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Research article

Theoretical and numerical analysis of solutions of some systems of nonlinear difference equations

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Abstract: In this paper, we obtain the form of the solutions of the following rational systems of difference equations

$$x_{n+1}=\frac{y_{n-1}z_n}{z_n\pm x_{n-2}},\ y_{n+1}=\frac{z_{n-1}x_n}{x_n\pm y_{n-2}},\ z_{n+1}=\frac{x_{n-1}y_n}{y_n\pm z_{n-2}},$$

with initial values are non-zero real numbers.

Keywords: difference equations; periodic solutions; stability; recursive sequences; system of difference equations
Mathematics Subject Classification: 39A10

1. Introduction

This paper is devoted to study the expressions forms of the solutions and periodic nature of the following third-order rational systems of difference equations

$$x_{n+1}=\frac{y_{n-1}z_n}{z_n\pm x_{n-2}},\ y_{n+1}=\frac{z_{n-1}x_n}{x_n\pm y_{n-2}},\ z_{n+1}=\frac{x_{n-1}y_n}{y_n\pm z_{n-2}},$$

with initial conditions are non-zero real numbers.

In the recent years, there has been great concern in studying the systems of difference equations. One of the most important reasons for this is a exigency for some mechanization which can be used in discussing equations emerge in mathematical models characterizing real life situations in economic, genetics, probability theory, psychology, population biology and so on.

Difference equations display naturally as discrete peer and as numerical solutions of differential equations having more applications in ecology, biology, physics, economy, and so forth. For all that the difference equations are quite simple in expressions, it is frequently difficult to realize completely the dynamics of their solutions see [1–19] and the related references therein.

There are some papers dealed with the difference equations systems, for example, The periodic nature of the solutions of the nonlinear difference equations system

$$A_{n+1} = \frac{1}{C_n},\ B_{n+1} = \frac{B_n}{A_{n-1}B_{n-1}},\ C_{n+1} = \frac{1}{A_{n-1}},$$

has been studied by Cinar in [7].

Almatrafi [3] determined the analytical solutions of the following systems of rational recursive equations

$$x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}(\pm 1 \pm x_{n-1}y_{n-3})},\ y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}(\pm 1 \pm y_{n-1}x_{n-3})}.$$

In [20], Khaliq and Shoaib studied the local and global asymptotic behavior of non-negative equilibrium points of a three-dimensional system of two order rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{\varepsilon + x_{n-1}y_{n-1}z_{n-1}},\ y_{n+1} = \frac{y_{n-1}}{\zeta + x_{n-1}y_{n-1}z_{n-1}},\ z_{n+1} = \frac{z_{n-1}}{\eta + x_{n-1}y_{n-1}z_{n-1}}.$$

In [9], Elabbasy et al. obtained the form of the solutions of some cases of the following system of difference equations

$$\begin{aligned} x_{n+1} &= \frac{a_1 + a_2 y_n}{a_3 z_n + a_4 x_{n-1} z_n},\ y_{n+1} = \frac{b_1 z_{n-1} + b_2 z_n}{b_3 x_n y_n + b_4 x_n y_{n-1}}, \\ z_{n+1} &= \frac{c_1 z_{n-1} + c_2 z_n}{c_3 x_{n-1} y_{n-1} + c_4 x_{n-1} y_n + c_5 x_n y_n}. \end{aligned}$$

In [12], Elsayed et al. have got the solutions of the systems of rational higher order difference equations

$$A_{n+1} = \frac{1}{A_{n-p}B_{n-p}},\quad B_{n+1} = \frac{A_{n-p}B_{n-p}}{A_{n-q}B_{n-q}},$$

and

$$A_{n+1} = \frac{1}{A_{n-p}B_{n-p}C_{n-p}},\quad B_{n+1} = \frac{A_{n-p}B_{n-p}C_{n-p}}{A_{n-q}B_{n-q}C_{n-q}},\ C_{n+1} = \frac{A_{n-q}B_{n-q}C_{n-q}}{A_{n-r}B_{n-r}C_{n-r}}.$$

