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Research article

New oscillation solutions of impulsive conformable partial differential equations

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Abstract: Partial fractional differential equations are fundamental in many physical and biological applications, engineering and medicine, in addition to their importance in the development of several mathematical and computer models. This study's main objective is to identify the necessary conditions for the oscillation of impulsive conformable partial differential equation systems with the Robin boundary condition. The important findings of the study are stated and demonstrated with a robust example at the end of the study.

Keywords: fractional integrals; nonlinear equations; conformable partial differential equations; impulse; oscillation; damping term; distributed deviating arguments
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1. Introduction

One of the most popular topics of theoretical studies in the vast area of mathematics is the theory of fractional derivatives. Theories such as fractional derivatives and fractional integrals play significant roles and they are apt theories for tackling the issues that prevail in the present world. They have been discussed and analyzed by many famous authors in their research works, and they are helpful in finding the solutions for real-life problems. The fractional equations, which are based on the properties of fractional derivatives, are used to solve problems in the fields of mathematical modeling and simulation of systems and processes.

The fields of science and engineering have gained importance and popularity by the documented applications of fractional differential equations, which are generalizations of the classical differential equations of integers in a diverse and widespread area. Fractional calculus is developing largely in the midst of science and engineering problems. Fractional derivatives are easily used to solve problems in interdisciplinary applications in an elegant manner. Most of the systems are constructed very accurately using fractional derivatives and integrals in an easy way, and fractional calculus is applicable in areas such as fluid flow, rheology, viscoelasticity, signal processing, economics, etc. Books on fractional derivatives and fractional integrals are largely available and published, such as [1–7].

The fundamental and the basic properties of the usual derivatives, such as the chain and product rules, have been lost and become more complicated in the obtained fractional derivatives in the present form of calculus. Khalil et al. [8]. introduced the conformable derivative, which is more similar to the classical derivative, in the year 2014. It was introduced as a new fractional derivative. The phenomena and the real-world scenario systems that are more aptly described with the help of fractional differential equations have been identified by many researchers in their works in recent times. The symmetries can be found by solving a related set of partial fractional differential equations. The real-world issues in the field of science are clearly understood with the help of an important mathematical tool. This tool is called the natural description of the evolution processes which us provided by the oscillation theory of differential equations. The monographs and the references mentioned [9–12] can be used by the readers to have a detailed discussion on the applications of impulsive differential equations in a very clear manner.

For the oscillation theory of impulsive differential equations, first investigation and research was published in the year 1989 [13], and a paper related to this topic was published in the year 1991 [14]. The simple and natural framework of mathematical modeling for population growth was provided by the impulsive differential equations that found by the authors mentioned in [14]. Several authors studied the oscillatory behavior of the differential equations with or without the module of impulse [15–28]. The concentration and attention are much less on systems of partial differential equations [29–36] and systems of impulsive partial differential equations [37–40]. Many researchers have found excellent results and outcomes, and significant attention has been given to analyzing the oscillation of the differential equations in the last few years. The references cited in this paper provide us with some notable results, with the help of [41–44], in the above discussed field in a detailed manner.

The current paper is organized as follows. In section 2, we introduce the proposed impulsive system and the boundary condition that will be discussed in the paper. In section 3, we present several preliminary definitions and notations we use through all the paper. In addition, we provide some needed auxiliary results. In section 4, we present the main results by establishing sufficient conditions for the oscillation of all solutions of the proposed problems. In section 5, we provide an example to illustrate the main results and to validate the proposed work. Finally, a brief conclusion and description of future work are provided at the end of this paper.

2. Impulsive system

In this paper, we will discuss the following impulsive system:

$$\left.\begin{array}{l}\dfrac{\partial^{\alpha}}{\partial\iota^{\alpha}}\left[r(\iota)\dfrac{\partial^{\alpha}}{\partial\iota^{\alpha}}\left(\vartheta_i(\omega,\iota)+\int_a^b g(\iota,\varsigma)\vartheta_i(\omega,\tau(\iota,\varsigma))d\eta(\varsigma)\right)\right]\\ +p(\iota)\dfrac{\partial^{\alpha}}{\partial\iota^{\alpha}}\left(\vartheta_i(\omega,\iota)+\int_a^b g(\iota,\varsigma)\vartheta_i(\omega,\tau(\iota,\varsigma))d\eta(\varsigma)\right)\\ +\sum_{n=1}^{m}\sum_{j=1}^{d}\int_a^b q_{inj}(\omega,\iota,\varsigma)f_{ij}(\vartheta_n(\omega,\sigma_j(\iota,\varsigma)))d\eta(\varsigma)=a_i(\iota)\Delta\vartheta_i(\omega,\iota)\\ +\sum_{n=1}^{m}\sum_{\hbar=1}^{l}a_{inh}(\iota)\Delta\vartheta_n(\omega,\rho_{\hbar}(\iota)),\quad \iota\neq\iota_{\ell},\ (\omega,\iota)\in\psi\times\mathbb{R}_+\equiv G\\ \vartheta_i(\omega,\iota_{\ell}^{+})=\ \alpha_{\ell_i}(\omega,\iota_{\ell},\vartheta_i(\omega,\iota_{\ell}))\\ \dfrac{\partial^{\alpha}\vartheta_i(\omega,\iota_{\ell}^{+})}{\partial\iota^{\alpha}}=\ \beta_{\ell_i}\left(\omega,\iota_{\ell},\dfrac{\partial^{\alpha}\vartheta_i(\omega,\iota_{\ell})}{\partial\iota^{\alpha}}\right),\quad \ell=1,2,\cdots,\ i=1,2,\cdots,m\end{array}\right\}\qquad (E)$$

where Δ represents the Laplacian in the Euclidean space $\mathbb{R}^N$, and ψ is a bounded domain in $\mathbb{R}^N$ with a piece-wise smooth boundary $\partial\psi$. $\frac{\partial^{\alpha}}{\partial\iota^{\alpha}}$ represents the conformable partial fractional derivative of order α, $0<\alpha\leq 1$, and $\mathbb{R}_+=[0,+\infty)$. Moreover, we study the boundary condition as follows:

$$\frac{\partial\vartheta_i(\omega,\iota)}{\partial\gamma}+\mu_i(\omega,\iota)\vartheta_i(\omega,\iota)=0,\quad (\omega,\iota)\in\partial\psi\times\mathbb{R}_+, \qquad (B)$$

where γ represents the outer surface normal vector to $\partial\psi$, and $\mu_i(\omega,\iota)\in C(\partial\psi\times\mathbb{R}_+,\mathbb{R}_+)$.

During this work, we let the following hypotheses hold.

(H_1) $r(\iota)\in C^{\alpha}(\mathbb{R}_+,(0,+\infty))$, $T_{\alpha}(r(\iota))\geq 0$, $p(\iota)\in C(\mathbb{R}_+,\mathbb{R})$, $g(\iota,\varsigma)\in C^{2\alpha}(\mathbb{R}_+\times[a,b],(0,+\infty))$, $\int_{\iota_0}^{+\infty}\frac{1}{s^{1-\alpha}\Lambda(s)}ds=+\infty$, where $\Lambda(\iota)=\exp\left(\int_{\iota_0}^{\iota}\frac{T_{\alpha}(r(s))+p(s)}{r(s)}ds\right)$.

(H_2) $a_i(\iota), a_{inh}(\iota)\in PC(\mathbb{R}_+,\mathbb{R}_+)$, $A_{\hbar}(\iota)=\min_{1\leq i\leq m}\left\{a_{iih}(\iota)-\sum_{n=1,\ n\neq i}^{m}|a_{nih}(\iota)|\right\}>0$, $i,n=1,2,\cdots,m,\hbar=1,2,\cdots,l$, where PC represents the functions that are piece-wise and continuous in ι which also have the discontinuities that take place in $\iota=\iota_{\ell}$, $\ell=1,2,\cdots$, and left continuous at $\iota=\iota_{\ell}$, $\ell=1,2,\cdots$.

(H_3) $\tau(\iota,\varsigma)\in C^{\alpha}(\mathbb{R}_+\times[a,b],\mathbb{R})$, $\sigma_j(\iota,\varsigma),\in C(\mathbb{R}_+\times[a,b],\mathbb{R})$, $\sigma_j(\iota,\varsigma)\leq\iota$, $\tau(\iota,\varsigma)\leq\iota$ for $\varsigma\in[a,b]$, $\sigma_j(\iota,\varsigma)$ and $\tau(\iota,\varsigma)$ are non-decreasing with respect to ι and ς respectively, and

$$\liminf_{\iota\to+\infty,\ \varsigma\in[a,b]}\sigma_j(\iota,\varsigma)=\liminf_{\iota\to+\infty,\ \varsigma\in[a,b]}\tau(\iota,\varsigma)=+\infty,\quad j=1,2,\cdots,d,$$

$\rho_{\hbar}(\iota)\in C(\mathbb{R}_+,\mathbb{R})$, $\rho_{\hbar}(\iota)\leq\iota$ and $\lim_{\iota\to+\infty}\rho_{\hbar}(\iota)=+\infty$, $\hbar=1,2,\cdots,l$, a,b are nonpositive constants with $a<b$.

