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Research article

Stability analysis of the implicit finite difference schemes for nonlinear Schrödinger equation

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Abstract: This paper analyzes the stability of numerical solutions for a nonlinear Schrödinger equation that is widely used in several applications in quantum physics, optical business, etc. One of the most popular approaches to solving nonlinear problems is the application of a linearization scheme. In this paper, two linearization schemes—Newton and Picard methods were utilized to construct systems of linear equations and finite difference methods. Crank-Nicolson and backward Euler methods were used to establish numerical solutions to the corresponding linearized problems. We investigated the stability of each system when a finite difference discretization is applied, and the convergence of the suggested approximation was evaluated to verify theoretical analysis.

Keywords: nonlinear Schrödinger equation; stability; linearization scheme; finite difference method
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1. Introduction

The Schrödinger equation is a key product of quantum mechanics, and its discovery was an important milestone in the development of the field. The one-dimensional nonlinear Schrödinger equation (NLSE) is a nonlinear variant of Schrödinger equation. This is primarily applied to the propagation of light in nonlinear optical fibers and planar waveguides. The equation appears in various studies, including studies on the propagation of a plane diffracted wave beam in the focusing region of the ionosphere [12], the analysis of the small amplitude gravity waves on the surface of inviscid water [21], and Davydov's analysis on the propagation of alpha-helix solitons that are responsible for the transfer of energy along molecular chains [7]. The equation is also solved by several different ways, for example, finite difference discretization of the cubic equation [1], fully discrete Galerkin methods of second-order temporal accuracy [2], conforming and nonconforming conservative virtual element methods [17], relaxation-type Galerkin FEM [18], and conservative numerical methods [25, 26] for the

nonlinear fractional Schrödinger equations. Considering [21], NLSE appears as one of the universal equations describing the evolution of slow-changing packets of quasi-monochromatic waves in a weakly nonlinear medium with dispersion.

Owing to its significant importance in quantum mechanics and its usage in several applications, the NLSE has been studied numerically. Considering the numerical perspectives, the stability of numerical techniques is one of the key factors in obtaining good approximations. To evaluate the stability of the utilized numerical approach, in [8], the NLSE is linearized using extrapolation and analyzed. The paper [10] eliminates nonlinearity by treating the nonlinear part as a constant and analyzes the stability. In [9], the Taylor's expansion at the equilibrium point is considered without a nonlinear term. In [3, 4, 14, 15], B-spline approximations are applied to a nonlinear term of differential equations. None of the previous approaches consider the application of the Newton/Picard methods and their resulting linear systems, considering stability analysis. This study explores the stability of Crank-Nicolson and backward Euler methods applied to the linearized NLSE using the Newton and Picard methods.

Consider the one-dimensional Schrödinger equation:

$$\begin{cases} i\partial_t u + \partial_{xx} u + V(x,t)u = 0, \quad x \in \mathbb{R},\ t > 0, \\ \lim_{|x|\to\infty} u(x,t) = 0, \\ u(x,t_0) = u_0(x), \end{cases} \tag{1.1}$$

where V denotes a potential. If the potential term is zero (i.e., $V(x,t) = 0$), then the equation represents the particle movement in a vacuum space, similar to a free particle. In this study, we consider the classical field equation, $V(x,t) = |u(x,t)|^2$, known as a cubic NLSE with a nonlinear potential [27]. We assume that the initial data are compactly supported in some interval I. Moreover, we assume that V is constant outside this interval. The solution to (1.1) is defined on the unbounded domain. To solve the problem numerically, the problem needs to be converted to that in a finite domain. If the truncation of the domain is considered, then some transformations can be used to handle the unbounded problem [5]. One way is to restrict the computational domain by absorbing some "energy" at the artificial boundary. This prevents unexpected reflection, known as the absorbing boundary conditions (ABC). Another way is to impose the transparent boundary condition (TBC) using the Dirichlet-Neumann operator, which only works well for a linear problem. However, it produces several reflections at the boundary in the presence of nonlinear perturbation [13]. Here, we consider the spatial domain as $[-\eta, \eta]$ for sufficiently large η to avoid the complication of handling ABC or TBC. Therefore, the equation retains the homogeneous Dirichlet boundary condition for u, such that $u(-\eta, t) = u(\eta, t) = 0$ for a large η.

The outline of this study is as follows. First, we propose a model problem, the one-dimensional nonlinear Schrödinger equation with complex valued function divided into two real functions. Thereafter, Newton and Picard methods are applied to generate the corresponding linear problems. Subsequently, Crank-Nicolson and the backward Euler methods are used to build discrete systems. Finally, we analyze the stability of each method, and several numerical test results are presented to confirm the analysis.

2. Model problem

Consider the one-dimensional nonlinear Schrödinger equation (NLSE) in a finite domain:

$$\begin{cases} i\partial_t u + \partial_{xx} u + |u|^2 u = 0, \\ u(-\eta, t) = u(\eta, t) = 0, \\ u(x, t_0) = u_0(x), \end{cases} \tag{2.1}$$

where $x \in [-\eta, \eta]$ for a sufficiently large η, $t \in \mathbb{R}$, $i = \sqrt{-1}$, and u_0 are initial conditions. We assume that $|u|^2$ is small, that is, only when the background density has a maximum. This is to disallow the background density value from changing greatly owing to dynamics [10]. The solution u can be divided into two parts, real and imaginary parts, such that

$$u(x, t) = p(x, t) + iq(x, t) \quad \text{for } x \in [0, 1],\ t \in \mathbb{R}, \tag{2.2}$$

where p and q are real-valued functions, and u_0 is decomposed as $p_0 + iq_0$. Substituting (2.2) into (2.1) yields

$$i\partial_t(p + iq) + \partial_{xx}(p + iq) + (p^2 + q^2)(p + iq) = 0.$$

This transforms the problem solving of the equation in (2.1) to a problem solving the following system of equations

$$\partial_t p + \partial_{xx} q + (p^2 + q^2)q = 0 \quad \text{and} \quad \partial_t q - \partial_{xx} p - (p^2 + q^2)p = 0, \tag{2.3}$$

where $p(\pm\eta, t) = 0$, $q(\pm\eta, t) = 0$ for $t > 0$ and $p(x, t_0) = p_0(x)$, $q(x, t_0) = q_0(x)$ for $x \in [-\eta, \eta]$.

The aim of this study is to analyze the stability issues in solving NLSE with various finite difference methods. Therefore, we consider the system of two real equations in (2.3) instead of dealing with complex variables and $i\partial_t u + \partial_{xx} u + |u|^2 u = 0$.

3. Linearization schemes

To apply the linearization schemes in (2.3), set

$$f_1(p, q) := \partial_t p + \partial_{xx} q + (p^2 + q^2)q \quad \text{and} \quad f_2(p, q) := \partial_t q - \partial_{xx} p - (p^2 + q^2)p. \tag{3.1}$$

Then, to eliminate the nonlinearity of $(p^2 + q^2)q$ and $(p^2 + q^2)p$ in (3.1), we employ the Newton and Picard methods and introduce two linearized iterate system for solving the model problem.