Kurbanli [25, 26] investigated the behavior of the solutions of the following systems

$$\begin{aligned} A_{n+1} &= \frac{A_{n-1}}{A_{n-1}B_n - 1},\ B_{n+1} = \frac{B_{n-1}}{B_{n-1}A_n - 1},\ \ C_{n+1} = \frac{1}{C_n B_n}, \\ A_{n+1} &= \frac{A_{n-1}}{A_{n-1}B_n - 1},\ B_{n+1} = \frac{B_{n-1}}{B_{n-1}A_n - 1},\ \ C_{n+1} = \frac{C_{n-1}}{C_{n-1}B_n - 1}. \end{aligned}$$

In [32], Yalçınkaya has obtained the conditions for the global asymptotically stable of the system

$$A_{n+1} = \frac{B_n A_{n-1} + a}{B_n + A_{n-1}},\ B_{n+1} = \frac{A_n B_{n-1} + a}{A_n + B_{n-1}}.$$

Zhang et al. [39] investigated the persistence, boundedness and the global asymptotically stable of the solutions of the following system

$$R_n = A + \frac{1}{Q_{n-p}}, \quad Q_n = A + \frac{Q_{n-1}}{R_{n-r}Q_{n-s}}.$$

Similar to difference equations and systems were studied see [21–24,27–38].

2. The system: $x_{n+1} = \frac{y_{n-1}z_n}{z_n + x_{n-2}},\ y_{n+1} = \frac{z_{n-1}x_n}{x_n + y_{n-2}},\ z_{n+1} = \frac{x_{n-1}y_n}{y_n + z_{n-2}}$

In this section, we obtain the expressions form of the solutions of the following three dimension system of difference equations

$$x_{n+1} = \frac{y_{n-1}z_n}{z_n + x_{n-2}},\ y_{n+1} = \frac{z_{n-1}x_n}{x_n + y_{n-2}},\ z_{n+1} = \frac{x_{n-1}y_n}{y_n + z_{n-2}}, \tag{1}$$

where $n \in \mathbb{N}_0$ and the initial conditions are non-zero real numbers.

Theorem 1. *We assume that $\{x_n, y_n, z_n\}$ are solutions of system (1). Then*

$$x_{6n-2} = \frac{ak^{3n}}{\prod_{i=0}^{n-1}(a+(6i)k)(a+(6i+2)k)(a+(6i+4)k)},$$

$$x_{6n-1} = \frac{bf^{3n}}{\prod_{i=0}^{n-1}(g+(6i+1)f)(g+(6i+3)f)(g+(6i+5)f)},$$

$$x_{6n} = \frac{c^{3n+1}}{\prod_{i=0}^{n-1}(d+(6i+2)c)(d+(6i+4)c)(d+(6i+6)c)},$$

$$x_{6n+1} = \frac{ek^{3n+1}}{(a+k)\prod_{i=0}^{n-1}(a+(6i+3)k)(a+(6i+5)k)(a+(6i+7)k)},$$

$$x_{6n+2} = \frac{f^{3n+2}}{(g+2f)\prod_{i=0}^{n-1}(g+(6i+4)f)(g+(6i+6)f)(g+(6i+8)f)},$$

$$x_{6n+3} = \frac{hc^{3n+2}}{(d+c)(d+3c)\prod_{i=0}^{n-1}(d+(6i+5)c)(d+(6i+7)c)(d+(6i+9)c)},$$

$$y_{6n-2} = \frac{dc^{3n}}{\prod_{i=0}^{n-1}(d+(6i)c)(d+(6i+2)c)(d+(6i+4)c)},$$

$$
\begin{aligned}
y_{6n-1} &= \frac{ek^{3n}}{\prod_{i=0}^{n-1}(a+(6i+1)k)(a+(6i+3)k)(a+(6i+5)k)}, \\
y_{6n} &= \frac{f^{3n+1}}{\prod_{i=0}^{n-1}(g+(6i+2)f)(g+(6i+4)f)(g+(6i+6)f)}, \\
y_{6n+1} &= \frac{hc^{3n+1}}{(d+c)\prod_{i=0}^{n-1}(d+(6i+3)c)(d+(6i+5)c)(d+(6i+7)c)}, \\
y_{6n+2} &= \frac{k^{3n+2}}{(a+2k)\prod_{i=0}^{n-1}(a+(6i+4)k)(a+(6i+6)k)(a+(6i+8)k)}, \\
y_{6n+3} &= \frac{bf^{3n+2}}{(g+f)(g+3f)\prod_{i=0}^{n-1}(g+(6i+5)f)(g+(6i+7)f)(g+(6i+9)f)},
\end{aligned}
$$