(H_4) There exists a function $\theta_j(\iota) \in C^{\alpha}(\mathbb{R}_+, \mathbb{R}_+)$ satisfying $\theta_j(\iota) \le \sigma_j(\iota, a)$, $T_\alpha(\theta_j(\iota)) > 0$ and $\lim_{\iota\to+\infty} \theta_j(\iota) = +\infty$, $j = 1, 2, \cdots, d$, $\eta(\varsigma) : [a, b] \to \mathbb{R}$ decreases, and the integral is of type Stieltjes in the BVP (E).

(H_5) $q_{inj}(\omega, \iota, \varsigma) \in C\left(\bar{\psi} \times \mathbb{R}_+ \times [a, b], \mathbb{R}\right)$, $q_{iij}(\iota, \varsigma) = \min_{\omega\in\bar{\psi}} q_{iij}(\omega, \iota, \varsigma)$,

$\bar{q}_{inj}(\iota, \varsigma) = \max_{\omega\in\bar{\psi}} \left|q_{inj}(\omega, \iota, \varsigma)\right|$, $Q_j(\iota, \varsigma) = \min_{1\le i\le m} \left\{q_{iij}(\iota, \varsigma) - \sum_{n=1,\ n\neq i}^{m} \bar{q}_{nij}(\iota, \varsigma)\right\} \ge 0$, $i, n = 1, 2, \cdots, m$, $j = 1, 2, \cdots, d$, $f_{ij}(\vartheta_n) \in C(\mathbb{R}, \mathbb{R})$ convex in $\mathbb{R}_+$, $\vartheta_n f_{ij}(\vartheta_n) > 0$ and $\dfrac{f_{ij}(\vartheta_n)}{\vartheta_n} \ge \epsilon > 0$, for $\vartheta_n \neq 0$, $i, n = 1, 2, \cdots, m$, $j = 1, 2, \cdots, d$.

(H_6) $\vartheta_i(\omega, \iota)$ and their derivatives $\dfrac{\partial^\alpha \vartheta_i(\omega, \iota)}{\partial\iota^\alpha}$ are piecewise continuous in ι with discontinuities of first kind only at $\iota = \iota_\ell$, $\ell = 1, 2, \cdots$, and left continuous at $\iota = \iota_\ell$, $\vartheta_i(\omega, \iota_\ell) = \vartheta_i(\omega, \iota_\ell^-)$, $\dfrac{\partial^\alpha \vartheta_i(\omega, \iota_\ell)}{\partial\iota^\alpha} = \dfrac{\partial^\alpha \vartheta_i(\omega, \iota_\ell^-)}{\partial\iota^\alpha}$, $\ell = 1, 2, \cdots$, $i = 1, 2, \cdots, m$.

(H_7) $\alpha_{\ell_i}\left(\omega, \iota_\ell, \vartheta_i(\omega, \iota_\ell)\right), \beta_{\ell_i}\left(\omega, \iota_\ell, \dfrac{\partial^\alpha \vartheta_i(\omega, \iota_\ell)}{\partial\iota^\alpha}\right) \in PC(\bar{\psi} \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, $\ell = 1, 2, \cdots$, $i = 1, 2, \cdots, m$, and there exist positive constants $a_{\ell_i}, a^*_{\ell_i}, b_{\ell_i}, b^*_{\ell_i}$ with $b_{\ell_i} \le a^*_{\ell_i}$ such that for $i = 1, 2, \cdots, m$, $\ell = 1, 2, \cdots$,

$$a^*_{\ell_i} \le \frac{\alpha_{\ell_i}\left(\omega, \iota_\ell, \vartheta_i(\omega, \iota_\ell)\right)}{\vartheta_i(\omega, \iota_\ell)} \le a_{\ell_i}, \quad b^*_{\ell_i} \le \frac{\beta_{\ell_i}\left(\omega, \iota_\ell, \frac{\partial^\alpha \vartheta_i(\omega, \iota_\ell)}{\partial\iota^\alpha}\right)}{\frac{\partial^\alpha \vartheta_i(\omega, \iota_\ell)}{\partial\iota^\alpha}} \le b_{\ell_i}.$$

3. Preliminaries

In this section, we present some definitions and review some noteworthy results from the literature which we will use throughout the paper.

Definition 1. *[45] A solution of system (E) means a vector function $(\vartheta_1(\omega, \iota), \cdots, \vartheta_m(\omega, \iota))$ such that $\vartheta_i(\omega, \iota) \in C^{2\alpha}(\bar{\psi} \times [\iota_{-1}, +\infty), \mathbb{R}) \cap C^{\alpha}(\bar{\psi} \times [\hat{\iota}_{-1}, +\infty), \mathbb{R}) \cap C(\bar{\psi} \times [\bar{\iota}_{-1}, +\infty), \mathbb{R})$ and $\vartheta_i(\omega, \iota)$, $i = 1, 2, \cdots, m$ are satisfying the BVP (E) in G such that*

$$\iota_{-1} := \min\left\{0, \min_{1\le\hbar\le l}\left\{\inf_{\iota\ge0} \rho_\hbar(\iota)\right\}\right\}$$
$$\hat{\iota}_{-1} := \min\left\{0, \min_{\varsigma\in[a,b]}\left\{\inf_{\iota\ge0} \tau(\iota, \varsigma)\right\}\right\}$$

and

$$\bar{\iota}_{-1} := \min\left\{0, \min_{1\le j\le d,\ \varsigma\in[a,b]}\left\{\inf_{\iota\ge0} \sigma_j(\iota, \varsigma)\right\}\right\}.$$

Definition 2. *[45] A nontrivial component $\vartheta_i(\omega, \iota)$ of the vector function $(\vartheta_1(\omega, \iota), \cdots, \vartheta_m(\omega, \iota))$ is said to be oscillatory in $\psi \times [\delta_0, +\infty)$ if for each $\delta > \delta_0$ there is a point $(\omega_0, \iota_0) \in \psi \times [\delta_0, +\infty)$ such that $\vartheta_i(\omega_0, \iota_0) = 0$.*

Definition 3. *[45] The vector solution $(\vartheta_1(\omega, \iota), \cdots, \vartheta_m(\omega, \iota))$ of the problem (E) and (B) is said to be oscillatory in the domain G if at least one of its nontrivial components oscillates in G. Otherwise, the vector solution $\vartheta_i(\omega, \iota)$ is said to be non-oscillatory in G.*

Definition 4. *[45] The vector solution* $(\vartheta_1(\omega,\imath),\cdots,\vartheta_m(\omega,\imath))$ *of the problem* (E) *and* (B) *is said to strongly oscillate in the domain G if each of its nontrivial components oscillates in G.*

We use some of the definitions given by the authors in [8].

Definition 5. *Let* $f:[0,\infty)\to\mathbb{R}$*. Then, the "conformable fractional derivative" of f of order* α *is defined by*

$$T_\alpha(f)(\imath)=\lim_{\varepsilon\to 0}\frac{f(\imath+\varepsilon\imath^{1-\alpha})-f(\imath)}{\varepsilon}$$

for all $\imath>0,\alpha\in(0,1]$.

If f is α-differentiable in some $(0,a), a>0$, and $\lim\limits_{\imath\to 0^+} f^{(\alpha)}(\imath)$ exists, then we define

$$f^{(\alpha)}(0)=\lim_{\imath\to 0^+} f^{(\alpha)}(\imath).$$

Definition 6. $I_\alpha^a(f)(\imath)=I_1^a(\imath^{\alpha-1}f)=\int_a^\imath \frac{f(\omega)}{\omega^{1-\alpha}}dx$*, such that the type of the integral is improper Riemann, and* $\alpha\in(0,1)$.

The following theorem defines the fundamental properties of the conformable fractional derivative.

Theorem 1. *Let* $\alpha\in(0,1]$ *and* f,g *be* α*-differentiable at a point* $\imath>0$*. Then,*

(*i*) $T_\alpha(af+bg)=aT_\alpha(f)+bT_\alpha(g)$*, for all* $a,b\in\mathbb{R}$.
(*ii*) $T_\alpha(\imath^p)=pt^{p-\alpha}$*, for all* $p\in\mathbb{R}$.
(*iii*) $T_\alpha(\kappa)=0$*, for all constant functions* $f(\imath)=\kappa$.
(*iv*) $T_\alpha(fg)=fT_\alpha(g)+gT_\alpha(f)$.
(*v*) $T_\alpha\left(\frac{f}{g}\right)=\dfrac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$.
(*vi*) *If f is differentiable, then* $T_\alpha(f)(\imath)=\imath^{1-\alpha}\dfrac{df}{dt}(\imath)$.

Definition 7. *[46] Let f be a function of n variables* $\omega_1,\omega_2,\cdots,\omega_n$*, and the conformable partial derivative of f of order* $0<\alpha\leq 1$ *in* ω_i *is defined as follows:*

$$\frac{\partial^\alpha}{\partial\omega_i^\alpha}f(\omega_1,\omega_2,\cdots,\omega_n)=\lim_{\varepsilon\to 0}\frac{f(\omega_1,\omega_2,\cdots,\omega_{i-1},\omega_i+\varepsilon\omega_i^{1-\alpha},\cdots,\omega_n)-f(\omega_1,\omega_2,\cdots,\omega_n)}{\varepsilon}.$$

Next, we state two results which will help us establish our main results.

