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Research article

The Cartesian closedness of c-spaces

Xiaolin Xie and Hui Kou*

Department of Mathematics, Sichuan University, Chengdu, 610064, China

* **Correspondence:** Email: kouhui@scu.edu.cn.

Abstract: Directed space was defined by Hui Kou in 2014 [21], which is equivalent to T_0 monotone determined space. Its main purpose is to build an extended framework for domain theory. In this paper, we study the category of c-spaces which is a subcategory of directed spaces. The main results are: (1) we will describe c-spaces using a new definition, which give us the convenience to construct new classes of spaces; (2) we give some conditions such that categorical products and topological products agree in **Dtop**; (3) the category of c-spaces is not Cartesian closed; (4) we define a new class of spaces, namely, FS-spaces, which forms a Cartesian closed category.

Keywords: dcpo; domain; directed spaces; continuous spaces; monotone determined spaces; c-spaces
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1. Introduction

Domain theory was first introduced by Dana Scott in the early 1970s, and the main purpose is to provide a mathematical tools for the semantics of functional programming languages. The most distinctive feature of domain theory is that it integrates order structures, topology structures and computer science. The main objects of domain theory are directed complete posets and domains. Directed space is defined by Hui Kou independently in [21]. It is worth noting that directed spaces are equivalent to T_0 monotone determined spaces, which is defined by Erné [5]. It was proved in [21] that directed spaces contain the basic objects of domain theory, all directed complete posets endowed with the Scott topology, which forms a Cartesian closed category. Thus, directed space is an extended framework of domain theory.

In Section 3, we will describe c-spaces by means of approximating, namely, continuous spaces, and this new definition leads us to construct a new class of spaces. Just like the category **Domain** in **Dcpo**, a c-space is a special directed space, we will also prove in Section 4 that the category **CS** (the category of all c-spaces and continuous functions) is not Cartesian closed. In domain theory, the products of two dcpos endowed with Scott topology may not equal to the topological products of two dcpos endowed with Scott topology respectively. This inspires us to explore conditions such that topological products and categorical products agree in **Dtop**. Since **CS** is not Cartesian closed, we shall continue to explore some Cartesian closed categories of **CS**, and furthermore, as we want to explore some maximal Cartesian closed full subcategories in **CS**, we will define a Cartesian closed category of **Dtop**, namely, FS-spaces.

2. Preliminaries

Now, we introduce the concepts needed in this article. More details, on domain theory, topology, and category theory, see [3, 7, 11]. Let P be a nonempty set. A relation $\leq$ on P is called a partial order, if $\leq$ satisfies reflexivity ($x \leq x$), transitivity ($x \leq y$ & $y \leq z \Rightarrow x \leq z$) and antisymmetry ($x \leq y$ & $y \leq x \Rightarrow x = y$). P is called a partially ordered set(poset) if P is endowed with some partial order $\leq$. Given $A \subseteq P$, denote $\downarrow A = \{x \in P : \exists a \in A,\ x \leq a\}$, $\uparrow A = \{x \in P : \exists a \in A,\ a \leq x\}$. We say A is a lower set (upper set) if $A = \downarrow A$ ($A = \uparrow A$). A nonempty set $D \subseteq P$ is called a directed set if each finite nonempty subset of D has an upper bound in D. Particularly, a poset is called a directed complete poset if each directed subset D has a supremum(denoted by $\bigvee D$), abbreviated as dcpo. The subset U of poset P is called a Scott open set if U is an upper set and for each directed set $D \subseteq P$, which $\bigvee D$ exists and belongs to U, then $U \cap D \neq \emptyset$. The set of all Scott open sets of poset P is a toplology on P, which is called the Scott topology and denoted by $\sigma(P)$. Suppose P, E are two posets, a function $f : P \longrightarrow E$ is called Scott continuous if it is continuous respect to Scott topology $\sigma(P)$ and $\sigma(E)$.

All topological spaces in this paper are T_0.

A net of a topological space X is a map $\xi : J \longrightarrow X$, where J is a directed set. Thus, each directed subset of a poset can be regarded as a net, and its index set is itself. Usually, we denote a net by $(x_j)_{j \in J}$ or (x_j). Let $x \in X$, saying (x_j) converges to x, denote by $(x_j) \to x$ or $x \equiv \lim x_j$, if (x_j) is eventually in every open neighborhood of x, that is, for each given open neighborhood U of x, there exists $j_0 \in J$ such that for every $j \in J$, $j \geq j_0 \Rightarrow x_j \in U$.

Let X be a T_0 topological space, its topology is denoted by $\mathcal{O}(X)$, the specialization order on X is defined as follows:

$$\forall x,\ y \in X,\ x \sqsubseteq y \Leftrightarrow x \in \overline{\{y\}}$$

here, $\overline{\{y\}}$ means the closure of $\{y\}$. From now on, the order of a T_0 topological space always indicates the specialization order "$\sqsubseteq$". Here are some basic properties of specialization order.

Proposition 2.1. ([3, 7]) For a T_0 topological space X, the following hold:

(1) For each open set $U \subseteq X, U = \uparrow U$;
(2) For each closed set $A \subseteq X, A = \downarrow A$;
(3) Suppose Y is another T_0 topological space, and $f : X \to Y$ is a continuous function from X to Y. Then for each $x,\ y \in X$, $x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$, i.e., every continuous function is monotone.

Suppose X is a T_0 space, then every directed set $D \subseteq X$ can be regarded as a net of X, we use $D \to x$ or $x \equiv \lim D$ to represent D converges to x. Define notation

$$D(X) = \{(D,\ x) : x \in X,\ D \text{ is a directed subset of } X \text{ and } D \to x\}.$$

It is easy to verify that, for each $x,\ y \in X$, $x \sqsubseteq y \Leftrightarrow \{y\} \to x$. Therefore, if $x \sqsubseteq y$ then $(\{y\},\ x) \in D(X)$. Next, we give the concept of directed space.