and

$$
\begin{aligned}
z_{6n-2} &= \frac{gf^{3n}}{\prod_{i=0}^{n-1}(g+(6i)f)(g+(6i+2)f)(g+(6i+4)f)}, \\
z_{6n-1} &= \frac{hc^{3n}}{\prod_{i=0}^{n-1}(d+(6i+1)c)(d+(6i+3)c)(d+(6i+5)c)}, \\
z_{6n} &= \frac{k^{3n+1}}{\prod_{i=0}^{n-1}(a+(6i+2)k)(a+(6i+4)k)(a+(6i+6)k)}, \\
z_{6n+1} &= \frac{bf^{3n+1}}{(g+f)\prod_{i=0}^{n-1}(g+(6i+3)f)(g+(6i+5)f)(g+(6i+7)f)},
\end{aligned}
$$

$$
\begin{aligned}
z_{6n+2} &= \frac{c^{3n+2}}{(d+2c)\prod_{i=0}^{n-1}(d+(6i+4)c)(d+(6i+6)c)(d+(6i+8)c)}, \\
z_{6n+3} &= \frac{ek^{3n+2}}{(a+k)(a+3k)\prod_{i=0}^{n-1}(a+(6i+5)k)(a+(6i+7)k)(a+(6i+9)k)},
\end{aligned}
$$

where $x_{-2} = a,\ x_{-1} = b, x_0 = c, y_{-2} = d, y_{-1} = e, y_0 = f, z_{-2} = g, z_{-1} = h$ *and* $z_0 = k$.

Proof. For $n = 0$ the result holds. Now assume that $n > 1$ and that our assumption holds for $n-1$, that is,

$$x_{6n-8} = \frac{ak^{3n-3}}{\prod_{i=0}^{n-2}(a+(6i)k)(a+(6i+2)k)(a+(6i+4)k)},$$
$$x_{6n-7} = \frac{bf^{3n-3}}{\prod_{i=0}^{n-2}(g+(6i+1)f)(g+(6i+3)f)(g+(6i+5)f)},$$
$$x_{6n-6} = \frac{c^{3n-2}}{\prod_{i=0}^{n-2}(d+(6i+2)c)(d+(6i+4)c)(d+(6i+6)c)},$$
$$x_{6n-5} = \frac{ek^{3n-2}}{(a+k)\prod_{i=0}^{n-2}(a+(6i+3)k)(a+(6i+5)k)(a+(6i+7)k)},$$
$$x_{6n-4} = \frac{f^{3n-1}}{(g+2f)\prod_{i=0}^{n-2}(g+(6i+4)f)(g+(6i+6)f)(g+(6i+8)f)},$$
$$x_{6n-3} = \frac{hc^{3n-1}}{(d+c)(d+3c)\prod_{i=0}^{n-2}(d+(6i+5)c)(d+(6i+7)c)(d+(6i+9)c)},$$

$$y_{6n-8} = \frac{dc^{3n-3}}{\prod_{i=0}^{n-2}(d+(6i)c)(d+(6i+2)c)(d+(6i+4)c)},$$
$$y_{6n-7} = \frac{ek^{3n-3}}{\prod_{i=0}^{n-2}(a+(6i+1)k)(a+(6i+3)k)(a+(6i+5)k)},$$
$$y_{6n-6} = \frac{f^{3n-2}}{\prod_{i=0}^{n-2}(g+(6i+2)f)(g+(6i+4)f)(g+(6i+6)f)},$$

$$y_{6n-5} = \frac{hc^{3n-2}}{(d+c)\prod_{i=0}^{n-2}(d+(6i+3)c)(d+(6i+5)c)(d+(6i+7)c)},$$
$$y_{6n-4} = \frac{k^{3n-1}}{(a+2k)\prod_{i=0}^{n-2}(a+(6i+4)k)(a+(6i+6)k)(a+(6i+8)k)},$$
$$y_{6n-3} = \frac{bf^{3n-1}}{(g+f)(g+3f)\prod_{i=0}^{n-2}(g+(6i+5)f)(g+(6i+7)f)(g+(6i+9)f)},$$

