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Research article

Magnetohydrodynamics approximation of the compressible full magneto-micropolar system

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Abstract: In this paper, we will use the Banach fixed point theorem to prove the uniform-in-ϵ existence of the compressible full magneto-micropolar system in a bounded smooth domain, where ϵ is the dielectric constant. Consequently, the limit as $\epsilon \to 0$ can be established. This approximation is usually referred to as the magnetohydrodynamics approximation and is equivalent to the neglect of the displacement current.

Keywords: magneto micropolar; uniform existence; dielectric constant
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1. Introduction

In this paper, we consider the compressible full magneto-micropolar system [1]

$$\partial_t\rho + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\begin{aligned}\partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) + \nabla(\rho\theta) - (\mu+\mu_r)\Delta u - (\lambda+\mu-\mu_r)\nabla\operatorname{div} u \\ = 2\mu_r\operatorname{rot} w + (E + u\times b)\times b,\end{aligned} \tag{1.2}$$

$$\partial_t(\rho w) + \operatorname{div}(\rho u w) - (c_a + c_d)\Delta w - (c_0 + c_d - c_a)\nabla\operatorname{div} w + 4\mu_r w = 2\mu_r\operatorname{rot} u, \tag{1.3}$$

$$\begin{aligned}\partial_t(\rho\theta) + \operatorname{div}(\rho u\theta) - k\Delta\theta + \rho\theta\operatorname{div} u = \frac{\mu}{2}(\nabla u + \nabla u^T) : (\nabla u + \nabla u^T) + \lambda(\operatorname{div} u)^2 \\ +4\mu_r\left|\frac{1}{2}\operatorname{rot} u - w\right|^2 + c_0(\operatorname{div} w)^2 + (c_a + c_d)\nabla w : \nabla w \\ +(c_d - c_a)\nabla w : \nabla w^T + |E + u\times b|^2,\end{aligned} \tag{1.4}$$

$$\epsilon\partial_t E - \operatorname{rot} b + E + u\times b = 0, \tag{1.5}$$

$$\partial_t b + \operatorname{rot} E = 0, \operatorname{div} b = 0, \tag{1.6}$$

in $Q_T := \Omega \times (0, T)$ for any $T > 0$, with the initial and boundary conditions

$$(\rho, u, w, \theta, E, b)(\cdot, 0) = (\rho_0, u_0, w_0, \theta_0, E_0, b_0) \text{ in } \Omega \subseteq \mathbb{R}^3, \tag{1.7}$$

$$u = 0,\ w = 0,\ \theta = 0,\ E \times n = 0,\ b \cdot n = 0 \text{ on } \partial\Omega \times (0, T). \tag{1.8}$$

Here, ρ is the density of the fluid, u is the fluid velocity field, w is the micro-rotational velocity, θ is the temperature, E is the electric field, and b is the magnetic field. Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, whose outward normal vector is denoted by n. The positive constant k is the heat conductivity, the physical constants μ and λ are the shear viscosity and bulk viscosity and satisfy $\mu > 0$ and $\lambda + \frac{2}{3}\mu \geq 0$. $\epsilon > 0$ is the dielectric constant. The positive constant μ_r represents the dynamic microrotation viscosity. c_0, c_a, c_d are constants called coefficients of angular viscosities, which satisfy $\lambda + \mu - \mu_r > 0, c_0 + c_d - c_a > 0$.

When $w = 0$, the above system is symmetric hyperbolic-parabolic. Kawashima-Shizuta [2–4] proved the local existence of smooth solutions for large data and global existence of smooth solutions for small data and studied the limit as $\epsilon \to 0$ when $\Omega := \mathbb{R}^2$. Jiang-Li [5, 6] studied the limit of $\epsilon \to 0$ when $\Omega := \mathbb{T}^3$. Similar results have been obtained in [7–13]. Li-Mu [14] studied the low Mach number limit of the problem (1.1)–(1.6) when $\Omega := \mathbb{R}^3$.

When $\epsilon = 0$ and the entropy is a constant, Wei-Guo-Li [15] and Wu-Wang [16] studied the long-time behavior of smooth solutions. Zhang [17] showed the local well-posedness (without proof) and a blow-up criterion.

The well-posedness of the problem has been studied in [18–21]. The numerical analysis of some related problems has been considered in [22–29].

The aim of this paper is to prove the uniform-in-ϵ existence of unique local strong solutions to the problem (1.1)–(1.8) when Ω is a bounded domain.

Here, we impose the following regularity conditions on the initial data:

$$\begin{aligned} &\theta_0 \geq 0,\ 0 < \tfrac{1}{C} \leq \rho_0 \leq C,\ \rho_0 \in W^{1,6},\ \operatorname{div} b_0 = 0 \text{ in } \Omega, \\ &E_0 \times n = 0,\ b_0 \cdot n = 0 \text{ on } \partial\Omega,\ E_0, b_0 \in H^2,\ u_0, w_0, \theta_0 \in H_0^1 \cap H^2. \end{aligned} \tag{1.9}$$

Theorem 1.1. *Let* (1.9) *hold true and* $0 < \epsilon < 1$*, and let* $k = 1$*. Then, there exist a small time* $\tilde{T} > 0$ *independent of* $\epsilon > 0$ *and a unique strong solution* $(\rho, u, w, \theta, E, b)$ *to the initial boundary value problem* (1.1)–(1.8) *such that*

$$\begin{aligned} &\theta \geq 0,\ \tfrac{1}{C} \leq \rho \leq C,\ \rho \in L^\infty(0, \tilde{T}; W^{1,6}),\ \partial_t\rho \in L^\infty(0, \tilde{T}; L^6), \\ &u, w \in L^\infty(0, \tilde{T}; H^2) \cap L^2(0, \tilde{T}; W^{2,6}),\ \theta \in L^\infty(0, \tilde{T}; H^2), \\ &u_t, w_t, \theta_t \in L^\infty(0, \tilde{T}; L^2) \cap L^2(0, \tilde{T}; H^1), \\ &b \in L^\infty(0, \tilde{T}; H^2), b_t \in L^\infty(0, \tilde{T}; H^1), \\ &E \in L^\infty(0, \tilde{T}; H^1) \cap L^2(0, \tilde{T}; H^2),\ E_t \in L^2(0, \tilde{T}; H^1), \end{aligned} \tag{1.10}$$

with the corresponding norms that are uniformly bounded with respect to $\epsilon > 0$.

We will prove Theorem 1.1. by the Banach fixed point theorem. We define the nonempty closed set

$$\mathcal{A} := \{(\tilde{u}, \tilde{w}) \in \mathcal{A}; \tilde{u}(\cdot, 0) = u_0, \tilde{w}(\cdot, 0) = w_0, \|(\tilde{u}, \tilde{w})\|_{\mathcal{A}} \leq A\}$$

with the norm

$$\|(\tilde{u}, \tilde{w})\|_{\mathcal{A}} := \|(\tilde{u}, \tilde{w})\|_{L^\infty(0,T;H^2)} + \|(\tilde{u}, \tilde{w})\|_{L^2(0,T;W^{2,6})} + \|\partial_t(\tilde{u}, \tilde{w})\|_{L^\infty(0,T;L^2)} + \|\partial_t(\tilde{u}, \tilde{w})\|_{L^2(0,T;H^1)}.$$