Definition 2.2. ([21]) Let X be a T_0 space.

(1) A subset U of X is called a directed open set if $\forall(D,\ x) \in D(X),\ x \in U \Rightarrow D \cap U \neq \emptyset$. Denote all directed open sets of X by $d(X)$.
(2) X is called a directed space if each directed open set of X is an open set, that is, $d(X) = O(X)$.

C-space was definded by Erné in [4]. A T_0 topological space X is a c-space if for each $x \in X$ and each open neighborhood U of x, there exists some $y \in U$ such that $x \in \mathrm{int}(\uparrow y) \subseteq U$.

A T_0 topological space X is called a locally finitary compact space if and only if, for every $x \in X$, for every open neighborhood U of x, there is a finitary compact $\uparrow E$ (i.e., with E finite) included in U such that x is in the interior of $\uparrow E$ (see [8]).

Obviously, every c-space is locally finitary compact, and the following proposition tell us that every locally finitary compact space is derected space. Thus, c-space and locally finitary compact space are both contained in directed space.

Proposition 2.3. *Suppose X is a locally finitary compact space, then X is a directed spaces.*

Proof. We only need to prove that for each $U \in d(X)$, $U \in O(X)$. For arbitrary $x \in U$, let

$$\mathcal{F} = \{F \subseteq X : x \in \mathrm{int}\,(\uparrow F) \subseteq \uparrow F \subseteq U \text{ and } U \text{ is finite}\}.$$

We claim that here exists some $F \in \mathcal{F}$ such that $F \subseteq U$.

Suppose not, that is for each $F \in \mathcal{F},\ F \nsubseteq U$. Then $\{F \backslash U : F \in \mathcal{F}\}$ is a directed family and $F_1 \leq F_2$ iff $\uparrow F_2 \subseteq \uparrow F_1$. According to Rudin's Lemma ([7]), there exists a directed set $D \subseteq \bigcup_{F \in \mathcal{F}}(F \backslash U)$, furthermore, for each $F \in \mathcal{F},\ D \cap (F \backslash U) \neq \emptyset$. It is obviously that D convergent to x in X. Since U is a directed open set, we may pick some $d \in D \cap U$. This is a contradiction. □

Remark 2.4.

(1) Each open set of a T_0 space is directed open, but the contrary is not necessarily true. For example, suppose Y is a non-discrete T_1 topological space, its specialization order is diagonal, that is, $\forall x,\ y \in Y,\ x \sqsubseteq y \Leftrightarrow x = y$. Thus, all subsets of Y are directed open. We notice that Y is non-discrete, at least one directed open set is not an open set.
(2) The definition of directed space here is equivalent to the T_0 monotone determined space defined in [5].

Example 2.5. Important examples of directed spaces:

- Alexandroff spaces (Posets endowed with the Alexandroff topology);
- Any poset with Scott topology (Posets endowed with the Scott topology);
- c-spaces;
- locally finitary compact spaces.

Next, we introduce the directed continuous function.

Definition 2.6. ([21]) Suppose $X,\ Y$ are two T_0 spaces. A function $f : X \longrightarrow Y$ is called directed continuous if it is monotone and preserves all limits of directed set of X; that is, $(D,\ x) \in D(X) \Rightarrow (f(D),\ f(x)) \in D(Y)$.

Here are some characterizations of the directed continuous functions.

Proposition 2.7. ([21]) Suppose $X,\ Y$ are two T_0 spaces. $f : X \longrightarrow Y$ is a function between X and Y.

(1) f is directed continuous if and only if $\forall U \in d(Y),\ f^{-1}(U) \in d(X)$.
(2) If $X,\ Y$ are directed spaces, then f is continuous if and only if it is directed continuous.

Now we introduce the product and exponential object of directed spaces.

Suppose $X,\ Y$ are two directed spaces. Let $X \times Y$ represents the Cartesian product of X and Y, then we have a natural partial order on it: $\forall (x_1,\ y_1),\ (x_2,\ y_2) \in X \times Y$,

$$(x_1,\ y_1) \leq (x_2,\ y_2) \iff x_1 \sqsubseteq x_2,\ y_1 \sqsubseteq y_2,$$

which is called the pointwise order on $X \times Y$. Now, we define a topological space $X \otimes Y$ as follows:

(1) The underlying set of $X \otimes Y$ is $X \times Y$;
(2) The topology on $X \times Y$ is generated as follows: For each given $\leq$- directed set $D \subseteq X \times Y$ and $(x,\ y) \in X \times Y$,
$$D \to (x,\ y) \in X \otimes Y \iff \pi_1 D \to x \in X,\ \pi_2 D \to y \in Y,$$
that is, a subset $U \subseteq X \times Y$ is open if and only if for every directed limit defined as above $D \to (x,\ y) \in U \Rightarrow U \cap D \neq \emptyset$.

Theorem 2.8. ([21]) Suppose X and Y are two directed spaces.

(1) The topological space $X \otimes Y$ defined above is a directed space and satisfies the following properties: The specialization order on $X \otimes Y$ equals to the pointwise order on $X \times Y$, that is, $\sqsubseteq = \leq$.
(2) Suppose Z is another directed space, then $f : X \otimes Y \longrightarrow Z$ is continuous if and only if it is continuous in each variable separately.

Let X, Y be two directed spaces. Denote the set of all continuous functions from X to Y by

$$Y^X = \{f : X \to Y | f \text{ is continuous from } X \text{ to } Y\}$$

There is a pointwise order on Y^X: $\forall f, g \in Y^X, \forall x \in X$

$$f \leq g \iff f(x) \sqsubseteq g(x).$$

Next, we define a topological space $[X \to Y]$ by the following way:

(1) The underlying set of $[X \to Y]$ is Y^X;
(2) A subset $\mathcal{U} \subseteq Y^X$ is open if and only if for arbitrary $\leq$ - directed subset $\{f_i\}_{i \in I} \subseteq Y^X$ and $f \in \mathcal{U}$, if for arbitrary $x \in X, \{f_i(x)\}_{i \in I} \to f(x)$, then $\mathcal{U} \cap \{f_i\}_{i \in I} \neq \emptyset$.