and

$$z_{6n-8} = \frac{gf^{3n-3}}{\prod_{i=0}^{n-2}(g+(6i)f)(g+(6i+2)f)(g+(6i+4)f)},$$

$$z_{6n-7} = \frac{hc^{3n-3}}{\prod_{i=0}^{n-2}(d+(6i+1)c)(d+(6i+3)c)(d+(6i+5)c)},$$

$$z_{6n-6} = \frac{k^{3n-2}}{\prod_{i=0}^{n-2}(a+(6i+2)k)(a+(6i+4)k)(a+(6i+6)k)},$$

$$z_{6n-5} = \frac{bf^{3n-2}}{(g+f)\prod_{i=0}^{n-2}(g+(6i+3)f)(g+(6i+5)f)(g+(6i+7)f)},$$

$$z_{6n-4} = \frac{c^{3n-1}}{(d+2c)\prod_{i=0}^{n-2}(d+(6i+4)c)(d+(6i+6)c)(d+(6i+8)c)},$$

$$z_{6n-3} = \frac{ek^{3n-1}}{(a+k)(a+3k)\prod_{i=0}^{n-2}(a+(6i+5)k)(a+(6i+7)k)(a+(6i+9)k)}.$$

It follows from Eq (1) that

$$\begin{aligned}
x_{6n-2} &= \frac{y_{6n-4}z_{6n-3}}{z_{6n-3}+x_{6n-5}} \\
&= \frac{\left(\frac{k^{3n-1}}{(a+2k)\prod_{i=0}^{n-2}(a+(6i+4)k)(a+(6i+6)k)(a+(6i+8)k)}\right)\left(\frac{ek^{3n-1}}{(a+k)(a+3k)\prod_{i=0}^{n-2}(a+(6i+5)k)(a+(6i+7)k)(a+(6i+9)k)}\right)}{\left(\frac{ek^{3n-1}}{(a+k)(a+3k)\prod_{i=0}^{n-2}(a+(6i+5)k)(a+(6i+7)k)(a+(6i+9)k)}\right)+\left(\frac{ek^{3n-2}}{(a+k)\prod_{i=0}^{n-2}(a+(6i+3)k)(a+(6i+5)k)(a+(6i+7)k)}\right)} \\
&= \frac{\left(\frac{k^{3n}}{(a+2k)\prod_{i=0}^{n-2}(a+(6i+4)k)(a+(6i+6)k)(a+(6i+8)k)}\right)}{(a+3k)\prod_{i=0}^{n-2}(a+(6i+9)k)\left[\left(\frac{k}{(a+3k)\prod_{i=0}^{n-2}(a+(6i+9)k)}\right)+\left(\frac{1}{\prod_{i=0}^{n-2}(a+(6i+3)k)}\right)\right]} \\
&= \frac{\left(\frac{k^{3n}}{(a+2k)\prod_{i=0}^{n-2}(a+(6i+4)k)(a+(6i+6)k)(a+(6i+8)k)}\right)}{\left[k+\left(\frac{(a+3k)\prod_{i=0}^{n-2}(a+(6i+9)k)}{\prod_{i=0}^{n-2}(a+(6i+3)k)}\right)\right]} \\
&= \frac{\left(\frac{k^{3n}}{(a+2k)\prod_{i=0}^{n-2}(a+(6i+4)k)(a+(6i+6)k)(a+(6i+8)k)}\right)}{[k+(a+(6n-3)k)]}
\end{aligned}$$

$$= \frac{ak^{3n}}{a(a+2k)\,(a+(6n-2)k)\prod_{i=0}^{n-2}(a+(6i+4)k)(a+(6i+6)k)(a+(6i+8)k)}.$$

