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Research article

Structural stability for the Darcy model in double diffusive convection flow with Magnetic field effect

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Abstract: In this paper, we consider the Darcy model with magnetic field affect which is used to describe the double diffusive flow of a fluid containing a solute. Using the energy estimate methods, we derive the prior bounds of the solutions. By using these a prior bounds, the continuous dependence of the solutions to Darcy model on the magnetic coefficient and the boundary parameter is established.

Keywords: continuous dependence; a priori bounds; Darcy flow; double diffusive
Mathematics Subject Classification: 49J20, 65N30

1. Introduction

There have been many literatures on continuous dependence and structural stability for the past few years, including those of Aulisa et al. [1], Celebi et al. [2,3], Liu et al. [4–6], Chen et al. [7,8],Ames and Payne [9, 10], Ames and Straughan [11],Ciarletta and Straughan [12], Franchi and Straughan [13–16], Lin and Payne [17, 18], Li et al. [19–21], Straughan et al. [22, 23] and Zhou et al. [24, 25]. Particularly, most researches focus on the continuous dependence on the boundary data, domain geometry, initial time geometry, and the model itself. Hirsch and Smale [26] pointed out the necessity of studying the continuous dependence of solutions. They emphasized the physical significance of this type of research. This means that changes in the coefficients of partial differential equations may be physically reflected through changes in constitutive parameters. We trust that mathematical analysis of these equations will help to disclose their applicability in physics. Since inevitable errors occur in both numerical calculations and physical measurements of data, continuous correlation results are very important. It is relevant to understand the extent to which such errors affect the solution.

Harfash [27] researched a system of equations to describe the double-diffusion convection in Darcy flow with magnetic field effect. The author assumed the magnetic fields with only the vertical component which was a specific magnetic field. By establishing a priori results, the author illustrates that the solution of the equations depends continuously on changes in the magnetic force and gravity vector coefficients. Some authors have paid attentions to similar problems. By employing Payne's [28] highly innovative procedure for obtaining a priori estimates, Ames and Payne [9] have established a similar result for the Navier-Stokes equations. But it is necessary to restrict the size of the interval or the size of the initial data in their result. A similar result for a Brinkman porous material and for the Darcy equations of flow in porous media has been derived by Franchi and Straughan [29] and Payne and Straughan [30], respectively.

In this paper, we assume that the Darcy flow with magnetic field effect occupies a bounded region Ω in R^3 and that the boundary of the region is denoted by $\partial\Omega$ which is sufficient smooth to use the divergence theorem. The variables v_i, T, C and p are the fluid velocity vector, the temperature, the salt concentration and the pressure, respectively. The governing equations for Darcy flow with magnetic field effect may be written as

$$v_i = -p_{,i} + g_i T + h_i C + \sigma[(\mathbf{v}\times\mathbf{B}_0)\times\mathbf{B}_0]_i, \tag{1.1}$$

$$T_{,t} + v_i T_{,i} = \Delta T, \tag{1.2}$$

$$C_{,t} + v_i C_{,i} = \Delta C + \gamma\Delta T, \tag{1.3}$$

$$v_{i,i} = 0, \tag{1.4}$$

where g_i and h_i are gravity vector terms arising in the density equation of state, Δ is Laplacian operator, γ is the Soret coefficient, σ is magnetic coefficient, and $\mathbf{B}_0 = (0, 0, B_0)$ is a magnetic field with only the vertical component and $\mathbf{v} = (v_1, v_2, v_3)$. In (1.1), we take a particular magnetic field, as in [27, 31].

On the boundary, we impose

$$v_i n_i = 0, \quad \frac{\partial T}{\partial n} + kT = F(x,t), \quad \frac{\partial C}{\partial n} + \tau C = G(x,t), \ on\ \partial\Omega\times\{t > 0\}, \tag{1.5}$$

where F and G are positive functions, n_i is the unit outward normal to $\partial\Omega$ and k and τ are positive constants. Equation (1.5) may be thought of as expressing Newton's law of cooling with inhomogeneous outside temperature or inhomogeneous outside salt concentration, i.e.

$$\frac{\partial T}{\partial n} = -k(T - T_a), \quad \frac{\partial C}{\partial n} = -\kappa(C - C_a),$$

where T_a and C_a are the ambient outside temperature and the ambient outside salt concentration, respectively. The initial conditions are written as

$$T(x, 0) = T_0(x); \quad C(x, 0) = C_0(x); \ in\ \Omega, \tag{1.6}$$

for prescribed functions T_0 and C_0.

In our work, we still consider the same particular equations as in [27]. But our boundary conditions is Newton's law of cooling type with inhomogeneous outside temperature. Thus, the Sobolev inequalities which are used in [27] are not available in our paper. Compared with [9], we no longer need to impose special restrictions on the region Ω. So their method fails to handle the system in this paper. In this paper, we derive the upper bounds of $\int_\Omega T^4 dx$ and $\int_\Omega C^4 dx$ which are difficulty to obtain. By using the these priori results, we derive the continuous dependence on the magnetic coefficient and the boundary parameter. Throughout this paper, the usual summation convention is employed with repeated Latin subscripts summed from 1 to 3. The comma is used to indicate partial differentiation, i.e. $u_{i,j} = \frac{\partial u_i}{\partial x_j}$, $u_{i,j}u_{i,j} = \Sigma_{i,j=1}^3 \frac{\partial u_i}{\partial x_j}$.

2. A priori bounds

In this section, we want to derive bounds for various norms of v_i, T and C in term of known data which will be used in the next sections. Before we derive these bounds, we prove some lemmas firstly.

Lemma 2.1. *Let functions* $f_i, (i = 1, 2, 3)$*, defined on* $\partial\Omega$*, be some functions such that*

$$f_i n_i \geq f_0 > 0\ , \ \textit{on}\ \partial\Omega, \tag{2.1}$$

and

$$|f_{i,i}| \leq m_1, \quad |f_i| \leq m_2, \tag{2.2}$$

where $f_0 > 0$ *is a constant and* m_1*,* m_2 *are both positive constants. Then,*

$$f_0 \int_{\partial\Omega} \varphi^2 dA \leq m_3 \int_\Omega \varphi^2 dx + \alpha \int_\Omega \varphi_{,i}\varphi_{,i} dx, \tag{2.3}$$

for a function φ *which is defined on the closure of the domain* Ω*. In (2.3),* $\alpha > 0$ *is an arbitrary constant which may be very small and* $m_3 = (m_1 + \frac{m_2^2}{\alpha})$*.*

Proof. We began with the identity

$$(f_i\varphi^2)_{,i} = f_{i,i}\varphi^2 + 2f_i\varphi\varphi_{,i}. \tag{2.4}$$

Integrating (2.4) over Ω, using (2.1) and the divergence theorem, we have

$$f_0 \int_{\partial\Omega} \varphi^2 dA \leq \int_\Omega (f_i\varphi^2)_{,i} dx = \int_\Omega f_{i,i}\varphi^2 dx + 2\int_\Omega f_i\varphi\varphi_{,i} dx. \tag{2.5}$$

The Hölder inequality and (2.2) allow us to obtain

$$f_0 \int_{\partial\Omega} \varphi^2 dA \leq m_1 \int_\Omega \varphi^2 dx + 2m_2\Big(\int_\Omega \varphi^2 dx\Big)^{\frac{1}{2}}\Big(\int_\Omega \varphi_{,i}\varphi_{,i} dx\Big)^{\frac{1}{2}}, \tag{2.6}$$

from which it follows that

$$f_0 \int_{\partial\Omega} \varphi^2 dA \leq \Big(m_1 + \frac{m_2^2}{\alpha}\Big) \int_\Omega \varphi^2 dx + \alpha \int_\Omega \varphi_{,i}\varphi_{,i} dx. \tag{2.7}$$

Lemma 2.2. *Let* $T, \boldsymbol{v} \in H^1(\Omega)$*,* $T_0 \in L^{2P}(\Omega)$ *and* $F \in L^{2P}(\partial\Omega)$*. Then, the solution for* (1.2) *satisfies*

$$\sup_{\Omega\times[0,\varsigma]} |T| \leq T_m,$$

where $T_m = \max\{|T_0|, |F|\}$*.*

Proof. We began with

$$\frac{d}{dt}\int_\Omega T^{2p}dx = 2p\int_\Omega T^{2p-1}T_{,t}dx.$$

Using (1.2), the divergence theorem and the Young inequality, we are leaded to

$$\begin{aligned}\frac{d}{dt}\int_\Omega T^{2p}dx &\le 2p\int_{\partial\Omega} T^{2p-1}FdA - 2pk\int_{\partial\Omega}T^{2p}dA - 2p(2p-1)\int_\Omega T^{2p-2}T_{,i}T_{,i}dx\\ &\le \frac{(2p-1)^{2p-1}}{(2pk)^{2p-1}}\int_{\partial\Omega}F^{2p}dA.\end{aligned}$$

An integration of this inequality allows that

$$\Big(\int_\Omega T^{2p}dx\Big)^{\frac{1}{2p}} \le \Big(\frac{2p-1}{2pk}\int_{\partial\Omega}F^{2p}dA + \int_\Omega T_0^{2p}dx\Big)^{\frac{1}{2p}}.$$

Allowing $p \to \infty$, we obtain

$$\sup_{\Omega\times[0,\varsigma]}|T| \le T_m,$$

where T_m depends on the initial-boundary conditions of T.