Lemma 1. *[47] If X and Y are non-negative, then*

$$X^\kappa+(\kappa-1)Y^\kappa\geq\alpha XY^{\kappa-1},\quad \kappa>1,$$
$$X^\kappa-(1-\kappa)Y^\kappa\leq\kappa XY^{\kappa-1},\quad 0<\kappa<1,$$

if and only if $X=Y$.

It is known in [48] that the first eigenvalue κ_0 of the problem

$$\begin{cases}\Delta w(\omega)+\kappa w(\omega) = 0 & \text{in } \psi,\\ w(\omega) = 0 & \text{on } \partial\psi,\end{cases}$$

is positive, and the corresponding eigenfunction $\Phi(\omega)$ is positive in ψ.

4. Oscillation of the BVP (E) and (B)

In this section, we establish sufficient conditions for the oscillation of all solutions of the problem (E), (B).

Theorem 2. *If the functional impulsive conformable fractional differential inequality*

$$\left.\begin{array}{l} T_\alpha\left(r(\iota)T_\alpha(W(\iota))\right) + p(\iota)T_\alpha(W(\iota)) \\ +\sum\limits_{j=1}^{d}\int_a^b \epsilon Q_j(\iota,\varsigma)\left[1-\int_a^b g(\sigma_j(\iota,\varsigma),\varsigma)d\eta(\varsigma)\right]W(\theta_j(\iota))d\eta(\varsigma)\le 0, \quad \iota\ne\iota_\ell, \\ a^*_{\ell_i}\le \dfrac{W(\iota_\ell^+)}{W(\iota_\ell)}\le a_{\ell_i},\ \ b^*_{\ell_i}\le\dfrac{T_\alpha(W(\iota_\ell^+))}{T_\alpha(W(\iota_\ell))}\le b_{\ell_i}\ \ \ell=1,2,\cdots,\ \ i=1,2,\cdots,m, \end{array}\right\} \tag{4.1}$$

has only zero and non-negative solutions, then each solution of the BVPs (E) and (B) is an oscillation in G.

Proof. We use the contradiction technique and assume that there exists a non-oscillatory solution $(\vartheta_1(\omega,\iota),\cdots,\vartheta_m(\omega,\iota))$ of the BVP (E) and (B). We let $|\vartheta_i(\omega,\iota)|>0$ for $\iota\ge\iota_0,\ i=1,2,\cdots m$. Let $\delta_i = sgn\ \vartheta_i(\omega,\iota)$, $w_i(\omega,\iota)=\delta_i\vartheta_i(\omega,\iota)$, and then $w_i(\omega,\iota)>0$, $(\omega,\iota)\in\psi\times[\iota_0,+\infty)$, $i=1,2,\cdots m$. From (H_3), there exists an $\iota_1>\iota_0$ such that $\tau(\iota,\varsigma)\ge\iota_0$, $\sigma_j(\iota,\varsigma)\ge\iota_0$ for $(\iota,\varsigma)\in[\iota_1,+\infty)\times[a,b]$ and $\rho_\hbar(\iota)\ge\iota_0$ for $\iota\ge\iota_0$. Then,

$$\begin{aligned} & w_i(\omega,\tau(\iota,\varsigma))>0 \qquad \text{for}\quad (\omega,\iota,\varsigma)\in\psi\times[\iota_1,+\infty)\times[a,b], \\ & w_i(\omega,\sigma_j(\iota,\varsigma))>0 \qquad \text{for}\quad (\omega,\iota,\varsigma)\in\psi\times[\iota_1,+\infty)\times[a,b],\quad j=1,2,\cdots,d, \\ \text{and}\quad & w_i(\omega,\rho_\hbar(\iota))>0 \qquad \text{for}\quad (\omega,\iota)\in\psi\times[\iota_1,+\infty),\ \ \hbar=1,2,\cdots,l. \end{aligned}$$

For $\iota\ge\iota_0,\ \iota\ne\iota_\ell,\ \ell=1,2,\cdots$, multiplying both sides of equation (E) by δ_i and integrating with respect to ω over the domain ψ, we obtain

$$\left.\begin{array}{l} \iota^{1-\alpha}\dfrac{d}{dt}\left[r(\iota)\iota^{1-\alpha}\dfrac{d}{dt}\left(\int_\psi \delta_i\vartheta_i(\omega,\iota)dx+\int_\psi\int_a^b\delta_i g(\iota,\varsigma)\vartheta_i(\omega,\tau(\iota,\varsigma))d\eta(\varsigma)dx\right)\right] \\ +p(\iota)\iota^{1-\alpha}\dfrac{d}{dt}\left(\int_\psi \delta_i\vartheta_i(\omega,\iota)dx+\int_\psi\int_a^b\delta_i g(\iota,\varsigma)\vartheta_i(\omega,\tau(\iota,\varsigma))d\eta(\varsigma)dx\right) \\ +\sum\limits_{n=1}^{m}\sum\limits_{j=1}^{d}\int_\psi\int_a^b\delta_i q_{inj}(\omega,\iota,\varsigma)f_{ij}\left(\vartheta_n(\omega,\sigma_j(\iota,\varsigma))\right)d\eta(\varsigma)dx \\ = a_i(\iota)\int_\psi\delta_i\Delta\vartheta_i(\omega,\iota)dx+\sum\limits_{n=1}^{m}\sum\limits_{\hbar=1}^{l}\int_\psi a_{inh}(\iota)\delta_i\Delta\vartheta_n(\omega,\rho_\hbar(\iota))dx, \\ \hfill \iota\ge\iota_1, i=1,2,\cdots,m. \end{array}\right\} \tag{4.2}$$

We can see that

$$\int_\psi\int_a^b\delta_i q_{inj}(\omega,\iota,\varsigma)f_{ij}\left(\vartheta_n(\omega,\sigma_j(\iota,\varsigma))\right)d\eta(\varsigma)dx=\int_a^b\int_\psi\delta_i q_{inj}(\omega,\iota,\varsigma)f_{ij}\left(\vartheta_n(\omega,\sigma_j(\iota,\varsigma))\right)dxd\eta(\varsigma),$$

and

$$\int_\psi\int_a^b g(\iota,\varsigma)\delta_i u_i(\omega,\tau(\iota,\varsigma))d\eta(\varsigma)dx=\int_a^b\int_\psi g(\iota,\varsigma)\delta_i u_i(\omega,\tau(\iota,\varsigma))dxd\eta(\varsigma).$$

Therefore,

$$\left.\begin{aligned}&\imath^{1-\alpha}\frac{d}{dt}\left[r(\imath)\imath^{1-\alpha}\frac{d}{dt}\left(\int_\psi w_i(\omega,\imath)dx+\int_a^b\int_\psi g(\imath,\varsigma)w_i(\omega,\tau(\imath,\varsigma))dxd\eta(\varsigma)\right)\right]\\&+p(\imath)\imath^{1-\alpha}\frac{d}{dt}\left(\int_\psi w_i(\omega,\imath)dx+\int_a^b\int_\psi g(\imath,\varsigma)w_i(\omega,\tau(\imath,\varsigma))dxd\eta(\varsigma)\right)\\&+\sum_{j=1}^{d}\left\{\int_a^b\int_\psi q_{iij}(\omega,\imath,\varsigma)f_{ii}\left(w_i(\omega,\sigma_j(\imath,\varsigma))\right)dxd\eta(\varsigma)\right.\\&\qquad\left.+\sum_{n=1,\ n\neq i}^{m}\delta_i\delta_n\int_a^b\int_\psi q_{inj}(\omega,\imath,\varsigma)f_{in}\left(w_n(\omega,\sigma_j(\imath,\varsigma))\right)dxd\eta(\varsigma)\right\}\\&=a_i(\imath)\int_\psi\Delta w_i(\omega,\imath)dx+\sum_{\hbar=1}^{l}\left\{\int_\psi a_{iih}(\imath)\Delta w_i(\omega,\rho_\hbar(\imath))dx\right.\\&\left.+\sum_{n=1,\ n\neq i}^{m}\delta_i\delta_n\int_\psi a_{inh}(\imath)\Delta w_n(\omega,\rho_\hbar(\imath))dx\right\},\qquad \imath\geq\imath_1,i=1,2,\cdots,m.\end{aligned}\right\}\tag{4.3}$$

Using boundary condition (B) and Green's formula, it follows that

$$\int_\psi\Delta w_i(\omega,\imath)dx=\int_{\partial\psi}\frac{\partial w_i(\omega,\imath)}{\partial\gamma}dS=-\int_{\partial\psi}\mu_i(\omega,\imath)w_i(\omega,\imath)dS,\tag{4.4}$$

and

$$\int_\psi\Delta w_n(\omega,\rho_\hbar(\imath))dx=\int_{\partial\psi}\frac{\partial w_n(\omega,\rho_\hbar(\imath))}{\partial\gamma}dS=-\int_{\partial\psi}\mu_n(\omega,\rho_\hbar(\imath))w_n(\omega,\rho_\hbar(\imath))dS,\tag{4.5}$$

where $\hbar=1,2,\cdots,l; i=1,2,\cdots,m$, and dS is the surface element on $\partial\psi$. Using Jensen's inequality from (H_5) and assumptions,

$$\int_a^b\int_\psi q_{iij}(\omega,\imath,\varsigma)f_{ii}\left(w_i(\omega,\sigma_j(\imath,\varsigma))\right)dxd\eta(\varsigma)\geq\int_a^b\int_\psi\epsilon q_{iij}(\omega,\imath,\varsigma)w_i(\omega,\sigma_j(\imath,\varsigma))dxd\eta(\varsigma),\tag{4.6}$$

and

$$\int_a^b\int_\psi q_{inj}(\omega,\imath,\varsigma)f_{in}\left(w_n(\omega,\sigma_j(\imath,\varsigma))\right)dxd\eta(\varsigma)\geq\int_a^b\int_\psi\epsilon q_{inj}(\omega,\imath,\varsigma)w_n(\omega,\sigma_j(\imath,\varsigma))dxd\eta(\varsigma).\tag{4.7}$$