Then we see that

$$x_{6n-2} = \frac{k^{3n}}{\prod_{i=0}^{n-1}(a+(6i)k)(a+(6i+2)k)(a+(6i+4)k)}.$$

Also, we see from Eq (1) that

$$\begin{aligned}
y_{6n-2} &= \frac{z_{6n-4}x_{6n-3}}{x_{6n-3}+y_{6n-5}} \\
&= \frac{\left(\frac{c^{3n-1}}{(d+2c)\prod_{i=0}^{n-2}(d+(6i+4)c)(d+(6i+6)c)(d+(6i+8)c)}\right)\left(\frac{hc^{3n-1}}{(d+c)(d+3c)\prod_{i=0}^{n-2}(d+(6i+5)c)(d+(6i+7)c)(d+(6i+9)c)}\right)}{\left(\frac{hc^{3n-1}}{(d+c)(d+3c)\prod_{i=0}^{n-2}(d+(6i+5)c)(d+(6i+7)c)(d+(6i+9)c)}\right)+\left(\frac{hc^{3n-2}}{(d+c)\prod_{i=0}^{n-2}(d+(6i+3)c)(d+(6i+5)c)(d+(6i+7)c)}\right)} \\
&= \frac{\left(\frac{c^{3n}}{(d+2c)\prod_{i=0}^{n-2}(d+(6i+4)c)(d+(6i+6)c)(d+(6i+8)c)}\right)}{(d+3c)\prod_{i=0}^{n-2}(d+(6i+9)c)\left[\left(\frac{c}{(d+3c)\prod_{i=0}^{n-2}(d+(6i+9)c)}\right)+\left(\frac{1}{\prod_{i=0}^{n-2}(d+(6i+3)c)}\right)\right]} \\
&= \frac{\left(\frac{c^{3n}}{(d+2c)\prod_{i=0}^{n-2}(d+(6i+4)c)(d+(6i+6)c)(d+(6i+8)c)}\right)}{[c+d+(6n-3)c]} \\
&= \frac{c^{3n}}{[d+(6n-2)c]\,(d+2c)\prod_{i=0}^{n-2}(d+(6i+4)c)(d+(6i+6)c)(d+(6i+8)c)}.
\end{aligned}$$

Then

$$y_{6n-2} = \frac{dc^{3n}}{\prod_{i=0}^{n-1}(d+(6i)c)(d+(6i+2)c)(d+(6i+4)c)}.$$

Finally from Eq (1), we see that

$$\begin{aligned}
z_{6n-2} &= \frac{x_{6n-4}y_{6n-3}}{y_{6n-3}+z_{6n-5}} \\
&= \frac{\left(\frac{f^{3n-1}}{(g+2f)\prod_{i=0}^{n-2}(g+(6i+4)f)(g+(6i+6)f)(g+(6i+8)f)}\right)\left(\frac{bf^{3n-1}}{(g+f)(g+3f)\prod_{i=0}^{n-2}(g+(6i+5)f)(g+(6i+7)f)(g+(6i+9)f)}\right)}{\left(\frac{bf^{3n-1}}{(g+f)(g+3f)\prod_{i=0}^{n-2}(g+(6i+5)f)(g+(6i+7)f)(g+(6i+9)f)}\right)+\left(\frac{bf^{3n-2}}{(g+f)\prod_{i=0}^{n-2}(g+(6i+3)f)(g+(6i+5)f)(g+(6i+7)f)}\right)}
\end{aligned}$$

$$= \frac{\left(\frac{f^{3n}}{(g+2f)\prod_{i=0}^{n-2}(g+(6i+4)f)(g+(6i+6)f)(g+(6i+8)f)}\right)}{(g+3f)\prod_{i=0}^{n-2}(g+(6i+9)f)\left[\left(\frac{f}{(g+3f)\prod_{i=0}^{n-2}(g+(6i+9)f)}\right)+\left(\frac{1}{\prod_{i=0}^{n-2}(g+(6i+3)f)}\right)\right]}$$

$$= \frac{\left(\frac{f^{3n}}{(g+2f)\prod_{i=0}^{n-2}(g+(6i+4)f)(g+(6i+6)f)(g+(6i+8)f)}\right)}{\left[f+\left(\frac{(g+3f)\prod_{i=0}^{n-2}(g+(6i+9)f)}{\prod_{i=0}^{n-2}(g+(6i+3)f)}\right)\right]}$$

$$= \frac{\left(\frac{f^{3n}}{(g+2f)\prod_{i=0}^{n-2}(g+(6i+4)f)(g+(6i+6)f)(g+(6i+8)f)}\right)}{[f+(g+(6n-3)f)]}$$

$$= \frac{f^{3n}}{(g+(6n-2)f)(g+2f)\prod_{i=0}^{n-2}(g+(6i+4)f)(g+(6i+6)f)(g+(6i+8)f)}.$$

Thus

$$z_{3n-2} = \frac{gf^{3n}}{\prod_{i=0}^{n-1}(g+(6i)f)(g+(6i+2)f)(g+(6i+4)f)}.$$

By similar way, one can show the other relations. This completes the proof.