Lemma 2.3. *Let $T, \boldsymbol{v} \in H^1(\Omega)$ and C be the solutions for (1.2) and* (1.3) *and $T_0, C_0 \in C^2(\Omega)$, $F, G \in C^2(\partial\Omega\times\{t>0\})$. Then,*

$$\int_\Omega T^2dx \le A_1(t), \quad \int_\Omega C^2dx \le A_2(t), \tag{2.8}$$

where $A_1(t)$ and $A_2(t)$ are positive functions which will be given later.

Proof. Using (1.2) and the divergence theorem, we compute

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega T^2dx &= \int_\Omega TT_{,t}dx = \int_\Omega T[\Delta T - v_iT_{,i}]dx\\ &= \int_{\partial\Omega}TFdA - k\int_{\partial\Omega}T^2dA - \int_\Omega T_{,i}T_{,i}dx.\end{aligned} \tag{2.9}$$

By the Hölder inequality and the Young inequality, from (2.9) we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega T^2dx + \int_\Omega T_{,i}T_{,i}dx \le \frac{1}{4k}\int_{\partial\Omega}F^2dA. \tag{2.10}$$

Integrating (2.10) from 0 to t, we have

$$\int_\Omega T^2dx + 2\int_0^t\int_\Omega T_{,i}T_{,i}dxd\eta \le \frac{1}{2k}\int_0^t\int_{\partial\Omega}F^2dAd\eta + \int_\Omega T_0^2dx \doteq A_1(t). \tag{2.11}$$

From the identity

$$\int_\Omega C(C_{,t} + v_iC_{,i} - \Delta C - \gamma\Delta T)dx = 0,$$

we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\Omega}C^2dx+\int_{\Omega}C_{,i}C_{,i}dx\\&=\int_{\partial\Omega}GCdA-\tau\int_{\partial\Omega}C^2dA+\gamma\int_{\partial\Omega}FCdA-k\gamma\int_{\partial\Omega}TCdA-\gamma\int_{\Omega}T_{,i}C_{,i}dx.\end{aligned}\tag{2.12}$$

Upon employing the Cauchy-Schwarz inequality and the arithmetic-geometric mean inequality, we can get

$$\begin{aligned}\int_{\partial\Omega}GCdA&\le\frac{1}{\tau}\int_{\partial\Omega}G^2dA+\frac{\tau}{4}\int_{\partial\Omega}C^2dA,\\\gamma\int_{\partial\Omega}FCdA&\le\frac{\gamma^2}{\tau}\int_{\partial\Omega}F^2dA+\frac{\tau}{4}\int_{\partial\Omega}C^2dA,\\k\gamma\int_{\partial\Omega}TCdA&\le\frac{1}{2\tau}k^2\gamma^2\int_{\partial\Omega}T^2dA+\frac{\tau}{2}\int_{\partial\Omega}C^2dA,\\\gamma\int_{\Omega}T_{,i}C_{,i}dx&\le\frac{1}{2}\gamma^2\int_{\Omega}T_{,i}T_{,i}dx+\frac{1}{2}\int_{\Omega}C_{,i}C_{,i}dx.\end{aligned}$$

We use these inequalities together with (2.12) to arrive at

$$\begin{aligned}&\frac{d}{dt}\int_{\Omega}C^2dx+\int_{\Omega}C_{,i}C_{,i}dx\\&\le\frac{2}{\tau}\int_{\partial\Omega}G^2dA+\frac{2\gamma^2}{\tau}\int_{\partial\Omega}F^2dA+\frac{k^2\gamma^2}{\tau}\int_{\partial\Omega}T^2dA+\gamma^2\int_{\Omega}T_{,i}T_{,i}dx.\end{aligned}\tag{2.13}$$

Letting $\varphi = T$ in Lemma 2.1 and using (2.11), we have

$$f_0\int_{\partial\Omega}T^2dA\le m_3\int_{\Omega}T^2dx+\alpha\int_{\Omega}T_{,i}T_{,i}dx\le m_3A_1(t)+\alpha\int_{\Omega}T_{,i}T_{,i}dx.\tag{2.14}$$

Thus, (2.13) can be rewritten as

$$\frac{d}{dt}\int_{\Omega}C^2dx+\int_{\Omega}C_{,i}C_{,i}dx\le\frac{2}{\tau}\int_{\partial\Omega}G^2dA+\frac{2\gamma^2}{\tau}\int_{\partial\Omega}F^2dA+\frac{k^2m_3\gamma^2}{f_0\tau}A_1(t)+2\gamma^2\int_{\Omega}T_{,i}T_{,i}dx,\tag{2.15}$$

with $\alpha = \frac{f_0\tau}{k^2}$. An integration of (2.15) leads to

$$\begin{aligned}\int_{\Omega}C^2dx+\int_0^t\int_{\Omega}C_{,i}C_{,i}dxd\eta\le&\frac{2}{\tau}\int_0^t\int_{\partial\Omega}G^2dAd\eta+\frac{2\gamma^2}{\tau}\int_0^t\int_{\partial\Omega}F^2dAd\eta\\&+\frac{k^2m_3\gamma^2}{f_0\tau}\int_0^tA_1(\eta)d\eta+2\gamma^2\int_0^t\int_{\Omega}T_{,i}T_{,i}dxd\eta+\int_{\Omega}C_0^2dx.\end{aligned}\tag{2.16}$$

In light of (2.11), we have

$$\begin{aligned}\int_{\Omega}C^2dx+\int_0^t\int_{\Omega}C_{,i}C_{,i}dxd\eta\le&\frac{2}{\tau}\int_0^t\int_{\partial\Omega}G^2dAd\eta+\frac{2\gamma^2}{\tau}\int_0^t\int_{\partial\Omega}F^2dAd\eta\\&+\frac{k^2m_3\gamma^2}{f_0\tau}\int_0^tA_1(\eta)d\eta+\gamma^2A_1(t)+\int_{\Omega}C_0^2dx\doteq A_2(t).\end{aligned}\tag{2.17}$$

Lemma 2.4. *Let T and C be the solutions for (1.2) and* (1.3)*, and $T, \boldsymbol{v} \in H^1(\Omega)$, $T_0, C_0 \in C^4(\Omega)$, $F, G \in C^4(\partial\Omega \times \{t > 0\})$. Then,*

$$\int_\Omega T^4 dx \le A_3(t), \quad \int_\Omega C^4 dx \le A_4(t), \tag{2.18}$$

where $A_3(t)$ and $A_4(t)$ will be given later.

Proof. We first let H be a solution of the problem

$$\begin{aligned} &H_{,t} + v_i H_{,i} = \Delta H, \ in\ \Omega \times \{t > 0\}, \\ &\frac{\partial H}{\partial n} + \tau H = G(x,t), \quad on\ \partial\Omega \times \{t > 0\}, \\ &H(x,0) = C_0(x), \quad in\ \Omega. \end{aligned} \tag{2.19}$$

Using (2.19) and the divergence theorem, we find

$$\begin{aligned} \frac{1}{4}\frac{d}{dt}\int_\Omega H^4 dx &= \int_\Omega H^3 H_{,t} dx = \int_\Omega H^3[\Delta H - v_i H_{,i}]dx \\ &= \int_{\partial\Omega} H^3 G dA - \tau \int_{\partial\Omega} H^4 dA - \frac{3}{4}\int_\Omega (H^2)_{,i}(H^2)_{,i} dx. \end{aligned} \tag{2.20}$$

By the Hölder inequality, we have

$$\int_\Omega H^4 dx + 3\int_0^t \int_\Omega (H^2)_{,i}(H^2)_{,i} dx d\eta \le \frac{27}{64\tau^3}\int_{\partial\Omega} G^4 dA + \int_\Omega C_0^4 dx. \tag{2.21}$$

From (2.21), it is clear that $\int_\Omega H^4 dx$ can be bounded by known data. Now, we set

$$\psi(x,t) = C - H.$$

Then, ψ satisfies the initial-boundary condition problem

$$\begin{aligned} &\psi_{,t} + v_i \psi_{,i} = \Delta\psi + \gamma\Delta T, \quad in\ \Omega \times \{t > 0\}, \\ &\frac{\partial \psi}{\partial n} + \tau\psi = 0, \quad on\ \partial\Omega \times \{t > 0\}, \\ &\psi(x,0) = 0, \quad in\ \Omega. \end{aligned} \tag{2.22}$$