Denote all the open sets defined above by $\mathcal{O}(Y^X)$, let $[X \to Y] = (Y^X,\ \mathcal{O}(Y^X))$.

Lemma 2.9. ([21]) Suppose X and Y are two arbitrary directed spaces, then $[X \to Y]$ is a directed space and the following hold:

(1) The specialization order $\sqsubseteq$ of $[X \to Y]$ equals to the pointwise order $\leq$ of Y^X;
(2) For arbitrary $\sqsubseteq$ - directed set $\{f_i\}_{i \in I} \subseteq Y^X$ and $f \in Y^X$, the following holds in $[X \to Y]$:

$$\{f_i\}_{i \in I} \to f \iff \{f_i(x)\}_{i \in I} \to f(x).$$

Theorem 2.10. ([21]) The category **Dtop** is Cartesian closed. For arbitrary directed space X and Y, the categorical product and the exponential object are $X \otimes Y$ and $[X \to Y]$ respectively.

Denote the category of all directed spaces and continuous functions by **Dtop**. It was proved in [21] that, **Dtop** contains all posets endowed with the Scott topology and **Dtop** is a Cartesian closed category.

Proposition 2.11. *For arbitrary directed spaces, the composition map* $(f, g) \mapsto f \circ g : [X_1 \to Y_1] \otimes [X_2 \to Y_2] \to [X_2 \to Y_1]$ *is continuous.*

Proof. By Theorem 2.8 and Proposition 2.7, we only need to check that if we have a directed set $\mathcal{D} = \{f_i\}_{i\in I} \subseteq [X_1 \to Y_1]$ with $\{f_i\}_{i\in I} \to f$, then $\{f_i \circ g\}_{i\in I} \to f \circ g$, that is for arbitrary $x \in X_2$, $(f_i \circ g)(x) \to (f \circ g)(x)$. □

Let P be a dcpo, and $x,\ y \in P$. We say x way below y, if for each given directed set $D \subseteq P$, $y \le \bigvee D$ implies that there exists some $d \in D$ such that $x \le d$. We write $\Downarrow x = \{a \in P : a \ll x\}$, $\Uparrow x = \{a \in P : x \ll a\}$.

Definition 2.12. ([7]) A dcpo P is called a continuous domain if for each $x \in P$, $\Downarrow x$ is directed and $x = \bigvee \Downarrow x$.

Theorem 2.13. ([7]) Suppose P is a continuous domain. The followings hold:

(1) $\forall x,\ y \in P,\ x \ll y \Rightarrow \exists z \in P,\ x \ll z \ll y$.
(2) $\forall x \in P$, $\Uparrow x$ is a Scott open set. Particularly, $\{\Uparrow x : x \in P\}$ is a base of $(P,\ \sigma(P))$.

3. Continuous spaces

In this section, we use a equivalent definition to c-space, and this new definition leads us to construct a new class of spaces in Section 4.

Definition 3.1. ([18]) Suppose X is a directed space, define a relation on X: for arbitrary $x,\ y \in X$, $x \ll_d y$ if and only if for arbitrary directed set $D \subseteq X$ with $D \to y$, there exists some $d \in \mathcal{D}$ such that $x \le d$. An element x is said to be compact if $x \ll_d x$ holds. Denote all compact elements of X by $K(X)$.

It is easy to check the following propositions of $\ll_d$.

Proposition 3.2. *Suppose X is a directed space and* $\forall x, y, z, \omega \in X$, *then*

(1) $x \ll_d y \Rightarrow x \le y$.
(2) $x \le y \ll_d z \le w \Rightarrow x \ll_d w$.

Similarly to the continuity of dcpo, we can define the continuity of an arbitrary T_0 space, and when the definition is restriced to the directed space,we have the following definition.

Definition 3.3. ([18]) A directed space X is said to be continuous if for arbitrary $x \in X$, there exists a directed subset $D \subseteq \Downarrow_d x$ such that $D \to x$.

Proposition 3.4. *Suppose X is a continuous directed space, then for arbitrary* $x \in X$, $\Downarrow_d x$ *is a directed set and* $\Downarrow_d x \to x$. *Moreover, x is the supremum of* $\Downarrow_d x$.

Proof. For an arbitrary continuous directed space X, $\forall x \in X$, for arbitrary $x_1, x_2 \in \Downarrow_d x$, by the continuity of X, we have some directed subset $D \subseteq \Downarrow_d x$ with $D \to x$. By the definition of $\ll_d$, there exist $d_i \in D$ such that $x_i \leq d_i, i = 1, 2$. Since D is directed, we may choose a $d \in D$ with $d_i \leq d, i = 1, 2$. Thus, $\Downarrow_d x$ is directed and $\Downarrow_d x \to x$.

By Proposition 3.2, binary relation $\ll_d \implies \leq$, then x is an upper bound of $\Downarrow_d x$. Suppose y is another upper bound of $\Downarrow_d x$ and $x \nleq y$, that is $x \in X \setminus \downarrow y$, will $X \setminus \downarrow y$ be an open neighborhood of x. which leads a contradiciton. □

Lemma 3.5. *Suppose X is a continuous directed space, then for arbitrary $x, y \in X$ with $x \ll_d y$, there exists some $z \in X$ such that $x \ll_d z \ll_d y$.*

Proof. Let $D = \{\omega \in X : \exists z \in X, \omega \ll_d z \ll_d y\}$. It is obviously that D is not an empty set since X is continuous, then $\Downarrow_d y \neq \emptyset$, we can pick some $a \in \Downarrow_d y$, and again by the continuity of X, $\Downarrow_d a \neq \emptyset$. Thus $D \neq \emptyset$. We claim that D is directed, for arbitrary $\omega_1, \omega_2 \in D$, by the definition of D, there exist $z_i \in X$ with $\omega_i \ll_d z_i \ll_d y, i = 1, 2$. According to 3.4, $\Downarrow_d y$ is directed, thus we may have some $z \in \Downarrow_d y$ such that $z_i \leq z$. Since X is continuous, $\Downarrow_d z$ is directed, and $\omega_i \in \Downarrow_d z$, we may pick some $\omega \in \Downarrow_d z$ with $\omega_i \leq \omega, i = 1, 2$. Now we have $\omega \ll_d z \ll_d y$, and D is directed.

