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Research article

Estimation of eigenvalues for the α-Laplace operator on pseudo-slant submanifolds of generalized Sasakian space forms

Meraj Ali Khan[1,*], **Ali H. Alkhaldi**[2] **and Mohd. Aquib**[3]

[1] Department of Mathematics, University of Tabuk, Tabuk, Kingdom of Saudi Arabia
[2] Department of Mathematics, College of Science King Khalid University, Saudi Arabia
[3] Department of Mathematics, Sri Venkateshwara College, University of Delhi, New Delhi 110021, India

* **Correspondence:** Email: meraj79@gmail.com; Tel: +966582903874.

Abstract: In this study, we seek to establish new upper bounds for the mean curvature and constant sectional curvature of the first positive eigenvalue of the α-Laplacian operator on Riemannian manifolds. More precisely, various methods are used to determine the first eigenvalue for the α-Laplacian operator on the closed oriented pseudo-slant submanifolds in a generalized Sasakian space form. From our findings for the Laplacian, we extend many Reilly-like inequalities to the α-Laplacian on pseudo slant submanifold in a unit sphere.

Keywords: eigenvalues; Laplacian; pseudo-slant submanifolds; generalized Sasakian space form
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1. Introduction

One of the most important components of Riemannian geometry is determining the bound of the eigenvalue for the Laplacian on a particular manifold. The eigenvalue that occurs as a solutions of the Dirichlet or Neumann boundary value problems for the curvature functions is one of the main goals. Because different boundary conditions exist on a manifold, one can adopt a theoretical perspective to the Dirichlet boundary condition, using the upper bound for the eigenvalue as a technique of analysis for the Laplacians on a given manifold by using appropriate bound. Estimating the eigenvalue for the Laplacian and α-Laplacian operators has been increasingly popular over the years [18, 19, 21, 25–27, 31]. The generalization of the usual Laplacian operator, which is anisotropic mean curvature, was studied in [15]. Let K denotes a complete non compact Riemannian manifold, and B denotes the compact domain within K. Let $\lambda_1(B) > 0$ be a first eigenvalue of the Dirichlet boundary value problem:

$$-\Delta\alpha + \lambda\alpha = 0, \text{ in } B \text{ and } \alpha \text{ on } \partial B,$$

where Δ represents the Laplacian operator on the Riemannian manifold K^m. The Reilly formula is dedicated entirely with the fundamental geometry of a given manifold. This can be generally acknowledged with the following phrase.

Let (K^m, g) be a compact m-dimensional Riemannian manifold, and λ_1 denotes the first nonzero eigenvalue of the Neumann problem:

$$-\Delta\alpha + \lambda\alpha = 0, \text{ on } K^m \text{ and } \frac{\partial\alpha}{\partial\nu} = 0 \text{ on } \partial K^m,$$

where ν is the outward normal on ∂K^m.

Reilly [25] established the following inequality for a manifold K^m isometrically immersed in the Euclidean space R^k with $\partial K^m = 0$:

$$\lambda_1^{\nabla} \leq \frac{1}{Vol(K^m)} \int_{K^m} ||H||^2 dV, \tag{1.1}$$

where H is the mean curvature vector of immersion K^m into R^n, λ_1^{∇} denotes the first non-zero eigenvalue of the Laplacian on K^m and dV denotes the volume element of K^m.

The upper bounds for α-Laplace operator in the sense of first eigenvalue for Finsler submanifold in the setting of Minkowski space was computed by Zeng and He [32]. Seto and Wei [28] presented the first eigenvalue of the Laplace operator for a closed manifold. However, F. Du et al. [13] derived the generalized Reilly inequality (1.3) and the first nonzero eigenvalue of the α-Laplace operator. Having followed the very similar approach, Blacker and Seto [4] demonstrated a Lichnerowicz type lower limit for the first nonzero eigenvalue of the α-Laplacian for Neumann and Dirichlet boundary conditions. Further, Papageorgiou et al. [24] studied p-Laplacian for concave-convex problems. Recently, $p(x)$-Laplacian are studied in the papers [14, 17].

The first non-null eigenvalue of the Laplacian is demonstrated in [10, 12], which is deemed a generalization of work of Reilly [29]. The results of the distinct classes of Riemannian submanifolds for diverse ambient spaces show that the results of both first nonzero eigenvalues portray similar inequality and have same upper bounds [9, 10]. In the case of the ambient manifold, it is clear from the previous studies that Laplace operators on Riemannian manifolds played a significant role in achieving various advances in Riemannian geometry (see [3, 6, 8, 11, 15, 22, 23, 29, 32]).

The α-Laplacian on a m-dimensional Riemannian manifold K^m is defined as

$$\Delta_\alpha = div(|\nabla h|^{\alpha-2}\nabla h), \tag{1.2}$$

where $\alpha > 1$, if $\alpha = 2$, then the above formula becomes usual Laplacian operator.

The eigenvalue of Δh, from the other hand is Laplacian like. If a function $h \neq 0$ meets the following equation with dirichlet boundary condition or Neumann boundary condition as discussed earlier:

$$\Delta_\alpha h = -\lambda|h|^{\alpha-2}h,$$

where λ is a real number called Dirichlet eigenvalue. In the same way, the previous requirements apply to the Neumann boundary condition.