From (4.3)–(4.7), we get

$$\begin{aligned}&\imath^{1-\alpha}\frac{d}{dt}\left[r(\imath)\imath^{1-\alpha}\frac{d}{dt}\left(\int_\psi w_i(\omega,\imath)dx+\int_a^b\int_\psi g(\imath,\varsigma)w_i(\omega,\tau(\imath,\varsigma))dxd\eta(\varsigma)\right)\right]\\&+p(\imath)\imath^{1-\alpha}\frac{d}{dt}\left(\int_\psi w_i(\omega,\imath)dx+\int_a^b\int_\psi g(\imath,\varsigma)w_i(\omega,\tau(\imath,\varsigma))dxd\eta(\varsigma)\right)\\&+\sum_{j=1}^{d}\left\{\int_a^b\int_\psi\epsilon q_{iij}(\imath,\varsigma)w_i(\omega,\sigma_j(\imath,\varsigma))dxd\eta(\varsigma)-\sum_{n=1,\ n\neq i}^{m}\int_a^b\int_\psi\epsilon\bar{q}_{inj}(\imath,\varsigma)w_n(\omega,\sigma_j(\imath,\varsigma))dxd\eta(\varsigma)\right\}\\&\leq\sum_{\hbar=1}^{l}\left\{-\int_{\partial\psi}\mu_i(\omega,\rho_\hbar(\imath))a_{iih}(\imath)w_i(\omega,\rho_\hbar(\imath))dS\right.\\&\qquad\left.+\sum_{n=1,\ n\neq i}^{m}\int_\psi|a_{inh}(\imath)|\,\mu_n(\omega,\rho_\hbar(\imath))w_n(\omega,\rho_\hbar(\imath))dS\right\},\ \imath\geq\imath_1,i=1,2,\cdots,m.\end{aligned}$$

Setting

$$v_i(\iota) = \int_{\psi} w_i(\omega, \iota)dx, \quad z_i(\iota) = \int_{\partial\psi} \mu_i(\omega, \iota)w_i(\omega, \iota)dS, \ \iota \geq \iota_1, i = 1, 2, \cdots, m,$$

we obtain

$$\left.\begin{array}{l} \iota^{1-\alpha}\dfrac{d}{dt}\left[r(\iota)\iota^{1-\alpha}\dfrac{d}{dt}\left(v_i(\iota) + \int_a^b g(\iota,\varsigma)v_i(\tau(\iota,\varsigma))d\eta(\varsigma)\right)\right] \\ +p(\iota)\iota^{1-\alpha}\dfrac{d}{dt}\left(v_i(\iota) + \int_a^b g(\iota,\varsigma)v_i(\tau(\iota,\varsigma))d\eta(\varsigma)\right) \\ +\sum\limits_{j=1}^{d}\left\{\int_a^b \epsilon q_{iij}(\iota,\varsigma)v_i(\sigma_j(\iota,\varsigma))d\eta(\varsigma) - \sum\limits_{n=1,\ n\neq i}^{m}\int_a^b \epsilon\bar{q}_{inj}(\iota,\varsigma)v_n(\sigma_j(\iota,\varsigma))d\eta(\varsigma)\right\} \\ \leq \sum\limits_{\hbar=1}^{l}\left\{-z_i(\rho_\hbar(\iota))a_{iih}(\iota) + \sum\limits_{n=1,\ n\neq i}^{m}|a_{inh}(\iota)|\, z_n(\rho_\hbar(\iota))\right\}, \\ \hfill \iota \geq \iota_1, i = 1, 2, \cdots, m. \end{array}\right\} \tag{4.8}$$

Let $V(\iota) = \sum\limits_{i=1}^{m} v_i(\iota)$, $Z(\iota) = \sum\limits_{i=n}^{m} z_i(\iota)$, for $\iota \geq \iota_1$. It follows from (4.8) that

$$\left.\begin{array}{l} \iota^{1-\alpha}\dfrac{d}{dt}\left[r(\iota)\iota^{1-\alpha}\dfrac{d}{dt}\left(V(\iota) + \int_a^b g(\iota,\varsigma)V(\tau(\iota,\varsigma))d\eta(\varsigma)\right)\right] \\ +p(\iota)\iota^{1-\alpha}\dfrac{d}{dt}\left(V(\iota) + \int_a^b g(\iota,\varsigma)V(\tau(\iota,\varsigma))d\eta(\varsigma)\right) \\ +\sum\limits_{j=1}^{d}\epsilon\left\{\sum\limits_{i=1}^{m}\left(\int_a^b q_{iij}(\iota,\varsigma)v_i(\sigma_j(\iota,\varsigma))d\eta(\varsigma) - \sum\limits_{n=1,\ n\neq i}^{m}\int_a^b \bar{q}_{inj}(\iota,\varsigma)v_n(\sigma_j(\iota,\varsigma))d\eta(\varsigma)\right)\right\} \\ +\sum\limits_{\hbar=1}^{l}\left\{\sum\limits_{i=1}^{m}\left(a_{iih}(\iota)z_i(\rho_\hbar(\iota)) - \sum\limits_{n=1,\ n\neq i}^{m}|a_{inh}(\iota)|\, z_n(\rho_\hbar(\iota))\right)\right\}, \\ \leq 0, \qquad\qquad \iota \geq \iota_1, \quad i = 1, 2, \cdots, m. \end{array}\right\} \tag{4.9}$$

Note that

$$\begin{aligned} &\sum_{i=1}^{m}\int_a^b\left(q_{iij}(\iota,\varsigma)v_i(\sigma_j(\iota,\varsigma)) - \sum_{n=1,\ n\neq i}^{m}\bar{q}_{inj}(\iota,\varsigma)v_n(\sigma_j(\iota,\varsigma))\right)d\eta(\varsigma) \\ &= \int_a^b\left(q_{11j}(\iota,\varsigma)v_1(\sigma_j(\iota,\varsigma)) - \sum_{n=1,\ n\neq 1}^{m}\bar{q}_{1nj}(\iota,\varsigma)v_n(\sigma_j(\iota,\varsigma))\right)d\eta(\varsigma) \\ &\quad + \int_a^b\left(q_{22j}(\iota,\varsigma)v_2(\sigma_j(\iota,\varsigma)) - \sum_{n=1,\ n\neq 2}^{m}\bar{q}_{2nj}(\iota,\varsigma)v_n(\sigma_j(\iota,\varsigma))\right)d\eta(\varsigma) \\ &\quad + \cdots + \int_a^b\left(q_{mmj}(\iota,\varsigma)v_m(\sigma_j(\iota,\varsigma)) - \sum_{n=1,\ n\neq m}^{m}\bar{q}_{mnj}(\iota,\varsigma)v_n(\sigma_j(\iota,\varsigma))\right)d\eta(\varsigma) \\ &= \int_a^b\left(q_{11j}(\iota,\varsigma) - \sum_{n=1,\ n\neq 1}^{m}\bar{q}_{n1j}(\iota,\varsigma)\right)v_1(\sigma_j(\iota,\varsigma))d\eta(\varsigma) \\ &\quad + \int_a^b\left(q_{22j}(\iota,\varsigma) - \sum_{n=1,\ n\neq 2}^{m}\bar{q}_{n2j}(\iota,\varsigma)\right)v_2(\sigma_j(\iota,\varsigma))d\eta(\varsigma) \end{aligned}$$

$$+\cdots+\int_a^b\left(q_{mmj}(\iota,\varsigma)-\sum_{n=1,\ n\neq m}^{m}\bar{q}_{nmj}(\iota,\varsigma)\right)v_m(\sigma_j(\iota,\varsigma))d\eta(\varsigma)$$
$$\geq\int_a^b\min_{1\leq i\leq m}\left(q_{iij}(\iota,\varsigma)-\sum_{n=1,\ n\neq i}^{m}\bar{q}_{nij}(\iota,\varsigma)\right)\sum_{i=1}^{m}v_i(\sigma_j(\iota,\varsigma))d\eta(\varsigma)$$
$$=\int_a^b Q_j(\iota,\varsigma)V(\sigma_j(\iota,\varsigma))d\eta(\varsigma),\quad \iota\geq\iota_1,\quad j=1,2,\cdots,d,$$

and similarly,

$$\sum_{i=1}^{m}\left(a_{iih}(\iota)z_i(\rho_\hbar(\iota))-\sum_{n=1,\ n\neq i}^{m}|a_{inh}(\iota)|\,z_n(\rho_\hbar(\iota))\right)$$
$$\geq\min_{1\leq i\leq m}\left(a_{iih}(\iota)-\sum_{n=1,\ n\neq i}^{m}|a_{nih}(\iota)|\right)\sum_{i=1}^{m}z_i(\rho_\hbar(\iota))$$
$$=A_\hbar(\iota)z(\rho_\hbar(\iota)),\quad \iota\geq\iota_1,\ \hbar=1,2,\cdots,l.$$