For arbitrary open neighborhood U of x, and $\Downarrow_d \to x$ implies that there exists some $z \in \Downarrow_d \cap U$, and $\Downarrow_d z \to z$ implies there exists some $\omega \in \Downarrow_d z \cap U$, thus $\omega \in D \cap U$, that is, $D \to x$. By the definition of $x \ll_d y$. Thus exists some $\omega \in D$ such that $x \leq \omega \ll_d z \ll_d y$, by Proposition 3.2, we have $x \ll_d z \ll_d y$. □

Lemma 3.6. *Suppose X is a directed continuous space, then for each $x \in X$, $\Uparrow_d x$ is an open set.*

Proof. Suppose X is a continuous directed space, and each $x \in X$, we only need to check that $\Uparrow_d x$ is a directed open set. Let D be a directed subset of X with $D \to z \in \Uparrow_d x$, by Lemma 3.5, there exists some $y \in X$ such that $x \ll_d y \ll_d z$. By the definition of $\ll_d$, we may pick some $d \in D$ such that $y \leq d$. Now we have $x \ll_d y \leq d$. According to Proposition 3.2, $x \ll_d d$, that is $D \cap \Uparrow_d x \neq \emptyset$, $\Uparrow_d x$ is open.

The following theorem is the main result of this section.

Theorem 3.7. *Suppose X is a directed space, then X is continuous if and only if X is a c-space.*

Proof. If X is a continuous directed space, $\forall x \in X$ and U is an arbitrary open neighbourhood of x. Thus $\Downarrow_d x \to x$ implies that there exists some $z \in \Downarrow_d x \cap U$, then $x \in \Uparrow_d z \subseteq U$, and by Lemma 3.6, $\Uparrow_d z$ is open, so $x \in \mathrm{int}(\uparrow z) \subseteq U$, and X is a c-space.

In the other direction, Suppose X is a c-space, and hence a directed space. It is direct to check that $x \in \mathrm{int}(\uparrow d)$ implies $d \ll_d x, \forall x, d \in X$. Then $D = \{d \in X : x \in \mathrm{int}(\uparrow d)\} \subseteq \Downarrow_d x$ is a directed set and $D \to x$, that is, X is a continuous directed space. □

Definition 3.8. A T_0 topological space X is an algebraic space if for each $x \in X$, there exists some net $\{x_i\}_{i\in I} \subseteq K(X) \cap \downarrow x$ such that $\{x_i\}_{i\in I} \to x$.

Note that the notion of algebraic space is equivalent to *finitary space* (*or φ-space*) defined by Ershov [6].

It is worth noting that $X \times Y = X \otimes Y$ when X and Y are c-spaces (see [18]). In next section, we are going to explore more conditions such that $X \times Y = X \otimes Y$.

4. The failure of Cartesian closedness of the category of c-spaces

As mentioned in Section 1, directed space can be regarded as an extended model of domain theory. In this section, we will explore other conditions such that the categorical products coincides with topological products. Moreover, we will also explore the Cartesian closedness of **CS**.

Theorem 4.1. ([7]) Let X be a topological space. Then X is core-compact iff the relation $(\in) = \{(x, U) \in X \times \mathcal{O}(X) : x \in U\}$ is open in $X \times \Sigma(\mathcal{O}(X))$.

Theorem 4.2. *Let X be a directed space. The following statements are equivalent:*

(1) X is core-compact;
(2) For any directed space Y, $X \otimes Y = X \times Y$.

Proof. (1) $\implies$ (2). We only need to show that every open set U in $X \otimes Y$ is open in $X \times Y$. For any $(x_0, y_0) \in U$, consider a new set $V_{y_0} = \{x \in X : (x, y_0) \in U\}$, it is easy to see that $V_{y_0} \in \mathcal{O}(X)$, since if we have directed set $D \subseteq X$ and $x \in V_{y_0}$ with $D \to x$, then $\{(d, y_0) : d \in D\}$ is a directed set in $X \times Y$ and $\{(d, y_0) : d \in D\} \to (x, y_0)$. Thus we may pick some $d_0 \in D$ such that $(d_0, y_0) \in U$, that is, $d_0 \in V_{y_0}$, V_{y_0} is an open set. Since X is core-compact, there exists a family of open sets $\{V_n : n \in N\}$ such that

$$x \in V_0 \ll \cdots \ll V_{n+1} \ll V_n \ll \cdots \ll V_1 \ll V.$$

Claim: $W = \bigcup_{n\geq 1}\{y \in Y : V_n \times \{y\} \subseteq U\}$ is an open set of Y.