If we look at Riemannian manifold without boundary, the Reilly type inequality for first nonzero eigenvalue $\lambda_{1,\alpha}$ for α-Laplacian was computed in [30]:

$$\lambda_{1,\alpha} = inf\{\frac{\int_K |\nabla h|^q}{\int_K |h|^q} : h \in W^{1,\alpha}(K^1)\{0\}, \int_K |h|^{\alpha-2}h = 0\}. \tag{1.3}$$

However, Chen [7] pioneered the geometry of slant immersions as a natural extension of both holomorphic and totally real immersions. Further, Lotta [20] introduced the notion of slant submanifolds in the frame of almost contact metric manifolds, these submanifolds further explored by Cabrerizo et al. [5]. More precisely, Cabririzo et al. explored slant submanifolds in the setting of Sasakian manifolds. Another generalization of slant and contact CR-submanifolds was given by V. A. Khan and M. A. Khan [16], basically they proposed the notion of pseudo-slant submanifolds in the almost contact metric manifolds and provide an example of these submanifolds.

After reviewing the literature, a natural question emerges: Is it possible to obtain the Reilly type inequalities for submanifolds of spheres via almost contact metric manifolds, which were studied in [2, 10, 12]? To answer this question, we explore the Reilly type inequalities for pseudo-slant submanifolds isometrically immersed in a generalized Sasakian space form. To this end our aim is to compute the bound for first non zero eigenvalues via α-Laplacian. The present study is leaded by the application of Gauss equation and studies done in [9, 10, 13].

2. Preliminaries

A $(2n + 1)$-dimensional C^∞-manifold $\bar{K}$ is said to have an almost contact structure, if on $\bar{K}$ there exist a tensor field ϕ of type $(1, 1)$, a vector field ξ and a 1-form η satisfying the following properties:

$$\phi^2 = -I + \eta \otimes \xi, \ \ \phi\xi = 0, \ \ \eta \circ \phi = 0, \ \ \eta(\xi) = 1. \tag{2.1}$$

The manifold $\bar{K}$ with the structure (ϕ, ξ, η) is called almost contact metric manifold. There exists a Riemannian metric g on an almost contact metric manifold $\bar{K}$, satisfying the following:

$$\eta(e_1) = g(e_1, \xi), \ \ g(\phi e_1, \phi e_2) = g(e_1, e_2) - \eta(e_1)\eta(e_2), \tag{2.2}$$

for all $e_1, e_2 \in T\bar{K}$, where $T\bar{K}$ is the tangent bundle of $\bar{K}$.

In [1], Alegre et al. introduced the notion of generalized Sasakian space form as that an almost contact metric manifold $(\bar{K}, \phi, \xi, \eta, g)$ whose curvature tensor $\bar{R}$ satisfies

$$\begin{aligned}\bar{R}(e_1, e_2)e_3 =& f_1\{g(e_2, e_3)e_1 - g(e_1, e_3)e_2\} + f_2\{g(e_1, \phi e_3)\phi e_2 \\ & - g(e_2, \phi e_3)\phi e_1 + 2g(e_1, \phi e_2)\phi e_3\} + f_3\{\eta(e_1)\eta(e_3)e_2 \\ & - \eta(e_2)\eta(e_3)e_1 + g(e_1, e_3)\eta(e_2)\xi - g(e_2, e_3)\eta(e_1)\xi\},\end{aligned} \tag{2.3}$$

for all vector fields e_1–e_3 and certain differentiable functions f_1–f_3 on $\bar{K}$.

A generalized Sasakian space form with functions f_1–f_3 is denoted by $\bar{K}(f_1, f_2, f_3)$. If $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$, then $\bar{M}(f_1, f_2, f_3)$ becomes a Sasakian space form $\bar{M}(c)$ [1]. If $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$, then $\bar{M}(f_1, f_2, f_3)$ becomes a Kenmotsu space form $\bar{M}(c)$ [1], and if $f_1 = f_2 = f_3 = \frac{c}{4}$, then $\bar{K}(f_1, f_2, f_3)$ becomes a cosymplectic space form $\bar{K}(c)$ [1].

Let K be a submanifold of an almost contact metric manifold $\bar{K}$ with induced metric g. The Riemannian connection $\bar{\nabla}$ of $\bar{K}$ induces canonically the connections ∇ and $\nabla^{\perp}$ on the tangent bundle TK and the normal bundle $T^{\perp}K$ of K respectively, then the Gauss and Weingarten formulae are governed by

$$\bar{\nabla}_{e_1} e_2 = \nabla_{e_1} e_2 + \sigma(e_1, e_2), \tag{2.4}$$

$$\bar{\nabla}_{e_1} v = -A_v e_1 + \nabla^{\perp}_{e_1} v, \tag{2.5}$$

for each $e_1, e_2 \in TK$ and $v \in T^{\perp}K$, where σ and A_v are the second fundamental form and the shape operator respectively for the immersion of K into $\bar{K}$, they are related as

$$g(\sigma(e_1, e_2), v) = g(A_v e_1, e_2), \tag{2.6}$$

where g is the Riemannian metric on $\bar{K}$ as well as the induced metric on K.