Let $\tilde{u} \in \mathcal{A}$ be given, and we consider the following linear problems:

$$\partial_t \rho + \operatorname{div}(\rho \tilde{u}) = 0, \tag{1.11}$$

$$\rho(\cdot, 0) = \rho_0; \tag{1.12}$$

$$\epsilon \partial_t E - \operatorname{rot} b + E + \tilde{u} \times b = 0, \tag{1.13}$$

$$\partial_t b + \operatorname{rot} E = 0, \tag{1.14}$$

$$\operatorname{div} b = 0, \tag{1.15}$$

$$(E, b)(\cdot, 0) = (E_0, b_0), \tag{1.16}$$

$$E \times n = 0,\ b \cdot n = 0 \ \text{ on } \ \partial\Omega \times (0, T); \tag{1.17}$$

$$\begin{aligned} \rho\partial_t\theta + \rho\tilde{u}\cdot\nabla\theta - \Delta\theta + \rho\theta\operatorname{div}\tilde{u} &= \frac{\mu}{2}(\nabla\tilde{u} + \nabla\tilde{u}^T) : (\nabla\tilde{u} + \nabla\tilde{u}^T) + \lambda(\operatorname{div}\tilde{u})^2 \\ &\quad + 4\mu_r\left|\frac{1}{2}\operatorname{rot}\tilde{u} - \tilde{w}\right|^2 + c_0(\operatorname{div}\tilde{w})^2 + (c_a + c_d)\nabla\tilde{w} : \nabla\tilde{w} \\ &\quad + (c_d - c_a)\nabla\tilde{w} : \nabla\tilde{w}^T + |E + \tilde{u} \times b|^2, \end{aligned} \tag{1.18}$$

$$\theta(\cdot, 0) = \theta_0, \tag{1.19}$$

$$\theta = 0 \ \text{ on } \ \partial\Omega \times (0, T). \tag{1.20}$$

$$\begin{aligned} &\rho\partial_t u + \rho\tilde{u}\cdot\nabla u + \nabla(\rho\theta) - (\mu + \mu_r)\Delta u - (\lambda + \mu - \mu_r)\nabla\operatorname{div} u \\ &\quad = 2\mu_r\operatorname{rot}\tilde{w} + (E + u \times b) \times b, \end{aligned} \tag{1.21}$$

$$u(\cdot, 0) = u_0, \tag{1.22}$$

$$u = 0 \ \text{ on } \ \partial\Omega \times (0, T). \tag{1.23}$$

$$\rho\partial_t w + \rho\tilde{u}\cdot\nabla w - (c_a + c_d)\Delta w - (c_0 + c_d - c_a)\nabla\operatorname{div} w + 4\mu_r w = 2\mu_r\operatorname{rot} u, \tag{1.24}$$

$$w(\cdot, 0) = w_0, \tag{1.25}$$

$$w = 0 \ \text{ on } \ \partial\Omega \times (0, T). \tag{1.26}$$

Let (u, w) be the unique strong solution to the above problem, and we define the fixed point map $F : (\tilde{u}, \tilde{w}) \in \mathcal{A} \to (u, w) \in \mathcal{A}$ with $\tilde{u} = \tilde{w} = 0$ on $\partial\Omega \times (0, T)$. We will prove that the map F maps $\mathcal{A}$ into $\mathcal{A}$ for suitable constant A and small T, and F is a contraction mapping on $\mathcal{A}$, and thus F has a unique fixed point in $\mathcal{A}$. This proves Theorem 1.1.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1.

Lemma 2.1. *Let* $(\tilde{u}, \tilde{w}) \in \mathcal{A}$ *be given. Then, the problem* (1.11) *and* (1.12) *has a unique solution* ρ *satisfying*

$$\frac{1}{C} \le \rho \le C,\ \|\rho\|_{L^\infty(0,T;W^{1,6})} \le C,\ \|\rho_t\|_{L^\infty(0,T;L^6)} \le CA$$

for some small $0 < T \le 1$.

Here and later on, C will denote a constant independent of ϵ and A.

Proof. Since Eq (1.11) is linear with regular $\tilde{u}$, the existence and uniqueness are well-known, and we only need to establish a priori estimates.

Let
$$\frac{dx(X,t)}{dt} = \tilde{u}(x(X,t),t) \text{ and } x(X,0) = X,$$
and we see that
$$\frac{d\rho(x(X,t),t)}{dt} = -\rho \operatorname{div} \tilde{u},$$
whence
$$\rho(x,t) = \rho_0 \exp\left(-\int_0^t \operatorname{div} \tilde{u} ds\right). \tag{2.1}$$

It follows from (2.1) that
$$\begin{aligned}
\rho(x,t) &\le \rho \exp\left(\int_0^T \|\operatorname{div} \tilde{u}\|_{L^\infty} dt\right) \\
&\le \rho_0 \exp\left(\int_0^T C\|\tilde{u}\|_{W^{2,6}} dt\right) \\
&\le \rho_0 \exp(CA\sqrt{T}) \le C\|\rho_0\|_{L^\infty}
\end{aligned}$$
if $A\sqrt{T} \le 1$;
$$\begin{aligned}
\rho(x,t) &\ge \rho_0 \exp\left(-\int_0^T \|\operatorname{div} \tilde{u}\|_{L^\infty} dt\right) \\
&\ge \inf \rho_0 \exp(-CA\sqrt{T}) \\
&\ge C \inf \rho_0
\end{aligned}$$
if $A\sqrt{T} \le 1$;
$$\nabla\rho = \nabla\rho_0 \exp\left(-\int_0^t \operatorname{div} \tilde{u} ds\right) - \rho_0 \exp\left(-\int_0^t \operatorname{div} \tilde{u} ds\right) \int_0^t \nabla \operatorname{div} \tilde{u} ds,$$
whence
$$\begin{aligned}
\|\nabla\rho\|_{L^\infty(0,T;L^6)} &\le \exp\left(\int_0^T \|\operatorname{div} \tilde{u}\|_{L^\infty} dt\right)\left(\|\nabla\rho_0\|_{L^6} + \|\rho_0\|_{L^\infty} \int_0^T \|\nabla \operatorname{div} \tilde{u}\|_{L^6} dt\right) \\
&\le C \exp(CA\sqrt{T})(1 + A\sqrt{T}) \\
&\le C
\end{aligned}$$
if $A\sqrt{T} \le 1$.

It follows from (1.11) that
$$\rho_t = -\tilde{u}\nabla\rho - \rho \operatorname{div} \tilde{u},$$
and
$$\begin{aligned}
\|\rho_t\|_{L^\infty(0,T;L^6)} &\le \|\tilde{u}\|_{L^\infty(0,T;L^\infty)} \|\nabla\rho\|_{L^\infty(0,T;L^6)} + \|\rho\|_{L^\infty(0,T;L^\infty)} \|\operatorname{div} \tilde{u}\|_{L^\infty(0,T;L^6)} \\
&\le CA
\end{aligned}$$
if $A\sqrt{T} \le 1$.

This completes the proof.

□

Lemma 2.2. *Let* $(\tilde{u}, \tilde{w}) \in \mathcal{A}$ *be given. Then, the problem* (1.13)–(1.17) *has a unique solution* (E, b) *satisfying* (2.2)*,* (2.4)*,* (2.6)*,* (2.8)*,* (2.9)*,* (2.10)*,* (2.11) *and* (2.12) *for some* $0 < T \leq 1$.

Proof. Since Eqs (1.13)–(1.15) are linear with regular $(\rho, \tilde{u})$, the existence and uniqueness are well-known, and we only need to establish the a priori estimates.

Testing (1.13) and (1.14) by E and b and summing up the result, we see that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int(\epsilon E^2 + b^2)dx + \int E^2 dx \\
= \; & -\int(\tilde{u} \times b)E dx \\
\leq \; & \|\tilde{u}\|_{L^\infty}\|b\|_{L^2}\|E\|_{L^2} \leq CA\|b\|_{L^2}\|E\|_{L^2} \\
\leq \; & \frac{1}{2}\|E\|_{L^2}^2 + CA^2\|b\|_{L^2}^2,
\end{aligned}$$

which gives

$$\int(\epsilon E^2 + b^2)dx + \int_0^T\int E^2 dxdt \leq C \tag{2.2}$$

if $A^2 T \leq 1$.