Next, we also define a new function

$$\Phi(t) = \delta_1 \int_\Omega T^4 dx + \delta_2 \int_\Omega T^2\psi^2 dx + \int_\Omega \psi^4 dx, \tag{2.23}$$

where δ_1 and δ_2 are positive constants to be determined later. Now, it is easy to see that

$$\begin{aligned} \Phi'(t) =& 4\delta_1 \int_\Omega T^3(\Delta T - v_i T_{,i})dx + 2\delta_2 \int_\Omega T\psi^2(\Delta T - v_i T_{,i})dx \\ &+ 2\delta_2 \int_\Omega T^2\psi(\Delta\psi + \gamma\Delta T - v_i\psi_{,i})dx + 4\int_\Omega \psi^3(\Delta\psi + \gamma\Delta T - v_i\psi_{,i})dx, \end{aligned} \tag{2.24}$$

from which we may get that

$$\begin{aligned}\Phi'(t) =& -3\delta_1\int_\Omega (T^2)_{,i}(T^2)_{,i}dx - 3\int_\Omega(\psi^2)_{,i}(\psi^2)_{,i}dx - 2\delta_2\int_\Omega(\psi T_{,i}+\psi_{,i}T)(\psi T_{,i}+\psi_{,i}T)dx\\ &-4\delta_2\int_\Omega T\psi\psi_{,i}T_{,i}dx - 4\delta_2\gamma\int_\Omega T\psi T_{,i}T_{,i}dx - 2\delta_2\gamma\int_\Omega T^2\psi_{,i}T_{,i}dx - 12\gamma\int_\Omega\psi^2\psi_{,i}T_{,i}dx\\ &-4\delta_1 k\int_{\partial\Omega}T^4dA - 4\tau\int_{\partial\Omega}\psi^4dA + 4\delta_1\int_{\partial\Omega}T^3FdA + 2\delta_2\int_{\partial\Omega}\psi^2TFdA\\ &+2\delta_2\gamma\int_{\partial\Omega}\psi T^2FdA - 2\delta_2(k+\tau)\int_{\partial\Omega}\psi^2T^2dA - 2\delta_2k\gamma\int_{\partial\Omega}\psi T^3dA\\ &+4\gamma\int_{\partial\Omega}\psi^3FdA - 4k\gamma\int_{\partial\Omega}\psi^3TdA\\ =&\sum_1^{16}J_i.\end{aligned} \tag{2.25}$$

Now using the arithmetic-geometric mean and the Schwarz inequalities, we find that

$$J_4 \le \frac{1}{2}\delta_2\varepsilon_1\int_\Omega(T^2)_{,i}(T^2)_{,i}dx + \frac{\delta_2}{2\varepsilon_1}\int_\Omega(\psi^2)_{,i}(\psi^2)_{,i}dx, \tag{2.26}$$

and

$$\begin{aligned}J_5+J_6 =& -4\delta_2\gamma\int_\Omega TT_{,i}[T\psi_{,i}+T_{,i}\psi]dx + 2\delta_2\gamma\int_\Omega T^2\psi_{,i}T_{,i}dx\\ \le&\delta_2\varepsilon_2\int_\Omega(T^2)_{,i}(T^2)_{,i}dx + \frac{\delta_2\gamma^2}{\varepsilon_2}\int_\Omega[T\psi_{,i}+T_{,i}\psi][T\psi_{,i}+T_{,i}\psi]dx\\ &+2\delta_2T_m^2\gamma\Big(\int_\Omega|\nabla\psi|^2dx\int_\Omega|\nabla T|^2dx\Big)^{\frac{1}{2}},\end{aligned} \tag{2.27}$$

where T_m is defined in Lemma 2.2. Furthermore,

$$\begin{aligned}J_7 =& -12\gamma\int_\Omega\psi\psi_{,i}[\psi T_{,i}+\psi_{,i}T]dx + 12\gamma\int_\Omega T\psi|\nabla\psi|^2dx\\ \le&3\varepsilon_3\int_\Omega(\psi^2)_{,i}(\psi^2)_{,i}dx + \frac{3\gamma^2}{\varepsilon_3}\int_\Omega[T\psi_{,i}+T_{,i}\psi][T\psi_{,i}+T_{,i}\psi]dx\\ &+3\gamma^2\varepsilon_4T_m^2\int_\Omega|\nabla\psi|^2dx + \frac{3}{\varepsilon_4}\int_\Omega(\psi^2)_{,i}(\psi^2)_{,i}dx,\end{aligned} \tag{2.28}$$

Inserting (2.26)–(2.28) into (2.25), and using the Hölder and the Young inequalities to the integrals on the boundary, we have

$$\begin{aligned}\Phi'(t)\leq&-(3\delta_1-\frac{1}{2}\delta_2\varepsilon_1-\delta_2\varepsilon_2)\int_\Omega(T^2)_{,i}(T^2)_{,i}dx-(3-\frac{\delta_2}{2\varepsilon_1}-3\varepsilon_3-\frac{3}{\varepsilon_4})\int_\Omega(\psi^2)_{,i}(\psi^2)_{,i}dx\\&-(2\delta_2-\frac{\delta_2\gamma^2}{\varepsilon_2}-\frac{3\gamma^2}{\varepsilon_3})\int_\Omega(\psi T_{,i}+\psi_{,i}T)(\psi T_{,i}+\psi_{,i}T)dx\\&+2\delta_2T_m^2\gamma(\int_\Omega|\nabla\psi|^2dx\int_\Omega|\nabla T|^2dx)^{\frac{1}{2}}+3\gamma^2\varepsilon_4T_m^2\int_\Omega|\nabla\psi|^2dx\\&-(4\delta_1\gamma-3\delta_1\varepsilon_5-\frac{\delta_2\varepsilon_7}{2\varepsilon_6}-\delta_2\varepsilon_8-\frac{\delta_2(\kappa+\tau)}{\varepsilon_{10}}-\frac{3}{2}\delta_2k\gamma\varepsilon_{11}-\gamma\varepsilon_{13}^{-3})\oint_{\partial\Omega}T^4dA\\&-(4\kappa-\delta_2\varepsilon_6-\frac{\delta_2\varepsilon_9}{2\varepsilon_8}-\delta_2(\kappa+\tau)\varepsilon_{10}-\frac{1}{2}\delta_2k\varepsilon_{11}^{-3}-3\gamma\varepsilon_{12}-3\gamma\varepsilon_{13})\oint_{\partial\Omega}\psi^4dA\\&+(\delta_1\varepsilon_5^{-3}+\frac{\delta_2}{2\varepsilon_6\varepsilon_7}+\frac{\delta_2}{2\varepsilon_8\varepsilon_9}+\gamma\varepsilon_{12}^{-3})\oint_{\partial\Omega}F^4dA,\end{aligned}\tag{2.29}$$

where ε_i $(i=1,2,\cdots,13)$ are positive constants to be determined. To ensure that the coefficients of the first three terms and the sixth and seventh terms to be non-positive, we choose that

$$\begin{aligned}&\delta_1=\max\{5\gamma^4,\frac{27\gamma^3(k+\tau)^2}{k}+(\frac{9}{2})^{\frac{4}{3}}k\gamma^3+\frac{1}{2}(\frac{9}{2})^3\frac{\gamma^3}{k^3}\},\ \delta_2=6\gamma^2,\\&\varepsilon_1=3\gamma^2,\ \varepsilon_2=\gamma^2,\ \varepsilon_3=\frac{1}{2},\ \varepsilon_4=6,\ \varepsilon_5=\frac{\gamma}{3},\ \varepsilon_6=\frac{k}{9\gamma^2},\ \varepsilon_7=\frac{k\delta_1}{54\gamma^3},\ \varepsilon_8=\frac{\delta_1}{12\gamma},\\&\varepsilon_9=\frac{k\delta_1}{108\gamma^3},\ \varepsilon_{10}=\frac{k}{9(\kappa+\tau)\gamma^2},\ \varepsilon_{11}=\sqrt[3]{\frac{9}{2}}\gamma,\ \varepsilon_{12}=\varepsilon_{13}=\frac{2k}{9\gamma}.\end{aligned}$$

We drop the non-positive terms in (2.29) to have

$$\begin{aligned}\Phi'(t)\leq&2\delta_2T_m^2\gamma(\int_\Omega|\nabla\psi|^2dx\int_\Omega|\nabla T|^2dx)^{\frac{1}{2}}+6\gamma^2\varepsilon_4T_m^2\int_\Omega|\nabla\psi|^2dx\\&+(\delta_1\varepsilon_5^{-3}+\frac{\delta_2}{2\varepsilon_6\varepsilon_7}+\frac{\delta_2}{2\varepsilon_8\varepsilon_9}+\gamma\varepsilon_{12}^{-3})\oint_{\partial\Omega}F^4dA.\end{aligned}$$

Using the arithmetic-geometric mean inequality and integrating the above formula from 0 to t, we obtain

$$\Phi(t)\leq\widetilde{m}_1\int_0^t\int_\Omega|\nabla\psi|^2dxd\eta+\widetilde{m}_2\int_0^t\int_\Omega|\nabla T|^2dxd\eta+\widetilde{m}_3\int_0^t\oint_{\partial\Omega}F^4dAd\eta,\tag{2.30}$$

where $\widetilde{m}_1=\delta_2T_m^2\gamma+6\gamma^2\varepsilon_4T_m^2$, $\widetilde{m}_2=\delta_2T_m^2\gamma$ and $\widetilde{m}_3=(\delta_1\varepsilon_5^{-3}+\frac{\delta_2}{2\varepsilon_6\varepsilon_7}+\frac{\delta_2}{2\varepsilon_8\varepsilon_9}+\gamma\varepsilon_{12}^{-3})$.