If Te_1 and Ne_1 represent the tangential and normal part of ϕe_1 respectively, for any $e_1 \in TK$, one can write

$$\phi e_1 = Te_1 + Ne_1. \tag{2.7}$$

Similarly, for any $v \in T^{\perp}K$, we write

$$\phi v = tv + nv, \tag{2.8}$$

where tv and nv are the tangential and normal parts of ϕv, respectively. Thus, T (resp. n) is 1-1 tensor field on TK (resp. $T^{\perp}K$) and t (resp. n) is a tangential (resp. normal) valued 1-form on $T^{\perp}K$ (resp. TK).

The notion of slant submanifolds in contact geometry was first defined by Lotta [20]. Later, these submanifolds were studied by Cabrerizo et al. [5]. Now, we have following definition of slant submanifolds.

Definition 2.1. *A submanifold K of an almost contact metric manifold $\bar{K}$ is said to be slant submanifold if for any $x \in K$ and $X \in T_xK - \langle \xi \rangle$, the angle between X and ϕX is constant. The constant angle $\theta \in [0, \pi/2]$ is then called slant angle of K in $\bar{K}$. If $\theta = 0$, the submanifold is invariant submanifold, and if $\theta = \pi/2$, then it is anti-invariant submanifold. If $\theta \neq 0, \pi/2$, it is proper slant submanifold.*

Moreover, Cabrerizo et al. [5] proved the characterizing equation for slant submanifold. More precisely, they proved that a submanifold N^m is said to be a slant submanifold if and only if $\exists$ a constant $\tau \in [0, \pi/2]$ and a $(1, 1)$ tensor field T which satisfies the following relation:

$$T^2 = \tau(I - \eta \otimes \xi), \tag{2.9}$$

where $\tau = -\cos^2\theta$.

From (2.9), it is easy to conclude the following:

$$g(Te_1, Te_2) = \cos^2\theta\{g(e_1, e_2) - \eta(e_1)\eta(e_2)\}, \tag{2.10}$$

$\forall e_1, e_2 \in K$.

Now, we define the pseudo-slant submanifold, which was introduced by V. A. Khan and M. A. Khan [16].

A submanifold K of an almost contact metric manifold $\bar{K}$ is said to be pseudo-slant submanifold if there exist two orthogonal complementary distributions S_θ and $S_\perp$ such that

(1) $TK = S_\perp \oplus S_\theta \oplus \langle\xi\rangle$.
(2) The distribution $S_\perp$ is anti-invariant, i.e., $\phi S^\perp \subseteq T^\perp K$.
(3) The distribution S_θ is slant with slant angle $\theta \neq \pi/2$.

If $\theta = 0$, then the pseudo-slant submanifold is a semi-invariant submanifold. Now, we have the following example of pseudo-slant submanifold.

Example 2.1. *[16] Consider the 5-dimensional submanifold R^9 with usual Sasakian structure, such that*

$$x(u, v, w, s, t) = 2(u, 0, w, 0, 0, v, s\cos\theta, s\sin\theta, t),$$

for any $\theta \in (0, \pi/2)$. Then it is easy to see that this is an example of pseudo-slant submanifold. Moreover, it can be observed

$$e_1 = 2(\frac{\partial}{\partial x^1} + y^1\frac{\partial}{\partial z}),\ \ e_2 = 2\frac{\partial}{\partial y^2}\ \ e_3 = 2(\frac{\partial}{\partial x^3} + y^3\frac{\partial}{\partial z}),$$

$$e_4 = 2\cos\theta\frac{\partial}{\partial y^3} + 2\sin\theta\frac{\partial}{\partial y^4},\ \ e_5 = 2\frac{\partial}{\partial z} = \xi,$$

form a local orthonormal frame of TM. In which $S_\perp = \langle e_1, e_2\rangle$ and $S_\theta = \langle e_3, e_4\rangle$, where $D_\perp$ is anti-invariant and S_θ is slant distribution with slant angle θ.

Suppose $K^{m=p+2q+1}$ be a pseudo-slant submanifold of dimension m, in which p and $2q$ are the dimensions of the anti-invariant and slant distributions respectively. Moreover, let $\{u_1, u_2, \ldots, u_p, u_{p+1} = v_1, u_{p+2} = v_2, \ldots, u_{m-1} = v_{2q}, u_m = v_{2q+1} = \xi\}$ is an orthonormal frame of vectors which form a basis for the submanifold K^{p+2q+1}, such that $\{u_1, \ldots, u_p\}$ is tangential to the distribution $D_\perp$ and the set $\{v_1, v_2 = \sec\theta T v_1, v_3, v_4 = \sec\theta T v_3, \ldots v_{2q} = \sec\theta T v_{2q-1}\}$ is tangential to D_θ. By the Eq (2.3), the curvature tensor $\bar{R}$ for pseudo-slant submanifold N^{p+2q+1} is given by

$$\bar{R}(u_i, u_j, u_i, u_j) = f_1(m^2 - m) + f_2(3\sum_{i,j=1}^{m} g^2(\phi u_i, u_j) - 2(m-1)). \tag{2.11}$$