Note that (1.13), (1.17) and (1.23) give the boundary condition

$$\mathrm{rot} b \times n = 0 \;\text{ on }\; \partial\Omega \times (0, T). \tag{2.3}$$

Taking rot to (1.13) and (1.14), testing by $\mathrm{rot}E$ and $\mathrm{rot}b$ and using (2.3), summing up the result and integrating by parts, we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int(\epsilon|\mathrm{rot}E|^2 + |\mathrm{rot}b|^2)dx + \int|\mathrm{rot}E|^2 dx \\
= \; & -\int \mathrm{rot}(\tilde{u} \times b)\cdot \mathrm{rot}E dx \\
\leq \; & C\|\tilde{u}\|_{H^2}\|\mathrm{rot}b\|_{L^2}\|\mathrm{rot}E\|_{L^2} \\
\leq \; & \frac{1}{2}\int|\mathrm{rot}E|^2 dx + CA^2\|\mathrm{rot}b\|_{L^2}^2,
\end{aligned}$$

which yields

$$\int(\epsilon|\mathrm{rot}E|^2 + |\mathrm{rot}b|^2)dx + \int_0^T\int|\mathrm{rot}E|^2 dxdt \leq C \tag{2.4}$$

if $A^2 T \leq 1$.

Here, we have used the Poincaré inequality

$$\|b\|_{L^2} \leq C\|\mathrm{rot}b\|_{L^2}. \tag{2.5}$$

Taking ∂_t to (1.13) and (1.14), testing by $\partial_t E$ and $\partial_t b$ and summing up the result and using (2.4), we infer that

$$\frac{1}{2}\frac{d}{dt}\int(\epsilon|E_t|^2 + |\partial_t b|^2) + \int|E_t|^2 dx$$

$$
\begin{aligned}
&= \int \partial_t(b\times\tilde{u})\cdot\partial_t E dx \\
&\le (\|\partial_t b\|_{L^2}\|\tilde{u}\|_{L^\infty}+\|b\|_{L^6}\|\partial_t\tilde{u}\|_{L^3})\|E_t\|_{L^2} \\
&\le \frac{1}{2}\int|E_t|^2dx + CA^2\|\partial_t b\|_{L^2}^2 + C\|\partial_t\tilde{u}\|_{L^2}\|\nabla\partial_t\tilde{u}\|_{L^2},
\end{aligned}
$$

which implies

$$
\int(\epsilon|E_t|^2+|b_t|^2)dx+\int_0^T\int|E_t|^2dxdt\le C \tag{2.6}
$$

if $A^2T\le 1$.

(1.14) and (2.3) give the boundary condition

$$
\mathrm{rot}^2E\times n=0 \ \text{ on } \ \partial\Omega\times(0,T). \tag{2.7}
$$

Taking rot^2 to (1.13) and (1.14), testing by rot^2E and rot^2b and using (2.7) and summing up the result, we derive

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int(|\epsilon\mathrm{rot}^2E|^2+|\mathrm{rot}^2b|^2)dx+\int|\mathrm{rot}^2E|^2dx \\
&= \int \mathrm{rot}^2(b\times\tilde{u})\mathrm{rot}^2Edx \\
&\le C\|\tilde{u}\|_{H^2}\|\mathrm{rot}^2b\|_{L^2}\|\mathrm{rot}^2E\|_{L^2} \\
&\le \frac{1}{2}\int|\mathrm{rot}^2E|^2dx+CA^2\|\mathrm{rot}^2b\|_{L^2}^2,
\end{aligned}
$$

which implies

$$
\int(\epsilon|\mathrm{rot}^2E|^2+|\mathrm{rot}^2b|^2)dx+\int_0^T\int|\mathrm{rot}^2E|^2dxdt\le C \tag{2.8}
$$

if $A^2T\le 1$.

Taking $\nabla\mathrm{div}$ to (1.13), testing by $\nabla\mathrm{div}\,E$ and using (2.8), we get

$$
\begin{aligned}
&\frac{\epsilon}{2}\frac{d}{dt}\int|\nabla\mathrm{div}\,E|^2dx+\int|\nabla\mathrm{div}\,E|^2dx \\
&= \int\nabla\mathrm{div}\,(b\times\tilde{u})\cdot\nabla\mathrm{div}\,Edx \\
&\le C\|\tilde{u}\|_{H^2}\|\mathrm{rot}^2b\|_{L^2}\|\nabla\mathrm{div}\,E\|_{L^2} \\
&\le \frac{1}{2}\int|\nabla\mathrm{div}\,E|^2dx+CA^2,
\end{aligned}
$$

which leads to

$$
\epsilon\int|\nabla\mathrm{div}\,E|^2dx+\int_0^T\int|\nabla\mathrm{div}\,E|^2dxdt\le C \tag{2.9}
$$

if $A^2T\le 1$.

Taking $\partial_t\mathrm{rot}$ to (1.13) and (1.14), testing by $\partial_t\mathrm{rot}\,E$ and $\partial_t\mathrm{rot}\,b$ and using (2.3) and (2.8), we obtain

$$
\frac{1}{2}\frac{d}{dt}\int(\epsilon|\mathrm{rot}\,E_t|^2+|\mathrm{rot}\,b_t|^2)dx+\int|\mathrm{rot}\,E_t|^2dx
$$

$$
\begin{aligned}
&= \int \operatorname{rot}(b_t \times \tilde{u} + b \times \tilde{u}_t)\operatorname{rot} E_t dx \\
&\leq \frac{1}{2}\int |\operatorname{rot} E_t|^2 dx + C\|b_t\|_{L^6}^2\|\nabla\tilde{u}\|_{L^3}^2 + C\|\tilde{u}\|_{L^\infty}^2\|\operatorname{rot} b_t\|_{L^2}^2 \\
&\quad + C\|b\|_{L^\infty}^2\|\nabla\tilde{u}_t\|_{L^2}^2 + C\|\operatorname{rot} b\|_{L^6}^2\|\tilde{u}_t\|_{L^3}^2 \\
&\leq \frac{1}{2}\int |\operatorname{rot} E_t|^2 dx + CA^2\|\operatorname{rot} b_t\|_{L^2}^2 + C\|\nabla\tilde{u}_t\|_{L^2}^2,
\end{aligned}
$$

which yields

$$
\int (\epsilon|\operatorname{rot} E_t|^2 + |\operatorname{rot} b_t|^2)dx + \int_0^T \int |\operatorname{rot} E_t|^2 dxdt \leq CA^2 \tag{2.10}
$$

if $A^2T \leq 1$.