Next, we multiply $(2.22)_1$ with ψ, integrate in Ω and use Cauchy-Schwarz's inequality to obtain

$$\begin{aligned}\frac{d}{dt}\|\psi\|^2=&-2\int_\Omega\psi_{,i}\psi_{,i}dx-2\tau\int_{\partial\Omega}\psi^2dA-2\gamma\int_\Omega T_{,i}\psi_{,i}dx-2\gamma\int_{\partial\Omega}F\psi dA-2k\gamma\int_{\partial\Omega}T\psi dA\\\leq&-\int_\Omega\psi_{,i}\psi_{,i}dx+\gamma^2\int_\Omega T_{,i}T_{,i}dx+\frac{\gamma^2}{\tau}\int_{\partial\Omega}F^2dA+\frac{k^2\gamma^2}{\tau}\int_{\partial\Omega}T^2dA.\end{aligned}\tag{2.31}$$

In light of (2.14), (2.31) yields that

$$\frac{d}{dt}\int_\Omega \psi^2 dx \le -\int_\Omega \psi_{,i}\psi_{,i}dx + \Big(\frac{k^2\gamma^2\alpha}{f_0\tau}+\gamma^2\Big)\int_\Omega T_{,i}T_{,i}dx + \frac{\gamma^2}{\tau}\int_{\partial\Omega}F^2dA + \frac{k^2m_3\gamma^2}{f_0\tau}A_1(t). \tag{2.32}$$

Integrating (2.32) from 0 to t, we have

$$\begin{aligned}&\int_\Omega \psi^2dx + \int_0^t\int_\Omega \psi_{,i}\psi_{,i}dxd\eta\\ \le&\Big(\frac{k^2\gamma^2\alpha}{f_0\tau}+\gamma^2\Big)\int_0^t\int_\Omega T_{,i}T_{,i}dxd\eta + \frac{\gamma^2}{\tau}\int_0^t\int_{\partial\Omega}F^2dAd\eta + \frac{k^2m_3\gamma^2}{f_0\tau}\int_0^t A_1(\eta)d\eta.\end{aligned} \tag{2.33}$$

With the aid of (2.11), inequality (2.33) can be rewritten as

$$\begin{aligned}&\int_\Omega \psi^2dx + \int_0^t\int_\Omega \psi_{,i}\psi_{,i}dxd\eta\\ \le&\frac{1}{2}\Big(\frac{k^2\gamma^2\alpha}{f_0\tau}+\gamma^2\Big)A_1(t) + \frac{\gamma^2}{\tau}\int_0^t\int_{\partial\Omega}F^2dAd\eta + \frac{k^2m_3\gamma^2}{f_0\tau}\int_0^t A_1(\eta)d\eta.\end{aligned} \tag{2.34}$$

Inserting (2.34) into (2.30) and using (2.11) again, we have

$$\Phi(t) \le m(t), \tag{2.35}$$

where

$$\begin{aligned}m(t) =&\frac{1}{2}\widetilde{m_1}\Big(\frac{k^2\gamma^2\alpha}{f_0\tau}+\gamma^2\Big)A_1(t) + \frac{\widetilde{m_1}\gamma^2}{\tau}\int_0^t\int_{\partial\Omega}F^2dAd\eta\\ &+\frac{\widetilde{m_1}k^2m_3\gamma^2}{f_0\tau}\int_0^t A_1(\eta)d\eta + \frac{m_2}{2}A_1(t) + \widetilde{m}_3\int_0^t\int_{\partial\Omega}F^4dAd\eta.\end{aligned}$$

Recalling the definition of $\Phi(t)$ in (2.23), we may get

$$\int_\Omega |T|^4dx \le \frac{1}{\delta_1}m(t) \doteq A_3(t), \quad \int_\Omega |\psi|^4dx \le m(t). \tag{2.36}$$

By the triangle inequality, we have

$$\Big(\int_\Omega C^4dx\Big)^{\frac14} \le \Big(\int_\Omega \psi^4dx\Big)^{\frac14} + \Big(\int_\Omega H^4dx\Big)^{\frac14}.$$

Combining (2.21) and (2.36), we have

$$\int_\Omega C^4dx \le A_4(t), \tag{2.37}$$

where

$$A_4(t) = \Big\{m^{\frac14}(t) + \Big[\frac{27}{64\tau^3}\int_{\partial\Omega}G^4dA + \int_\Omega C_0^4dx\Big]^{\frac14}\Big\}^4.$$

Next, we pay our attention to seek the bound for L_2 norm of v_i as well as ∇v. We obtain the following lemma which will be used in the continuous dependence proof.

Lemma 2.5. *Let v_i, T and C are the solutions of (1.1)–(1.3) with the initial-boundary conditions (1.5) and (1.6), and $T_0, C_0 \in C^4(\Omega)$, $F, G \in C^4(\partial\Omega \times \{t > 0\})$. Then,*

$$\int_\Omega v_i v_i dx \leq A_5(t), \quad \int_0^t \int_\Omega v_{i,j} v_{i,j} dx d\eta \leq A_6(t), \tag{2.38}$$

where $A_5(t)$ and $A_6(t)$ are positive functions which will be derived later.

Proof. We start with the identity

$$\int_\Omega v_i v_i dx = \int_\Omega v_i \Big\{ -p_{,i} + g_i T + h_i C + \sigma[(\mathbf{v} \times \mathbf{B}_0) \times \mathbf{B}_0]_i \Big\} dx.$$

Since $\mathbf{B}_0 = (0, 0, B_0)$, it is clear that $[(\mathbf{v} \times \mathbf{B}_0) \times \mathbf{B}_0]_i = B_0^2(\bar{k}_i v_3 - v_i)$, where $\bar{\mathbf{k}} = (\bar{k}_1, \bar{k}_2, \bar{k}_3) = (0, 0, 1)$. Obviously,

$$[(\mathbf{v} \times \mathbf{B}_0) \times \mathbf{B}_0]\mathbf{v} = B_0^2(\bar{k}_i v_3 - v_i) v_i = -B_0^2[v_1^2 + v_2^2] \leq 0, \tag{2.39}$$

so by the Hölder inequality and the arithmetic-geometric mean inequality, we have

$$\int_\Omega v_i v_i dx \leq 2g^2 \int_\Omega T^2 dx + 2h^2 \int_\Omega C^2 dx.$$

Combining (2.8) and Lemma 2.3, we obtain

$$\int_\Omega v_i v_i dx \leq 2g^2 A_1(t) + 2h^2 A_2(t) \doteq A_5(t). \tag{2.40}$$

We commence bounding the L_2 norm for the velocity gradient. To do this, we split the velocity into symmetric and skew parts. We write

$$\int_\Omega v_{i,j} v_{i,j} dx = \int_\Omega v_{i,j}(v_{i,j} - v_{j,i}) dx + \int_\Omega v_{i,j} v_{j,i} dx. \tag{2.41}$$

To bound the first term of (2.41), we use the Eq (1.1) to have

$$\begin{aligned} \int_\Omega v_{i,j}(v_{i,j} - v_{j,i}) dx &= \int_\Omega \{-p_{,ij} + g_i T_{,j} + h_i C_{,j} + \sigma B_0^2(\bar{k}_i v_3 - v_i)_{,j}\} v_{i,j} dx \\ &\quad - \int_\Omega \{-p_{,ij} + g_j T_{,i} + h_j C_{,i} + \sigma B_0^2(\bar{k}_j v_3 - v_j)_{,i}\} v_{i,j} dx \\ &= \int_\Omega (g_i T_{,j} - g_j T_{,i}) v_{i,j} dx + \int_\Omega (h_i C_{,j} - h_j C_{,i}) v_{i,j} dx \\ &\quad + \sigma B_0^2 \int_\Omega (\bar{k}_i v_{3,j} - \bar{k}_j v_{3,i}) v_{i,j} dx - \sigma B_0^2 \int_\Omega (v_{i,j} - v_{j,i}) v_{i,j} dx. \end{aligned} \tag{2.42}$$

Using Hölder inequality and arithmetic-geometric inequality again in (2.42), we arrive at

$$\begin{aligned} \int_\Omega (g_i T_{,j} - g_j T_{,i}) v_{i,j} dx &\leq \int_\Omega (g_i T_{,j} - g_j T_{,i})(g_i T_{,j} - g_j T_{,i}) dx + \frac{1}{4} \int_\Omega v_{i,j} v_{i,j} dx \\ &= 2 \int_\Omega (g^2 T_{,i} T_{,i} - g_i T_{,i} g_j T_{,j}) dx + \frac{1}{4} \int_\Omega v_{i,j} v_{i,j} dx \\ &\leq 2 \int_\Omega (g^2 T_{,i} T_{,i} + \frac{1}{2} g_i g_i T_{,i} T_{,i} + \frac{1}{2} g_j g_j T_{,j} T_{,j}) dx + \frac{1}{4} \int_\Omega v_{i,j} v_{i,j} dx \\ &\leq 4g^2 \int_\Omega T_{,i} T_{,i} dx + \frac{1}{4} \int_\Omega v_{i,j} v_{i,j} dx. \end{aligned} \tag{2.43}$$

Similarly, we also have

$$\int_\Omega (h_iC_{,j} - h_jC_{,i})v_{i,j}dx \le 4h^2\int_\Omega C_{,i}C_{,i}dx + \frac{1}{4}\int_\Omega v_{i,j}v_{i,j}dx. \tag{2.44}$$

