \mathcal{H}(x)y - x\mathcal{H}(y), t) = 1 \quad \text{for all } x \in X \text{ and all } t > 0.$$

Thus $\mathcal{H}(x)\mathcal{H}(y) - \mathcal{H}(x)y - x\mathcal{H}(y) = 0$. So, the mapping $\mathcal{H} : X \to X$ is a hom-der.

We finally prove that $\mathcal{H}$ is unique hom-der in the sense that (2.4) holds. To see this, suppose that there exists a hom-der $\mathcal{H}' : X \to X$ such that

$$N(\mathcal{H}'(x) - f(x), t) \geq \frac{t}{t + K\varphi(0,x)} \quad \text{for all } x \in X, t > 0 \tag{2.7}$$

for some real number $K \geq 0$. This shows that $d(\mathcal{H}', f) \leq K < +\infty$. Since $\mathcal{H}'$ is additive, we have $\mathcal{H}'(\mathbf{k}x) = \mathbf{k}\mathcal{H}'(x)$ for all $x \in X$. This implies that $\mathcal{J}\mathcal{H}' = \mathcal{H}'$. Hence Theorem 1.1 shows that $\mathcal{H}' = \mathcal{H}$.

This completes the proof. □

By defining the function $\varphi : X \times X \to [0, +\infty)$ in a particular way, we obtain the following corollary.

Corollary 2.2. *Let $\eta : X \to [0, +\infty)$ be a function such that $\eta(\ell x) = |\ell|^p\eta(x)$ for all $x \in X$ and all $\ell \in \mathbb{R}$, for some $p \in (2, +\infty)$. Suppose that $f : X \to X$ is a mapping that satisfies the following inequalities:*

$$\begin{aligned}
N(f(\lambda x + \mathbf{k}y) - \lambda f(x) - \mathbf{k}f(y), t) &\geq \frac{t}{t + \eta(x) + \eta(y)},\\
N(f(x)f(y) - f(x)y - xf(y), t) &\geq \frac{t}{t + \eta(x) + \eta(y)},
\end{aligned}$$

for all $x, y \in X, t > 0$, and all $\lambda \in \mathbb{T}^1$. Then there exists a unique hom-der $\mathcal{H} : X \to X$ such that

$$N(\mathcal{H}(x) - f(x), t) \geq \frac{(\mathbf{k}^p - \mathbf{k})t}{(\mathbf{k}^p - \mathbf{k}^2)t + \mathbf{k}\eta(x)} \quad \textit{for all } x \in X, t > 0.$$

Proof. We first define $\varphi : X \times X \to [0, +\infty)$ by

$$\varphi(x, y) := \eta(x) + \eta(y) \quad \text{for all } x, y \in X.$$

For any $x, y \in X$, we see that

$$\varphi\left(\frac{x}{\mathbf{k}}, \frac{y}{\mathbf{k}}\right) = \frac{1}{\mathbf{k}^p}\varphi(x, y) = \frac{\mathbf{k}^{2-p}}{\mathbf{k}^2}\varphi(x, y).$$

Hence we let $L := \mathbf{k}^{p-2}$. Note that $L < 1$ since $p > 2$. By Theorem 2.1, there exists a unique hom-der $\mathcal{H} : X \to X$ such that

$$N(\mathcal{H}(x) - f(x), t) \geq \frac{(\mathbf{k} - \mathbf{k}L)t}{(\mathbf{k} - \mathbf{k}L)t + L\varphi(0, x)} = \frac{(\mathbf{k}^p - \mathbf{k})t}{(\mathbf{k}^p - \mathbf{k}^2)t + \mathbf{k}\eta(x)}$$

for all $x \in X$.

This completes the proof. □

Remark 2.3. According to Theorem 2.1, we have the following remarkable observations:

1) It suffices to prove the existence of the $\mathbb{C}$-linear mapping $\mathcal{H}$ by only assuming that $\varphi\left(\frac{x}{\mathbf{k}}, \frac{y}{\mathbf{k}}\right) \leq \frac{L}{\mathbf{k}}\varphi(x, y)$ for all $x, y \in X$.
2) We note from (2.6) that the result still holds if we replace the expression $\frac{t}{t+\varphi(x,y)}$ in (2.3) by the expression $\phi(x, y, t)$ where $\phi : X \times X \times (0, +\infty) \to [0, 1]$ satisfies $\lim_{n\to+\infty} \phi\left(\frac{x}{\mathbf{k}^n}, \frac{y}{\mathbf{k}^n}, \frac{t}{\mathbf{k}^{2n}}\right) = 1$ for all $x, y \in X, t > 0$.