3.1. Newton's method

Newton's linearization scheme converts a nonlinear problem (2.3) to a linear problem by adopting the first-order Taylor expansion of f_1 and f_2 at a given (p_0, q_0)

$$f_j(p, q) = f_j(p_0, q_0) + \nabla f_j(p_0, q_0)(\tilde{p}, \tilde{q}), \quad j = 1, 2$$

in which $\tilde{p} = \tau p_0 + (1 - \tau)p$ and $\tilde{q} = \theta q_0 + (1 - \theta)q$ for some $\tau, \theta \in [0, 1]$. Therefore, it can solve

$$\nabla f_j(p_0, q_0)(\delta p, \delta q) = -f_j(p_0, q_0), \quad j = 1, 2 \tag{3.2}$$

to obtain an approximation $(\hat{p}, \hat{q}) := (p_0 + \delta p, q_0 + \delta q)$ for the solution (p, q) of (2.3) under the assumption (p_0, q_0) is close to (p, q). Practically, obtaining a good initial guess (p_0, q_0) acquires an extra preliminary process (e.g., considering several Picard iterations first to obtain an initial guess). Regarding some cases, it is known that the solution of the linear problem (the problem without a nonlinear term) can provide a good initial guess. Therefore, solving (2.3) is recast into solving

$$\begin{cases} \partial_t p + \partial_{xx} q + (p_0^2 q + 2p_0 q_0 p + 3q_0^2 q) = -f_1(p_0, q_0), \\ \partial_t q - \partial_{xx} p - (3p_0^2 p + 2p_0 q_0 q + q_0^2 p) = -f_2(p_0, q_0), \\ p(-\eta, t) = p(\eta, t) = 0, \quad q(-\eta, t) = q(\eta, t) = 0, \\ p(x, t_0) = 0, \quad q(x, t_0) = 0, \end{cases} \tag{3.3}$$

where δp and δq are denoted as p and q, respectively, for notational simplicity. The boundary and initial conditions for δp and δq are induced from (2.1).

3.2. Picard method

The Picard method is an iterative scheme that replaces the nonlinear term with the approximation from the previous iteration. Considering a given nonlinear problem, $F(U) = \mathcal{L}U + \mathcal{N}(U) = G$ with $\mathcal{L}$ (the linear operator) and $\mathcal{N}$ (the nonlinear operator), the conventional approach obtains an approximation by iteratively solving

$$\mathcal{L}U^{n+1} = G - \mathcal{N}(U^n)$$

with the previously obtained U^n. Therefore, regarding a given (p_0, q_0), the system (2.3) is transformed into the linear system

$$\partial_t p + \partial_{xx} q = -(p_0^2 + q_0^2)\, q_0 \quad \text{and} \quad \partial_t q - \partial_{xx} p = (p_0^2 + q_0^2)\, p_0.$$

Here, we substitute the subscripts denoting the iteration index n for notational simplicity. Moreover, (p, q) and (p_0, q_0) denote (p^{n+1}, q^{n+1}) and (p^n, q^n), respectively). To modify the Picard iteration, set $\mathcal{N}(u, v, w) = (u^2 + v^2)w$ and apply the Taylor expansion of $\mathcal{N}$ at (u_0, v_0, w_0) in the direction of (u_0, v_0, w) such as

$$\mathcal{N}(u_0, v_0, w_0) = \mathcal{N}(u_0, v_0, w) + (w_0 - w)\mathcal{N}_w(u_0, v_0, w_0) + O(|w_0 - w|^2) \approx \mathcal{N}(u_0, v_0, w)$$

considering the assumption that w and w_0 are very close. Thus, the modified Picard iteration transforms the problem (2.3) into the linear problem

$$\partial_t p + \partial_{xx} q + (p_0^2 + q_0^2)\, q = 0 \quad \text{and} \quad \partial_t q - \partial_{xx} p - (p_0^2 + q_0^2)\, p = 0, \tag{3.4}$$

where $p(\pm\eta, t) = 0$, $q(\pm\eta, t) = 0$ for $t > t_0$ and $p(x, t_0) = p_0(x)$, $q(x, t_0) = q_0(x)$ for $x \in [-\eta, \eta]$.

Remark 1. *The goal of this paper is analyzing the stability of linearized system* (3.4)*. However, there are several equally important issues, including the convergence of the solution of the linearized problem toward the solution of the original nonlinear problem* (2.1)*, in linearization schemes. Since the convergence analysis of linearization schemes is totally different regime and it has to be handled carefully with thorough analysis, we do not handle the convergence of linearization schemes here. We refer the interested reader to related analysis in [19, 20].*

4. Stability analysis

In this section, we analyze the stability of the finite difference schemes to solve the systems (3.3) and (3.4). This study focuses on the numerical stabilities when two implicit methods are applied for t-derivative, ∂_t. Considering ∂_{xx}, the centered difference scheme is chosen and is consistently used throughout the study.

We define the domain $\Omega = [-\eta, \eta] \times [t_0, T]$ for the fixed constants η and $T > t_0$. Let the interval $[-\eta, \eta]$ be divided into K-subintervals, $[x_k, x_{k+1}]$ for $k = 0, 1, \cdots, K-1$ with a uniform mesh $\Delta x := x_{k+1} - x_k$, and let $[t_0, T]$ be divided into N-subintervals, such that $t_0 < t_1 < \cdots < t_N = T$ with a uniform step size $\Delta t = t_{n+1} - t_n$. We denote w_k^n as an approximation for the exact value of 'w', which is evaluated at (x_k, t_n), $w(x_k, t_n)$.

4.1. Newton's method

4.1.1. Newton's method: Crank-Nicolson

It is well known that when a partial differential equation is provided as

$$\partial_t w = F(w), \tag{4.1}$$

the Crank-Nicolson (CN) method discretizes the above such as

$$\frac{w^{n+1} - w^n}{\Delta t} = \frac{1}{2}\Big(F(w^n) + F(w^{n+1})\Big).$$

Therefore, the resulting discrete system corresponding to (3.3) is

$$\begin{cases} p^{n+1} + \dfrac{\Delta t}{2}\Big(\partial_{xx} q^{n+1} + p_0^2 q^{n+1} + 2p_0 q_0 p^{n+1} + 3q_0^2 q^{n+1}\Big) \\ \qquad = p^n - \dfrac{\Delta t}{2}\Big(\partial_{xx} q^n + p_0^2 q^n + 2p_0 q_0 p^n + 3q_0^2 q^n\Big) - \Delta t f_1(p_0, q_0), \\ q^{n+1} - \dfrac{\Delta t}{2}\Big(\partial_{xx} p^{n+1} + 3p_0^2 p^{n+1} + 2p_0 q_0 q^{n+1} + q_0^2 p^{n+1}\Big) \\ \qquad = q^n + \dfrac{\Delta t}{2}\Big(\partial_{xx} p^n + 3p_0^2 p^n + 2p_0 q_0 q^n + q_0^2 p^n\Big) - \Delta t f_2(p_0, q_0), \end{cases} \tag{4.2}$$

where the discretization in the x-variable is implicitly embedded in the above to simplify the notation. More precisely, the left hand side of first equation in (4.2) is

$$p_k^{n+1} + \frac{\Delta t}{2}\left(\frac{q_{k-1}^{n+1} - 2q_k^{n+1} + q_{k+1}^{n+1}}{\Delta x^2} + p_0^2 q_k^{n+1} + 2p_0 q_0 p_k^{n+1} + 3q_0^2 q_k^{n+1}\right) \tag{4.3}$$

in which $p_0 = p^n = p^n(x_k, t_n)$ and $q_0 = q^n = q^n(x_k, t_n)$ where p_0, q_0 are obtained from the previous Newton step. The right hand side can also be delineated analogously. Thus, the full system is recast into the following algebraic system:

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} \mathbf{p}^{n+1} \\ \mathbf{q}^{n+1} \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} \mathbf{p}^n \\ \mathbf{q}^n \end{bmatrix} + \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}, \tag{4.4}$$

where $\mathbf{p}^n = (p_k^n)_{k=1}^{K-1}$ and $\mathbf{q}^n = (q_k^n)_{k=1}^{K-1}$. Here, the right hand sides $\mathbf{f}_1$ and $\mathbf{f}_2$ are composed of $-\Delta t f_1(p_0, q_0)$, $-\Delta t f_2(p_0, q_0)$ and the terms from the left hand side, involving the initial conditions provided for p and q. Considering (4.4), each block matrix, $A_j \in \mathbb{R}^{(K-1)\times(K-1)}$, presents

$$A_1 = I + \Delta t I_{p_0 q_0},\ A_2 = \frac{\Delta t}{2}\left(S + I_{p_0^2 + 3q_0^2}\right),\ A_3 = -\frac{\Delta t}{2}\left(S + I_{3p_0^2 + q_0^2}\right),\ A_4 = I - \Delta t I_{p_0 q_0}$$

with $I_\alpha = \mathrm{diag}(\alpha(x_1, t_n), \alpha(x_2, t_n), \cdots, \alpha(x_{K-1}, t_n))$, a diagonal matrix satisfying the condition that the diagonal components are filled with the evaluated α. These are the functional values at each node, and S, the tridiagonal symmetric matrix discretizing ∂_{xx}, has $-2/(\Delta x^2)$ on the diagonal and $1/(\Delta x^2)$ on one lower/upper diagonal part. The right hand side matrix resembles the left hand side, such that

$$B_1 = A_4,\ \ B_2 = -A_2,\ \ B_3 = -A_3,\ \ B_4 = A_1.$$

By setting $\mathbf{u}^n = [\mathbf{p}^n; \mathbf{q}^n]$ and $\mathbf{F} = [\mathbf{f}_1; \mathbf{f}_2]$, the system (4.4) can be simply denoted as

$$A\mathbf{u}^{n+1} = B\mathbf{u}^n + \mathbf{F}, \tag{4.5}$$

where $A = [A_1, A_2; A_3, A_4]$ and $B = [B_1, B_2; B_3, B_4]$. If $\mathbf{u} = [\mathbf{p}; \mathbf{q}]$ is the exact solution of (3.3) with $\mathbf{p}$, and $\mathbf{q}$ is the exact function value evaluated at the corresponding nodes of $\mathbf{p}^n$ and $\mathbf{q}^n$, then combining them with (4.5) yields

$$\mathbf{E}^{n+1} = A^{-1}B\mathbf{E}^n = \cdots = \left(A^{-1}B\right)^{n+1}\mathbf{E}^0, \tag{4.6}$$

where $\mathbf{E}^n := \mathbf{u} - \mathbf{u}^n$, the error of the scheme (4.5) is dominated by $A^{-1}B$ at each iteration. Assume that A is invertible; therefore, the following inequality indicates that the stability of the numerical scheme can be analyzed by exploring the eigenvalues of $A^{-1}B$

$$\|\mathbf{E}^{n+1}\|_{\ell_2} \le \|A^{-1}B\|_{\ell_2}^{n+1}\|\mathbf{E}^0\|_{\ell_2} \le |\lambda_{A^{-1}B,\max}|^{n+1}\|\mathbf{E}^0\|_{\ell_2}, \tag{4.7}$$

where $\|\mathbf{w}\|_{\ell_2} = \sqrt{w_1^2 + \cdots + w_{2K-2}^2}$, $\|A^{-1}B\|_{\ell_2} = \sup_{\mathbf{0}\neq\mathbf{x}\in\mathbb{R}^{2K-2}} \|A^{-1}B\mathbf{x}\|_{\ell_2}/\|\mathbf{x}\|_{\ell_2}$, and $\lambda_{A^{-1}B,\max}$ represents the eigenvalues of $A^{-1}B$, which has the largest magnitude. Therefore, the rest of this subsection focuses on finding the eigenvalue of $A^{-1}B$. The submatrices A_j's are mostly composed of diagonal matrices, excluding the matrix S. The eigenvalues of matrix S is well-known.

Remark 2. *The matrix S is a Toeplitz matrix in which each descending diagonal from left to right is a constant, $-2/(\Delta x^2)$. Moreover, the eigenvalue of this tridiagonal symmetric Toeplitz matrix S is well-known [16], and we denote the eigenvalues of S as*

$$\lambda_{S_k} = \frac{2}{\Delta x^2}\left(\cos(k\pi\Delta x) - 1\right),\ \ k = 1, 2, \cdots, K-1.$$

Therefore, the corresponding eigenvector is $\mathbf{x}_k = [\sin(k\pi\Delta x), \sin(2k\pi\Delta x), \cdots, \sin((K-1)k\pi\Delta x)]^{\mathrm{T}}$ with superscript $^{\mathrm{T}}$, and the vector transpose here.

Remark 3. *Because $A + B = 2I$, A and B share the eigenvectors and their corresponding eigenvalues satisfy $\lambda_A = 2 - \lambda_B$.*

Subsequently, refer to the well-known formula [6] for the inverse matrix of a 2×2 block matrix before we further investigate the structure of A and B.

Lemma 1. *Let $P \in \mathbb{R}^{n\times n}, Q \in \mathbb{R}^{n\times m}, R \in \mathbb{R}^{m\times n}$ and $T \in \mathbb{R}^{m\times m}$. If P and $T - RP^{-1}Q$ are nonsingular, therefore,*

$$\begin{bmatrix} P & Q \\ R & T \end{bmatrix}^{-1} = \begin{bmatrix} P^{-1} + P^{-1}Q(T - RP^{-1}Q)^{-1}RP^{-1} & -P^{-1}Q(T - RP^{-1}Q)^{-1} \\ -(T - RP^{-1}Q)^{-1}RP^{-1} & (T - RP^{-1}Q)^{-1} \end{bmatrix}.$$

Finding a specific eigenvalue of A or B in (4.4) is impossible in the current form; thus, we first simplify the systems by assuming that

$$p_0(x,t) \approx p_0 \quad \text{and} \quad q_0(x,t) \approx q_0 \tag{4.8}$$

with p_0 and q_0 constants at each iteration for detailed analysis. Subsequently, we eliminate the above assumption and retain the analysis. Applying (4.8) yields the matrices A and B of

$$A = \begin{bmatrix} (1+a)\,I & \frac{1}{2}L_1 \\ -\frac{1}{2}L_2 & (1-a)\,I \end{bmatrix}, \quad B = \begin{bmatrix} (1-a)\,I & -\frac{1}{2}L_1 \\ \frac{1}{2}L_2 & (1+a)\,I \end{bmatrix},$$

respectively, where $a = \Delta t p_0 q_0$, $L_1 = \Delta t(S + (p_0^2 + 3q_0^2)I)$ and $L_2 = \Delta t(S + (3p_0^2 + q_0^2)I)$.