Given any direted net $(y_i) \to y \in W$, there is some n such that $V_n \times \{y\} \subseteq U$. For any $x \in V_n$, $\{(x, y_i)\}_i \to (x, y) \in U$. So there is some i with $(x, y_i) \in U$. Hence there exists an open neighborhood V_x of x with $V_x \times \{y_i\} \subseteq U$. Notice that $V_{n+1} \ll V_n \subseteq \bigcup_{x\in V_n} V_x$, it follows that $V_{n+1} \subseteq \bigcup_{i=1}^{n} V_{x_i}$ for some finite set of V_n. It is easy to find some y_k such that $V_{n+1} \times \{y_k\} \subseteq U$. It means that $y_k \in W$. The claim is proved. Now we can see that $(x_0, y_0) \in V_0 \times W \subseteq U$. Therefore, U is an open set of $X \times Y$.
(2) $\implies$ (1). Take $Y = \Sigma(\mathcal{O}(X))$, which is a directed space. Then we have $X \otimes \Sigma(\mathcal{O}(X)) = X \times \Sigma(\mathcal{O}(X))$. Hence we only need to show that the relation $(\in) = \{(x, U) \in X \times \mathcal{O}(X) : x \in U\}$ is an open set of $X \otimes \Sigma(\mathcal{O}(X))$. For any directed net (of $X \otimes \Sigma(\mathcal{O}(X))$) $\{(x_i, U_i)\}_i \to (x, U) \in (\in)$. This is equivalent to say that $(x_i)_i \to x$ in X, $(U_i)_i \to U$ in $\Sigma(\mathcal{O}(X))$. It follows that $x \in U \subseteq \bigcup_i U_i$. It is easy to find some i_0 such that $x_{i_0} \in U_{i_0}$. □

Theorem 4.3. *Let X, Y be directed spaces. If both X and Y are first countable, then $X \otimes Y = X \times Y$.*

Proof. We only to show that every open set U of $X\otimes Y$ is open in $X\times Y$. For any $(x_0, y_0) \in U$, assume the countable basis $(V_n)_n$ of x_0, and $(W_n)_n$ of y_0. We want to show that there is some n such that $V_n\times W_n \subseteq U$. By contradiction, assume $V_n \times W_n \nsubseteq U$ for any $n \in N$. Then there exists $(x_n, y_n) \in (V_n \times W_n)\backslash U$ for any $n \geq 1$. Let $K = \{x_i \in X : i \geq 1\} \cup \{x_0\}$. Obviously K is a compact subset.

Claim: $W = \{y \in Y : K \times \{y\} \subseteq U\}$ is an open neighborhood of y_0.

Given any directed net $(y_i)_i \to y \in W$. For any $x \in K$, $(x, y_i) \to (x, y) \in U$. It is obtained that $(x, y_{i_x}) \in U$ for some i_x. It is easy to check that $V_x = \{\hat{x} \in X : (\hat{x}, y_{i_x}) \in U\}$ is an open set. It follows that $K \subseteq \bigcup_{x\in K} V_x$. Then there is a finite set $\{x_i : 1 \leq i \leq n\}$ such that $K \subseteq \bigcup_{i=1}^{n} V_{x_i}$. Hence we can find some y_k which belongs to W. The claim is proved. Since W is an open neighborhood of y_0, there exists some W_n such $W_m \subseteq W$. It implies that $(x_m, y_m) \in K \times W_m \subseteq U$. Which is a contradiction. □

The following example shows that a first countable directed space need not to be core-compact.

Example 4.4. There exists a first countable but not core-compact directed space.

For an arbitrary topological space X, let $Q(X)$ be the set of all compact saturated sets of X. Let $\sigma(Q(X))$ be the Scott topology, and $\mathcal{V}(Q(X))$ be the topology generated by $\{\Box U : U \in O(X)\}$, here $\Box U = \{K \in Q(X) : K \subseteq U\}$. According to [10], we know that

$$(Q(X), \mathcal{V}(Q(X))) \text{ is core-compact } \iff X \text{ is locally compact.}$$

Let $X = \mathbb{Q}$ (the set of all rational numbers endowed with the relative topology of all real numbers $\mathbb{R}$, endowed with the usual topology). Then we claim that $(Q(X), \sigma(Q(X))$ is first countable but not core-compact. Since X is sober and countable based, $Q(X)$ is a dcpo. According to [17], we have $(Q(X), \sigma(Q(X)) = (Q(X), \mathcal{V}(Q(X))$. Since X is not locally compact, then the directed space $(Q(X), \sigma(Q(X))$ is first countable but not core-compact.

Actually, this example can be easily verified, Clearly $\mathbb{Q}$ is firstly countable, T_2, non-locally compact, and hence non-core compact. (In fact, in the lattice $O(\mathbb{Q})$, the set of elements way-below $\mathbb{Q}$ is empty.)

To prove the main result of this section, we need to first have some preparations.

Definition 4.5. ([8]) (Application map) For each pair of topological spaces X, Y, the application map App maps pairs (f, x) of a continuous map $f : X \to Y$ and of an element $x \in X$ to $f(x)$.

Theorem 4.6. ([8]) Let **C** be any full subcategory of **Top** with finite products, and assume that $1 = \{\star\}$ is an object of **C**. Let X, Y be two objects of **C** that have an exponential object Y^X in **C**. Then there is a unique homeomorphism $\theta : Y^X \to [X \to Y]$, for some unique topology on $[X \to Y]$, such that $App(h, x) = \theta(h)(x)$ for all $h \in Y^X, x \in X$.

Proposition 4.7. *If a d-space is also a directed space, then it is a Scott space.*

Proof. Suppose X is a d-space, then X is a dcpo endowed with a topology coarser than the Scott topology. We need only to check that for each $U \in \sigma(X)$, $U \in d(X)$. Suppose we have a directed set $D \subseteq X$ and $x \in U$ with $D \to x \in U$, we need to prove that $D \cap U \neq \emptyset$. We only need to show that $\vee D \in U$. By contradiction, if we have $\vee D \notin U$, then $x \in X \setminus \downarrow\vee D$. Since $D \to x \in U$, then $D \cap X \setminus \downarrow\vee D \neq \emptyset$. Thus, we have some $d \in D$ such that $d \notin \downarrow\vee D$, which is a contradiction. □

The following theorem is a main result of this paper.