In view of $\overline{\mathbf{k}} = (0,0,1)$, the third term of (2.42) yields

$$\begin{aligned}\sigma B_0^2\int_\Omega(\overline{k}_iv_{3,j} - \overline{k}_jv_{3,i})v_{i,j}dx =& \frac{1}{2}\sigma B_0^2\int_\Omega(\overline{k}_iv_{3,j} - \overline{k}_jv_{3,i})(v_{i,j} - v_{j,i})dx\\ =&\sigma B_0^2\int_\Omega \overline{k}_iv_{3,j}(v_{i,j} - v_{j,i})dx\\ =&\sigma B_0^2\int_\Omega v_{3,j}(v_{3,j} - v_{j,3})dx \le \sigma B_0^2\int_\Omega(v_{i,j} - v_{j,i})v_{i,j}dx.\end{aligned} \tag{2.45}$$

Inserting (2.43)–(2.45) into (2.42), we have

$$\int_\Omega v_{i,j}(v_{i,j} - v_{j,i})dx \le 4g^2\int_\Omega T_{,i}T_{,i}dx + 4h^2\int_\Omega C_{,i}C_{,i}dx + \frac{1}{2}\int_\Omega v_{i,j}v_{i,j}dx. \tag{2.46}$$

To handle the second term of (2.41), we use the divergence theorem and integrate by parts to obtain

$$\int_\Omega v_{i,j}v_{j,i}dx = \int_{\partial\Omega} v_{i,j}v_jn_idA = \int_{\partial\Omega}(v_in_i)_{,j}v_jdA - \int_{\partial\Omega} v_iv_jn_{i,j}dA. \tag{2.47}$$

The first term of (2.47) is zero, since $v_in_i = 0$ on $\partial\Omega$. If the region Ω is convex, Lin and Payne [18] state $\int_{\partial\Omega} v_iv_jn_{i,j}dA \ge 0$ which leads to

$$\int_\Omega v_{i,j}v_{j,i}dx \le 0.$$

For non-convex Ω,

$$\int_\Omega v_{i,j}v_{j,i}dx \le k_0\int_{\partial\Omega} v_iv_idA.$$

Using Lemma 2.1 with $\varphi = v_i$, we conclude that

$$\int_\Omega v_{i,j}v_{j,i}dx \le \frac{k_0m_3}{f_0}\int_\Omega v_iv_idx + \frac{k_0}{f_0}\alpha\int_\Omega v_{i,j}v_{i,j}dx. \tag{2.48}$$

Choosing $\alpha = \frac{f_0}{4k_0}$ and then inserting (2.46) and (2.48) into (2.41), we have

$$\int_\Omega v_{i,j}v_{i,j}dx \le 4g^2\int_\Omega T_{,i}T_{,i}dx + 4h^2\int_\Omega C_{,i}C_{,i}dx + \frac{k_0m_3}{f_0}\int_\Omega v_iv_idx + \frac{3}{4}\int_\Omega v_{i,j}v_{i,j}dx,$$

from which it follows that

$$\int_\Omega v_{i,j}v_{i,j}dx \le 16g^2\int_\Omega T_{,i}T_{,i}dx + 16h^2\int_\Omega C_{,i}C_{,i}dx + \frac{4k_0m_3}{f_0}\int_\Omega v_iv_idx.$$

By (2.11), (2.19) and (2.48), we have

$$\int_0^t\int_\Omega v_{i,j}v_{i,j}dxd\eta \le 8g^2A_1(t) + 16h^2A_2(t) + \frac{4k_0m_3}{f_0}\int_0^t A_5(\eta)d\eta \doteq A_6(t),$$

where we have used (2.11), (2.17) and (2.40).

3. Continuous dependence on the coefficient σ

Let (v_i, p, T, C) and (v_i^*, p^*, T^*, C^*) be the solutions to the problem (1.1)–(1.6) for the same initial-boundary data, but for different magnetic coefficients σ_1 and σ_2, respectively. Differential variables w_i, π, θ, Σ and σ are defined by

$$w_i = v_i - v_i^*, \quad \theta = T - T^*, \quad \Sigma = C - C^*, \quad \pi = p - p^*, \quad \sigma = \sigma_1 - \sigma_2.$$

Then,

$$w_i = -\pi_{,i} + g_i\theta + h_i\Sigma + \sigma[(v^* \times \mathbf{B}_0) \times \mathbf{B}_0]_i + \sigma_1[(w \times \mathbf{B}_0) \times \mathbf{B}_0]_i, \tag{3.1}$$

$$\theta_{,t} + v_i^*\theta_{,i} + w_i T_{,i} = \Delta\theta, \tag{3.2}$$

$$\Sigma_{,t} + v_i^*\Sigma_{,i} + w_i C_{,i} = \Delta\Sigma + \gamma\Delta\theta, \tag{3.3}$$

$$w_{i,i} = 0, \tag{3.4}$$

with the initial-boundary conditions

$$w_i n_i = 0, \quad \frac{\partial\theta}{\partial n} = -k\theta, \quad \frac{\partial\Sigma}{\partial n} = -\tau\Sigma, \quad on \ \partial\Omega \times \{t > 0\}, \tag{3.5}$$

$$\theta(x, 0) = \Sigma(x, 0) = 0, \ x \in \Omega. \tag{3.6}$$

We have the following theorem.

Theorem 3.1. *If $T_0, C_0 \in L^\infty(\Omega)$, $F, G \in C^4(\partial\Omega \times \{t > 0\})$, then the solutions of (1.1)–(1.6) depend continuously on the magnetic coefficient σ, as shown explicit in inequalities (3.26) and (3.27) which derives a relation of the form*

$$\beta \int_\Omega \theta^2 dx + \int_\Omega \Sigma^2 dx \le L_1\sigma^2,$$

and

$$\int_\Omega w_i w_i dx \le L_2\sigma^2,$$

where L_1 and L_2 are priori constants and $\beta > 0$ is a computable constant.

Proof. Multiplying (3.16) with w_i and integrating over Ω, then using Cauchy-Schwarz's inequality and the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned}\int_\Omega w_i w_i dx \le & g\Big(\int_\Omega \theta^2 dx\Big)^{\frac{1}{2}}\Big(\int_\Omega w_i w_i dx\Big)^{\frac{1}{2}} + h\Big(\int_\Omega \Sigma^2 dx\Big)^{\frac{1}{2}}\Big(\int_\Omega w_i w_i dx\Big)^{\frac{1}{2}} \\ & + \sigma B_0^2 \int_\Omega (\bar{k}_i v_3^* - v_i^*) w_i dx + \sigma_1 B_0^2 \int_\Omega (\bar{k}_i w_3 - w_i) w_i dx,\end{aligned} \tag{3.7}$$

where $g = \max\{\sqrt{g_i g_i}\}$, $h = \max\{\sqrt{h_i h_i}\}$. Since $\bar{\mathbf{k}} = (0, 0, 1)$, it is easy to find

$$\sigma_1 B_0^2 \int_\Omega (\bar{k}_i w_3 - w_i) w_i dx \le 0 \tag{3.8}$$

as in (2.39). By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sigma B_0^2\int_\Omega(\overline{k}_i v_3^* - v_i^*)w_i dx &\leq \sigma B_0^2\Big(\int_\Omega (v_3^*)^2dx\Big)^{\frac{1}{2}}\Big(\int_\Omega w_iw_idx\Big)^{\frac{1}{2}} + \sigma B_0^2\Big(\int_\Omega v_i^*v_i^*dx\Big)^{\frac{1}{2}}\Big(\int_\Omega w_iw_idx\Big)^{\frac{1}{2}}\\ &\leq 2\sigma B_0^2\Big(\int_\Omega v_i^*v_i^*dx\Big)^{\frac{1}{2}}\Big(\int_\Omega w_iw_idx\Big)^{\frac{1}{2}}.\end{aligned}\tag{3.9}$$

Inserting (3.8) and (3.9) into (3.7) and applying the arithmetic-geometric mean inequality, we have

$$\int_\Omega w_iw_idx \leq 4g^2\int_\Omega \theta^2dx + 4h^2\int_\Omega \Sigma^2dx + 8\sigma^2B_0^4\int_\Omega v_i^*v_i^*dx.\tag{3.10}$$

In view of (2.38) in Lemma 2.5, from (3.10) we have

$$\int_\Omega w_iw_idx \leq 4g^2\int_\Omega \theta^2dx + 4h^2\int_\Omega \Sigma^2dx + 8\sigma^2B_0^4A_5(t).\tag{3.11}$$

Next, we compute

$$\begin{aligned}&\frac{d}{dt}\Big(\beta\int_\Omega\theta^2dx + \int_\Omega\Sigma^2dx\Big)\\ =&2\beta\int_\Omega\theta\theta_{,t}dx + 2\int_\Omega\Sigma\Sigma_{,t}dx\\ =&2\beta\int_\Omega\theta[\Delta\theta - v_i^*\theta_{,i} - w_iT_{,i}]dx + 2\int_\Omega\Sigma[\Delta\Sigma + \gamma\Delta\theta - v_i^*\Sigma_{,i} - w_iC_{,i}]dx\\ =&-2\beta\int_\Omega\theta_{,i}\theta_{,i}dx - 2\int_\Omega\Sigma_{,i}\Sigma_{,i}dx - 2\beta k\int_{\partial\Omega}\theta^2dA - 2\tau\int_{\partial\Omega}\Sigma^2dA\\ &+2\beta\int_\Omega\theta_{,i}w_iTdx + 2\int_\Omega\Sigma_{,i}w_iCdx - 2\gamma\int_\Omega\theta_{,i}\Sigma_{,i}dx - 2k\gamma\int_{\partial\Omega}\theta\Sigma dA.\end{aligned}\tag{3.12}$$