To discuss the stability of the system (4.4), we focus on the structure of the eigensystem of the matrices A and B in (4.5) using Remark 2. The above sub-matrices A_j and $B_j, j = 1, \cdots, 4$ are nonsingular. Because $|p_0| + |q_0| < l$ for some constants l and $\Delta t, \Delta h \ll 1$, we can demonstrate that $A_4 - A_3A_1^{-1}A_2 = (1-a)\,I + \frac{1}{4}(1+a)^{-1}L_2L_1$ is positively definite under the assumption that $\lambda_S + 4l^2 < 0$ as follows. Because L_1 and L_2 are commutative, the eigenvalues of L_1L_2 are the same with $\lambda_{L_1}\lambda_{L_2}$ [24] and λ_{L_1}. The eigenvalue of L_1 and λ_{L_2} is the eigenvalue of L_2, such that

$$\begin{aligned} \lambda_{L_1}\lambda_{L_2} - 4a^2 &= \Delta t^2(\lambda_S + (p_0^2 + 3q_0^2))(\lambda_S + (3p_0^2 + q_0^2)) - 4\Delta t^2 p_0^2 q_0^2 \\ &= \Delta t^2\left(\lambda_S + 3(p_0^2 + q_0^2)\right)(\lambda_S + p_0^2 + q_0^2) > 0 \end{aligned} \tag{4.9}$$

in which I and S share the eigenvectors and $(1-a)(1+a) + \frac{1}{4}\lambda_{L_1}\lambda_{L_2} = 1 + \frac{1}{4}(\lambda_{L_1}\lambda_{L_2} - 4a^2) > 0$. Thus, Lemma 1 yields

$$A^{-1} = \begin{bmatrix} \left(\frac{1}{1+a}\right)I - \frac{1}{4}\left(\frac{1}{1+a}\right)^2 L_1ML_2 & -\frac{1}{2}\frac{1}{1+a}L_1M \\ \frac{1}{2}\frac{1}{1+a}ML_2 & M \end{bmatrix},$$

where $M = \left((1-a)\,I + \frac{1}{4}\frac{1}{1+a}L_1L_2\right)^{-1}$; hence, we obtain

$$A^{-1}B = \begin{bmatrix} (A^{-1}B)_{11} & (A^{-1}B)_{12} \\ (A^{-1}B)_{21} & (A^{-1}B)_{22} \end{bmatrix}, \tag{4.10}$$

where

$$\begin{aligned} (A^{-1}B)_{11} &= (1-a)\left(\frac{1}{1+a}I - \frac{1}{4}\left(\frac{1}{1+a}\right)^2 L_1ML_2\right) - \frac{1}{4}\frac{1}{1+a}L_1ML_2, \\ (A^{-1}B)_{12} &= -\frac{1}{2}L_1\left(\frac{1}{1+a}I - \frac{1}{4}\left(\frac{1}{1+a}\right)^2 L_1ML_2\right) - \frac{1}{2}\frac{1+a}{1+a}L_1M, \\ (A^{-1}B)_{21} &= \frac{1}{2}\frac{1-a}{1+a}ML_2 + \frac{1}{2}ML_2, \\ (A^{-1}B)_{22} &= -\frac{1}{4}\frac{1}{1+a}L_1ML_2 + (1+a)\,M. \end{aligned}$$

As mentioned in Remark 2, the matrix S has $K-1$ distinct eigenvalues, λ_S, and a corresponding eigenvector, for example $\mathbf{x}$. Because I and S share the same eigenvector $\mathbf{x}$, we obtain

$$\lambda_{L_1} = \Delta t(\lambda_S + (p_0^2 + 3q_0^2)), \quad \lambda_{L_2} = \Delta t(\lambda_S + (3p_0^2 + q_0^2)),$$
$$\lambda_M = \frac{1+a}{(1+a)(1-a) + \frac{1}{4}\lambda_{L_1}\lambda_{L_2}} = \frac{1+a}{1 + \frac{1}{4}\Delta t^2\left(\lambda_S + 3(p_0^2+q_0^2)\right)\left(\lambda_S + (p_0^2+q_0^2)\right)},$$

where λ_B denotes the eigenvalue of the matrix B. Furthermore, one can easily observe that

$$L_1 M L_2 \mathbf{x} = \lambda_{L_2}(L_1 M \mathbf{x}) = \lambda_{L_2}\lambda_M(L_1\mathbf{x}) = \lambda_{L_2}\lambda_M\lambda_{L_1}\mathbf{x}, \tag{4.11}$$

which yields

$$A^{-1}B\begin{bmatrix}\mathbf{x}\\ \alpha\mathbf{x}\end{bmatrix} = \begin{bmatrix}(A^{-1}B)_{11} & (A^{-1}B)_{12}\\ (A^{-1}B)_{21} & (A^{-1}B)_{22}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \alpha\mathbf{x}\end{bmatrix} = \begin{bmatrix}(a_1+\alpha a_2)\mathbf{x}\\ (a_3+\alpha a_4)\mathbf{x}\end{bmatrix} = \lambda\begin{bmatrix}\mathbf{x}\\ \alpha\mathbf{x}\end{bmatrix}. \tag{4.12}$$

The goal is to find α to make the above hold for some λ when

$$a_1 = \tfrac{1}{N_1}\left(1 - 2a - \tfrac{1}{4}(\lambda_{L_1}\lambda_{L_2} - 4a^2)\right), \quad a_2 = -\tfrac{\lambda_{L_1}}{N_1}, \quad a_3 = \tfrac{\lambda_{L_2}}{N_1},$$
$$a_4 = \tfrac{1}{N_1}\left(1 + 2a - \tfrac{1}{4}(\lambda_{L_1}\lambda_{L_2} - 4a^2)\right) \quad \text{with } N_1, \text{ the denominator of } \lambda_M.$$

Considering (4.12), α must satisfy $a_1 + a_2\alpha = \lambda$ and $a_3 + a_4\alpha = \alpha\lambda$, implying that

$$a_2\alpha^2 + (a_1 - a_4)\alpha - a_3 = 0. \tag{4.13}$$

Moreover, $N_1 = (1+a)(1-a) + \frac{1}{4}\lambda_{L_1}\lambda_{L_2} > 0$ from $\lambda_S + 4l^2 < 0$ implies that the denominator of λ_M is not 0 and the coefficient of α^2 is

$$a_2 = -\frac{1}{N_1}\Delta t(\lambda_S + p_0^2 + 3q_0^2) > 0.$$

Therefore, the Eq (4.13) yields

$$\alpha_\pm = \frac{a_4 - a_1 \pm \sqrt{(a_1-a_4)^2 + 4a_2a_3}}{2a_2}. \tag{4.14}$$