Thus, from (4.9), we have

$$\iota^{1-\alpha}\frac{d}{dt}\left[r(\iota)\iota^{1-\alpha}\frac{d}{dt}\left(V(\iota)+\int_a^b g(\iota,\varsigma)V(\tau(\iota,\varsigma))d\eta(\varsigma)\right)\right]$$
$$+p(\iota)\iota^{1-\alpha}\frac{d}{dt}\left(V(\iota)+\int_a^b g(\iota,\varsigma)V(\tau(\iota,\varsigma))d\eta(\varsigma)\right)$$
$$+\sum_{j=1}^{d}\epsilon\int_a^b Q_j(\iota,\varsigma)V(\sigma_j(\iota,\varsigma))d\eta(\varsigma)+\sum_{\hbar=1}^{l}A_\hbar(\iota)Z(\rho_\hbar(\iota))\leq 0,\quad \iota\geq\iota_1, i=1,2,\cdots,m.$$

We obtain

$$Z(\rho_\hbar(\iota))=\sum_{i=1}^{m}z_i(\rho_\hbar(\iota))\geq 0,\quad \iota\geq\iota_1,\ \hbar=1,2,\cdots,l.$$

Hence,

$$\iota^{1-\alpha}\frac{d}{dt}\left[r(\iota)\iota^{1-\alpha}\frac{d}{dt}\left(V(\iota)+\int_a^b g(\iota,\varsigma)V(\tau(\iota,\varsigma))d\eta(\varsigma)\right)\right]$$
$$+p(\iota)\iota^{1-\alpha}\frac{d}{dt}\left(V(\iota)+\int_a^b g(\iota,\varsigma)V(\tau(\iota,\varsigma))d\eta(\varsigma)\right)$$
$$+\sum_{j=1}^{d}\epsilon\int_a^b Q_j(\iota,\varsigma)V(\sigma_j(\iota,\varsigma))d\eta(\varsigma)\leq 0,\quad \iota\geq\iota_1, i=1,2,\cdots,m.$$

Set $W(\iota)=V(\iota)+\int_a^b g(\iota,\varsigma)V(\tau(\iota,\varsigma))d\eta(\varsigma)$. Then,

$$T_\alpha\left(r(\iota)T_\alpha(W(\iota))\right)+p(\iota)T_\alpha(W(\iota))$$
$$+\sum_{j=1}^{d}\epsilon\int_a^b Q_j(\iota,\varsigma)V(\sigma_j(\iota,\varsigma))d\eta(\varsigma)\leq 0,\quad \iota\geq\iota_1, i=1,2,\cdots,m. \tag{4.10}$$

It is easy to get that $W(\iota) > 0$ for $\iota \geq \iota_1$. Next, we show that $T_\alpha(W(\iota)) > 0$ for $\iota \geq \iota_2$. As a matter of fact, assume the opposite, that there exists $T \geq \iota_2$ such that $T_\alpha(W(T)) \leq 0$.

$$\begin{aligned} T_\alpha\left(r(\iota)T_\alpha(W(\iota))\right) + p(\iota)T_\alpha(W(\iota)) \leq 0, \quad \iota \geq \iota_2, \\ T_\alpha(r(\iota))T_\alpha(W(\iota)) + r(\iota)T_\alpha(T_\alpha(W(\iota))) + p(\iota)T_\alpha(W(\iota)) \leq 0, \quad \iota \geq \iota_2. \end{aligned} \tag{4.11}$$

From (H_1), we have $T_\alpha(\Lambda(\iota)) = \Lambda(\iota)\left(\dfrac{T_\alpha(r(\iota)) + p(\iota)}{r(\iota)}\right)$ and $T_\alpha(\Lambda(\iota)) \geq 0$, $\Lambda(\iota) > 0$ for $\iota \geq \iota_2$. We multiply $\dfrac{\Lambda(\iota)}{r(\iota)}$ on both sides of (4.11), and we obtain

$$\Lambda(\iota)T_\alpha(T_\alpha(W(\iota))) + T_\alpha(\Lambda(\iota))T_\alpha(W(\iota)) = T_\alpha\left(\Lambda(\iota)T_\alpha(W(\iota))\right) \leq 0, \quad \iota \geq \iota_2. \tag{4.12}$$

From (4.12), we have $\Lambda(\iota)(T_\alpha(W(\iota))) \leq \Lambda(T)T_\alpha(W(T)) \leq 0$, $\iota \geq T$. Thus,

$$\begin{aligned} \int_T^\iota T_\alpha(W(s))ds &\leq \int_T^\iota \frac{\Lambda(T)T_\alpha(W(T))}{s^{1-\alpha}\Lambda(s)}ds, \quad \iota \geq T, \\ W(\iota) &\leq W(T) + \Lambda(T)T_\alpha(W(T))\int_T^\iota \frac{ds}{s^{1-\alpha}\Lambda(s)}, \quad \iota \geq T. \end{aligned}$$

From the hypothesis (H_1), we get $\lim_{\iota\to+\infty} W(\iota) = -\infty$. This contradicts $W(\iota) > 0$ for $\iota \geq 0$. Thus, $T_\alpha(W(\iota)) > 0$ and $\tau(\iota, \varsigma) \leq \iota$ for $\iota \geq \iota_1$. Hence,

$$\begin{aligned} V(\iota) = &\ W(\iota) - \int_a^b g(\iota,\varsigma)V(\tau(\iota,\varsigma))d\eta(\varsigma) \\ \geq &\ W(\iota) - c(\iota)W(\iota) \\ \geq &\ W(\iota)\left(1 - \int_a^b g(\iota,\varsigma)d\eta(\varsigma)\right) \end{aligned}$$

and

$$V(\sigma_j(\iota,\varsigma)) \geq W(\sigma_j(\iota,\varsigma))\left(1 - \int_a^b g(\sigma_j(\iota,\varsigma),\varsigma)d\eta(\varsigma)\right), \quad j = 1, 2, \cdots, d.$$

Therefore, from (4.10), we have

$$\begin{aligned} &T_\alpha\left(r(\iota)T_\alpha(W(\iota))\right) + p(\iota)T_\alpha(W(\iota)) \\ &+ \sum_{j=1}^d \int_a^b \epsilon Q_j(\iota,\varsigma)\left[1 - \int_a^b g(\sigma_j(\iota,\varsigma),\varsigma)d\eta(\varsigma)\right]W(\sigma_j(\iota,\varsigma))d\eta(\varsigma) \leq 0, \quad \iota \geq \iota_1. \end{aligned}$$

From (H_3) and (H_4), we have

$$W[\sigma_j(\iota,\varsigma)] \geq W[\sigma_j(\iota,a)] > 0, \quad \varsigma \in [a,b] \quad \text{and} \quad \theta_j(\iota) \leq \sigma_j(\iota,a) \leq \iota,$$

and consequently, $W(\theta_j(\iota)) \leq W(\sigma_j(\iota,a))$ for $\iota \geq \iota_1$. Therefore,

$$T_\alpha\left(r(\iota)T_\alpha(W(\iota))\right) + p(\iota)T_\alpha(W(\iota))$$

$$+\sum_{j=1}^{d}\epsilon\int_a^b Q_j(\iota,\varsigma)\left[1-\int_a^b g(\sigma_j(\iota,\varsigma),\varsigma)d\eta(\varsigma)\right]W(\theta_j(\iota))d\eta(\varsigma)\le 0,\quad \iota\ne\iota_\ell.$$

For $\iota\ge\iota_0,\ \iota=\iota_\ell,\ \ell=1,2,\cdots$, multiplying both sides of the equation (E) by δ_i, and integrating with respect to ω over the domain ψ and from (H_7), we get

$$a_\ell^*\le\frac{\alpha_\ell(\omega,\iota_\ell,\vartheta(\omega,\iota_\ell))}{\vartheta(\omega,\iota_\ell)}\le a_\ell,\qquad b_\ell^*\le\frac{\beta_\ell(\omega,\iota_\ell,\vartheta_\iota(\omega,\iota_\ell))}{\vartheta_\iota(\omega,\iota_\ell)}\le b_\ell,$$
$$a_{\ell_i}^*\le\frac{\vartheta_i(\omega,\iota_\ell^+)}{\vartheta_i(\omega,\iota_\ell)}\le a_{\ell_i},\qquad b_{\ell_i}^*\le\frac{\frac{\partial^\alpha\vartheta_i(\omega,\iota_\ell^+)}{\partial\iota^\alpha}}{\frac{\partial^\alpha\vartheta_i(\omega,\iota_\ell)}{\partial\iota^\alpha}}\le b_{\ell_i}.$$

According to $w_i(\iota)=\delta_i\int_\psi\vartheta_i(\omega,\iota_\ell)dx$, we have

$$a_{\ell_i}^*\le\frac{V(\iota_\ell^+)}{V(\iota_\ell)}\le a_{\ell_i},\qquad b_{\ell_i}^*\le\frac{T_\alpha(V(\iota_\ell^+))}{T_\alpha(V(\iota_\ell))}\le b_{\ell_i}.$$