Applying $\partial_t\operatorname{div}$ to (1.13), testing by $\partial_t\operatorname{div} E$, and using (2.10) and (2.8), we have

$$
\begin{aligned}
&\frac{\epsilon}{2}\frac{d}{dt}\int |\operatorname{div} E_t|^2 dx + \int |\operatorname{div} E_t|^2 dx \\
&= \int \operatorname{div}(b_t \times \tilde{u} + b \times \tilde{u}_t)\partial_t\operatorname{div} E dx \\
&= \int (\tilde{u}\operatorname{rot} b_t - b_t\operatorname{rot}\tilde{u} + \tilde{u}_t\operatorname{rot} b - b\operatorname{rot}\tilde{u}_t)\partial_t\operatorname{div} E dx \\
&\leq (\|\tilde{u}\|_{L^\infty}\|\operatorname{rot} b_t\|_{L^2} + \|b_t\|_{L^6}\|\operatorname{rot}\tilde{u}\|_{L^3} + \|\tilde{u}_t\|_{L^6}\|\operatorname{rot} b\|_{L^3} + \|b\|_{L^\infty}\|\operatorname{rot}\tilde{u}_t\|_{L^2})\|\operatorname{div} E_t\|_{L^2} \\
&\leq C(A^2 + \|\nabla\tilde{u}_t\|_{L^2})\|\operatorname{div} E_t\|_{L^2} \\
&\leq \frac{1}{2}\|\operatorname{div} E_t\|_{L^2}^2 + CA^4 + C\|\nabla\tilde{u}_t\|_{L^2}^2,
\end{aligned}
$$

which gives

$$
\epsilon\int (\operatorname{div} E_t)^2 dx + \int_0^T \int (\operatorname{div} E_t)^2 dxdt \leq CA^2 \tag{2.11}
$$

if $A^2T \leq 1$.

(2.6), (2.10) and (2.11) imply

$$
\begin{aligned}
&\frac{d}{dt}\int (E^2 + (\operatorname{div} E)^2 + |\operatorname{rot} E|^2)dx \\
&= 2\int (E \cdot E_t + \operatorname{div} E\operatorname{div} E_t + \operatorname{rot} E\operatorname{rot} E_t)dx \\
&\leq \left(\int (E^2 + (\operatorname{div} E)^2 + |\operatorname{rot} E|^2)\right)^{\frac{1}{2}} \left(\int (E_t^2 + (\operatorname{div} E_t)^2 + (\operatorname{rot} E_t)^2)dx\right)^{\frac{1}{2}},
\end{aligned}
$$

whence

$$
\frac{dy}{dt} \leq C(\|E_t\|_{L^2} + \|\operatorname{div} E_t\|_{L^2} + \|\operatorname{rot} E_t\|_{L^2}),
$$

with

$$
y(t) := \left(\int (E^2 + (\operatorname{div} E)^2 + (\operatorname{rot} E)^2)dx\right)^{\frac{1}{2}}.
$$

Integrating the above inequality, we have

$$y(t) \le y(0) + CAT \le C \tag{2.12}$$

if $AT \le 1$.

This completes the proof.

□

Lemma 2.3. *Let* $(\tilde{u}, \tilde{w}) \in \mathcal{A}$ *be given. Then, the problem* (1.18) *and* (1.19) *has a unique solution* θ *satisfying* $\theta \ge 0$*,* (2.14)*,* (2.15) *and* (2.16).

Proof. Since Eq (1.18) is linear with regular $(\rho, \tilde{u}, \tilde{w}, E, b)$, the existence and uniqueness are well-known, and we only need to establish a priori estimates.

Testing (1.18) by θ and using (1.11) and Lemmas 2.1–2.2, we infer that

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\int \rho\theta^2 dx + \int |\nabla\theta|^2 dx \\
= \ & -\int \rho\theta^2 \operatorname{div}\tilde{u} dx + \int \Big[\frac{\mu}{2}(\nabla\tilde{u} + \nabla\tilde{u}^T) : (\nabla\tilde{u} + \nabla\tilde{u}^T) + \lambda(\operatorname{div}\tilde{u})^2 \\
& +4\mu_r \left|\frac{1}{2}\operatorname{rot}\tilde{u} - \tilde{w}\right|^2 + c_0(\operatorname{div}\tilde{w})^2 + (c_a + c_d)\nabla\tilde{w} : \nabla\tilde{w} \\
& +(c_d - c_a)\nabla\tilde{w} : \nabla\tilde{w}^T + |E + \tilde{u} \times b|^2\Big]\theta dx \\
\le \ & \|\operatorname{div}\tilde{u}\|_{L^\infty}\int \rho\theta^2 dx + C\|\nabla\tilde{u}\|_{L^6}\|\nabla\tilde{u}\|_{L^3}\|\theta\|_{L^2} + C\|\tilde{w}\|_{L^6}\|\tilde{w}\|_{L^3}\|\theta\|_{L^2} \\
& +C\|\nabla\tilde{w}\|_{L^6}\|\nabla\tilde{w}\|_{L^3}\|\theta\|_{L^2} + C\|E\|_{L^3}\|E\|_{L^6}\|\theta\|_{L^2} + C\|\tilde{u}\|_{L^\infty}^2\|b\|_{L^4}^2\|\theta\|_{L^2} \\
\le \ & \|\operatorname{div}\tilde{u}\|_{L^\infty}\int \rho\theta^2 dx + CA^2\|\theta\|_{L^2} + C\|\theta\|_{L^2},
\end{aligned}$$

which yields

$$\int \theta^2 dx + \int_0^T \int |\nabla\theta|^2 dx dt \le C \tag{2.13}$$

if $A^2 T \le 1$.

Testing (1.18) by θ_t and using Lemmas 2.1–2.2, we deduce that

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\int |\nabla\theta|^2 dx + \int \rho\theta_t^2 dx \\
= \ & -\int (\rho\theta \operatorname{div}\tilde{u} + \rho\tilde{u}\cdot\nabla\theta)\theta_t dx \\
& + \int \Big[\frac{\mu}{2}(\nabla\tilde{u} + \nabla\tilde{u}^T) : (\nabla\tilde{u} + \nabla\tilde{u}^T) + \lambda(\operatorname{div}\tilde{u})^2 \\
& +4\mu_r \left|\frac{1}{2}\operatorname{rot}\tilde{u} - \tilde{w}\right|^2 + c_0(\operatorname{div}\tilde{w})^2 + (c_a + c_d)\nabla\tilde{w} : \nabla\tilde{w} \\
& +(c_d - c_a)\nabla\tilde{w} : \nabla\tilde{w}^T + |E + \tilde{u} \times b|^2\Big]\theta_t dx \\
\le \ & \|\operatorname{div}\tilde{u}\|_{L^6}\|\theta\|_{L^3}\|\sqrt{\rho}\theta_t\|_{L^2}\|\sqrt{\rho}\|_{L^\infty} + \|\tilde{u}\|_{L^\infty}\|\nabla\theta\|_{L^2}\|\sqrt{\rho}\theta_t\|_{L^2}\|\sqrt{\rho}\|_{L^\infty}
\end{aligned}$$

$$
\begin{aligned}
&+C(\|\nabla\tilde{u}\|_{L^4}^2+\|\tilde{w}\|_{L^4}^2+\|\nabla\tilde{w}\|_{L^4}^2+\|E\|_{L^4}^2+\|\tilde{u}\|_{L^\infty}^2\|b\|_{L^4}^2)\|\theta_t\|_{L^2}\\
\le\ & \frac{1}{2}\int\rho\theta_t^2dx+CA^2(\|\theta\|_{L^3}^2+\|\nabla\theta\|_{L^2}^2)+CA^4+C,
\end{aligned}
$$

which gives

$$
\int|\nabla\theta|^2dx+\int_0^T\int\theta_t^2dxdt\le C \tag{2.14}
$$

if $A^4T\le 1$.