So, we obtain a pair of eigenvalues $\lambda_\pm$ such as

$$\lambda_\pm = a_1 + a_2\alpha_\pm$$

and the corresponding eigenvectors $\begin{bmatrix}\mathbf{x}\\ \alpha_\pm\mathbf{x}\end{bmatrix}$. Here, the eigenvalues $\lambda_\pm$ are determined by p_0, q_0 and λ_{S_k} in Remark 2. As there are λ_{S_k}, $k = 1, \cdots, K-1$ for each α_j, $j = 1, 2$, we have found $2(K-1)$ distinct eigenvalues of $A^{-1}B$ and the eigenspace $\left\{\begin{bmatrix}\mathbf{x}_k\\ \alpha_+\mathbf{x}_k\end{bmatrix}, \begin{bmatrix}\mathbf{x}_k\\ \alpha_-\mathbf{x}_k\end{bmatrix} | k = 1, \cdots, K-1\right\}$. Finally, we focus on the size of the eigenvalue λ of $A^{-1}B$. Considering the formula $\alpha_\pm$ in (4.14),

$$\lambda = a_1 + a_2\alpha = \frac{a_1 + a_4}{2} \pm \frac{1}{2}\sqrt{(a_1-a_4)^2 + 4a_2a_3}. \tag{4.15}$$

Because $a_1 + a_4 = \frac{2}{N_1}\left(1 - \frac{1}{4}(\lambda_{L_1}\lambda_{L_2} - 4a^2)\right)$ and $(a_1 - a_4)^2 + 4a_2a_3 = -\frac{4}{N_1^2}(\lambda_{L_1}\lambda_{L_2} - 4a^2)$, we obtain

$$\lambda = \frac{1}{N_1}\left(1 - \frac{1}{4}(\lambda_{L_1}\lambda_{L_2} - 4a^2)\right) \pm i\frac{\sqrt{\lambda_{L_1}\lambda_{L_2} - 4a^2}}{N_1};$$

hence,

$$|\lambda|^2 = \frac{1}{N_1^2}\left(\left(1 - \frac{1}{4}(\lambda_{L_1}\lambda_{L_2} - 4a^2)\right)^2 + (\lambda_{L_1}\lambda_{L_2} - 4a^2)\right) = 1. \tag{4.16}$$

The above (4.16) shows the convergence of the iteration (4.6). The assumption (4.8) identifies an explicit eigenvalue form of the iteration matrix $A^{-1}B$; however, it is a strong constraint.

Subsequently, we discuss the stability of the Crank-Nicolson method without the assumption (4.8). We adopt the perturbation theory for the eigenvalues of Hamiltonian matrices. A matrix $\mathcal{H} \in \mathbb{F}^{2(K-1)\times 2(K-1)}$ is known as Hamiltonian if $(J\mathcal{H})^* = J\mathcal{H}$, where $J = \begin{bmatrix} \mathbf{0} & I \\ -I & \mathbf{0} \end{bmatrix}$ and * represents the conjugate transpose if $\mathbb{F} = \mathbb{C}$, which is the transpose if $\mathbb{F} = \mathbb{R}$). The matrices A and B can be rewritten as the sum of the Hamiltonian matrices H and P, such that

$$A = I + \frac{\Delta t}{2}(H + P), \quad B = I - \frac{\Delta t}{2}(H + P), \tag{4.17}$$

where

$$H = \begin{bmatrix} \mathbf{0} & S \\ -S & \mathbf{0} \end{bmatrix}, \qquad P = \begin{bmatrix} I_{p_0q_0} & I_{p_0^2 + 3q_0^2} \\ -I_{3p_0^2 + q_0^2} & -I_{p_0q_0} \end{bmatrix},$$

and S is the discrete Laplacian defined above. The submatrices in P are diagonal matrices composed of $p_0(x_k, t_n)$ and $q_0(x_k, t_n)$ on the diagonal for $k = 1, \cdots, K-1$. It can be observed that both H and P are Hamiltonian. First, we observe that the eigenvalue λ_H of H satisfies that $\lambda_H^2 = -\lambda_S^2$. Therefore,

$$\lambda_H = \pm i\lambda_S$$

for each eigenvalue λ_S of S, implying that H has all the distinct purely imaginary eigenvalues. Second, it can be observed that P has conjugate pairs of purely imaginary eigenvalues $\lambda_P = \pm i\sqrt{\lambda}$, where λ is an eigenvalue of the diagonal matrix $I_{3(p_0^2+q_0^2)^2}$ because

$$P^2 = \begin{bmatrix} I_{p_0q_0}^2 - I_{p_0^2 + 3q_0^2}I_{3p_0^2 + q_0^2} & \mathbf{0} \\ \mathbf{0} & I_{p_0q_0}^2 - I_{p_0^2 + 3q_0^2}I_{3p_0^2 + q_0^2} \end{bmatrix} = \begin{bmatrix} -I_{3(p_0^2+q_0^2)^2} & \mathbf{0} \\ \mathbf{0} & -I_{3(p_0^2+q_0^2)^2} \end{bmatrix}.$$

There is no guarantee that the eigenvalues of the sum of two matrices H and P maintain pure imaginary values although both H and P have pure imaginary eigenvalues. To further investigate the eigenvalues of $H + P$, we consider the assumption $|u|^2$ in Eq (2.1), making p_0^2 and q_0^2 small to imply that

$$\|P\| < \epsilon$$

for sufficiently small $\epsilon > 0$. Third, the general perturbation theory ([22, Theorem 3.2]) of the Hamiltonian matrix indicates that $H + P$ comprises purely imaginary eigenvalues from all the distinct eigenvalues λ_H of H for sufficiently small $\epsilon > 0$. Therefore, we consider the following.

Theorem 1. *Consider $A = I + \frac{\Delta t}{2}(H + P)$ and $B = I - \frac{\Delta t}{2}(H + P)$, where H and P are defined by* (4.17). *Suppose $\|P\|$ is sufficiently small, then A is nonsingular and*

$$|\lambda_{A^{-1}B}| = 1$$

for all the eigenvalues of $A^{-1}B$.

Proof. Considering [22, Theorem 3.2], a Hamiltonian matrix H with pure imaginary eigenvalues of algebraic multiplicity 1 yields $H + P$ has all the pure imaginary eigenvalues for sufficiently small $\|P\|$. Furthermore, A and B have a complete set of common eigenvectors, and $A^{-1}B$ has the form $\lambda_{A^{-1}B} = (1 - \beta i)/(1 + \beta i)$ for any eigenvalue βi of $H + P$. Therefore, the eigenvalues of $A^{-1}B$ have a magnitude of one. □

Nevertheless, if the assumption of Theorem 1 is not satisfied, system instability (4.4) may occur, which is discussed subsequently.