The dimension of the pseudo-slant submanifold K^m can be decomposed as $m = p + 2q + 1$, then using the formula (2.9) for slant and anti-invariant distributions, we have

$$g^2(\phi u_i, u_{i+1}) = 0,\ \ for\ \ i \in \{1, \ldots, p-1\},$$

and

$$g^2(\phi u_i, u_{i+1}) = \cos^2\theta,\ \ \text{for}\ \ i \in \{p+1, \ldots, 2q-1\}.$$

Then

$$\sum_{i,j=1}^{m} g^2(\phi u_i, u_j) = 2q\cos^2\theta.$$

The relation (2.11) implies that

$$\bar{R}(u_i, u_j, u_i, u_j) = f_1(m^2 - m) + f_2(6q\cos^2\theta - 2(m-1)). \tag{2.12}$$

From the relation (2.12) and Gauss equation, one has

$$f_1 m(m-1) + f_2(6q\cos^2\theta - 2(m-1)) = 2\tau - n^2\|H\|^2 + \|\sigma\|^2$$

or

$$2\tau = n^2\|H\|^2 - \|\sigma\|^2 + f_1 m(m-1) + f_2(6q\cos^2\theta - 2(m-1)). \tag{2.13}$$

In the paper [2], one of the present author Ali H. Alkhaldi with others studied the effect of the conformal transformation on the curvature and second fundamental form. More precisely, assume that $\bar{K}^{2n+1}$ consists a conformal metric $g = e^{2\rho}\bar{g}$, where $\rho \in C^\infty(\bar{K})$. Then $\bar{\Gamma}_a = e^\rho \Gamma_a$ stands for the dual coframe of $(\bar{K}, \bar{g})$, $\bar{e}_a == e^\rho e_a$ represents the orthogonal frame of $(\bar{K}, \bar{g})$. Moreover, we have

$$\bar{\Gamma}_{ab} = \Gamma_{ab} + \rho_a\Gamma_b - \rho_b\Gamma_a, \tag{2.14}$$

where ρ_a is the covariant derivative of ρ along the vector e_a, i.e., $d\rho = \sum_a \rho_a e_a$.

$$\begin{aligned} e^{2\rho}\bar{R}_{pqrs} =& R_{pqrs} - (\rho_{pr}\delta_{qs} + \rho_{qs}\delta_{pr} - \rho_{ps}\delta_{qr} - \rho_{qr}\delta_{ps}) \\ &+ (\rho_p\rho_r\delta_{qs} + \rho_q\rho_s\delta_{pr} - \rho_q\rho_t\delta_{ps} - \rho_p\rho_s\delta_{qr}) - |\nabla_\alpha|^2(\delta_{pr}\delta_{qs} - \delta_{il}\delta_{qr}). \end{aligned} \tag{2.15}$$

Applying pullback property in (2.14) to K^m via point x, we get

$$\bar{\sigma}^\alpha_{pq} = e^{-\rho}(\sigma^\alpha_{pq} - \rho_\alpha\delta_{qp}), \tag{2.16}$$

$$\bar{H}^\alpha = e^\alpha(H^\alpha - \rho_\alpha). \tag{2.17}$$

The following significant relation was proved in [1]:

$$e^{2\rho}(\|\bar{\sigma}\|^2 - m\|\bar{H}\|^2) + m\|H\|^2 = \|\sigma\|^2. \tag{2.18}$$

3. Main Theorem

Initially, some basic results and formulas will be discussed which are compatible with the papers [2, 22]. Now, we have the following result.

Lemma 3.1. *[2] Let K^m be a slant submanifold of a Sasakian space form $\bar{K}^{2t+1}(c)$ which is closed and oriented with dimension ≥ 2. If $f : K^m \to \bar{K}^{2t+1}(c)$ is embedding from K^m to $\bar{K}^{2t+1}(c)$. Then there is a standard conformal map $x : \bar{K}^{2t+1}(c) \to S^{2t+1}(1) \subset R^{2t+2}$ such that the embedding $\Gamma = x \circ f = (\Gamma^1, \ldots, \Gamma^{2t+2})$ satisfies that*

$$\int_{K^m} |\Gamma^a|^{\alpha-2}\Gamma^a dV_K = 0, \quad a = 1, \ldots, 2(t+1),$$

for $\alpha > 1$.

In the next result, we obtain a result which is analogous to Lemma 2.7 of [22]. Indeed, in Lemma 3.1, by the application of test function, we obtain the higher bound for $\lambda_{1,\alpha}$ in terms of conformal function.

Proposition 3.1. *Let K^m be a m-dimensional pseudo slant submanifold, which is closed orientable isometrically immersed in a generalized Sasakian space form $\bar{K}^{2t+1}(f_1, f_2, f_3)$, then we have*

$$\lambda_{1,\alpha} Vol(K^m) \leq 2^{|1-\frac{\alpha}{2}|}(t+1)^{|1-\frac{\alpha}{2}|} m^{\frac{\alpha}{2}} \int_{K^m} (e^{2\rho})^{\frac{\alpha}{2}} dV, \tag{3.1}$$

where x is the conformal map used in Lemma 3.1, and $\alpha > 1$. The standard metric is identified by L_c and consider $x^* L_1 = e^{2p} L_c$.