Theorem 4.8. *The category of c-spaces and continuous maps (**CS** for short) is not Cartesian closed.*

Proof. Let $\mathbb{Z}^-$ be the set of non-positive integers with Scott topology. Assume **CS** is a ccc. It is easy to see that the topological product $X \times Y$ is the categorical product because $X \times Y$ is a c-space. Since **CS** is Cartesian closed, according to Theorem 4.6, there exists exponential topology τ on $[\mathbb{Z}^- \to \mathbb{Z}^-]$, which we denote by $[\mathbb{Z}^- \to \mathbb{Z}^-]_\tau$. Then for any c-space Y and any map $f\colon Y \times \mathbb{Z}^- \to \mathbb{Z}^-$, f is continuous iff $\bar{f}\colon Y \to [\mathbb{Z}^- \to \mathbb{Z}^-]_\tau$ is continuous.

Claim 1**:** The specialization order on $[\mathbb{Z}^- \to \mathbb{Z}^-]_\tau$ is equal to the pointwise order. For any $g_1, g_2 \in [\mathbb{Z}^- \to \mathbb{Z}^-]_\tau$ with $g_1 \leq_\tau g_2$ ($g_1 \neq g_2$), take $Y = \mathbb{S}$ with Scott topology. A map $\theta\colon \mathbb{S} \to [\mathbb{Z}^- \to \mathbb{Z}^-]_\tau$ is defined as $\theta(1) = g_2, \theta(0) = g_1$. It is easy to see that θ is continuous. Hence $\hat{\theta} : \mathbb{S} \times \mathbb{Z}^- \to \mathbb{Z}^-$ is continuous. It follows that $g_1(x) = \hat{\theta}(0, x) \leq \hat{\theta}(1, x) = g_2(x)$ for any $x \in X$.

For any $g_1, g_2 \in [\mathbb{Z}^- \to \mathbb{Z}^-]_\tau$ with $g_1 \le g_2$, consider a continuous map $f\colon \mathbb{S} \times \mathbb{Z}^- \to \mathbb{Z}^-$ which is defined as $f(0, x) = g_1(x), f(1, x) = g_2(x)\ \forall x \in X$. It follows that the transpose map $\bar{f}$ is continuous hence monotone. It implies that $g_1 = \bar{f}(0) \le_\tau \bar{f}(1) = g_2$.

Claim 2**:** $[\mathbb{Z}^- \to \mathbb{Z}^-]_\tau$ is a d-space.

We only need to show that for any directed family $(g_i)_{i\in I}$ of $[\mathbb{Z}^- \to \mathbb{Z}^-]_\tau$, $(g_i)_{i\in I}$ converges to its supremum $g = \bigvee^{\uparrow}_{i\in I} g_i$. Let Y be a set $I \cup \{\infty\}$ with a topology generated by $\{\uparrow i \cup \{\infty\} : i \in I\}$, and obviously Y is a c-space. Consider a map $f\colon Y \times \mathbb{Z}^- \to \mathbb{Z}^-$ which is defined as $f(\infty, x) = g(x), f(i, x) = g_i(x)$. It is direct to verify that f is continuous (Actually, according to Theorem 2.8, f is continuous iff it is separately continuous). It follows that $\bar{f}\colon Y \to [\mathbb{Z}^- \to \mathbb{Z}^-]_\tau$ is continuous. It implies that $(g_i = \bar{f}(i))_i$ converges to $\bar{f}(\infty) = g$.

Therefore, according to Proposition 4.7, τ is just the Scott topology on $[\mathbb{Z}^- \to \mathbb{Z}^-]$. But $[\mathbb{Z}^- \to \mathbb{Z}^-]$ is not a continuous domain [1], it is not a c-space, which is a contradiction. □

Since $[\mathbb{Z}^- \to \mathbb{Z}^-]$ is meet continous but not continuous, according to Proposition III-3.10 in [7], a meet continuous quasicontinuous domain is a domain, we claim that $[\mathbb{Z}^- \to \mathbb{Z}^-]$ is not a quasi continuous space. Therefore, according to the proof of Theorem 4.8, we have the following corollary.

Corollary 4.9. *The category of all locally finitary compact spaces and continuous functions is not Cartesian closed.*

5. FS-spaces

As mentioned in Section 4, **CS** is not a Cartesian closed category of **Dtop**. This inspires us to find some other Cartesian closed subcartegory of **Dtop**. In this section, we define a new class of spaces, namely, FS-spaces, which forms a Cartesian closed subcategory of **CS**.

Definition 5.1. An approximate identity for a directed space X is a directed set $\mathcal{D} \subseteq [X \to X]$ satisfing $\mathcal{D} \to 1_X$ (pointwise convergence), the identity on X.

Lemma 5.2. *Approximate identities are preserved under the following constructions.*

(1) If $\mathcal{D} \subseteq [X \to X]$ is an approximate identity for X, then $\mathcal{D}' = \{\delta^2 = \delta \circ \delta : \delta \in \mathcal{D}\}$ is also an approximate identity.

(2) If $\mathcal{D} \subseteq [X \to X]$ is an approximate identity for X and $\mathcal{E} \subseteq [Y \to Y]$ is an approximate identity for Y, then $[\mathcal{D} \to \mathcal{E}]$ is an approximate identity for $[X \to Y]$, where members of $[\mathcal{D} \to \mathcal{E}]$ are denoted by $[\delta \to \varepsilon]$ for $\delta \in \mathcal{D}$ and $\varepsilon \in \mathcal{E}$ and definded by $[\delta \to \varepsilon](g) = \varepsilon g \delta$ for $g \in [X \to Y]$.

(3) If a directed space X has an approximate identity $\mathcal{D}$ such that $\delta(x) \ll_d x$ for all $\delta \in \mathcal{D}$ and for all $x \in X$, then X is a c-space.

Proof.