Remark 4. *Suppose $\|P\|$ is not small enough, some complex eigenvalue of $H + P$ have non-zero real part. Generally, the characteristic polynomial of a real Hamiltonian matrix is even. Therefore, if a Hamiltonian matrix has a complex eigenvalue λ, then its corresponding complex conjugate quadruple $-\lambda$, λ^*, and $-\lambda^*$ are also eigenvalues [23]. This indicates that if $H+P$ has an eigenvalue $\alpha+\beta i$, $(\alpha, \beta \in \mathbb{R})$, then $\{1 + \frac{\Delta t}{2}(\alpha \pm \beta i), 1 - \frac{\Delta t}{2}(\alpha \pm \beta i)\}$ are also the eigenvalues of A and B. Therefore, considering Remark 3, one has an eigenvalue of $A^{-1}B$ satisfying*

$$|\lambda_{A^{-1}B}| = \left|\frac{\lambda_B}{\lambda_A}\right| = \frac{(1 - \frac{\Delta t}{2}\alpha)^2 + \frac{\Delta t^2}{4}\beta^2}{(1 + \frac{\Delta t}{2}\alpha)^2 + \frac{\Delta t^2}{4}\beta^2},$$

and another has an eigenvalue satisfying

$$|\lambda_{A^{-1}B}| = \left|\frac{\lambda_B}{\lambda_A}\right| = \frac{(1 + \frac{\Delta t}{2}\alpha)^2 + \frac{\Delta t^2}{4}\beta^2}{(1 - \frac{\Delta t}{2}\alpha)^2 + \frac{\Delta t^2}{4}\beta^2}$$

in which either of the eigenvalues has a magnitude greater than one. Therefore, instability may occur when $\|P\|$ is not sufficiently small when using the Crank-Nicolson method for ∂_t-discretization.

4.1.2. Newton's method: Backward-Euler

Regarding this subsection, we investigate the stability when the backward-Euler method is applied to (3.3) for discretization of ∂_t, such that

$$p_k^{n+1} + \Delta t\left(\frac{q_{k-1}^{n+1} - 2q_k^{n+1} + q_{k+1}^{n+1}}{\Delta x^2} + \left(p_0^2 q_k^{n+1} + 2p_0 q_0 p_k^{n+1} + 3q_0^2 q_k^{n+1}\right)\right) = p_k^n - \Delta t f_1(p_0, q_0),$$
$$q_k^{n+1} - \Delta t\left(\frac{p_{k-1}^{n+1} - 2p_k^{n+1} + p_{k+1}^{n+1}}{\Delta x^2} + (3p_0^2 p_k^{n+1} + 2p_0 q_0 q_k^{n+1} + q_0^2 p_k^{n+1})\right) = q_k^n - \Delta t f_2(p_0, q_0),$$

where $p_0 = p^n = p^n(x_k, t_n)$ and $q_0 = q^n = q^n(x_k, t_n)$. Considering the assumption that p_0, q_0 function values are approximated with constants, the above can be rewritten as the form in (4.5), $A\mathbf{u}^{n+1} = \mathbf{u}^n + \mathbf{F}$, that is,

$$\begin{bmatrix} (1+a)I & L_1 \\ -L_2 & (1-a)I \end{bmatrix}\begin{bmatrix} \mathbf{p}^{n+1} \\ \mathbf{q}^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{p}^n \\ \mathbf{q}^n \end{bmatrix} - \Delta t \begin{bmatrix} f_1(p_0, q_0) \\ f_2(p_0, q_0) \end{bmatrix}, \tag{4.18}$$

where $a = 2\Delta t p_0 q_0$, $L_1 = \Delta t(S + (p_0^2 + 3q_0^2)I)$ and $L_2 = \Delta t(S + (3p_0^2 + q_0^2)I)$. Analogously, the invertibility of A can be proven by the invertibility of the submatrices

$$(1 + a)I \quad \text{and} \quad (1 - a)I + 11 + aL_1L_2$$

under the assumptions that $\lambda_S + 4l^2 < 0$ and (4.9). $(1 - a)(1 + a) + \lambda_{L_1}\lambda_{L_2} = 1 + (\lambda_{L_1}\lambda_{L_2} - a^2)$ is always positive for sufficiently small p_0, q_0 and $|p_0| + |q_0| < l$. Regarding Lemma 1, the inverse of A is constructed as

$$A^{-1} = \begin{bmatrix} \frac{1}{1+a}I - \frac{1}{(1+a)^2}L_1ML_2 & -\frac{1}{1+a}L_1M \\ \frac{1}{1+a}ML_2 & M \end{bmatrix},$$

where $M = \left((1 - a)I + \frac{1}{1+a}L_1L_2\right)^{-1}$. Let $\mathbf{x}$ be an eigenvector corresponding to the eigenvalue λ_S of matrix S. Therefore, the relation (4.11) still holds with the eigenvalue of M, $\lambda_M = (1 + a)/\left(1 + (\lambda_{L_1}\lambda_{L_2} - a^2)\right)$. We rewrite (4.18) as $\mathbf{u}^{n+1} = A^{-1}\mathbf{u}^n - \Delta t A^{-1}\mathbf{F}$. Based on the formulation (4.6), the stability of the backward Euler method can be suggested by analyzing the eigenvalue of A^{-1}. Considering the process (4.12)–(4.15), we let

$$A^{-1}\begin{bmatrix} \mathbf{x} \\ \alpha\mathbf{x} \end{bmatrix} = \begin{bmatrix} (A^{-1})_{11} & (A^{-1})_{12} \\ (A^{-1})_{21} & (A^{-1})_{22} \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \alpha\mathbf{x} \end{bmatrix} = \lambda\begin{bmatrix} \mathbf{x} \\ \alpha\mathbf{x} \end{bmatrix},$$

where

$$(A^{-1})_{11} = \frac{1 - a}{N_2}, \quad (A^{-1})_{12} = -\frac{\lambda_{L_1}}{N_2}, \quad (A^{-1})_{21} = \frac{\lambda_{L_2}}{N_2}, \quad (A^{-1})_{22} = \frac{1 + a}{N_2}$$

with $N_2 = 1 + (\lambda_{L_1}\lambda_{L_2} - a^2)$. As $\lambda_S + 4l^2 < 0$ for $|p_0| + |q_0| < l$ and $\lambda_{L_1} = \Delta t(\lambda_S + p_0^2 + 3q_0^2) \neq 0$, similar to (4.15), we obtain

$$\lambda = \frac{a_1 + a_4}{2} \pm \frac{1}{2}\sqrt{(a_1 - a_4)^2 + 4a_2a_3} = \frac{1}{N_2} \pm \frac{1}{2}\sqrt{-4(\lambda_{L_1}\lambda_{L_2} - a^2)} = \frac{1}{N_2} \pm i\frac{1}{N_2}\sqrt{\lambda_{L_1}\lambda_{L_2} - a^2}.$$

Therefore, the above yields

$$|\lambda|^2 = \frac{1}{N_2} = \frac{1}{1 + \Delta t^2\left(\lambda_S + 3(p_0^2 + q_0^2)\right)(\lambda_S + p_0^2 + q_0^2)} < 1,$$

indicating that NLSE is unconditionally stable when it is linearized by Newton and is discretized by the backward Euler in time.

Analogous to the Crank-Nicolson approach, the analysis can be extended easily to the case where p_0 and q_0 are not constants; however, they are the approximations obtained from a previous iteration that corresponds to the approximations of p and q evaluated at x_k, $k = 1, \cdots, K - 1$, respectively. Moreover, the matrix A in (4.18) is composed of

$$A = I + \Delta t(H + P), \quad H = \begin{bmatrix} \mathbf{0} & S \\ -S & \mathbf{0} \end{bmatrix}, \quad P = \begin{bmatrix} I_{p_0q_0} & I_{p_0^2 + 3q_0^2} \\ -I_{3p_0^2 + q_0^2} & -I_{p_0q_0} \end{bmatrix}. \tag{4.19}$$

Theorem 2. *Considering the same hypothesis in Theorem 1, we obtain*

$$|\lambda_{A^{-1}}| < 1$$

for all the eigenvalues $\lambda_{A^{-1}}$ of A^{-1}.