Using Cauchy-Schwarz inequality and the arithmetic-geometric mean inequality and Lemma 2.4, we have

$$\begin{aligned}2\beta\int_\Omega\theta_{,i}w_iTdx &\leq 2\beta\Big(\int_\Omega\theta_{,i}\theta_{,i}dx\Big)^{\frac{1}{2}}\Big(\int_\Omega(w_iw_i)^2dx\Big)^{\frac{1}{4}}\Big(\int_\Omega T^4dx\Big)^{\frac{1}{4}}\\ &\leq \beta\int_\Omega\theta_{,i}\theta_{,i}dx + \beta\Big(\int_\Omega(w_iw_i)^2dx\Big)^{\frac{1}{2}}A_3^{\frac{1}{2}}(t),\end{aligned}\tag{3.13}$$

and

$$2\int_\Omega\Sigma_{,i}w_iCdx \leq \int_\Omega\Sigma_{,i}\Sigma_{,i}dx + \Big(\int_\Omega(w_iw_i)^2dx\Big)^{\frac{1}{2}}A_4^{\frac{1}{2}}(t).\tag{3.14}$$

Inserting these two inequalities into (3.12) and using the Cauchy-Schwarz inequality in the last two terms on the right of (3.12), we have

$$\begin{aligned}\frac{d}{dt}\Big(\beta\int_\Omega\theta^2dx + \int_\Omega\Sigma^2dx\Big) \leq& -\Big(\beta - \frac{\gamma}{\beta_1}\Big)\int_\Omega\theta_{,i}\theta_{,i}dx - (1-\gamma\beta_1)\int_\Omega\Sigma_{,i}\Sigma_{,i}dx\\ &- k(2\beta - \frac{\gamma}{\beta_2})\int_{\partial\Omega}\theta^2dA - (2\tau - k\gamma\beta_2)\int_{\partial\Omega}\Sigma^2dA\\ &+ \Big(\int_\Omega(w_iw_i)^2dx\Big)^{\frac{1}{2}}\Big[\beta A_3^{\frac{1}{2}}(t) + A_4^{\frac{1}{2}}(t)\Big],\end{aligned}\tag{3.15}$$

for some arbitrary positive constants β_1 and β_2.

Now, we use the bound for L_4 norm of w_i which has been derived in [18] (see (B.17)). We write here as the form

$$\Big(\int_\Omega (w_iw_i)^2dx\Big)^{\frac{1}{2}} \leq M\Big\{(1+\frac{\delta}{4})\int_\Omega w_iw_idx + \frac{3}{4}\delta^{-\frac{1}{3}}\int_\Omega w_{i,j}w_{i,j}dx\Big\}, \tag{3.16}$$

where M is a positive computable constant and $\delta > 0$ is an arbitrary constant. To get the bound for $\int_\Omega w_{i,j}w_{i,j}dx$, we use a similar methods which were used in (2.41) and (2.48) with $\alpha = \frac{f_0}{2k_0}$ to have

$$\int_\Omega w_{i,j}w_{i,j}dx \leq 2\int_\Omega w_{i,j}(w_{i,j}-w_{j,i})dx + \frac{2k_0m_3}{f_0}\int_\Omega w_iw_idx. \tag{3.17}$$

To handle the first term of (3.17), we compute

$$\begin{aligned}
&\int_\Omega (w_{i,j}-w_{j,i})(w_{i,j}-w_{j,i})dx\\
=&2\int_\Omega w_{i,j}(w_{i,j}-w_{j,i})dx\\
=&2\int_\Omega w_{i,j}[-\pi_{,ij}+g_i\theta_{,j}+h_i\Sigma_{,j}+\sigma B_0^2(\overline{k}_iv^*_{3,j}-v^*_{i,j})+\sigma_1B_0^2(\overline{k}_iw_{3,j}-w_{i,j})]dx\\
&-2\int_\Omega w_{i,j}[-\pi_{,ij}+g_j\theta_{,i}+h_j\Sigma_{,i}+\sigma B_0^2(\overline{k}_jv^*_{3,i}-v^*_{j,i})+\sigma_1B_0^2(\overline{k}_jw_{3,j}-w_{j,i})]dx\\
=&2\int_\Omega [g_i\theta_{,j}-g_j\theta_{,i}]w_{i,j}dx + 2\int_\Omega [g_j\Sigma_{,i}-g_i\Sigma_{,j}]w_{i,j}dx\\
&+2\sigma B_0^2\int_\Omega [\overline{k}_iv^*_{3,j}-\overline{k}_jv^*_{3,i}]w_{i,j}dx - 2\sigma B_0^2\int_\Omega [v^*_{i,j}-v^*_{j,i}]w_{i,j}dx\\
&+2\sigma_1B_0^2\int_\Omega [\overline{k}_iw_{3,j}-\overline{k}_jw_{3,i}]w_{i,j}dx - 2\sigma_1B_0^2\int_\Omega [w_{i,j}-w_{j,i}]w_{i,j}dx.
\end{aligned} \tag{3.18}$$

Using the Cauchy-Schwarz inequality and the arithmetic-geometric mean inequality, we have

$$\begin{aligned}
2\int_\Omega [g_i\theta_{,j}-g_j\theta_{,i}]w_{i,j}dx &= \int_\Omega [g_i\theta_{,j}-g_j\theta_{,i}][w_{i,j}-w_{j,i}]dx = 2\int_\Omega g_i\theta_{,j}[w_{i,j}-w_{j,i}]dx\\
&\leq 8g^2\int_\Omega \theta_{,j}\theta_{,j}dx + \frac{1}{8}\int_\Omega (w_{i,j}-w_{j,i})(w_{i,j}-w_{j,i})dx\\
&\leq 8g^2\int_\Omega \theta_{,j}\theta_{,j}dx + \frac{1}{4}\int_\Omega (w_{i,j}-w_{j,i})w_{i,j}dx,
\end{aligned} \tag{3.19}$$

and

$$2\int_\Omega [h_i\Sigma_{,j}-h_j\Sigma_{,i}]w_{i,j}dx \leq 8h^2\int_\Omega \Sigma_{,j}\Sigma_{,j}dx + \frac{1}{4}\int_\Omega (w_{i,j}-w_{j,i})w_{i,j}dx. \tag{3.20}$$

Using the Cauchy-Schwarz inequality and the arithmetic-geometric mean inequality, we have

$$\begin{aligned}
&2\sigma B_0^2\int_\Omega [\overline{k}_iv^*_{3,j}-\overline{k}_jv^*_{3,i}]w_{i,j}dx - 2\sigma B_0^2\int_\Omega [v^*_{i,j}-v^*_{j,i}]w_{i,j}dx\\
=&2\sigma B_0^2\int_\Omega \overline{k}_iv^*_{3,j}[w_{i,j}-w_{j,i}]dx - 2\sigma B_0^2\int_\Omega v^*_{i,j}[w_{i,j}-w_{j,i}]dx\\
\leq&8\sigma^2B_0^4\int_\Omega v^*_{3,j}v^*_{3,j}dx + 8\sigma^2B_0^4\int_\Omega v^*_{i,j}v^*_{i,j}dx + \frac{1}{2}\int_\Omega (w_{i,j}-w_{j,i})w_{i,j}dx.
\end{aligned} \tag{3.21}$$

Since $\underline{\overline{\mathbf{k}}} = (0, 0, 1)$, we have

$$\begin{aligned}
&2\sigma_1 B_0^2 \int_\Omega [\overline{k}_i w_{3,j} - \overline{k}_j w_{3,i}] w_{i,j} dx \\
&= 2\sigma_1 B_0^2 \int_\Omega \overline{k}_i w_{3,j}(w_{i,j} - w_{j,i}) dx \\
&= 2\sigma_1 B_0^2 \int_\Omega w_{3,j}(w_{3,j} - w_{j,3}) dx \\
&\leq 2\sigma_1 B_0^2 \int_\Omega w_{i,j}(w_{i,j} - w_{j,i}) dx.
\end{aligned} \tag{3.22}$$

Inserting (3.19)–(3.21) and (3.22) into (3.18), we obtain

$$\int_\Omega w_{i,j}(w_{i,j} - w_{j,i}) dx \leq 8g^2 \int_\Omega \theta_{,j}\theta_{,j} dx + 8h^2 \int_\Omega \Sigma_{,j}\Sigma_{,j} dx + 8\sigma^2 B_0^4 \int_\Omega v^*_{3,j} v^*_{3,j} dx + 8\sigma^2 B_0^4 \int_\Omega v^*_{i,j} v^*_{i,j} dx.$$