Proof. Consider Γ^a as a test function along with Lemma 3.1, we have

$$\lambda_{1,\alpha} \int_{K^m} |\Gamma^a|^\alpha \leq |\nabla\Gamma^a|^\alpha dV, \quad 1 \leq a \leq 2(t+1), \tag{3.2}$$

observing that $\sum_{a=1}^{2t+2} |\Gamma^a|^2 = 1$, then $|\Gamma^a| \leq 1$, we get

$$\sum_{a=1}^{2t+2} |\nabla\Gamma^a|^2 = \sum_{i=1}^{m} |\nabla_{e_i}\Gamma|^2 = me^{2\rho}. \tag{3.3}$$

On using $1 < \alpha \leq 2$, we conclude

$$|\Gamma^a|^2 \leq |\Gamma^a|^\alpha. \tag{3.4}$$

By the application of Hölder's inequality, together with (3.2)–(3.4), we get

$$\begin{aligned} \lambda_{1,\alpha} Vol(K^m) &= \lambda_{1,\alpha} \sum_{a=1}^{2t+2} \int_{K^m} |\Gamma^a|^2 dV \leq \lambda_{1,\alpha} \sum_{a=1}^{2t+2} \int_{K^m} |\Gamma^a|^\alpha dV \\ &\leq \lambda_{1,\alpha} \int_{K^m} \sum_{a=1}^{2t+2} |\nabla\Gamma^a|^\alpha dV \leq (2t+2)^{1-\alpha/2} \int_{K^m} (\sum_{a=1}^{2t+1} |\nabla\Gamma^a|^2)^{\alpha/2} dV \\ &= 2^{1-\frac{\alpha}{2}}(t+1)^{1-\frac{\alpha}{2}} \int_{K^m} (me^{2\rho})^{\frac{\alpha}{2}} dV, \end{aligned} \tag{3.5}$$

which is (3.1). On the other hand, if we assume $\alpha \geq 2$, then, by Hölder inequality,

$$I = \sum_{a=1}^{2t+2} |\Gamma^a|^2 \leq (2t+2)^{1-\frac{2}{\alpha}} (\sum_{a=1}^{2t+2} |\Gamma^a|^\alpha)^{\frac{2}{\alpha}}. \tag{3.6}$$

As a result, we get

$$\lambda_{1,\alpha} Vol(N^m) \leq (2t+2)^{\frac{\alpha}{2}-1} (\sum_{a=1}^{2t+2} \lambda_{1,\alpha} \int_{N^m} |\Gamma^a|^\alpha dV). \tag{3.7}$$

The Minkowski inequality provides

$$\sum_{a=1}^{2t+2} |\nabla\Gamma^a|^\alpha \leq (\sum_{a=1}^{2t+2} |\nabla\Gamma^a|^2)^{\frac{\alpha}{2}} = (me^{2\rho})^{\frac{\alpha}{2}}. \tag{3.8}$$

By the application of (3.2), (3.7) and (3.8), it is easy to get (3.1).

□

In the next theorem, we are going to provide a sharp estimate for the first eigenvalue of the α-Laplace operator on the pseudo-slant submanifold of a generalized Sasakian space form $\bar{K}^{2t+1}(f_1, f_2, f_3)$.

Theorem 3.1. *Let K^m be a m-dimensional pseudo-slant submanifold of a generalized Sasakian space form $\bar{K}^{2t+1}(f_1, f_2, f_3)$, then*

(1) The first non-null eigenvalue $\lambda_{1,\alpha}$ of the α-Laplacian satisfies

$$\lambda_{1,\alpha} \leq \frac{2^{(1-\frac{\alpha}{2})}(t+1)^{(1-\frac{\alpha}{2})}m^{\frac{\alpha}{2}}}{(Vol(K))^{\alpha/2}} \times \{ \int_{K^m} (f_1 + f_2(\frac{6q\cos^2\theta}{m(m-1)} - \frac{2}{m}) + ||H||^2)dV\}^{\alpha/2} \tag{3.9}$$

for $1 < \alpha \leq 2$, and

$$\lambda_{1,\alpha} \leq \frac{2^{(1-\frac{\alpha}{2})}(t+1)^{(1-\frac{\alpha}{2})}m^{\frac{\alpha}{2}}}{(Vol(K))^{\alpha/2}} \times \{ \int_{K^m} (f_1 + f_2(\frac{6q\cos^2\theta}{m(m-1)} - \frac{2}{m}) + ||H||^2)dV\}^{\alpha/2} \tag{3.10}$$

for $2 < \alpha \leq \frac{m}{2}+1$, where p and $2q$ are the dimensions of the anti-invariant and slant distributions.