Proof. If $\|P\|$ is sufficiently small, then A is nonsingular, and it satisfies $|\lambda_A| = |1 + i\beta| > 1$ with non-zero β; hence, the scheme is unconditionally stable. □

Similarly, as discussed in Remark 4, the stability of the scheme (4.18) is no longer guaranteed when the assumption does not hold.

4.2. Picard method: Crank-Nicolson and Backward-Euler

Applying the Crank-Nicolson(CN) method for ∂_t and the centered difference method for ∂_{xx} to the linearized Eq (3.4), it is discretized into

$$\begin{cases} p_k^{n+1} + \dfrac{\Delta t}{2}\left(\dfrac{q_{k-1}^{n+1} - 2q_k^{n+1} + q_{k+1}^{n+1}}{\Delta x^2} + (p_0^2 + q_0^2)q_k^{n+1}\right) \\ \qquad = p_k^n - \dfrac{\Delta t}{2}\left(\dfrac{q_{k-1}^n - 2q_k^n + q_{k+1}^n}{\Delta x^2} + (p_0^2 + q_0^2)q_k^n\right), \\ q_k^{n+1} - \dfrac{\Delta t}{2}\left(\dfrac{p_{k-1}^{n+1} - 2p_k^{n+1} + p_{k+1}^{n+1}}{\Delta x^2} + (p_0^2 + q_0^2)p_k^{n+1}\right) \\ \qquad = q_k^n + \dfrac{\Delta t}{2}\left(\dfrac{p_{k-1}^n - 2p_k^n + p_{k+1}^n}{\Delta x^2} + (p_0^2 + q_0^2)p_k^n\right) \end{cases} \tag{4.20}$$

with $p_0 = p^n(x_k, t_n)$ and $q_0 = q^n(x_k, t_n)$. The above system (4.20) recasts in the form of (4.4) with

$$A_1 = A_4 = I, \quad A_2 = \tfrac{\Delta t}{2}\left(S + I_{p_0^2 + q_0^2}\right), \quad A_3 = -A_2,$$
$$B_1 = B_4 = I, \quad B_2 = -A_2, \quad B_3 = A_2,$$

I is the identity matrix, $I_\alpha = \mathrm{diag}(\alpha(x_1, t_n), \alpha(x_2, t_n), \cdots, \alpha(x_{K-1}, t_n))$ is a diagonal matrix satisfies that the diagonal components are filled with the evaluated α-functional values at each node, and S is the tridiagonal symmetric matrix discretizing ∂_{xx}, which has $-2/(\Delta x^2)$ on the diagonal and $1/(\Delta x^2)$ on the lower/upper diagonal part.

Furthermore, the matrices $A = [A_1, A_2; A_3, A_4]$ and $B = [B_1, B_2; B_3, B_4]$ are composed of the identity matrix and a skew-symmetric matrix such as

$$A = I + \frac{\Delta t}{2}\begin{bmatrix} \mathbf{0} & S + I_{p_0^2 + q_0^2} \\ -S - I_{p_0^2 + q_0^2} & \mathbf{0} \end{bmatrix} \text{ and } B = I - \frac{\Delta t}{2}\begin{bmatrix} \mathbf{0} & S + I_{p_0^2 + q_0^2} \\ -S - I_{p_0^2 + q_0^2} & \mathbf{0} \end{bmatrix}.$$

We obtain the condition that the matrix A is non-singular and can be automatically obtained from the structure. This can be used to investigate the magnitude of eigenvalues of A. Because a real valued skew-symmetric matrix has only pure imaginary eigenvalues and A, B share the same eigenvector, the eigenvalues of two matrices A and B are complex conjugate to each other. Thus, $|\lambda_A| = |\lambda_B| \geq 1$ for each conjugate eigenvalue of A and B. Therefore, we obtain

$$|\lambda_{A^{-1}B}| = 1$$

for any eigenvalue of $A^{-1}B$, making the Crank-Nicolson (CN) method is stable.

Remark 5. *No constraints on the matrices are applied; however, Picard iteration usually converges very slowly.*

Remark 6. *The backward-Euler method is applied to* (3.4) *to obtain*

$$p_k^{n+1} + \Delta t\left(\frac{q_{k-1}^{n+1} - 2q_k^{n+1} + q_{k+1}^{n+1}}{\Delta x^2} + (p_0^2 + q_0^2)q_k^{n+1}\right) = p_k^n,$$

$$q_k^{n+1} - \Delta t\left(\frac{p_{k-1}^{n+1} - 2p_k^{n+1} + p_{k+1}^{n+1}}{\Delta x^2} + (p_0^2 + q_0^2)p_k^{n+1}\right) = q_k^n,$$

where p_0 *and* q_0 *are evaluated from the previous Picard step. The above system yields the iteration* $A\boldsymbol{u}^{n+1} = \boldsymbol{u}^n + \mathbf{F}$*, where*

$$A = I + \Delta t\begin{bmatrix} \mathbf{0} & S + I_{p_0^2+q_0^2} \\ -S - I_{p_0^2+q_0^2} & \mathbf{0} \end{bmatrix}.$$

Considering a similar argument as in the case of the CN method, one has $|\lambda_A| \geq 1$*; therefore,* $|\lambda_{A^{-1}}| \leq 1$*, which shows that the iteration is also unconditionally stable.*

5. Numerical results

In the previous sections, the nonlinear Schrödinger equation (2.1) is transformed to the system of real nonlinear equations (2.3) in which it is linearized using the Newton and Picard methods. Once the system is linearized, the Crank-Nicolson and backward Euler methods are used to build an approximation of the solution to (2.3). The analysis suggests that both the Crank-Nicolson and backward Euler methods provide the unconditional stability, independent of the mesh and time step sizes. To verify this stability analysis, we examine the convergence of the approximations.

We choose a theoretical solution of the model problem (1.1) given by

$$u(x,t) = \sqrt{2\rho}\exp\left\{i\left(\frac{1}{2}\xi x - \left(\frac{1}{4}\xi^2 - \rho\right)t\right)\right\}\operatorname{sech}\left(\sqrt{\rho}(x - \xi t)\right), \tag{5.1}$$

which yields the solution of (2.3) as

$$p(x,t) = \sqrt{2\rho}\cos\left(0.5\xi x - \left(0.25\xi^2 - \rho\right)t\right)\operatorname{sech}\left(\sqrt{\rho}(x - \xi t)\right),$$
$$q(x,t) = \sqrt{2\rho}\sin\left(0.5\xi x - \left(0.25\xi^2 - \rho\right)t\right)\operatorname{sech}\left(\sqrt{\rho}(x - \xi t)\right),$$

where ρ and ξ are positive constants [11]. This solution u does not satisfy the boundary condition in (2.1); therefore, we modify u with $w := u - u_b$, where

$$u_b(x,t) = u(x_1,t)\left(1 + (x - x_1)/(x_1 - x_2)\right) + u(x_2,t)\left(1 - (x - x_2)/(x_1 - x_2)\right) \tag{5.2}$$

for $[x_1, x_2] := [-\eta, \eta]$. Therefore, the Eq (2.1) transforms to the equation considering w, which has a non-zero right hand side and zero boundary condition. Once a numerical solution w_{app} for w is sought, the final approximation for u is obtained as $w_{\text{app}} + u_b$.