Lemma 1. Let $\{x_n, y_n, z_n\}$ be a positive solution of system (1), then all solution of (1) is bounded and approaching to zero.

Proof. It follows from Eq (1) that

$$\begin{aligned} x_{n+1} &= \frac{y_{n-1}z_n}{z_n + x_{n-2}} \leq y_{n-1}, \quad y_{n+1} = \frac{z_{n-1}x_n}{x_n + y_{n-2}} \leq z_{n-1}, \\ z_{n+1} &= \frac{x_{n-1}y_n}{y_n + z_{n-2}} \leq x_{n-1}, \end{aligned}$$

we see that

$$\begin{aligned} x_{n+4} &\leq y_{n+2}, \quad y_{n+2} \leq z_n, \quad z_n \leq x_{n-2}, \quad \Rightarrow x_{n+4} < x_{n-2}, \\ y_{n+4} &\leq z_{n+2}, \quad z_{n+2} \leq x_n, \quad x_n \leq y_{n-2}, \quad \Rightarrow y_{n+4} < y_{n-2}, \\ z_{n+4} &\leq x_{n+2}, \quad x_{n+2} \leq y_n, \quad y_n \leq z_{n-2}, \quad \Rightarrow z_{n+4} < z_{n-2}, \end{aligned}$$

Then all subsequences of $\{x_n, y_n, z_n\}$ (i.e., for $\{x_n\}$ are $\{x_{6n-2}\}$, $\{x_{6n-1}\}$, $\{x_{6n}\}$, $\{x_{6n+1}\}$, $\{x_{6n+2}\}$, $\{x_{6n+3}\}$ are decreasing and at that time are bounded from above by K, L and M since $K = \max\{x_{-2}, x_{-1}, x_0, x_1, x_2, x_3\}$, $L = \max\{y_{-2}, y_{-1}, y_0, y_1, y_2, y_3\}$ and $M = \max\{z_{-2}, z_{-1}, z_0, z_1, z_2, z_3\}$.

Example 1. We assume an interesting numerical example for the system (1) with $x_{-2} = -.22,\ x_{-1} = -.4,\ x_0 = .12,\ y_{-2} = -.62, y_{-1} = 4,\ y_0 = .3,\ z_{-2} = .4,\ z_{-1} = .53$ *and* $z_0 = -2$. (See Figure 1).

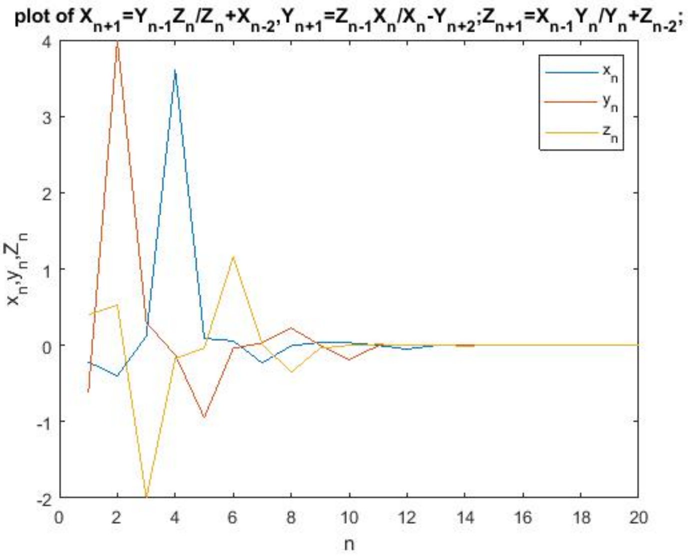

Figure 1. This figure shows the behavior of the solutions of the system (1) with the initial conditions $x_{-2} = -.22,\ x_{-1} = -.4,\ x_0 = .12,\ y_{-2} = -.62, y_{-1} = 4,\ y_0 = .3,\ z_{-2} = .4,\ z_{-1} = .53$ *and* $z_0 = -2$. (We see from this figure that all solutions converges to zero).