(2) The equality satisfies in (3.9) and (3.10) if and only if $\alpha = 2$ and K^m is minimally immersed in a geodesic sphere of radius r_c of $\bar{K}^{2t+1}(f_1, f_2, f_3)$ with the following relations:

$$r_0 = (\frac{m}{\lambda_1^{\Delta}})^{1/2}, \quad r_1 = \sin^{-1} r_0, \quad r_{-1} = \sinh^{-1} r_0.$$

Proof. $1 < \alpha \leq 2 \implies \frac{\alpha}{2} \leq 1$. Proposition 3.1 together with Hölder inequality provides

$$\begin{aligned}\lambda_{1,\alpha} Vol(K^m) &\leq 2^{1-\frac{\alpha}{2}}(t+1)^{1-\frac{\alpha}{2}}m^{\frac{\alpha}{2}} \int_{K^m} (e^{2\rho})^{\frac{\alpha}{2}} dV \\ &\leq 2^{1-\frac{\alpha}{2}}(t+1)^{|1-\frac{\alpha}{2}|}\, m^{\frac{\alpha}{2}}(Vol(K^m))^{1-\frac{\alpha}{2}}(\int_{K^m} e^{2\rho}dV)^{\frac{\alpha}{2}}.\end{aligned} \tag{3.11}$$

We can calculate $e^{2\rho}$ with the help of conformal relations and Gauss equation. Let $\bar{K}^{2k+1} = \bar{K}^{2k+1}(f_1, f_2, f_3)$, and $\bar{g} = e^{-2\rho}L_c$, $\bar{g} = c^*L_1$. From (2.13), the Gauss equation for the embedding f and the pseudo slant embedding $\Gamma = x \circ f$, we have

$$R = (f_1)m(m-1) + (f_2)(m-1)\{6q\cos^2\theta - 2(m-1)\} + m(m-1)||H||^2 + m||H||^2 - S\,||\sigma|^2, \tag{3.12}$$

$$\bar{R} - m(m-1) = m(m-1)||\bar{H}||^2 + (m||\bar{H}||^2 - ||\bar{\sigma}|^2). \tag{3.13}$$

On tracing (2.15), we have

$$e^{2\rho}\bar{R} = R - (m-2)(m-1)|\nabla_{\rho}|^2 - 2(m-1)\Delta_{\rho}. \tag{3.14}$$

Using (3.12) and (3.13) in (3.14), we get

$$\begin{aligned}&e^{2\rho}(m(m-1) + m(m-1)||\bar{H}||^2 + (m||\bar{H}||^2 - ||\bar{\sigma}|^2)) \\ =&(f_1)m(m-1) + (f_2)\{6q\cos^2\theta - 2(m-1)\} + m(m-1)||H||^2 \\ &+ (m||H||^2 - ||\sigma|^2) - (m-2)(m-1)||\nabla\rho||^2 - 2(m-1)\Delta_{\rho}.\end{aligned} \tag{3.15}$$

□

The above relation implies

$$\begin{aligned}&e^{2\rho}\|\bar{\sigma}|^2-(m-2)(m-1)|\nabla_\rho|^2-2(m-1)\Delta_\rho\\=&m(m-1)[\{e^{2\rho}-f_1-(f_2)(\frac{6q\cos^2\theta}{m(m-1)}-\frac{2}{m})\}(e^{2\rho}\|\bar{H}\|^2-\|H\|^2)]+m(e^{2\rho}\|\bar{H}\|^2-\|H\|^2).\end{aligned}\tag{3.16}$$

From (2.17) and (2.18), we derive

$$\begin{aligned}&m(m-1)\{e^{2\rho}-(f_1)-(f_2)(\frac{6q\cos^2\theta}{m(m-1)}-\frac{2}{m})\}+m(m-1)\sum_\alpha(H^\alpha-\rho\alpha)^2\\=&m(m-1)\|H\|^2-(m-2)(m-1)|\nabla_\rho|^2-2(m-1)\Delta_\rho.\end{aligned}\tag{3.17}$$

Further, on simplification we get

$$e^{2\rho}=\{(f_1)+(f_2)(\frac{6q\cos^2\theta}{m(m-1})-\frac{2}{m})+\|H\|^2\}-\frac{2}{m}\Delta_\rho-\frac{m-2}{m}|\Delta_\rho|^2-\|(\nabla_\rho)^\perp-H\|^2.\tag{3.18}$$

On integrating along dV, it is easy to see that

$$\begin{aligned}\lambda_{1,\alpha}Vol(K^m)&\leq 2^{|1-\frac{\alpha}{2}|}(t+1)^{|1-\frac{\alpha}{2}|}m^{\frac{\alpha}{2}}(Vol(K^m))^{1-\frac{\alpha}{2}}(\int_{K^m}e^{2\rho}dV)^{\frac{\alpha}{2}}.\\&\leq\frac{2^{|1-\frac{\alpha}{2}|}(t+1)^{|1-\frac{\alpha}{2}|}m^{\frac{\alpha}{2}}}{(Vol(K^m))^{\frac{\alpha}{2}-1}}\{\int_{K^m}\{f_1+f_2(\frac{6q\cos^2\theta}{m(m-1)}-\frac{2}{m})+\|H\|^2\}dV\}^{\alpha/2},\end{aligned}\tag{3.19}$$