Hence we have the following proposition.

Proposition 2.4. *Let $\varphi : X \times X \to [0, +\infty)$ be a function and suppose that there exists a positive real number $L < 1$ satisfying*

$$\varphi\left(\frac{x}{\mathbf{k}}, \frac{y}{\mathbf{k}}\right) \leq \frac{L}{\mathbf{k}}\varphi(x, y) \quad \textit{for all } x, y \in X.$$

Suppose that $f : X \to X$ is a mapping that satisfies

$$N(f(\lambda x + \mathbf{k}y) - \lambda f(x) - \mathbf{k}f(y), t) \geq \frac{t}{t + \varphi(x, y)},$$
$$N(f(x)f(y) - f(x)y - xf(y), t) \geq \phi(x, y, t),$$

for all $x, y \in X, t > 0$, and all $\lambda \in \mathbb{T}^1$ where $\phi : X \times X \times (0, +\infty) \to [0, 1]$ is a function satisfying

$$\lim_{n\to+\infty} \phi\left(\frac{x}{\mathbf{k}^n}, \frac{y}{\mathbf{k}^n}, \frac{t}{\mathbf{k}^{2n}}\right) = 1 \quad \textit{for all } x, y \in X \textit{ and all } t > 0.$$

Then there exists a unique hom-der $\mathcal{H} : X \to X$ such that

$$N(\mathcal{H}(x) - f(x), t) \geq \frac{(\mathbf{k} - \mathbf{k}L)t}{(\mathbf{k} - \mathbf{k}L)t + L\varphi(0, x)} \quad \textit{for all } x \in X, t > 0.$$

Proof. By following the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping $\mathcal{H} : X \to X$ such that (2.4) holds. In fact, $\mathcal{H}(x) = \lim_{n\to+\infty} \mathbf{k}^n f\left(\frac{x}{\mathbf{k}^n}\right)$ for all $x \in X$. We now show that $\mathcal{H}$ is a hom-der. Given $x, y \in X$ and $t > 0$, we see that

$$\begin{aligned}
&N\left(\mathbf{k}^{2n} f\left(\frac{x}{\mathbf{k}^n}\right) f\left(\frac{y}{\mathbf{k}^n}\right) - \mathbf{k}^n f\left(\frac{x}{\mathbf{k}^n}\right) \cdot \frac{\mathbf{k}^n y}{\mathbf{k}^n} - \frac{\mathbf{k}^n x}{\mathbf{k}^n} \cdot \mathbf{k}^n f\left(\frac{y}{\mathbf{k}^n}\right), t\right) \\
=&N\left(f\left(\frac{x}{\mathbf{k}^n}\right) f\left(\frac{y}{\mathbf{k}^n}\right) - f\left(\frac{x}{\mathbf{k}^n}\right) \cdot \frac{y}{\mathbf{k}^n} - \frac{x}{\mathbf{k}^n} \cdot f\left(\frac{y}{\mathbf{k}^n}\right), \frac{t}{\mathbf{k}^{2n}}\right) \\
\geq&\phi\left(\frac{x}{\mathbf{k}^n}, \frac{y}{\mathbf{k}^n}, \frac{t}{\mathbf{k}^{2n}}\right).
\end{aligned}$$

Letting $n \to +\infty$ gives the result.

This completes the proof. □

Corollary 2.5. *Let $p, q > 0$. Suppose that $\eta, \eta' : X \to [0, +\infty)$ are functions such that*

$$\eta(\ell x) = |\ell|^p \eta(x) \quad \textit{and} \quad \eta'(\ell x) = |\ell|^q \eta'(x) \quad \textit{for all } x \in X \textit{ and all } \ell \in \mathbb{R}.$$

Suppose that $f : X \to X$ is a mapping that satisfies

$$\begin{aligned}
N\left(f(\lambda x + \mathbf{k}y) - \lambda f(x) - \mathbf{k} f(y), t\right) &\geq \frac{t}{t + \eta(x) + \eta(y)}, \\
N\left(f(x)f(y) - f(x)y - x f(y), t\right) &\geq \frac{t}{t + \eta'(x) + \eta'(y)},
\end{aligned}$$

for all $x, y \in X, t > 0$, and all $\lambda \in \mathbb{T}^1$. If $p > 1$ and $q > 2$, then there exists a unique hom-der $\mathcal{H} : X \to X$ such that

$$N\left(\mathcal{H}(x) - f(x), t\right) \geq \frac{(\mathbf{k}^p - \mathbf{k})t}{(\mathbf{k}^p - \mathbf{k})t + \eta(x)} \quad \textit{for all } x \in X, t > 0.$$