Because $W(\iota)=V(\iota)+\int_a^b g(\iota,\varsigma)V(\tau(\iota,\varsigma))d\eta(\varsigma)$, we obtain

$$a_{\ell_i}^*\le\frac{W(\iota_\ell^+)}{W(\iota_\ell)}\le a_{\ell_i},\qquad b_{\ell_i}^*\le\frac{T_\alpha(W(\iota_\ell^+))}{T_\alpha(W(\iota_\ell))}\le b_{\ell_i}.$$

Therefore, $W(\iota)$ is an eventually positive solution of (4.1). This contradicts the hypothesis and completes the proof. □

Theorem 3. *If there exist some* $j_0\in\{1,2,\cdots,d\}$ *and* $\varphi(\iota)\in C^\alpha(\mathbb{R}_+,(0,+\infty))$ *such that*

$$\int_{\iota_0}^{+\infty}\prod_{\iota_0\le\iota_\ell<s}\left(\frac{b_{\ell_i}}{a_{\ell_i}^*}\right)^{-1}s^{\alpha-1}\left[\varphi(s)B(s)-\frac{A^2(s)}{4C(s)}\right]ds=+\infty,\tag{4.13}$$

where

$$A(\iota)=\frac{T_\alpha(\varphi(\iota))}{\varphi(\iota)}-\frac{p(\iota)}{r(\iota)},\quad B(\iota)=\epsilon\int_a^b Q_{j_0}(\iota,\varsigma)\left[1-\int_a^b g(\sigma_{j_0}(\iota,\varsigma),\varsigma)d\eta(\varsigma)\right]d\eta(\varsigma),$$

and

$$C(\iota)=\frac{T_\alpha(\theta_{j_0}(\iota))}{\varphi(\theta_{j_0}(\iota))r(\theta_{j_0}(\iota))}$$

then each solution of the (BVPs) (E) *and* (B) *represents an oscillation in G.*

Proof. From the proof of Theorem 2, we suppose that $W(\iota)$ is a non-zero and non-negative solution of the inequality (4.1). Then, a number $\iota_1\ge\iota_0$ is introduced in a way that $W(\theta_{j_0}(\iota))>0,\ j=1,2,\cdots,d$ for $\iota\ge\iota_1$. Thus, we obtain

$$T_\alpha\left(r(\iota)T_\alpha(W(\iota))\right)+p(\iota)T_\alpha(W(\iota))$$

$$+\epsilon\int_a^b Q_{j_0}(\iota,\varsigma)\left[1-\int_a^b g(\sigma_{j_0}(\iota,\varsigma),\varsigma)d\eta(\varsigma)\right]W(\theta_{j_0}(\iota))d\eta(\varsigma)\le 0,\quad \iota\ge\iota_1. \tag{4.14}$$

Define

$$Z(\iota):=\varphi(\iota)\frac{r(\iota)T_\alpha(W(\iota))}{W(\theta_{j_0}(\iota))},\quad \iota\ge\iota_0.$$

Then, $Z(\iota)\ge 0$ for $\iota\ge\iota_0$, and

$$\begin{aligned}T_\alpha(Z(\iota))\le\ &\left(\frac{T_\alpha(\varphi(\iota))}{\varphi(\iota)}-\frac{p(\iota)}{r(\iota)}\right)Z(\iota)-\epsilon\varphi(\iota)\int_a^b Q_{j_0}(\iota,\varsigma)\left[1-\int_a^b g(\sigma_{j_0}(\iota,\varsigma),\varsigma)d\eta(\varsigma)\right]\\ &-\frac{Z^2(\iota)}{\varphi(\theta_{j_0}(\iota))}\frac{T_\alpha(\theta_{j_0}(\iota))}{r(\theta_{j_0}(\iota))}.\end{aligned}$$

Thus,

$$\begin{aligned}T_\alpha(Z(\iota))\le\ &A(\iota)Z(\iota)-B(\iota)\varphi(\iota)-Z^2(\iota)C(\iota),\\ Z(\iota_\ell^+)\le\ &\frac{b_{\ell_i}}{a^*_{\ell_i}}Z(\iota_\ell).\end{aligned}$$

Define

$$U(\iota):=\prod_{\iota_0\le\iota_\ell<\iota}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1}Z(\iota).$$

In fact, $Z(\iota)$ is continuous on each interval $(\iota_\ell,\iota_{\ell+1}]$, and we take into account that $Z(\iota_\ell^+)\le\frac{b_{\ell_i}}{a^*_{\ell_i}}Z(\iota_\ell)$. It follows that for $\iota\ge\iota_0$,

$$U(\iota_\ell^+)=\prod_{\iota_0\le\iota_j\le\iota_\ell}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1}Z(\iota_\ell^+)\le\prod_{\iota_0\le\iota_j<\iota_\ell}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1}Z(\iota_\ell)=U(\iota_\ell),$$

and for all $\iota\ge\iota_0$,

$$U(\iota_\ell^-)=\prod_{\iota_0\le\iota_j\le\iota_{\ell-1}}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1}Z(\iota_\ell^-)\le\prod_{\iota_0\le\iota_j<\iota_\ell}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1}Z(\iota_\ell)=U(\iota_\ell),$$

which implies that $U(\iota)$ is continuous on $[\iota_0,+\infty)$. Also,

$$\begin{aligned}&T_\alpha(U(\iota))+\prod_{\iota_0\le\iota_\ell<\iota}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)U^2(\iota)C(\iota)+\prod_{\iota_0\le\iota_\ell<\iota}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1}B(\iota)\varphi(\iota)-A(\iota)U(\iota)\\ &=\prod_{\iota_0\le\iota_\ell<\iota}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1}T_\alpha(Z(\iota))+\prod_{\iota_0\le\iota_\ell<\iota}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)\prod_{\iota_0\le\iota_\ell<\iota}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-2}C(\iota)Z^2(\iota)\\ &\quad+\prod_{\iota_0\le\iota_\ell<\iota}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1}B(\iota)\varphi(\iota)-\prod_{\iota_0\le\iota_\ell<\iota}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1}A(\iota)Z(\iota)\end{aligned}$$

$$= \prod_{\imath_0 \le \imath_\ell < \imath} \left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1} \left[T_\alpha(Z(\imath)) + Z^2(\imath)C(\imath) - Z(\imath)A(\imath) + B(\imath)\varphi(\imath)\right] \le 0,$$

that is,

$$T_\alpha(U(\imath)) \le -\prod_{\imath_0 \le \imath_\ell < \imath} \left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right) C(\imath)U^2(\imath) + A(\imath)U(\imath) - \prod_{\imath_0 \le \imath_\ell < \imath} \left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1} B(\imath)\varphi(\imath). \tag{4.15}$$

Taking

$$X = \sqrt{\prod_{\imath_0 \le \imath_\ell < \imath} \left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right) C(\imath)} U(\imath), \quad Y = \frac{A(\imath)}{2} \sqrt{\prod_{\imath_0 \le \imath_\ell < \imath} \left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1} \frac{1}{C(\imath)}},$$

from Lemma 1, we have

$$A(\imath)U(\imath) - \prod_{\imath_0 \le \imath_\ell < \imath} \left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right) C(\imath)U^2(\imath) \le \frac{A^2(\imath)}{4C(\imath)} \prod_{\imath_0 \le \imath_\ell < \imath} \left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1}.$$

Thus,

$$T_\alpha(U(\imath)) \le -\prod_{\imath_0 \le \imath_\ell < \imath} \left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1} \left[B(\imath)\varphi(\imath) - \frac{A^2(\imath)}{4C(\imath)}\right].$$

Using the technique of integrating both sides from $\imath_0$ to $\imath$, we get

$$U(\imath) \le U(\imath_0) - \int_{\imath_0}^{\imath} \prod_{\imath_0 \le \imath_\ell < s} \left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1} s^{\alpha-1} \left[B(s)\varphi(s) - \frac{A^2(s)}{4C(s)}\right] ds.$$

Letting $\imath \to +\infty$, from (4.13), we have $\lim_{\imath \to +\infty} U(\imath) = -\infty$, which contradicts $U(\imath) \ge 0$. □

Theorem 4. *Suppose that* $\varphi(\imath)$, $\phi(\imath) \in C^\alpha(\mathbb{R}_+, (0, +\infty))$, *and* $E(\imath, s), e(\imath, s) \in C^\alpha(D, \mathbb{R})$, *in a way that* $D = \{(\imath, s) | \imath \ge s \ge \imath_0 > 0\}$ *where*

(H_8) $E(\imath, \imath) = 0, \quad \imath \ge \imath_0; \quad E(\imath, s) > 0, \quad \imath > s \ge \imath_0$;

(H_9) $\dfrac{\partial^\alpha E(\imath, s)}{\partial \imath^\alpha} \ge 0; \quad \dfrac{\partial^\alpha E(\imath, s)}{\partial s^\alpha} \le 0$;

(H_{10}) $-\dfrac{\partial^\alpha E(\imath, s)}{\partial s^\alpha} = e(\imath, s)\sqrt{E(\imath, s)}$.