It follows from (3.17) that

$$\begin{aligned}
\int_\Omega w_{i,j} w_{i,j} dx \leq & 16g^2 \int_\Omega \theta_{,j}\theta_{,j} dx + 16h^2 \int_\Omega \Sigma_{,j}\Sigma_{,j} dx \\
& + 16\sigma^2 B_0^4 \int_\Omega v^*_{3,j} v^*_{3,j} dx + 16\sigma^2 B_0^4 \int_\Omega v^*_{i,j} v^*_{i,j} dx + \frac{2k_0 m_3}{f_0} \int_\Omega w_i w_i dx.
\end{aligned} \tag{3.23}$$

Combining (3.15), (3.16) and (3.23), we conclude

$$\begin{aligned}
\frac{d}{dt}\Big(\beta \int_\Omega \theta^2 dx + \int_\Omega \Sigma^2 dx\Big) \leq & - M_1 \int_\Omega \theta_{,i}\theta_{,i} dx - M_2 \int_\Omega \Sigma_{,i}\Sigma_{,i} dx - M_3 \int_{\partial\Omega} \theta^2 dA \\
& - M_4 \int_{\partial\Omega} \Sigma^2 dA + M_5 \int_\Omega w_i w_i dx [\beta A_3^{\frac{1}{2}}(t) + A_4^{\frac{1}{2}}(t)] \\
& + M_6 \sigma^2 \Big[\int_\Omega v^*_{3,j} v^*_{3,j} dx + \int_\Omega v^*_{i,j} v^*_{i,j} dx \Big]\Big[\beta A_3^{\frac{1}{2}}(t) + A_4^{\frac{1}{2}}(t)\Big],
\end{aligned} \tag{3.24}$$

where

$$\begin{aligned}
M_1 &= \beta - \frac{\gamma}{\beta_1} - 12g^2 M \delta^{-\frac{1}{3}} [\beta A_3^{\frac{1}{2}}(t) + A_4^{\frac{1}{2}}(t)], \\
M_2 &= 1 - \gamma\beta_1 - 12h^2 M \delta^{-\frac{1}{3}} [\beta A_3^{\frac{1}{2}}(t) + A_4^{\frac{1}{2}}(t)], \\
M_3 &= k(2\beta - \frac{\gamma}{\beta_2}),\ M_4 = 2\tau - k\gamma\beta_2, \\
M_5 &= M(1 + \frac{1}{4}\delta + \frac{3}{4}\delta^{-\frac{1}{3}}),\ M_6 = 12M\delta^{-\frac{1}{3}} B_0^4.
\end{aligned}$$

Choosing $\beta_1 = \frac{1}{2\gamma}$, $\beta_2 = \frac{2\tau}{k\gamma}$ and $\beta = \max\{\frac{k\gamma^2}{4\tau}, 2\gamma^2\}$, we note that $M_3 > 0$, $M_4 = 0$, $\beta - \frac{\gamma}{\beta_1} > 0$ and $1 - \gamma\beta_1 > 0$. Since the constant δ is at our disposal then provided $A_3(t)$ and $A_4(t)$ are bounded, we may choose δ so large that $M_1 \geq 0$ and $M_2 \geq 0$. Dropping the non-positive terms in (3.24) and using Lemma 2.5 and (3.11), we have

$$\frac{d}{dt}\Big(\beta \int_\Omega \theta^2 dx + \int_\Omega \Sigma^2 dx\Big) \leq \mathcal{F}_1(t)\Big(\beta \int_\Omega \theta^2 dx + \int_\Omega \Sigma^2 dx\Big) + \sigma^2 \mathcal{F}_2(t), \tag{3.25}$$

where

$$\mathcal{F}_1(t) = 4M_5(\beta A_3^{\frac{1}{2}}(t) + A_4^{\frac{1}{2}}(t)) \max\{\frac{g^2}{\beta}, h^2\},$$
$$\mathcal{F}_2(t) = 8M_5(\beta A_3^{\frac{1}{2}}(t) + A_4^{\frac{1}{2}}(t))B_0^4 A_5(t) + 2M_6(\beta A_3^{\frac{1}{2}}(t) + A_4^{\frac{1}{2}}(t))B_0^4 A_6(t).$$

From (3.25), we have

$$\frac{d}{dt}\Big\{\Big(\beta\int_\Omega \theta^2 dx + \int_\Omega \Sigma^2 dx\Big)\exp\Big(-\int_0^t \mathcal{F}_1(\eta)d\eta\Big)\Big\} \le \sigma^2\mathcal{F}_2(t)\exp\Big(-\int_0^t \mathcal{F}_1(\eta)d\eta\Big),$$

which follows that

$$\beta\int_\Omega \theta^2 dx + \int_\Omega \Sigma^2 dx \le \sigma^2\int_0^t \mathcal{F}_2(t)\exp\Big(-\int_\eta^t \mathcal{F}_1(\zeta)d\zeta\Big)d\eta. \tag{3.26}$$

This is the continuous dependence result we want to prove. By (3.11), we may obtain the continuous dependence for $\mathbf{v}$,

$$\int_\Omega w_i w_i dx \le \sigma^2\Big[\int_0^t \mathcal{F}_2(t)\exp\Big(-\int_\eta^t \mathcal{F}_1(\zeta)d\zeta\Big)d\eta + 8B_0^4 A_5(t)\Big]. \tag{3.27}$$

4. Continuous dependence on the cooling coefficients

In this section, we derive the continuous dependence on the cooling coefficients and we let (u_i, p, T, C) and (u_i^*, p^*, T^*, C^*) be the solutions to the problem (1.1)–(1.3) for the same initial-boundary data and the same F and G, but for different the cooling coefficients k_1, k_2, τ_1 and τ_2, respectively. As in Section 3, we still set

$$w_i = v_i - v_i^*, \quad \theta = T - T^*, \quad \Sigma = C - C^*, \quad \pi = p - p^*, \quad k = k_1 - k_2, \quad \tau = \tau_1 - \tau_2.$$

Then $(w_i, \theta, \Sigma, \pi)$ satisfy

$$w_i = -\pi_{,i} + g_i\theta + h_i\Sigma + \sigma[(\mathbf{w}\times\mathbf{B}_0)\times\mathbf{B}_0]_i, \tag{4.1}$$
$$\theta_{,t} + v_i^*\theta_{,i} + w_i T_{,i} = \Delta\theta, \tag{4.2}$$
$$\Sigma_{,t} + v_i^*\Sigma_{,i} + w_i C_{,i} = \Delta\Sigma + \gamma\Delta\theta, \tag{4.3}$$
$$w_{i,i} = 0, \tag{4.4}$$

with the initial-boundary conditions

$$w_i n_i = 0, \quad \frac{\partial\theta}{\partial n} + k_1\theta = -kT^*, \quad \frac{\partial\Sigma}{\partial n} + \tau_1\Sigma = -\tau C^*, \quad on\ \ \partial\Omega\times\{t>0\}, \tag{4.5}$$

$$\theta(x,0) = \Sigma(x,0) = 0,\ \ x\in\Omega. \tag{4.6}$$

We now prove the following theorem.

Theorem 4.1. *If* $T_0, C_0 \in L^\infty(\Omega)$*,* $F, G \in C^4(\partial\Omega \times \{t > 0\})$*, then the solution of Eq (1.1)–*(1.3) *with initial-boundary conditions (1.5) and (1.6) depends continuously on the boundary parameters* k *and* τ *in the sense that*

$$\beta \int_\Omega \theta^2 dx + \int_\Omega \Sigma^2 dx \le L_3 k^2 + L_4 \tau^2.$$

Further, $\mathbf{v}$ *depends continuously on* k *and* τ *in the manner*

$$\int_\Omega w_i w_i dx \le L_5 k^2 + L_6 \tau^2,$$

where L_3*–*L_6 *are a priori constants.*

Proof. Employing a similar methods of the last section, we have

$$\int_\Omega w_i w_i dx \le 4g^2 \int_\Omega \theta^2 dx + 4h^2 \int_\Omega \Sigma^2 dx, \tag{4.7}$$

and

$$\int_\Omega w_{i,j} w_{i,j} dx \le 16g^2 \int_\Omega \theta_{,j}\theta_{,j} dx + 16h^2 \int_\Omega \Sigma_{,j}\Sigma_{,j} dx + \frac{8k_0 m_3}{f_0}\Big(g^2 \int_\Omega \theta^2 dx + h^2 \int_\Omega \Sigma^2 dx\Big). \tag{4.8}$$

By using (4.2), (4.3) and the divergence theorem, as the calculation in (3.12), we get

$$\begin{aligned}
&\frac{d}{dt}\Big(\beta \int_\Omega \theta^2 dx + \int_\Omega \Sigma^2 dx\Big)\\
&= -2\beta \int_\Omega \theta_{,i}\theta_{,i} dx - 2\int_\Omega \Sigma_{,i}\Sigma_{,i} dx - 2\beta k_1 \int_{\partial\Omega} \theta^2 dA\\
&\quad - 2\beta k \int_{\partial\Omega} \theta T^* dA - 2\tau_1 \int_{\partial\Omega} \Sigma^2 dA - 2\tau \int_{\partial\Omega} \Sigma C^* dA + 2\beta \int_\Omega \theta_{,i} w_i T dx\\
&\quad + 2\int_\Omega \Sigma_{,i} w_i C dx - 2\gamma \int_\Omega \theta_{,i}\Sigma_{,i} dx - 2k_1\gamma \int_{\partial\Omega} \theta\Sigma dA - 2k\gamma \int_{\partial\Omega} T^*\Sigma dA.
\end{aligned} \tag{4.9}$$