which is equivalent to (3.9). If $\alpha>2$, then it is not possible to apply Hölder inequality to govern $\int_{K^m}(e^{2\rho}dV)^{\frac{\alpha}{2}}$ by using $\int_{K^m}(e^{2\rho})$. Now, multiply both sides of (3.18) by $e^{(\alpha-2)\rho}$ and integrating on K^m,

$$\begin{aligned}\int_{K^m}e^{\alpha\rho}dV&\leq\int_{K^m}\{f_1+f_2(\frac{6q\cos^2\theta}{m(m-1)}-\frac{2}{m})+\|H\|^2\}e^{(\alpha-2)\rho}dV-(\frac{m-2-2\alpha+4}{m})\int_{K^m}e^{(\alpha-2)}|\Delta\rho|^2dV\\&\leq\int_{K^m}\{f_1+f_2(\frac{6q\cos^2\theta}{m(m-1)}-\frac{2}{m})+\|H\|^2\}e^{(\alpha-2)\rho}dV.\end{aligned}\tag{3.20}$$

From the assumption, it is evident that $m\geq 2\alpha-2$. On applying Young's inequality, we arrive

$$\begin{aligned}&\int_{K^m}\{f_1+f_2(\frac{6q\cos^2\theta}{m(m-1)}-\frac{2}{m})+\|H\|^2\}e^{(\alpha-2)\rho}dV\\\leq&\frac{2}{\alpha}\int_{K^m}\{|f_1+f_2(\frac{6q\cos^2\theta}{m(m-1)}-\frac{2}{m})+\|H\|^2|\}^{\alpha/2}dV+\frac{\alpha-2}{\alpha}\int_{K^m}e^{\frac{\alpha}{\rho}}dV.\end{aligned}\tag{3.21}$$

From (3.20) and (3.21), we conclude the following:

$$\int_{K^m}e^{\alpha\rho}dV\leq\int_{K^m}\{|f_1+f_2(\frac{6q\cos^2\theta}{m(m-1)}-\frac{2}{m})+\|H\|^2|\}^{\alpha/2}dV.\tag{3.22}$$

Substituting (3.22) in (3.1), we obtain (3.10). For the pseudo slant submanifolds, the equality case holds in (3.9), the equality cases of (3.2) and (3.4) imply that

$$|\Gamma^a|^2=|\Gamma^a|^\alpha,\ \ \Delta_\alpha\Gamma^a=\lambda_{1,\alpha}|\Gamma^a|^{\alpha-2}\Gamma^a,$$

for $a = 1, \ldots, 2t+2$. For $1 < \alpha < 2$, we have $|\Gamma^a| = 0$ or 1. Therefore, there exists only one a for which $|\Gamma^a| = 1$ and $\lambda_{i,\alpha} = 0$, and it can not be possible since eigenvalue $\lambda_{i,\alpha} \neq 0$. This leads to use the value of α equal to 2, therefore, we can apply Theorem 1.5 of [15].

For $\alpha > 2$, the equality in (3.10) still holds, this indicates that equalities in (3.7) and (3.8) satisfy, and this leads to

$$|\Gamma^1|^\alpha = \cdots = |\Gamma^{2t+2}|^\alpha,$$

and there exists a such that $|\nabla\Gamma^a| = 0$. It shows that Γ^a is a constant and $\lambda_{1,\alpha} = 0$, this again contradicts with the fact that $\lambda_{1,\alpha} \neq 0$, this completes the proof.

Note 3.1. If $\alpha = 2$, then the α-Laplacian operator becomes the Laplacian operator. Therefore, we have the following corollary.

Corollary 3.1. *Let K^m be a m-dimensional pseudo-slant submanifold of a generalized Sasakian space form $\bar{K}^{2t+1}(f_1, f_2, f_3)$, then the first non-null eigenvalue λ_1^Δ of the Laplacian satisfies*

$$\lambda_1^\Delta \leq \frac{m}{(Vol(K))} \int_{K^m} \{f_1 + f_2(\frac{6q\cos^2\theta - 2}{m}) + ||H||^2)\} dV. \tag{3.23}$$

By the application of Theorem 3.1 for $1 < \alpha \leq 2$, we have the following result.

Theorem 3.2. *Let K^m be a m-dimensional pseudo-slant submanifold of a generalized Sasakian space form $\bar{K}^{2t+1}(f_1, f_2, f_3)$, then the first non-null eigenvalue $\lambda_{1,\alpha}$ of the α-Laplacian satisfies*

$$\lambda_{1,\alpha} \leq \frac{2^{(1-\frac{\alpha}{2})}(t+1)^{(1-\frac{\alpha}{2})} m^{\frac{\alpha}{2}}}{(Vol(K))^{\alpha/2}} \times [\int_{K^m} (f_1 + f_2(\frac{6q\cos^2\theta - 2}{m}) + ||H||^2)^{\frac{\alpha}{2(\alpha-1)}}]^{\alpha-1} dV \tag{3.24}$$

for $1 < \alpha \leq 2$, and

$$\lambda_{1,\alpha} \leq \frac{2^{(1-\frac{\alpha}{2})}(t+1)^{(1-\frac{\alpha}{2})} m^{\frac{\alpha}{2}}}{(Vol(K))^{\alpha/2}} \times [\int_{K^m} (f_1 + f_2(\frac{6q\cos^2\theta - 2}{m}) + ||H||^2)^{\frac{\alpha}{2(\alpha-1)}}]^{\alpha-1} dV \tag{3.25}$$

for $2 < \alpha \leq \frac{m}{2} + 1$.