(1) According to Proposition 2.11, the map $(\delta, \delta) \mapsto \delta^2 : [X \to X] \otimes [X \to X] \to [X \to X]$ is continuous, and $\mathcal{D} \to id_X$ implies $\{\delta^2 : \delta \in \mathcal{D}\} \to id_X$.

(2) Firstly, for each $\delta \in \mathcal{D}, \varepsilon \in \mathcal{E}$, the map $g \mapsto \varepsilon g \delta : [X \to Y] \to [X \to Y]$ is continuous. If we have a directed subset $\{g_i\}_{i\in I} \subseteq [X \to Y]$ with $\{g_i\}_{i\in I} \to g$, then for each $x \in X$, $\{g_i(\varepsilon(x))\} \to g(\varepsilon(x))$, and hence $\delta(g(\varepsilon(x))) \to \delta(g(\varepsilon(x)))$. That is, $\{\varepsilon g_i \delta\}_{i\in I} \to \varepsilon g \delta$, the map is continuous.
Secondly, the directed set $[\mathcal{D} \to \mathcal{E}] \to id_{[X\to Y]}$, equivalently, for each $g \in [X \to Y]$, $\{\varepsilon g \delta : \varepsilon \in \mathcal{E}, \delta \in \mathcal{D}\} \to g$, equivalently, $\forall x \in X$, $\forall g \in [X \to Y]$, $\{(\varepsilon g \delta)(x) : \varepsilon \in \mathcal{E}, \delta \in \mathcal{D}\} \to g(x)$. By

hypothesis, $\mathcal{D}$ is an approximate identity for X, then $\{\delta(x) : \delta \in \mathcal{D}\} \to x$, hence $\{g(\delta(x)) : \delta \in \mathcal{D}\} \to g(x)$. Again, by the hypothesis that $\mathcal{E}$ is an approximate identity of Y, we have $\{(\varepsilon g\delta)(x) : \varepsilon \in \mathcal{E}, \delta \in \mathcal{D}\} \to g(x)$.

(3) If the supposed conditions are satisfied, then for each $x \in X$, $\{\delta(x) : \delta \in \mathcal{D}\} \subseteq \Downarrow_d x$ is directed and $\{\delta(x) : \delta \in \mathcal{D}\} \to x$, X is a continuous space, by Theorem 3.7, X is a c-space.

Definition 5.3. A continuous function $\delta : X \to X$ on a directed space X is *finitely seperating* if there exists a finite set F_δ such that for each $x \in X$, there exists $y \in F_\delta$ such that $\delta(x) \leq y \leq x$. A directed space is *finitely seperated* if there is an approximate identity for X consisting of finitely seperating functions. A finitely seperated directed space that is also a c-space will be called an *FS-space*.

Lemma 5.4. *Let X be a directed space, if $\delta \in [X \to X]$ is finitely seperating, then $\delta(x) \ll_d x$ for all $x \in X$. Thus a finitely seperated directed space is an FS-space.*

Proof. Let D be a directed set such that $D \to x$. Since δ is a finitely seperating function, for each $d \in D$ there exists some $y_d \in F_\delta$ such that $\delta(d) \leq y_d \leq d$. But F_δ is finite, denoted by $\{y_1, \cdots, y_n\}$, we may pick finite elements $d_1, \ldots, d_n \in D$ such that $\delta(d_i) \leq y_i \leq d_i, i = 1, \ldots, n$. Since D is directed, we have an upper bound d for $d_1, \ldots, d_n$. We claim that $\delta(x) \leq d$, since $\delta(x) \leq y_i \leq d_i$ for some $i \in \{1, \ldots, n\}$, then $\delta(x) \leq d$.

By (iii) of Lemma 5.2, a finitely seperated directed space is an FS-space. □

Denoting the category of all FS-spaces and continuous functions by **FS**, the following theorem indicates that **FS** is Cartesian closed.

Theorem 5.5.

(1) A finite product of FS-space is again an FS-space.
(2) Let X and Y be FS-space, then $[X \to Y]$ is an FS-space.
(3) The category **FS** *is is a full Cartesian closed subcategory of* **Dtop**.

Proof.

(1) We only need to prove (2). Suppose X and Y are two FS-spaces and $\mathcal{D}, \mathcal{E}$ are approximate identity of X and Y respectively which consist of finitely seperating functions. Then we claim that the directed family $\mathcal{D} \times \mathcal{E}$ is an approximate identity for $X \times Y$ such that $X \times Y$ is an FS-space. Firstly, $\forall (x, y) \in X \times Y,\ \mathcal{D} \times \mathcal{E}(x, y) = \{(\delta(x), \varepsilon(y)) : \delta \in \mathcal{D}, \varepsilon \in \mathcal{E}\} \to (x, y)$, that is, $\mathcal{D} \times \mathcal{E} \to \mathrm{id}_X \times \mathrm{id}_Y$. For finitely seperating property, we only need to take $F_\delta \times F_\varepsilon$ for each $\delta \in \mathcal{D}, \varepsilon \in \mathcal{E}$.

(2) W define a directed family $\mathcal{D} \otimes \mathcal{E}$ on $[X \to Y]$ by $g \mapsto \varepsilon^2 g \delta^2$ for $\varepsilon \in \mathcal{E}$ and $\delta \in \mathcal{D}$. By (i) and (ii) of Lemma 5.2, $\mathcal{D} \otimes \mathcal{E}$ is an approximate identity for $[X \to Y]$. Next, we show that each such function is finitely separating.

Let F_δ and F_ε be the finite sets guaranteed for δ and ε respectively. Define a relation $\sim$ on $[X \to Y]$: $\forall x \in F_\delta, y \in F_\varepsilon$,$f \sim g$ if

$$\varepsilon f(x) \leq y \leq f(x) \iff \varepsilon g(x) \leq y \leq g(x).$$

Since F_δ and F_ε are finite, we conclude that there are only finitely many equivalence classes for $\sim$. Pick one representative from each class, say $\{f_1, \ldots, f_n\}$. We claim that the finite family $\{\varepsilon f_1 \delta, \ldots, \varepsilon f_n \delta\}$ is the one neeed to establish finite separation.