If a consistent numerical scheme is stable, the resulting approximation converges to the solution. In the absence of nonlinear terms in the system (2.3), the backward Euler and Crank-Nicolson methods provide $O(\Delta t + \Delta x^2)$ and $O(\Delta t^2 + \Delta x^2)$-convergence, respectively (see [8] and references therein). Prior

to solving (3.3) and (3.4), we consider the following linear problem to confirm these known results by constructing the right hand sides, f and g, using the above p and q induced from (5.1) and (5.2)

$$\partial_t p + \partial_{xx} q = f, \quad \partial_t q - \partial_{xx} p = g \quad \text{in } \Omega = [x_1, x_2] \times [0, T], \tag{5.3}$$

where $p(x_1, t) = p(x_2, t) = 0$, $q(x_1, t) = q(x_2, t) = 0$ for $0 \le t \le T$ and $p(x, 0) = p_0(x)$, $q(x, 0) = q_0(x)$ for $x_1 \le x \le x_2$. Here, we set $x_1 = -10$, $x_2 = 30$, $T = 40$, $\rho = 1/2$, and $\xi = 0.7$. It is not necessary to choose x-interval as $[-\eta, \eta]$; therefore, all the previous theoretical analysis still holds for any interval $[x_1, x_2]$. Table 1 shows the convergence of ℓ^∞-error at the final time for each methods, where

$$\begin{aligned} \text{err} &:= \|p(\cdot, T) - p_{\text{app}}(\cdot, T)\|_\infty + \|q(\cdot, T) - q_{\text{app}}(\cdot, T)\|_\infty, \\ \text{rate} &:= \Big(\log(\text{err}_{\text{(k+1)-th}})\text{-}\log(\text{err}_{\text{(k)-th}})\Big) \Big/ \Big(\log(\Delta x_{\text{(k+1)-th}})\text{-}\log(\Delta x_{\text{(k)-th}})\Big), \end{aligned}$$

and $\|\cdot\|_\infty$ denotes the discrete ℓ^∞-norm.

Table 1. ℓ^∞-errors when $\Delta t = \Delta x/2$ and $\Delta t = \Delta x^2$, BE: backward Euler, CN: Crank-Nicolson.

$1/\Delta x$	$\Delta t = \Delta x/2$				$\Delta t = \Delta x^2$			
	BE		CN		BE		CN	
	err	rate	err	rate	err	rate	err	rate
5	1.678e-2		3.088e-3		6.587e-3		3.158e-3	
10	8.561e-3	0.97	8.154e-4	1.92	1.668e-3	1.99	8.511e-4	1.89
20	4.406e-3	0.96	2.011e-4	2.02	4.299e-4	1.96	2.156e-4	1.98
30	2.986e-3	0.96	8.944e-5	2.00	1.935e-4	1.97	9.652e-5	1.98
40	2.259e-3	0.97	5.115e-5	1.94	1.094e-4	1.99	5.442e-5	1.99

The numbers show $O(\Delta t + \Delta x^2)$-convergence for the backward Euler method, whereas the Crank-Nicolson method has $O(\Delta t^2 + \Delta x^2)$-convergence.

We observe the numerical results when the systems (3.3) and (3.4) are solved with the backward Euler and Crank-Nicolson methods with various mesh sizes and time steps. To avoid the phenomena that the error due to the linearization process dominates the whole behavior of the approximation, the initial guess in the linearization process is chosen as the exact solution perturbed with 0.01%-random noise, and one linearization step is employed. If the proposed combinations of the numerical approaches lead to convergence, then that can be one of the evidences verifying the stability of the method.

Table 2 shows $O(\Delta t + \Delta x^2)$- and $O(\Delta t^2 + \Delta x^2)$-convergence with the backward Euler and Crank-Nicolson methods, respectively. This confirms the analysis in the previous sections, which indicates that the backward Euler and Crank-Nicolson methods applied to the linearized system of (2.3) are unconditionally stable. In addition, we measure the respective elapsed computational times of the proposed methods, Newton method with backward Euler, Newton method with Crank-Nicolson, Picard method with backward Euler, and Picard method with Crank-Nicolson, which are 26.8, 28.9, 26.0, and 27.5 seconds when $\Delta t = \Delta x/2$ with $\Delta x = 30$ and 4996.1, 4527.1, 4527.5, and 4500.7 seconds when $\Delta t = \Delta x^2$ with $\Delta x = 30$.

Table 2. ℓ^∞-errors when $\Delta t = \Delta x/2$ and $\Delta t = \Delta x^2$, BE: backward Euler, CN: Crank-Nicolson.

	$\Delta t = \Delta x/2$							
	Newton				Picard			
$1/\Delta x$	BE		CN		BE		CN	
	err	rate	err	rate	err	rate	err	rate
5	3.546e-1		6.166e-1		5.017e-1		4.279e-2	
10	8.935e-2	1.99	1.605e-1	1.94	2.793e-1	0.84	1.059e-2	2.01
20	3.696e-2	1.27	4.825e-2	1.74	1.475e-1	0.92	2.619e-3	2.02
30	2.026e-2	1.48	2.123e-2	2.02	1.002e-1	0.95	1.127e-3	2.08
40	1.377e-2	1.34	1.137e-2	2.18	7.589e-2	0.97	6.160e-4	2.10
	$\Delta t = \Delta x^2$							
5	5.079e-1		6.460e-1		2.275e-1		4.849e-2	
10	1.325e-1	1.94	1.488e-1	2.12	6.066e-2	1.91	1.228e-2	1.98
20	4.861e-2	1.45	4.521e-2	1.72	1.549e-2	1.97	3.044e-3	2.01
30	2.508e-2	1.63	2.038e-2	1.97	6.963e-3	1.97	1.321e-3	2.06
40	1.585e-2	1.60	1.218e-2	1.79	3.967e-3	1.96	7.216e-4	2.10

6. Conclusions

To solve the one-dimensional nonlinear Schrödinger equation over a complex field, we transform the equation to a system of differential equations over a real field by separating the complex valued function into two real functions. Thereafter, the Newton and Picard methods were applied to build corresponding linear problems. The Crank-Nicolson and backward Euler methods were used to discretize the linear problems. When solving the problems numerically with finite difference methods, it is important to analyze the stability of the resulting discrete system to obtain a converging approximation. This study analyzes the stability of numerical solutions of the nonlinear Schrödinger equation (2.1) when a finite difference method is combined with a linearization scheme. The stability analyses of the Crank-Nicolson and backward Euler methods when combined with the Newton's or Picard method show that both approaches provide unconditionally stable numerical solutions. The convergence of the numerical tests confirms the theoretical analysis.

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Conflict of interest

The authors declare no conflict of interest.

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