Taking ∂_t to (1.18), testing by θ_t, and using (1.11) and Lemmas 2.1–2.2, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\rho\theta_t^2dx+\int|\nabla\theta_t|^2dx\\
\le\ & -\int\rho_t\theta_t^2dx-\int\partial_t(\rho\tilde{u})\cdot\nabla\theta\cdot\theta_tdx-\int\partial_t(\rho\theta\operatorname{div}\tilde{u})\cdot\theta_tdx\\
&+C\int|\nabla\tilde{u}||\nabla\tilde{u}_t||\theta_t|dx+C\int|\tilde{w}||\tilde{w}_t||\theta_t|dx\\
&+C\int|\nabla\tilde{w}||\nabla\tilde{w}_t||\theta_t|dx+C\int|EE_t\theta_t|dx\\
&+C\left|\int(\tilde{u}\times b)(\tilde{u}_t\times b+\tilde{u}\times b_t)\theta_tdx\right|\\
\le\ & \|\rho_t\|_{L^6}\|\theta_t\|_{L^2}\|\theta_t\|_{L^3}+\|\rho_t\|_{L^6}\|\tilde{u}\|_{L^\infty}\|\nabla\theta\|_{L^2}\|\theta_t\|_{L^3}\\
&+C\|\tilde{u}_t\|_{L^3}\|\nabla\theta\|_{L^2}\|\theta_t\|_{L^6}+C\|\rho_t\|_{L^6}\|\theta\|_{L^6}\|\operatorname{div}\tilde{u}\|_{L^6}\|\theta_t\|_{L^2}\\
&+C\|\sqrt{\rho}\theta_t\|_{L^2}^2\|\operatorname{div}\tilde{u}\|_{L^\infty}+C\|\theta\|_{L^6}\|\nabla\tilde{u}_t\|_{L^2}\|\theta_t\|_{L^3}\\
&+C\|\nabla\tilde{u}\|_{L^6}\|\nabla\tilde{u}_t\|_{L^2}\|\theta_t\|_{L^3}+C\|\tilde{w}\|_{L^6}\|\tilde{w}_t\|_{L^2}\|\theta_t\|_{L^3}\\
&+C\|\nabla\tilde{w}\|_{L^6}\|\nabla\tilde{w}_t\|_{L^2}\|\theta_t\|_{L^3}+C\|E\|_{L^6}\|E_t\|_{L^2}\|\theta_t\|_{L^3}\\
&+C\|\tilde{u}\|_{L^\infty}\|b\|_{L^\infty}^2\|\tilde{u}_t\|_{L^2}\|\theta_t\|_{L^2}+C\|\tilde{u}\|_{L^\infty}^2\|b\|_{L^\infty}\|b_t\|_{L^2}\|\theta_t\|_{L^2}\\
\le\ & CA\|\theta_t\|_{L^2}^{\frac{3}{2}}\|\nabla\theta_t\|_{L^2}^{\frac{1}{2}}+CA^2\|\theta_t\|_{L^2}^{\frac{1}{2}}\|\nabla\theta_t\|_{L^2}^{\frac{1}{2}}\\
&+C\|\tilde{u}_t\|_{L^2}^{\frac{1}{2}}\|\nabla\tilde{u}_t\|_{L^2}^{\frac{1}{2}}\|\nabla\theta_t\|_{L^2}+CA^2\|\theta_t\|_{L^2}+C\|\operatorname{div}\tilde{u}\|_{L^\infty}\int\rho\theta_t^2dx\\
&+CA\|\nabla\tilde{u}_t\|_{L^2}\|\theta_t\|_{L^2}^{\frac{1}{2}}\|\nabla\theta_t\|_{L^2}^{\frac{1}{2}}+CA\|\nabla\tilde{w}_t\|_{L^2}\|\theta_t\|_{L^2}^{\frac{1}{2}}\|\nabla\theta_t\|_{L^2}^{\frac{1}{2}}+C\|E_t\|_{L^2}\|\theta_t\|_{L^3},
\end{aligned}
$$

which yields

$$
\int\theta_t^2dx+\int_0^T\int|\nabla\theta_t|^2dxdt\le C \tag{2.15}
$$

if $(A^4+A^3+A^2)T\le 1$. Here, we bound

$$
\begin{aligned}
CA\|\nabla\tilde{u}_t\|_{L^2}\|\theta_t\|_{L^2}^{\frac{1}{2}}\|\nabla\theta_t\|_{L^2}^{\frac{1}{2}} &\le \frac{1}{16}\|\nabla\theta_t\|_{L^2}^2+CA^{\frac{4}{3}}\|\nabla\tilde{u}_t\|_{L^2}^{\frac{4}{3}}\|\theta_t\|_{L^2}^{\frac{2}{3}}\\
&\le \frac{1}{16}\|\nabla\theta_t\|_{L^2}^2+CA^{\frac{4}{3}}\|\nabla\tilde{u}_t\|_{L^2}^{\frac{4}{3}}(1+\|\theta_t\|_{L^2}^2)\\
CA\|\nabla\tilde{w}_t\|_{L^2}\|\theta_t\|_{L^2}^{\frac{1}{2}}\|\nabla\theta_t\|_{L^2}^{\frac{1}{2}} &\le \frac{1}{16}\|\nabla\theta_t\|_{L^2}^2+CA^{\frac{4}{3}}\|\nabla\tilde{w}_t\|_{L^2}^{\frac{4}{3}}\|\theta_t\|_{L^2}^{\frac{2}{3}}
\end{aligned}
$$

$$\leq \frac{1}{16}\|\nabla\theta_t\|_{L^2}^2 + CA^{\frac{4}{3}}\|\nabla\tilde{w}_t\|_{L^2}^{\frac{4}{3}}(1+\|\theta_t\|_{L^2}^2),$$

$$C\|E_t\|_{L^2}\|\theta_t\|_{L^3} \leq C\|E_t\|_{L^2}\|\theta_t\|_{L^2}^{\frac{1}{2}}\|\nabla\theta_t\|_{L^2}^{\frac{1}{2}}$$
$$\leq \frac{1}{16}\|\nabla\theta_t\|_{L^2}^2 + C\|E_t\|_{L^2}^{\frac{4}{3}}(1+\|\theta_t\|_{L^2}^2).$$

It follows from (1.18), (2.15) and (2.14) that

$$\|\theta\|_{L^\infty(0,T;H^2)} \leq CA^2 + C. \tag{2.16}$$

This completes the proof. □

Lemma 2.4. *Let* $(\tilde{u}, \tilde{w}) \in \mathcal{A}$ *be given. Then, the problem* (1.21)–(1.23) *has a unique solution* u *satisfying*

$$\|u\|_{L^\infty(0,T;H^2)} + \|u\|_{L^2(0,T;W^{2,6})} + \|u_t\|_{L^\infty(0,T;L^2)} + \|u_t\|_{L^2(0,T;H^1)} \leq C_1 \tag{2.17}$$

for some small $0 < T \leq 1$. *Here,* C_1 *is a positive constant independent of* ϵ *and* $\mathcal{A}$.

Proof. Since Eq (1.21) is linear with regular $(\rho, \tilde{u}, \tilde{w}, \theta, E, b)$, the existence and uniqueness are well-known, and we only need to establish (2.17).

Testing (1.21) by u and using (1.11) and Lemmas 2.1–2.3, we see that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|u|^2dx + \int((\mu+\mu_r)|\nabla u|^2 + (\lambda+\mu-\mu_r)(\operatorname{div} u)^2)dx + \int |u\times b|^2dx \\
= {} & \int \rho\theta \operatorname{div} u dx + 2\mu_r\int u \operatorname{rot}\tilde{w} dx + \int (E\times b)u dx \\
\leq {} & \|\rho\|_{L^\infty}\|\theta\|_{L^2}\|\operatorname{div} u\|_{L^2} + C\|u\|_{L^2}\|\operatorname{rot}\tilde{w}\|_{L^2} + \|E\|_{L^6}\|b\|_{L^3}\|u\|_{L^2} \\
\leq {} & C\|\operatorname{div} u\|_{L^2} + C\|u\|_{L^2} + CA\|u\|_{L^2} \\
\leq {} & \frac{\lambda+\mu-\mu_r}{2}\|\operatorname{div} u\|_{L^2}^2 + C\|u\|_{L^2} + C + CA\|u\|_{L^2},
\end{aligned}$$

which gives

$$\int |u|^2dx + \int_0^T\int |\nabla u|^2dxdt \leq C \tag{2.18}$$

if $A^2T \leq 1$.