Proof. To obtain the result, we define two functions $\varphi : X \times X \to [0, +\infty)$ and $\phi : X \times X \times (0, +\infty) \to [0, 1]$ by

$$\begin{aligned}
\varphi(x, y) &:= \eta(x) + \eta(y) \qquad \text{for all } x, y \in X; \\
\phi(x, y, t) &:= \frac{t}{t + \eta'(x) + \eta'(y)} \qquad \text{for all } x, y \in X, t > 0.
\end{aligned}$$

It is noted that, for any $x, y \in X$ and $t > 0$,

$$\begin{aligned}
\varphi\left(\frac{x}{\mathbf{k}}, \frac{y}{\mathbf{k}}\right) &= \frac{1}{\mathbf{k}^p} \varphi(x, y) = \frac{\mathbf{k}^{1-p}}{\mathbf{k}} \varphi(x, y), \\
\phi\left(\frac{x}{\mathbf{k}^n}, \frac{y}{\mathbf{k}^n}, \frac{t}{\mathbf{k}^{2n}}\right) &= \frac{\frac{t}{\mathbf{k}^{2n}}}{\frac{t}{\mathbf{k}^{2n}} + \eta'\left(\frac{x}{\mathbf{k}^n}\right) + \eta'\left(\frac{y}{\mathbf{k}^n}\right)} = \frac{t}{t + \mathbf{k}^{(2-q)n}\left(\eta'(x) + \eta'(y)\right)},
\end{aligned}$$

for all $n \in \mathbb{N}$. Note that $L := \mathbf{k}^{1-p} < 1$. Since $q > 2$, letting $n \to +\infty$ gives

$$\lim_{n\to+\infty} \phi\left(\frac{x}{\mathbf{k}^n}, \frac{y}{\mathbf{k}^n}, \frac{t}{\mathbf{k}^{2n}}\right) = 1.$$

This completes the proof. □

The following theorem is a supplement of Theorem 2.1.

Theorem 2.6. *Let $\varphi : X \times X \to [0, +\infty)$ be a function and suppose that there exists a positive real number $L < 1$ satisfying*

$$\varphi(\mathbf{k}x, \mathbf{k}y) \leq L\mathbf{k}\varphi(x, y) \quad \text{for all } x, y \in X.$$

If $f : X \to X$ is a function such that $f(0) = 0$ and satisfies two inequalities

$$N\left(f(\lambda x + \mathbf{k}y) - \lambda f(x) - \mathbf{k}f(y), t\right) \geq \frac{t}{t + \varphi(x, y)}, \tag{2.8}$$

$$N\left(f(x)f(y) - f(x)y - xf(y), t\right) \geq \frac{t}{t + \varphi(x, y)}, \tag{2.9}$$

for all $x, y \in X, t > 0$, and all $\lambda \in \mathbb{T}^1$, then there exists a unique hom-der $\mathcal{H} : X \to X$ such that

$$N\left(\mathcal{H}(x) - f(x), t\right) \geq \frac{(1 - L)t}{(1 - L)t + \varphi(0, x)} \quad \text{for all } x \in X, t > 0. \tag{2.10}$$

Proof. Since $f(0) = 0$, we have by letting $\lambda = 1$ in (2.8) that

$$N\left(f(y) - \frac{1}{\mathbf{k}}f(\mathbf{k}y), \frac{t}{\mathbf{k}}\right) \geq \frac{\frac{t}{\mathbf{k}}}{\frac{t}{\mathbf{k}} + \varphi(0, y)} = \frac{t}{t + \mathbf{k}\varphi(0, y)}$$

for all $x, y \in X$ and $t > 0$. Equivalently,

$$N\left(f(y) - \frac{1}{\mathbf{k}}f(\mathbf{k}y), t\right) \geq \frac{t}{t + \varphi(0, y)} \quad \text{for all } y \in X, t > 0.$$

By applying the fixed point theorem, we now consider the complete generalized metric space (S, d) defined as in the proof of Theorem 2.1.