We note that (3.13) and (3.14) are still valid in this section. We inserting them into (4.9) and use Cauchy-Schwarz inequality in the other terms on the right of (4.9) to have

$$\begin{aligned}
&\frac{d}{dt}(\beta \int_\Omega \theta^2 dx + \int_\Omega \Sigma^2 dx)\\
&\le -(\beta - \frac{\gamma}{\beta_1}) \int_\Omega \theta_{,i}\theta_{,i} dx - (1 - \gamma\beta_1) \int_\Omega \Sigma_{,i}\Sigma_{,i} dx\\
&\quad - (2\beta k_1 - \beta\beta_3 - \frac{k_1\gamma}{\beta_2}) \int_{\partial\Omega} \theta^2 dA - (2\tau_1 - \beta_4 - k_1\gamma\beta_2 - \gamma\beta_5) \int_{\partial\Omega} \Sigma^2 dA\\
&\quad + \Big(\int_\Omega (w_i w_i)^2 dx\Big)^{\frac{1}{2}} [\beta A_3^{\frac{1}{2}}(t) + A_4^{\frac{1}{2}}(t)] + k^2(\frac{\beta}{\beta_3} + \frac{\gamma}{\beta_5}) \int_{\partial\Omega} (T^*)^2 dA + \frac{\tau^2}{\beta_4} \int_{\partial\Omega} (C^*)^2 dA.
\end{aligned} \tag{4.10}$$

We use the inequality (3.16) again and use (4.8) to have

$$\int_\Omega (w_i w_i)^2 dx \le M\Big\{\int_\Omega w_i w_i dx + \delta^{-\frac{1}{3}}\Big[\int_\Omega \theta_{,i}\theta_{,i} dx + \int_\Omega \Sigma_{,i}\Sigma_{,i} dx\Big]\Big\}, \tag{4.11}$$

where M is a positive computable constant. Inserting (4.11) into (4.10) and letting

$$\beta_1 = \frac{1}{\gamma},\ \beta_2 = \frac{\tau_1}{2k_1\gamma},\ \beta_3 = k_1,\ \beta_4 = \tau_1,\ \beta_5 = \frac{\tau_1}{2\gamma},$$

and then choosing β and δ large enough such that the coefficients of the first four terms of (4.10) are non-positive, we have

$$\begin{aligned}&\frac{d}{dt}\Big(\beta\int_\Omega \theta^2 dx + \int_\Omega \Sigma^2 dx\Big)\\ \le& M\int_\Omega w_i w_i dx\Big[\beta A_3^{\frac{1}{2}}(t) + A_4^{\frac{1}{2}}(t)\Big] + k^2\Big(\frac{\beta}{\beta_3}+\frac{\gamma}{\beta_5}\Big)\int_{\partial\Omega}(T^*)^2 dA + \frac{\tau^2}{\beta_4}\int_{\partial\Omega}(C^*)^2 dA,\end{aligned} \tag{4.12}$$

where we have dropped the non-positive terms. Now, we derive bounds for the integrals on $\partial\Omega$. Using Lemma 2.1, we find

$$\int_{\partial\Omega}(T^*)^2 dA \le \frac{m_3}{f_0}\int_\Omega (T^*)^2 dx + \int_\Omega T^*_{,i}T^*_{,i}dx, \tag{4.13}$$

and

$$\int_{\partial\Omega}(C^*)^2 dA \le \frac{m_3}{f_0}\int_\Omega (C^*)^2 dx + \int_\Omega C^*_{,i}C^*_{,i}dx, \tag{4.14}$$

where we have chosen $\alpha = 1$. Inserting (4.13) and (4.14) into (4.12) and recalling (2.8) and (4.7), we have

$$\begin{aligned}\frac{d}{dt}\Big(\beta\int_\Omega \theta^2 dx + \int_\Omega \Sigma^2 dx\Big) \le& \widetilde{\mathcal{F}}_1(t)\Big[\beta\int_\Omega \theta^2 dx + \int_\Omega \Sigma^2 dx\Big] + k^2\Big(\frac{\beta}{\beta_3}+\frac{\gamma}{\beta_5}\Big)\frac{m_3}{f_0}A_1(t)\\ &+ k^2\Big(\frac{\beta}{\beta_3}+\frac{\gamma}{\beta_5}\Big)\int_\Omega T^*_{,i}T^*_{,i}dx + \frac{\tau^2 m_3}{f_0\beta_4}A_2(t) + \frac{\tau^2}{\beta_4}\int_\Omega C^*_{,i}C^*_{,i}dx,\end{aligned} \tag{4.15}$$

where

$$\widetilde{\mathcal{F}}_1(t) = 4M\max\{\frac{g^2}{\beta}, h^2\}\Big[\beta A_3^{\frac{1}{2}}(t) + A_4^{\frac{1}{2}}(t)\Big].$$

It is obvious that (4.15) yields that

$$\begin{aligned}&\frac{d}{dt}\Big[\Big(\beta\int_\Omega \theta^2 dx + \int_\Omega \Sigma^2 dx\Big)\cdot \exp\Big(-\int_0^t \widetilde{\mathcal{F}}_1(\eta)d\eta\Big)\Big]\\ \le&\Big\{k^2\Big(\frac{\beta}{\beta_3}+\frac{\gamma}{\beta_5}\Big)\frac{m_3}{f_0}A_1(t) + k^2\Big(\frac{\beta}{\beta_3}+\frac{\gamma}{\beta_5}\Big)\int_\Omega T^*_{,i}T^*_{,i}dx\\ &+\frac{\tau^2 m_3}{f_0\beta_4}A_2(t) + \frac{\tau^2}{\beta_4}\int_\Omega C^*_{,i}C^*_{,i}dx\Big\}\cdot \exp\Big(-\int_0^t \widetilde{\mathcal{F}}_1(\eta)d\Big)\\ \le& k^2\Big(\frac{\beta}{\beta_3}+\frac{\gamma}{\beta_5}\Big)\frac{m_3}{f_0}A_1(t) + k^2\Big(\frac{\beta}{\beta_3}+\frac{\gamma}{\beta_5}\Big)\int_\Omega T^*_{,i}T^*_{,i}dx + \frac{\tau^2 m_3}{f_0\beta_4}A_2(t) + \frac{\tau^2}{\beta_4}\int_\Omega C^*_{,i}C^*_{,i}dx,\end{aligned} \tag{4.16}$$

where we have used the fact $\exp\Big(-\int_0^t \widetilde{\mathcal{F}}_1(\eta)d\eta\Big) \le 1$ for $t > 0$.

Integrating (4.16) from 0 to t leads to

$$
\begin{aligned}
&\Big(\beta\int_\Omega \theta^2 dx+\int_\Omega \Sigma^2 dx\Big)\cdot\exp\Big(-\int_0^t \widetilde{\mathcal{F}}_1(\eta)d\eta\Big)\\
\leq &k^2\Big(\frac{\beta}{\beta_3}+\frac{\gamma}{\beta_5}\Big)\frac{m_3}{f_0}\int_0^t A_1(\eta)d\eta+k^2\Big(\frac{\beta}{\beta_3}+\frac{\gamma}{\beta_5}\Big)\int_0^t\int_\Omega T^*_{,i}T^*_{,i}dxd\eta\\
&+\frac{\tau^2 m_3}{f_0\beta_4}\int_0^t A_2(\eta)d\eta+\frac{\tau^2}{\beta_4}\int_0^t\int_\Omega C^*_{,i}C^*_{,i}dxd\eta.
\end{aligned}
\tag{4.17}
$$

Using (2.11) and (2.17) in (4.17) and setting

$$
\widetilde{\mathcal{F}}_2(t)=\Big(\frac{\beta}{\beta_3}+\frac{\gamma}{\beta_5}\Big)\Big[\frac{m_3}{f_0}\int_0^t A_1(\eta)d\eta+\frac{1}{2}A_1(t)\Big],\quad \widetilde{\mathcal{F}}_3(t)=\frac{1}{\beta_4}\Big[\frac{m_3}{f_0 A_2(t)}+\int_0^t A_2(\eta)d\eta\Big], \tag{4.18}
$$

we obtain

$$
\beta\int_\Omega \theta^2 dx+\int_\Omega \Sigma^2 dx\leq k^2\widetilde{\mathcal{F}}_2(t)\cdot\exp\Big(\int_0^t\widetilde{\mathcal{F}}_1(\eta)d\eta\Big)+\tau^2\widetilde{\mathcal{F}}_3(t)\cdot\exp\Big(\int_0^t\widetilde{\mathcal{F}}_1(\eta)d\eta\Big). \tag{4.19}
$$

This is the continuous dependence result for T and C. The continuous dependence for v_i follows directly from (4.7).

5. Conclusions

In this paper, the continuous dependence of the solution is obtained by using the methods of energy estimate and a priori estimates. The main innovation is to deal with the influence of boundary conditions and magnetic field. The structural stability of boundary parameters and magnetic field coefficients is proved.

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Conflict of interest

The authors declare that they have no competing interests.

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