Testing (1.21) by u_t and using Lemmas 2.1–2.3, we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int[(\mu+\mu_r)|\nabla u|^2 + (\lambda+\mu-\mu_r)(\operatorname{div} u)^2]dx + \int\rho|u_t|^2dx \\
= {} & -\int\rho\tilde{u}\cdot\nabla u\cdot u_t dx - \int\nabla(\rho\theta)\cdot u_t dx + 2\mu_r\int u_t \operatorname{rot}\tilde{w} dx + \int[(E+u\times b)\times b]u_t dx \\
\leq {} & \frac{1}{2}\int\rho|u_t|^2dx + C\|\tilde{u}\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 + C\|\rho\|_{L^\infty}^2\|\nabla\theta\|_{L^2}^2 + C\|\theta\|_{L^3}^2\|\nabla\rho\|_{L^6}^2 \\
& + C\|\operatorname{rot}\tilde{w}\|_{L^2}^2 + C\|E\|_{L^6}^2\|b\|_{L^3}^2 + C\|u\|_{L^\infty}^2\|b\|_{L^4}^2 \\
\leq {} & \frac{1}{2}\int\rho|u_t|^2dx + CA^2\|\nabla u\|_{L^2}^2 + C + CA^2,
\end{aligned}$$

which implies

$$\int |\nabla u|^2 dx + \int_0^T \int |u_t|^2 dxdt \le C \tag{2.19}$$

if $A^2 T \le 1$.

Applying ∂_t to (1.21), testing by u_t, and using (1.11), Lemmas 2.1–2.3 and (2.19), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho |u_t|^2 dx + \int [(\mu+\mu_r)|\nabla u_t|^2 + (\lambda+\mu-\mu_r)(\operatorname{div} u_t)^2]dx + \int |u_t \times b|^2 dx \\
= \;& -\int \rho_t u_t^2 dx - \int (\rho\tilde{u})_t \cdot \nabla u \cdot u_t dx + \int (\rho\theta)_t \operatorname{div} u_t dx + 2\mu_r \int \tilde{w}_t \operatorname{rot} u_t dx \\
& + \int (E_t \times b + E \times b_t) u_t dx + \int [(u \times b_t) \times b + (u \times b) \times b_t] u_t dx \\
\le \;& \|\rho_t\|_{L^6}\|u_t\|_{L^3}\|u_t\|_{L^2} + \|\rho_t\|_{L^6}\|\tilde{u}\|_{L^\infty}\|\nabla u\|_{L^2}\|u_t\|_{L^3} \\
& + \|\rho\|_{L^\infty}\|\tilde{u}_t\|_{L^3}\|\nabla u\|_{L^2}\|u_t\|_{L^6} + \|\rho_t\|_{L^6}\|\theta\|_{L^3}\|\operatorname{div} u_t\|_{L^2} \\
& + \|\rho\|_{L^\infty}\|\theta_t\|_{L^2}\|\operatorname{div} u_t\|_{L^2} + 2\mu_r\|\tilde{w}_t\|_{L^2}\|\operatorname{rot} u_t\|_{L^2} + \|E_t\|_{L^2}\|b\|_{L^\infty}\|u_t\|_{L^2} \\
& + \|E\|_{L^6}\|b_t\|_{L^3}\|u_t\|_{L^2} + \|u\|_{L^6}\|b_t\|_{L^3}\|b\|_{L^\infty}\|u_t\|_{L^2} \\
\le \;& C\|u_t\|_{L^2}\|u_t\|_{L^3} + CA\|u_t\|_{L^3} + C\|\tilde{u}_t\|_{L^3}\|u_t\|_{L^6} + C\|\operatorname{div} u_t\|_{L^2} \\
& + CA\|\operatorname{rot} u_t\|_{L^2} + C\|E_t\|_{L^2}\|u_t\|_{L^2} + C\|b_t\|_{L^3}\|u_t\|_{L^2} \\
\le \;& \frac{\mu}{2}\int |\nabla u_t|^2 dx + C\|u_t\|_{L^2}^2 + CA^2 + C\|\tilde{u}_t\|_{L^2}\|\nabla \tilde{u}_t\|_{L^2} + C \\
& + C\|E_t\|_{L^2}\|u_t\|_{L^2} + C\|b_t\|_{L^3}\|u_t\|_{L^2},
\end{aligned}$$

which gives

$$\int |u_t|^2 dx + \int_0^T \int |\nabla u_t|^2 dxdt \le C_1 \tag{2.20}$$

if $A^2 T \le 1$.

Since

$$\tilde{u}(x,t) = u_0(x) + \int_0^t \partial_t \tilde{u} ds,$$

and

$$\|\nabla \tilde{u}\|_{L^\infty(0,T;L^2)} \le C + \int_0^T \|\nabla \partial_t \tilde{u}\|_{L^2} dt \le C + C\sqrt{T}A \le C \tag{2.21}$$

if $A^2 T \le 1$.

Similarly, we have

$$\|\nabla \tilde{w}\|_{L^\infty(0,T;L^2)} \le C \tag{2.22}$$

if $A^2 T \le 1$.

We rewrite (1.21) as

$$-\mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u = f := 2\mu_r \operatorname{rot} \tilde{w} + (E + u \times b) \times b - \rho\partial_t u - \rho\tilde{u} \cdot \nabla u - \nabla(\rho\theta).$$

By the H^2-theory of the elliptic system, we get

$$\|u\|_{H^2} \le C\|f\|_{L^2}$$

$$
\begin{aligned}
&\leq C\|\nabla\tilde{w}\|_{L^2} + C\|E\|_{L^6}\|b\|_{L^3} + C\|u\|_{L^6}\|b\|_{L^6}^2 + C\|\rho\partial_t u\|_{L^2} \\
&\quad + C\|\tilde{u}\|_{L^6}\|\nabla u\|_{L^3} + C\|\rho\nabla\theta\|_{L^2} + C\|\theta\|_{L^3}\|\nabla\rho\|_{L^6} \\
&\leq C + C\|\nabla u\|_{L^3},
\end{aligned}
$$

which yields

$$\|u\|_{L^\infty(0,T;H^2)} \leq C_1. \tag{2.23}$$

Similarly, by the $W^{2,6}$-theory of the elliptic system, we have

$$
\begin{aligned}
\|u\|_{W^{2,6}} &\leq C\|f\|_{L^6} \\
&\leq C\|\mathrm{rot}\,\tilde{w}\|_{L^6} + C\|E\|_{L^6}\|b\|_{L^\infty} + C\|u\|_{L^\infty}\|b\|_{L^\infty}\|b\|_{L^6} \\
&\quad + C\|u_t\|_{L^6} + C\|\tilde{u}\|_{L^6}\|\nabla u\|_{L^\infty} + C\|\nabla\theta\|_{L^6} + C\|\nabla\rho\|_{L^6} \\
&\leq C + C\|\nabla u\|_{L^\infty} + CA^2 + C\|u_t\|_{L^6} \\
&\leq C + C\|\nabla u\|_{L^2}^{\frac{1}{4}}\|u\|_{W^{2,6}}^{\frac{3}{4}} + CA^2 + C\|\nabla u_t\|_{L^2},
\end{aligned}
$$

whence

$$\|u\|_{W^{2,6}} \leq C + CA^2 + C\|\nabla u_t\|_{L^2},$$

which yields

$$\|u\|_{L^2(0,T;W^{2,6})} \leq C_1 \tag{2.24}$$

if $A^4 T \leq 1$.