Also, we define the mapping $\mathcal{S} : S \to S$ by, for each $g \in S$,

$$(\mathcal{S}g)(x) := \frac{1}{\mathbf{k}}g(\mathbf{k}y) \quad \text{for all } x \in X.$$

It is not hard to verify that $\mathcal{S}$ is a strict contraction with the Lipschitz constant $L < 1$. We also note that $d(f, \mathcal{S}f) \leq 1$. It follows from Theorem 1.1 that there exists a mapping $\mathcal{H} : X \to X$ such that $\mathcal{H}(\mathbf{k}x) = \mathbf{k}\mathcal{H}(x)$ for all $x \in X$ and $\mathcal{H}$ is determined by

$$\mathcal{H}(x) = N\text{-}\lim_{n \to +\infty} \frac{1}{\mathbf{k}^n} f(\mathbf{k}^n x) \quad \text{for all } x \in X.$$

Moreover, we also have $d(\mathcal{H}, f) \leq \frac{1}{1-L}d(f, \mathcal{S}f) \leq \frac{1}{1-L}$ and hence for any $x \in X, t > 0$;

$$\begin{aligned} N\left(\mathcal{H}(x) - f(x), t\right) &= N\left(\mathcal{H}(x) - f(x), \frac{1}{1 - L}(1 - L)t\right) \\ &\geq N\left(\mathcal{H}(x) - f(x), d(\mathcal{H}, f)(1 - L)t\right) \\ &\geq \frac{(1 - L)t}{(1 - L)t + \varphi(0, x)}. \end{aligned}$$

This implies that the inequality (2.10) holds.

We show that $\mathcal{H}$ is $\mathbb{C}$-linear. Given $x, y \in X, \lambda \in \mathbb{T}^1$, and $n \in \mathbb{N}_0$ we see that

$$\begin{aligned}
&N\left(\frac{1}{\mathbf{k}^n} f\left(\mathbf{k}^n(\lambda x+\mathbf{k} y)\right)-\lambda \frac{1}{\mathbf{k}^n} f(\mathbf{k}^n x)+\mathbf{k} \frac{1}{\mathbf{k}^n} f(\mathbf{k}^n y), t\right) \\
=&N\left(f\left(\lambda(\mathbf{k}^n x)+\mathbf{k}(\mathbf{k}^n y)\right)-\lambda f(\mathbf{k}^n x)+\mathbf{k} f(\mathbf{k}^n y), \mathbf{k}^n t\right) \\
\geq&\frac{\mathbf{k}^n t}{\mathbf{k}^n t+\varphi\left(\mathbf{k}^n x, \mathbf{k}^n y\right)} \\
\geq&\frac{t}{t+L^n \varphi(x, y)}.
\end{aligned}$$

Since $\lim_{n \to +\infty} L^n \varphi(X, X)=0$, we have that $\mathcal{H}(\lambda x+\mathbf{k} y)=\lambda \mathcal{H}(x)+\mathbf{k} \mathcal{H}(y)=0$ for all $x, y \in X$. Using the same method as in the proof of [21, Theorem 2.1], one can conclude that $\mathcal{H}$ is $\mathbb{C}$-linear.

We finally prove that $\mathcal{H}$ is a hom-der, we see from (2.9) and $\mathbf{k}<\mathbf{k}^2$ that

$$\begin{aligned}
&N\left(\frac{1}{\mathbf{k}^n} f(\mathbf{k}^n x) \frac{1}{\mathbf{k}^n} f(\mathbf{k}^n y)-\frac{1}{\mathbf{k}^n} f(\mathbf{k}^n x) y+x \frac{1}{\mathbf{k}^n} f(\mathbf{k}^n y), t\right) \\
=&N\left(f(\mathbf{k}^n x) f(\mathbf{k}^n(\mathbf{k}^n y))-f(\mathbf{k}^n x)(\mathbf{k}^n y)+(\mathbf{k}^n x) f(\mathbf{k}^n y), \mathbf{k}^{2n} t\right) \\
\geq&\frac{\mathbf{k}^{2n} t}{\mathbf{k}^{2n} t+\varphi\left(\mathbf{k}^n x, \mathbf{k}^n y\right)} \\
\geq&\frac{\mathbf{k}^{2n} t}{\mathbf{k}^{2n} t+\mathbf{k}^n L^n \varphi(x, y)} \\
\geq&\frac{t}{t+L^n \varphi(x, y)}.
\end{aligned}$$

for all $x, y \in X, t>0$, and $n \in \mathbb{N}_0$. So, the mapping $\mathcal{H}: X \to X$ is a hom-der in X.