If

$$\limsup_{\imath \to +\infty} \frac{1}{E(\imath, \imath_0)} \int_{\imath_0}^{\imath} \prod_{\imath_0 \le \imath_\ell < s} \left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1} \left[\frac{B(s)\varphi(s)E(\imath, s)\phi(s)}{s^{1-\alpha}} - \frac{s^{1-\alpha}}{4C(s)\phi(s)} \left[e(\imath, s)\phi(s) - T_\alpha(\phi(s))\sqrt{E(\imath, s)} - \frac{A(s)\phi(s)\sqrt{E(\imath, s)}}{s^{1-\alpha}}\right]^2\right] ds = +\infty, \tag{4.16}$$

then all the solutions of the BVP of both (E) *and* (B) *are oscillatory in G.*

Proof. From the proof of Theorem 3,

$$T_\alpha(U(\imath)) \leq -\prod_{\imath_0\leq\imath_\ell<\imath}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)C(\imath)U^2(\imath)+A(\imath)U(\imath)-\prod_{\imath_0\leq\imath_\ell<\imath}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1}B(\imath)\varphi(\imath).$$

We multiply the above inequality by $H(\imath, s)\phi(s)$ for $\imath \geq s \geq T$ and integrate from T to $\imath$, and we get

$$\begin{aligned}\int_T^\imath \frac{T_\alpha(U(s))E(\imath,s)\phi(s)}{s^{1-\alpha}}ds \leq & -\int_T^\imath\prod_{\imath_0\leq\imath_\ell<s}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)\frac{C(s)U^2(s)E(\imath,s)\phi(s)}{s^{1-\alpha}}ds\\ &+\int_T^\imath\frac{A(s)U(s)E(\imath,s)\phi(s)}{s^{1-\alpha}}ds\\ &-\int_T^\imath\prod_{\imath_0\leq\imath_\ell<s}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1}\frac{B(s)\varphi(s)E(\imath,s)\phi(s)}{s^{1-\alpha}}ds.\end{aligned}$$

Thus,

$$\begin{aligned}\int_T^\imath\prod_{\imath_0\leq\imath_\ell<s}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1}\frac{1}{s^{1-\alpha}}B(s)\varphi(s)E(\imath,s)\phi(s)ds \leq U(T)E(\imath,T)\phi(T)\\ -\int_T^\imath\left[-\frac{\partial^\alpha E(\imath,s)}{\partial s^\alpha}\phi(s)-E(\imath,s)T_\alpha(\phi(s))-\frac{A(s)E(\imath,s)\phi(s)}{s^{1-\alpha}}\right]U(s)ds\\ -\int_T^\imath\prod_{\imath_0\leq\imath_\ell<s}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)\frac{C(s)U^2(s)E(\imath,s)\phi(s)}{s^{1-\alpha}}ds.\end{aligned}$$

$$\begin{aligned}&\int_T^\imath\prod_{\imath_0\leq\imath_\ell<s}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1}\frac{B(s)\varphi(s)E(\imath,s)\phi(s)}{s^{1-\alpha}}ds\\ &-\frac{1}{4}\int_T^\imath\prod_{\imath_0\leq\imath_\ell<s}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1}\frac{s^{1-\alpha}}{C(s)\phi(s)}\left[e(\imath,s)\phi(s)-T_\alpha(\phi(s))\sqrt{E(\imath,s)}-\frac{A(s)\phi(s)\sqrt{E(\imath,s)}}{s^{1-\alpha}}\right]^2ds\\ &\leq U(T)E(\imath,T)\phi(T)\end{aligned}\tag{4.17}$$

From (4.17) for $\imath \geq T \geq \imath_0$, we have

$$\begin{aligned}&\frac{1}{E(\imath,\imath_0)}\int_{\imath_0}^\imath\prod_{\imath_0\leq\imath_\ell<s}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1}\Bigg[\frac{B(s)\varphi(s)E(\imath,s)\phi(s)}{s^{1-\alpha}}\\ &\qquad-\frac{s^{1-\alpha}}{4C(s)\phi(s)}\left[e(\imath,s)\phi(s)-T_\alpha(\phi(s))\sqrt{E(\imath,s)}-\frac{A(s)\phi(s)\sqrt{E(\imath,s)}}{s^{1-\alpha}}\right]^2\Bigg]ds\\ &=\frac{1}{E(\imath,\imath_0)}\left[\int_{\imath_0}^T+\int_T^\imath\right]\Bigg\{\prod_{\imath_0\leq\imath_\ell<s}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1}\Bigg[\frac{B(s)\varphi(s)E(\imath,s)\phi(s)}{s^{1-\alpha}}\\ &\qquad-\frac{s^{1-\alpha}}{4C(s)\phi(s)}\left[e(\imath,s)\phi(s)-T_\alpha(\phi(s))\sqrt{E(\imath,s)}-\frac{A(s)\phi(s)\sqrt{E(\imath,s)}}{s^{1-\alpha}}\right]^2\Bigg]\Bigg\}ds\end{aligned}$$

$$\leq \frac{1}{E(\imath,\imath_0)}\int_{\imath_0}^{T}\prod_{\imath_0\leq\imath_\ell<s}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1}\frac{B(s)\varphi(s)E(\imath,s)\phi(s)}{s^{1-\alpha}}ds+\phi(T)U(T)$$
$$\leq \int_{\imath_0}^{T}\prod_{\imath_0\leq\imath_\ell<s}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1}\frac{B(s)\varphi(s)\phi(s)}{s^{1-\alpha}}ds+\phi(T)U(T).$$

Letting $\imath\to+\infty$, we get

$$\begin{aligned}\limsup_{\imath\to+\infty}\frac{1}{E(\imath,\imath_0)}\int_{\imath_0}^{\imath}\prod_{\imath_0\leq\imath_\ell<s}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1}&\left[\frac{B(s)\varphi(s)E(\imath,s)\phi(s)}{s^{1-\alpha}}\right.\\&\left.-\frac{s^{1-\alpha}}{4C(s)\phi(s)}\left[e(\imath,s)\phi(s)-T_\alpha(\phi(s))\sqrt{E(\imath,s)}-\frac{A(s)\phi(s)\sqrt{E(\imath,s)}}{s^{1-\alpha}}\right]^2\right]ds\\&\leq\int_{\imath_0}^{T}\prod_{\imath_0\leq\imath_\ell<s}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1}\frac{E(\imath,s)B(s)\varphi(s)\phi(s)}{s^{1-\alpha}}ds+\phi(T)U(T)\\&<+\infty,\end{aligned}$$

which implies a contradiction with (4.16). □

Remark 1. *In Theorem 4, by choosing $\phi(s)=\varphi(s)\equiv 1$, we have the following corollary.*

Corollary 1. *Suppose that*

$$\limsup_{\imath\to+\infty}\frac{1}{E(\imath,\imath_0)}\int_{\imath_0}^{\imath}\prod_{\imath_0\leq\imath_\ell<s}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1}\left[\frac{B(s)E(\imath,s)}{s^{1-\alpha}}-\frac{s^{1-\alpha}}{4C(s)}\left[e(\imath,s)-\frac{A(s)\sqrt{E(\imath,s)}}{s^{1-\alpha}}\right]^2\right]ds=+\infty.$$

Then, all the solutions of the boundary value problem mentioned in $(E),(B)$ *are oscillatory in G.*

Remark 2. *Using Theorem 4 and Corollary 1, by varying the weighted functions' parameters $E(\imath,s)$ we can attain various oscillatory conditions. We shall give an example, by choosing $E(\imath,s)=(\imath-s)^{\kappa-1}$, $\imath\geq s\geq\imath_0$, in which $\kappa>2$ is an integer, and then $e(\imath,s)=s^{1-\alpha}(\kappa-1)(\imath-s)^{(\kappa-3)/2}$, $\imath\geq s\geq\imath_0$. From Corollary 1, we get the following*

Corollary 2. *If an integer $\kappa>2$ such that*

$$\begin{aligned}\limsup_{\imath\to+\infty}\frac{1}{(\imath-\imath_0)^{\kappa-1}}\int_{\imath_0}^{\imath}\prod_{\imath_0\leq\imath_\ell<s}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1}(\imath-s)^{\kappa-1}&\left[\frac{B(s)}{s^{1-\alpha}}-\frac{s^{1-\alpha}}{4C(s)}\times\right.\\&\left.\left[\frac{A^2(s)}{s^{2-2\alpha}}-\frac{2(\kappa-1)A(s)}{(\imath-s)}+\frac{s^{2-2\alpha}(\kappa-1)^2}{(\imath-s)^2}\right]\right]ds=+\infty,\end{aligned}$$

then all the solutions of the BVP mentioned in both (E) *and* (B) *are oscillatory in G.*

Now, we study $E(\imath,s)=[R(\imath)-R(s)]^\kappa$, $\imath\geq s\geq\imath_0$, where $R(\imath)=\int_{\imath_0}^{\imath}\frac{1}{r(s)}ds$ and $\lim_{\imath\to+\infty}R(\imath)=+\infty$, and then $e(\imath,s)=s^{1-\alpha}\kappa[R(\imath)-R(s)]^{\frac{\kappa-2}{2}}$. From Corollary 1, one can get the following

Corollary 3. *If an integer $\kappa > 2$, such that*

$$\limsup_{\iota\to+\infty}\frac{1}{[R(\iota)-R(s)]^{\kappa}}\int_{\iota_0}^{\iota}\prod_{\iota_0\le\iota_\ell<s}\left(\frac{b_{\ell_i}}{a^*_{\ell_i}}\right)^{-1}[R(\iota)-R(s)]^{\kappa}\Bigg[\frac{B(s)}{s^{1-\alpha}}-\frac{s^{1-\alpha}}{4C(s)}\times \left[\frac{A^2(s)}{s^{2-2\alpha}}-\frac{2\kappa A(s)}{(R(\iota)-R(s))}+\frac{s^{2-2\alpha}\kappa^2}{(R(\iota)-R(s))^2}\right]\Bigg]ds=+\infty,$$

then all the solutions of the BVP of both (E) and (B) are oscillatory in G.