This completes the proof. □

Similarly to Lemma 2.4, we have the following.

Lemma 2.5. *Let* $(\tilde{u}, \tilde{w}) \in \mathcal{A}$ *be given. Then, the problem* (1.24)–(1.26) *has a unique solution* w *satisfying* (2.17) *with* $u := w$.

Proof. Since the proof is very similar to that of Lemma 2.4, we omit the details here. □

Due to the above Lemmas 2.1–2.5, we can take $A := C_1$, and thus F maps $\mathcal{A}$ into $\mathcal{A}$. The following lemma tells us that F is a contraction mapping in the sense of weaker norm.

Lemma 2.6. *There is a constant* $0 < \delta < 1$ *such that for any* $\tilde{u}_i$ $(i = 1, 2)$*,*

$$\|F(\tilde{u}_1, \tilde{w}_1) - F(\tilde{u}_2, \tilde{w}_2)\|_{L^2(0,T;H^1)} \leq \delta\|(\tilde{u}_1 - \tilde{u}_2, \tilde{w}_1 - \tilde{w}_2)\|_{L^2(0,T;H^1)} \tag{2.25}$$

for some small $0 < T \leq 1$.

Proof. Suppose $(\rho_i, u_i, w_i, \theta_i, E_i, b_i)$ $(i = 1, 2)$ are the solutions to the problem (1.11)–(1.26) corresponding to $\tilde{u}_i$ $(i = 1, 2)$. Define

$$
\begin{aligned}
&\rho := \rho_1 - \rho_2, u := u_1 - u_2, w := w_1 - w_2, \theta := \theta_1 - \theta_2, \\
&E := E_1 - E_2, b := b_1 - b_2, \tilde{u} := \tilde{u}_1 - \tilde{u}_2, \tilde{w} := \tilde{w}_1 - \tilde{w}_2.
\end{aligned}
$$

Then, we have

$$\rho_t + \mathrm{div}\,(\rho\tilde{u}_1) = -\mathrm{div}\,(\rho_2\tilde{u}), \tag{2.26}$$

$$\epsilon\partial_t E - \operatorname{rot} b + E + \tilde{u}\times b_1 + \tilde{u}_2 \times b = 0, \tag{2.27}$$

$$\partial_t b + \operatorname{rot} E = 0, \operatorname{div} b = 0, \tag{2.28}$$

$$\begin{aligned}&\rho_1\partial_t\theta + \rho_1\tilde{u}_1\cdot\nabla\theta - \Delta\theta + \rho_1\theta_1\operatorname{div}\tilde{u}_1 - \rho_2\theta_2\operatorname{div}\tilde{u}_2 + \rho\partial_t\theta_2 + (\rho_1\tilde{u}_1 - \rho_2\tilde{u}_2)\nabla\theta_2\\ &\quad = Q_1 - Q_2,\end{aligned} \tag{2.29}$$

$$\begin{aligned}&\rho_1\partial_t u + \rho_1\tilde{u}_1\cdot\nabla u + \nabla(\rho_1\theta_1 - \rho_2\theta_2) - (\mu+\mu_r)\Delta u - (\lambda+\mu-\mu_r)\nabla\operatorname{div} u\\ &\quad +\rho\partial_t u_2 + (\rho_1\tilde{u}_1 - \rho_2\tilde{u}_2)\nabla u_2\\ &\quad = 2\mu_r\operatorname{rot}\tilde{w} + (E_1 + u_1\times b_1)\times b_1 - (E_2 + u_2\times b_2)\times b_2,\end{aligned} \tag{2.30}$$

with

$$\begin{aligned}Q_i := &\ \frac{\mu}{2}(\nabla\tilde{u}_i + \nabla\tilde{u}_i^T):(\nabla\tilde{u}_i + \nabla\tilde{u}_i^T) + \lambda(\operatorname{div}\tilde{u}_i)^2 + 4\mu_r\left|\frac{1}{2}\operatorname{rot}\tilde{u}_i - \tilde{w}_i\right|^2 + c_0(\operatorname{div}\tilde{w}_i)^2\\ &+(c_a+c_d)\nabla\tilde{w}_i:\nabla\tilde{w}_i + (c_d - c_a)\nabla\tilde{w}_i:\nabla\tilde{w}_i^T + |E_i + \tilde{u}_i\times b_i|^2\ \ (i=1,2)\end{aligned}$$

$$\begin{aligned}&\rho_1\partial_t w + \rho_1\tilde{u}_1\cdot\nabla w - (c_a+c_d)\Delta w - (c_0 + c_d - c_a)\nabla\operatorname{div} w + 4\mu_r w\\ = &\ 2\mu_r\operatorname{rot} u - \rho\partial_t w_2 - (\rho_1\tilde{u}_1 - \rho_2\tilde{u}_2)\cdot\nabla w_2.\end{aligned} \tag{2.31}$$

Testing (2.26) by ρ, we see that

$$\begin{aligned}\frac{d}{dt}\int\rho^2dx &\le C\|\nabla\tilde{u}_1\|_{L^\infty}\int\rho^2dx + C(\|\rho_2\|_{L^\infty}\|\nabla\tilde{u}\|_{L^2} + \|\tilde{u}\|_{L^6}\|\nabla\rho_2\|_{L^3}\|\rho\|_{L^2}\\ &\le C\|\tilde{u}_1\|_{W^{2,6}}\int\rho^2dx + C\|\nabla\tilde{u}\|_{L^2}\|\rho\|_{L^2}\\ &\le \eta_1\|\nabla\tilde{u}\|_{L^2}^2 + C(1+\|\tilde{u}_1\|_{W^{2,6}})\|\rho\|_{L^2}^2\end{aligned} \tag{2.32}$$

for any $0 < \eta_1 < 1$.

Testing (2.27) and (2.28) by E and b and summing up the result, we find that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int(\epsilon E^2 + b^2)dx + \int E^2dx\\ = &\int(b_1\times\tilde{u} + b\times\tilde{u}_2)Edx\\ \le &\ \frac{1}{4}\int E^2dx + C\|b_1\|_{L^\infty}^2\|\tilde{u}\|_{L^2}^2 + C\|\tilde{u}_2\|_{L^\infty}^2\|b\|_{L^2}^2\\ \le &\ \frac{1}{4}\int E^2dx + C\|\tilde{u}\|_{L^2}^2 + C\|b\|_{L^2}^2.\end{aligned} \tag{2.33}$$