This completes the proof. □

Corollary 2.7. *Let $\eta: X \to [0,+\infty)$ be a function such that $\eta(\ell x)=|\ell|^p \eta(x)$ for all $x \in X$ and all $\ell \in \mathbb{R}$, for some $p \in (0,1)$. If $f: X \to X$ is a mapping that satisfies*

$$\begin{aligned}
N(f(\lambda x+\mathbf{k} y)-\lambda f(x)-\mathbf{k} f(y), t) &\geq \frac{t}{t+\eta(x)+\eta(y)}, \\
N(f(x) f(y)-f(x) y-x f(y), t) &\geq \frac{t}{t+\eta(x)+\eta(y)},
\end{aligned}$$

for all $x, y \in X, t>0$, and all $\lambda \in \mathbb{T}^1$, then there exists a unique hom-der $\mathcal{H}: X \to X$ such that

$$N(\mathcal{H}(x)-f(x), t) \geq \frac{(\mathbf{k}-\mathbf{k}^p) t}{(\mathbf{k}-\mathbf{k}^p) t+\mathbf{k} \eta(x)} \quad \text{for all } x \in X, t>0.$$

Proof. It can be proved similary to the proof of Corollary 2.5. □

Based on the similar observations as in Remark 2.3 and Proposition 2.4, we present the following proposition.

Proposition 2.8. *Let $\varphi : X \times X \to [0, +\infty)$ be a function and suppose that there exists a positive real number $L < 1$ satisfying*

$$\varphi(\mathbf{k}x, \mathbf{k}y) \le L\mathbf{k}\varphi(x, y) \quad \text{for all } x, y \in X.$$

Suppose that $f : X \to X$ is a mapping that satisfies $f(0) = 0$ and

$$N\left(f(\lambda x + \mathbf{k}y) - \lambda f(x) - \mathbf{k}f(y), t\right) \ge \frac{t}{t + \varphi(x, y)},$$
$$N\left(f(x)f(y) - f(x)y - xf(y), t\right) \ge \phi(x, y, t),$$

for all $x, y \in X, t > 0$, and all $\lambda \in \mathbb{T}^1$ where $\phi : X \times X \times (0, +\infty) \to [0, 1]$ has the property

$$\lim_{n\to+\infty} \phi\left(\mathbf{k}^n x, \mathbf{k}^n y, \mathbf{k}^{2n} t\right) = 1 \quad \text{for all } x, y \in X, t > 0$$

Then there exists a unique hom-der $H : X \to X$ such that

$$N\left(\mathcal{H}(x) - f(x), t\right) \ge \frac{(1 - L)t}{(1 - L)t + \varphi(0, x)} \quad \text{for all } x \in X, t > 0.$$

Proof. The proof is similar to the proof of Proposition 2.4. □

Corollary 2.9. *Let $p, q > 0$ be given real numbers. Suppose that $\eta, \eta' : X \to [0, +\infty)$ are functions satisfying*

$$\eta(\ell x) = |\ell|^p \eta(x) \quad \text{and} \quad \eta'(\ell x) = |\ell|^q \eta'(x) \quad \text{for all } x \in X \text{ and all } \ell \in \mathbb{R}.$$

Suppose that $f : X \to X$ is a mapping that satisfies

$$N\left(f(\lambda x + \mathbf{k}y) - \lambda f(x) - \mathbf{k}f(y), t\right) \ge \frac{t}{t + \eta(x) + \eta(y)},$$
$$N\left(f(x)f(y) - f(x)y - xf(y), t\right) \ge \frac{t}{t + \eta'(x) + \eta'(y)},$$

for all $x, y \in X, t > 0$, and all $\lambda \in \mathbb{T}^1$. If $p < 1$ and $q < 2$, then there exists a unique hom-der $\mathcal{H} : X \to X$ such that

$$N\left(\mathcal{H}(x) - f(x), t\right) \ge \frac{(\mathbf{k} - \mathbf{k}^p)t}{(\mathbf{k} - \mathbf{k}^p)t + \mathbf{k}\eta(x)} \quad \text{for all } x \in X, t > 0.$$

3. Conclusions and future work

We investigated a new type of derivations in fuzzy Banach algebras. Moreover, by using the fixed point method, we obtained some stability results of hom-ders in fuzzy Banach algebras associated with the functional equation

$$f(x + \mathbf{k}y) = f(x) + \mathbf{k}f(y)$$

where $\mathbf{k}$ is a fixed positive integer greater than 1. The next work is to study the relationship among homomorphism, derivations and hom-ders in various normed algebras.

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Conflict of interest

The authors declare that they have no competing interests.

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