5. An example

In this section, we illustrate our main result with an example.

Example 1. *We give the following system:*

$$\left.\begin{aligned}
&\frac{\partial^{1/2}}{\partial\iota^{1/2}}\left[4\frac{\partial^{1/2}}{\partial\iota^{1/2}}\left(\vartheta_1(\omega,\iota)+\int_{-\pi/2}^{-\pi/4}\frac{1}{2}\vartheta_1(\omega,\iota+2\varsigma)d\varsigma\right)\right]\\
&+\left(-\frac{4}{5}\right)\frac{\partial^{1/2}}{\partial\iota^{1/2}}\left(\vartheta_1(\omega,\iota)+\int_{-\pi/2}^{-\pi/4}\frac{1}{2}\vartheta_1(\omega,\iota+2\varsigma)d\varsigma\right)\\
&+6\iota\int_{-\pi/2}^{-\pi/4}\vartheta_1(\omega,\iota+2\varsigma)d\varsigma+12\iota\int_{-\pi/2}^{-\pi/4}\vartheta_2(\omega,\iota+2\varsigma)d\varsigma=\frac{\iota^{1/2}}{5}\Delta\vartheta_1(\omega,\iota)\\
&+\left(8\iota+\frac{3\iota^{1/2}}{5}-\frac{3}{2}\right)\Delta\vartheta_1(\omega,\iota-3\pi/2)+\frac{1}{2}\Delta\vartheta_2(\omega,\iota-3\pi/2),\quad \iota\neq\iota_\ell,\ \ell=1,2,\cdots,\\
&\frac{\partial^{1/2}}{\partial\iota^{1/2}}\left[4\frac{\partial^{1/2}}{\partial\iota^{1/2}}\left(\vartheta_2(\omega,\iota)+\int_{-\pi/2}^{-\pi/4}\frac{1}{2}\vartheta_2(\omega,\iota+2\varsigma)d\varsigma\right)\right]\\
&+\left(-\frac{4}{5}\right)\frac{\partial^{1/2}}{\partial\iota^{1/2}}\left(\vartheta_2(\omega,\iota)+\int_{-\pi/2}^{-\pi/4}\frac{1}{2}\vartheta_2(\omega,\iota+2\varsigma)d\varsigma\right)\\
&+12\iota\int_{-\pi/2}^{-\pi/4}\vartheta_1(\omega,\iota+2\varsigma)d\varsigma+14\iota\int_{-\pi/2}^{-\pi/4}\vartheta_2(\omega,\iota+2\varsigma)d\varsigma=\left(10\iota-\frac{1}{2}\right)\Delta\vartheta_2(\omega,\iota)\\
&+\left(10\iota+\frac{\iota^{1/2}}{5}\right)\Delta\vartheta_1(\omega,\iota-3\pi/2)+\left(\frac{3\iota^{1/2}}{5}-\frac{3}{2}\right)\Delta\vartheta_2(\omega,\iota-3\pi/2),\quad \iota\neq\iota_\ell,\ \ell=1,2,\cdots,\\
&\vartheta_i(\omega,\iota_\ell^+)=\frac{\ell+1}{\ell}\vartheta_i(\omega,\iota_\ell),\\
&\frac{\partial^\alpha}{\partial\iota^\alpha}\vartheta_i(\omega,\iota_\ell^+)=\frac{\partial^\alpha}{\partial\iota^\alpha}\vartheta_i(\omega,\iota_\ell),\quad \ell=1,2,\cdots,\ \ i=1,2,
\end{aligned}\right\}\tag{5.1}$$

for $(\omega,\iota)\in(0,\pi)\times\mathbb{R}_+$, with the boundary condition

$$\frac{\partial}{\partial\omega}\vartheta_i(0,\iota)=\frac{\partial}{\partial\omega}\vartheta_i(\pi,\iota)=0,\quad \iota\neq\iota_\ell,\ \ i=1,2.\tag{5.2}$$

Here, $\psi = (0,\pi)$, $N = 2$, $m = 2$, $d = 1$, $l = 1$, $\alpha = \frac{1}{2}$, $a_{\ell_i} = a^*_{\ell_i} = \dfrac{\ell+1}{\ell}$, $b_{\ell_i} = b^*_{\ell_i} = 1$, $i = 1,2$, $r(\iota) = 4$, $g(\iota,\varsigma) = \dfrac{1}{2}$, $\rho_1(\iota) = \iota - 3\pi/2$, $p(\iota) = -\dfrac{4}{5}$, $\sigma_1(\iota,\varsigma) = \tau(\iota,\varsigma) = \iota + 2\varsigma$, $\eta(\varsigma) = \varsigma$, $f_{ij}(\vartheta_n) = \vartheta_n$, $\epsilon = 1$, $q_{111}(\omega,\iota,\varsigma) = 6\iota$, $q_{121}(\omega,\iota,\varsigma) = 12\iota$, $a_1(\iota) = \dfrac{\iota^{1/2}}{5}$, $a_{111}(\iota) = 8\iota + \dfrac{3\iota^{1/2}}{5} - \dfrac{3}{2}$, $a_{121}(\iota) = \dfrac{1}{2}$, $q_{211}(\omega,\iota,\varsigma) = 12\iota$, $q_{221}(\omega,\iota,\varsigma) = 14\iota$, $a_2(\iota) = 8\iota - \dfrac{1}{2}$, $a_{211}(\iota) = 8\iota + \dfrac{\iota^{1/2}}{5}$, $a_{221}(\iota) = \dfrac{3\iota^{1/2}}{5} - \dfrac{3}{2}$, $Q_1(\iota,\varsigma) = -6\iota$, $[a,b] = [-\pi/2,-\pi/4]$, $\kappa = 3$, $\theta_1(\iota) = \iota$, $T_\alpha(\theta_1(\iota)) = \iota^{1-\alpha}$. *Since* $\iota_0 = 1$, $\iota_\ell = 2^\ell$, $A(s) = \dfrac{1}{5}$, $B(s) = -\dfrac{3s(8\pi-\pi^2)}{16}$, $E(s) = \dfrac{s^{1/2}}{4}$.

Then, hypotheses $(H_1)-(H_7)$ *hold; moreover,*

$$\begin{aligned}\lim_{\iota\to+\infty}\int_{\iota_0}^{\iota}\prod_{\iota_0\le\iota_\ell<s}\frac{b^*_{\ell_i}}{a_{\ell_i}}ds &= \int_1^{+\infty}\prod_{1<\iota_\ell<s}\frac{\ell}{\ell+1}ds\\ &= \int_1^{\iota_1}\prod_{1<\iota_\ell<s}\frac{\ell}{\ell+1}ds+\int_{\iota_1^+}^{\iota_2}\prod_{1<\iota_\ell<s}\frac{\ell}{\ell+1}ds+\int_{\iota_2^+}^{\iota_3}\prod_{1<\iota_\ell<s}\frac{\ell}{\ell+1}ds+\cdots\\ &= 1+\frac{1}{2}\times 2+\frac{1}{2}\times\frac{2}{3}\times 2^2+\cdots\\ &= \sum_{n=0}^{+\infty}\frac{2^n}{n+1} = +\infty.\end{aligned}$$

Thus,

$$\begin{aligned}&\limsup_{\iota\to+\infty}\frac{1}{(\iota-1)^2}\left\{\int_1^{\iota}\prod_{1<\iota_\ell<s}\frac{\ell+1}{\ell}(\iota-s)^2\left[-\frac{3s^{3/2}}{16}[8\pi-\pi^2]-\frac{4s}{(\iota-s)^2}+\frac{4}{5(\iota-s)}-\frac{1}{25s}\right]ds\right\}\\ &= +\infty.\end{aligned}$$

Hence, all the mentioned conditions of Corollary 2 hold, meaning that all the solutions of the problem (5.1)-(5.2) *are oscillatory in G. As a matter of fact,* $\vartheta_1(\omega,\iota) = \cos\omega\sin\iota$, $\vartheta_2(\omega,\iota) = \cos\omega\cos\iota$ *is such a solution.*

6. Conclusions

In this work, we have discussed several systems of impulsive conformable partial fractional differential equations and some of their oscillatory solutions under the Robin boundary condition. In addition, we used several modified techniques to find some sufficient conditions for the solutions. To validate the work, we worked on illustrating the main results by providing a section of an example. In our future work, we will discuss some oscillatory solutions for systems of impulsive conformable partial fractional differential equations of neutral type.

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Conflict of interest

The authors declare that they have no competing interests.

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