Proof. Suppose $1 < \alpha \leq 2$, we have $\frac{\alpha}{2(\alpha-1)} \geq 1$, then the Hölder inequality provides

$$\begin{aligned} &\int_{K^m} \{(f_1) + (f_2)(\frac{3\cos^2\theta - 2}{m}) + ||H||^2\} dV \\ \leq &((Vol(K^m))^{1-\frac{2(\alpha-1)}{\alpha}}) \times [\int_{K^m} (f_1 + f_2(\frac{6q\cos^2\theta}{m(m-1)} - \frac{2}{m}) + ||H||^2)^{\frac{\alpha}{2(\alpha-1)}}]^{\frac{2(\alpha-1)}{\alpha}}. \end{aligned} \tag{3.26}$$

On combining (3.9) and (3.26), we get the required inequality, this completes the proof. □

Note 3.2. If $\theta = 0$, then the pseudo-slant submanifolds become the semi-invariant submanifolds.

By the application of above findings, we can deduce the following results for semi-invariant submanifolds in the setting of Sasakian manifolds.

Corollary 3.2. *Let K^m be a m-dimensional semi-invariant submanifold of a generalized Sasakian space form $\bar{K}^{2t+1}(f_1, f_2, f_3)$, then*

(1) *The first non-null eigenvalue* $\lambda_{1,\alpha}$ *of the* α*-Laplacian satisfies*

$$\lambda_{1,\alpha} \leq \frac{2^{(1-\frac{\alpha}{2})}(t+1)^{(1-\frac{\alpha}{2})}m^{\frac{\alpha}{2}}}{(Vol(K))^{\alpha/2}} \times \{ \int_{K^m} (f_1 + f_2\frac{(6q-2)}{m} + \|H\|^2)\}^{\alpha/2} dV \tag{3.27}$$

for $1 < \alpha \leq 2$, *and*

$$\lambda_{1,\alpha} \leq \frac{2^{(1-\frac{\alpha}{2})}(t+1)^{(1-\frac{\alpha}{2})}m^{\frac{\alpha}{2}}}{(Vol(K))^{\alpha/2}} \times \{ \int_{K^m} (f_1 + f_2\frac{(6q-2)}{m} + \|H\|^2)\}^{\alpha/2} dV \tag{3.28}$$

for $2 < \alpha \leq \frac{m}{2}+1$, *where* p *and* $2q$ *are the dimensions of the anti-invariant and slant distributions.*

(2) *The equality satisfies in (3.27) and (3.28) if and only if* $\alpha = 2$, *and* K^m *is minimally immersed in a geodesic sphere of radius* r_c *of* $\bar{K}^{2t+1}(f_1, f_2, f_3)$ *with the following relations:*

$$r_0 = (\frac{m}{\lambda_1^{\Delta}})^{1/2}, \quad r_1 = \sin^{-1} r_0, \quad r_{-1} = \sinh^{-1} r_0.$$

Further, by Corollary 3.4 and Note 3.1, we deduce the following.

Corollary 3.3. *Let* K^m *be a m-dimensional semi-invariant submanifold of a generalized Sasakian space form* $\bar{K}^{2t+1}(f_1, f_2, f_3)$, *then the first non-null eigenvalue* λ_1^{Δ} *of the Laplacian satisfies*

$$\lambda_1^{\Delta} \leq \frac{m}{(Vol(K))} \int_{K^m} \{f_1 + \frac{f_2(6q-2)}{m} + \|H\|^2)\} dV. \tag{3.29}$$

In addition, we also have the following corollary, which can be derived by Theorem 3.2.

Corollary 3.4. *Let* K^m *be a m-dimensional semi-invariant submanifold of a generalized Sasakian space form* $\bar{K}^{2t+1}(f_1, f_2, f_3)$, *then the first non-null eigenvalue* $\lambda_{1,\alpha}$ *of the* α*-Laplacian satisfies*

$$\lambda_{1,\alpha} \leq \frac{2^{(1-\frac{\alpha}{2})}(t+1)^{(1-\frac{\alpha}{2})}m^{\frac{\alpha}{2}}}{(Vol(K))^{\alpha/2}} \times [\int_{K^m} (f_1 + \frac{f_2(6q-2)}{4} + \|H\|^2)^{\frac{\alpha}{2(\alpha-1)}}]^{\alpha-1} dV \tag{3.30}$$

for $1 < \alpha \leq 2$.

4. Conclusions

In this paper, we established the upper bounds for the first eigenvalues of the α-Laplacian operator for the pseudo-slant submanifolds in the setting of generalized Sasakian space forms. The class of pseudo-slant submanifold includes the class of semi-invariant, invariant, anti-invariant, and slant submanifolds. Therefore, the results obtained in this paper generalize the results for the first eigenvalues for these particular submanifolds.

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Conflict of interest

The authors state that there is no conflict of interest.

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