Let $g \in [X \to Y]$. Pick $f_i \sim g$. Given $x \in X$, there exists $m \in F_\delta$ such that $\delta(x) \leq m \leq x$, then $g\delta(x) \leq g(m)$. There exists $n \in F_\varepsilon$ such that $\varepsilon g(m) \leq n \leq g(m)$. Then $\varepsilon f_i(m) \leq n \leq f_i(m)$. $\delta(x) \leq m$ implies $\varepsilon f_i\delta(x) \leq \varepsilon f_i(m)$, $m \leq x$ implies $g(m) \leq g(x)$. Combining these two inequalities, we have

$$\varepsilon f_i\delta(x) \leq \varepsilon f_i(m) \leq n \leq g(m) \leq g(x),$$

that is $\varepsilon f_i\delta \leq g$. A symmetric argument yields that $\varepsilon g\delta \leq f_i$, and hence $\varepsilon^2 g\delta^2 \leq \varepsilon f_i\delta \leq g$.

(3) Immediately from (1) and (2).

FS-domain is an important object in Domain theory. It is obviously that every FS-domain is FS-space, however, the following example tell us that FS-spaces are not exactly FS-domains (for the concept of FS-domain, refer to [7]).

Example 5.6. Let $\mathbb{N}$ denote all natural numbers with usual order endowed with the Alexandroff topology, which can be shown to be an FS-space. For each $n \in \mathbb{N}$, we can define $f_n : \mathbb{N} \to \mathbb{N}$,

$$f_n(x) = \begin{cases} x, & x \leq n; \\ n, & x > n. \end{cases}$$

Since for each $n \in \mathbb{N}$, f_n has finite range $\{1, 2, \ldots, n.\}$, and so it is finitely seperating. Furthermore, $\{f_n\}_{n\in\mathbb{N}} \to id$. According to Definition 5.3, $\mathbb{N}$ is a FS-space but not FS-domain.

Similar to algebraic FS-domain, we consider the algebraic FS-space, which is a direct generalization of BF-domain, and so we omit the detailed proofs.

Proposition 5.7. *For a directed space X, the following properties are equivalent:*

(1) X is an algebraic FS-space;
(2) X is an algebraic space and has an approximate identity consisiting of maps with finite range;
(3) X has an approximate identity consisiting of kernel operators with finite range, a kernel operator δ means idempotent and for each $x \in X, \delta(x) \leq x$.

Definition 5.8. A c-space satisfying any of the equivalent conditions of Proposition 5.7 is called a bifinite space. We denote by **BF** the category of all bifinite spaces and continuous functions between them.

Theorem 5.9. *If X and Y are both bifinite c-spaces, then*

(1) $X \times Y$ is an bifinite c-space;
(2) $[X \to Y]$ is an bifinite c-space.

Corollary 5.10. *The category* **BF** *of bifinite space is a full Cartesian closed subcategory of* **Dtop**.

Remark 5.11. In Domain theory, we have two maximal full Cartesian closed subcategory of $\mathbf{Domain}_\perp$ (domains with least element), namely, L-domain and FS-domain. This leads us to find some maximal full Cartesian closed subcategory of **CS**. In [2], Kou definded a full Cartesian closed category of **CS** by adding each c-space a continuous join operation, denoted by **SCTop**. Here we study the maximality of **FS**. However, there exists many differences. Let **Poset** be the category of all posets and monotone maps, and **Alex** be the category of all Alexandroff spaces and continuous functions, then **Poset** $\cong$ **Alex**. Since **Poset** is a Cartesian slosed category, then **Alex** is a Cartesian closed subcategory of **CS**. It is not difficult to prove that **Alex** is not contained in neither **FS** nor **SCTop**.

The main reason for this phenomenon is that the exponential topology of **Dtop** on some objects may be different from that of **Alex**. For example, the exponential topology on $\mathbb{N}^{\mathbb{N}}$ in **Alex** is the poset $\mathbb{N}^{\mathbb{N}}$ (pointwise order) endowed with the topology such that each $h \in \mathbb{N}^{\mathbb{N}}$, $\uparrow h$ is open, denoted by $\mathcal{A}(\mathbb{N}^{\mathbb{N}})$. The exponential topology in **Dtop** is definded as in Theorem 2.10, $[\mathbb{N} \to \mathbb{N}]$. We claim that $\uparrow id \in \mathcal{A}(\mathbb{N}^{\mathbb{N}})$ is not open in $[\mathbb{N} \to \mathbb{N}]$. Define $f_n : \mathbb{N} \to \mathbb{N}$,

$$f_n(x) = \begin{cases} x, & x \leq n; \\ n, & x > n. \end{cases}$$

It is straightforward to check that $\{f_n\}$ is a directed set and $f_n \to id$.However, there is no $n \in \mathbb{N}$ such that $f_n \in \uparrow id$. Then $\uparrow id$ is not open in $[\mathbb{N} \to \mathbb{N}]$, thus $[\mathbb{N} \to \mathbb{N}] \neq (\mathbb{N}^{\mathbb{N}}, \mathcal{A}(\mathbb{N}^{\mathbb{N}}))$.

Finally, we leave a conjecture: **Alex** is a maximal full Cartesian closed subcategory of **CS**.

6. Conclusions

The category of c-spaces and locally finitary compact spaces are both not Cartesian closed in **Dtop**. FS-spaces and BF-spaces are definded by approximation relation $\ll_d$, and they both form Cartesian closed category of c-spaces. We also geive two conditions that the finitary categorical products and topological products coincide. These works extended the Domain theory and provide strong support for directed space to become an extended mathematical model of Domain theory.

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Conflict of interest

The authors declare that there is no conflict of interest in this paper.

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