Testing (2.29) by θ and using $\partial_t\rho_1 + \operatorname{div}(\rho_1\tilde{u}_1) = 0$, we infer that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\rho_1\theta^2dx + \int|\nabla\theta|^2dx\\ = &\int[\rho\theta_1\operatorname{div}\tilde{u}_1 + \rho_2\theta\operatorname{div}\tilde{u}_1 + \rho_2\theta_2\operatorname{div}\tilde{u} + \rho\partial_t\theta_2 + (\rho\tilde{u}_1 + \rho_2\tilde{u})\nabla\theta_2]\theta dx + \int(Q_1 - Q_2)\theta dx\\ \le &\ \|\rho\|_{L^2}\|\theta_1\|_{L^\infty}\|\operatorname{div}\tilde{u}_1\|_{L^\infty}\|\theta\|_{L^2} + \|\rho_2\|_{L^\infty}\|\operatorname{div}\tilde{u}_1\|_{L^\infty}\|\theta\|_{L^2}^2\end{aligned}$$

$$
\begin{aligned}
&+\|\rho_2\|_{L^\infty}\|\theta_2\|_{L^\infty}\|\mathrm{div}\,\tilde u\|_{L^2}\|\theta\|_{L^2}+\|\rho\|_{L^2}\|\partial_t\theta_2\|_{L^3}\|\theta\|_{L^6}\\
&+\|\rho\|_{L^2}\|\tilde u_1\|_{L^\infty}\|\nabla\theta_2\|_{L^6}\|\theta\|_{L^3}+\|\rho_2\|_{L^\infty}\|\tilde u\|_{L^2}\|\nabla\theta_2\|_{L^6}\|\theta\|_{L^3}\\
&+C(\|\tilde w_1\|_{W^{1,6}}+\|\tilde w_2\|_{W^{1,6}})\|\nabla\tilde w\|_{L^2}\|\theta\|_{L^3}\\
&+C(\|\nabla\tilde u_1\|_{L^6}+\|\nabla\tilde u_2\|_{L^6})\|\nabla\tilde u\|_{L^2}\|\theta\|_{L^3}+C(\|E_1\|_{L^6}+\|E_2\|_{L^6})\|E\|_{L^2}\|\theta\|_{L^3}\\
&+C(\|\tilde u_1\|_{L^\infty}\|b_1\|_{L^\infty}+\|\tilde u_2\|_{L^\infty}\|b_2\|_{L^\infty})(\|\tilde u\|_{L^2}\|b_1\|_{L^\infty}+\|\tilde u_2\|_{L^\infty}\|b\|_{L^2})\|\theta\|_{L^2}\\
\le\ & C\|\mathrm{div}\,\tilde u_1\|_{L^\infty}(\|\rho\|_{L^2}^2+\|\theta\|_{L^2}^2)+\eta_1\|\nabla\tilde u\|_{L^2}^2+\eta_1\|\nabla\tilde w\|_{L^2}^2+C\|\theta\|_{L^2}^2+\frac{1}{24}\|\nabla\theta\|_{L^2}^2\\
&+C\|\partial_t\theta_2\|_{L^3}^2\|\rho\|_{L^2}^2+C\|\rho\|_{L^2}^2+\eta_1\|\tilde u\|_{L^2}^2+\eta_2\|E\|_{L^2}^2+\eta_1\|b\|_{L^2}^2 \qquad (2.34)
\end{aligned}
$$

for any $0<\eta_1,\eta_2<1$.

Testing (2.30) by u and using $\partial_t\rho_1+\mathrm{div}\,(\rho_1\tilde u_1)=0$, we deduce that

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int\rho_1|u|^2dx+\int[(\mu+\mu_r)|\nabla u|^2+(\lambda+\mu-\mu_r)(\mathrm{div}\,u)^2]dx+\int(u\times b_1)^2dx\\
=\ & -\int[\rho\partial_t u_2+(\rho\tilde u_1+\rho_2\tilde u)\cdot\nabla u_2]udx+\int(\rho_1\theta_1-\rho_2\theta_2)\mathrm{div}\,udx+2\mu_r\int\mathrm{rot}\,\tilde w\cdot udx\\
&+\int[(E_1+u_1\times b_1)\times b_1-(E_2+u_2\times b_2)\times b_2]udx\\
\le\ & \|\rho\|_{L^2}\|\partial_t u_2\|_{L^3}\|u\|_{L^6}+(\|\rho\|_{L^2}\|\tilde u_1\|_{L^\infty}+\|\rho_2\|_{L^\infty}\|\tilde u\|_{L^2})\|\nabla u_2\|_{L^6}\|u\|_{L^3}\\
&+(\|\rho\|_{L^2}\|\theta_1\|_{L^\infty}+\|\rho_2\|_{L^\infty}\|\theta\|_{L^2})\|\mathrm{div}\,u\|_{L^2}+C\|\mathrm{rot}\,\tilde w\|_{L^2}\|u\|_{L^2}\\
&+\|E\|_{L^2}\|b_1\|_{L^\infty}\|u\|_{L^2}+\|b\|_{L^2}\|E_2\|_{L^6}\|u\|_{L^3}\\
&+(\|u_2\|_{L^\infty}\|b\|_{L^2}\|b_1\|_{L^\infty}+\|u_2\|_{L^\infty}\|b_2\|_{L^\infty}\|b\|_{L^2})\|u\|_{L^2}\\
\le\ & \frac{\mu}{16}\|\nabla u\|_{L^2}^2+C\|\partial_t u_2\|_{L^3}^2\|\rho\|_{L^2}^2+C(\|\rho\|_{L^2}+\|\tilde u\|_{L^2})\|u\|_{L^3}\\
&+C\|\rho\|_{L^2}^2+C\|\theta\|_{L^2}^2+C\|E\|_{L^2}\|u\|_{L^2}+C\|b\|_{L^2}\|u\|_{L^3}+C\|b\|_{L^2}\|u\|_{L^2}+C\|\mathrm{rot}\,\tilde w\|_{L^2}\|u\|_{L^2}\\
\le\ & \frac{\mu}{8}\|\nabla u\|_{L^2}^2+C\|\partial_t u_2\|_{L^3}^2\|\rho\|_{L^2}^2+\eta_1\|\tilde u\|_{L^2}^2+C\|u\|_{L^2}^2+C\|\rho\|_{L^2}^2\\
&+C\|\theta\|_{L^2}^2+C\eta_2\|E\|_{L^2}^2+\eta_1\|b\|_{L^2}^2+\eta_1\|\mathrm{rot}\,\tilde w\|_{L^2}^2 \qquad (2.35)
\end{aligned}
$$

for any $0<\eta_1,\eta_2<1$.

Testing (2.31) by w and using $\partial_t\rho_1+\mathrm{div}\,(\rho_1\tilde u_1)=0$, we compute

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int\rho_1|w|^2dx+(c_a+c_d)\int|\nabla w|^2dx+(c_0+c_d-c_a)\int(\mathrm{div}\,w)^2dx+4\mu_r\int|w|^2dx\\
=\ & 2\mu_r\int w\mathrm{rot}\,udx-\int[\rho\partial_t w_2+(\rho\tilde u_1+\rho_2\tilde u)\cdot\nabla w_2]wdx\\
\le\ & C\|w\|_{L^2}^2+\frac{\mu}{8}\|\nabla u\|_{L^2}^2+\|\rho\|_{L^2}\|\partial_t w_2\|_{L^3}\|w\|_{L^6}\\
&+(\|\rho\|_{L^2}\|\tilde u_1\|_{L^\infty}+\|\rho_2\|_{L^\infty}\|\tilde u\|_{L^2})\|\nabla w_2\|_{L^6}\|w\|_{L^3}\\
\le\ & C\|w\|_{L^2}^2+\frac{\mu}{8}\|\nabla u\|_{L^2}^2+\frac{c_a+c_d}{8}\|\nabla w\|_{L^2}^2+C\|\partial_t w_2\|_{L^3}^2\|\rho\|_{L^2}^2\\
&+C\|\rho\|_{L^2}^2+\eta_1\|\tilde u\|_{L^2}^2. \qquad (2.36)
\end{aligned}
$$

Taking (2.32)+$\eta_1\times$(2.33)+(2.34)+(2.35)+(2.36), taking $\eta_2<<\eta_1$ and using the Gronwall inequality, we arrive at (2.25) for small $0<T\le1$.

This completes the proof.

□

Proof of Theorem 1.1.

By Lemmas 2.1–2.6 and the Banach fixed point theorem, we finish the proof.

□

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Conflict of interest

The authors declare no conflict of interest.

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