3. The system: $x_{n+1} = \frac{y_{n-1}z_n}{z_n + x_{n-2}},\ y_{n+1} = \frac{z_{n-1}x_n}{x_n + y_{n-2}},\ z_{n+1} = \frac{x_{n-1}y_n}{y_n - z_{n-2}}$

In this section, we get the solution's form of the following system of difference equations

$$x_{n+1} = \frac{y_{n-1}z_n}{z_n + x_{n-2}},\ y_{n+1} = \frac{z_{n-1}x_n}{x_n + y_{n-2}},\ z_{n+1} = \frac{x_{n-1}y_n}{y_n - z_{n-2}}, \tag{2}$$

where $n \in \mathbb{N}_0$ and the initial values are non-zero real numbers with $x_{-2} \neq \pm z_0, \neq -2z_0,\ z_{-2} \neq y_0, \neq 2y_0, \neq 3y_0$ and $y_{-2} \neq 2x_0, \neq \pm x_0$.

Theorem 2. *Assume that* $\{x_n, y_n, z_n\}$ *are solutions of (2). Then for* $n = 0, 1, 2, ...,$

$$\begin{aligned} x_{6n-2} &= \frac{(-1)^n k^{3n}}{a^{2n-1}(a+2k)^n},\ x_{6n-1} = \frac{(-1)^n b f^{3n}}{(f-g)^{2n}(3f-g)^n},\ x_{6n} = \frac{(-1)^n c^{3n+1}}{d^{2n}(2c-d)^n}, \\ x_{6n+1} &= \frac{e k^{3n+1}}{(a-k)^n(a+k)^{2n+1}},\ x_{6n+2} = \frac{(-1)^n f^{3n+2}}{g^n(2f-g)^{2n+1}},\ x_{6n+3} = \frac{(-1)^n h c^{3n+2}}{(c-d)^{2n+1}(c+d)^{n+1}}, \end{aligned}$$

$$\begin{aligned} y_{6n-2} &= \frac{(-1)^n c^{3n}}{d^{2n-1}(2c-d)^n},\ y_{6n-1} = \frac{e k^{3n}}{(a-k)^n(a+k)^{2n}},\ y_{6n} = \frac{(-1)^n f^{3n+1}}{g^n(2f-g)^{2n}}, \\ y_{6n+1} &= \frac{(-1)^n h c^{3n+1}}{(c-d)^{2n}(c+d)^{n+1}},\ y_{6n+2} = \frac{(-1)^n k^{3n+2}}{a^{2n}(a+2k)^{n+1}},\ y_{6n+3} = \frac{(-1)^n b f^{3n+2}}{(f-g)^{2n+1}(3f-g)^{n+1}}, \end{aligned}$$

and

$$z_{6n-2} = \frac{(-1)^n f^{3n}}{g^{n-1}(2f-g)^{2n}},\ z_{6n-1} = \frac{(-1)^n h c^{3n}}{(c-d)^{2n}(c+d)^n},\ z_{6n} = \frac{(-1)^n k^{3n+1}}{a^{2n}(a+2k)^n},$$

$$z_{6n+1} = \frac{(-1)^n b f^{3n+1}}{(f-g)^{2n+1}(3f-g)^n},\ z_{6n+2} = \frac{(-1)^{n+1} c^{3n+2}}{d^{2n+1}(2c-d)^n},\ z_{6n+3} = \frac{-ek^{3n+2}}{(a-k)^n(a+k)^{2n+2}},$$

where $x_{-2} = a,\ x_{-1} = b,\ x_0 = c,\ y_{-2} = d,\ y_{-1} = e,\ y_0 = f,\ z_{-2} = g,\ z_{-1} = h$ *and* $z_0 = k$.

Proof. The result is true for $n = 0$. Now suppose that $n > 0$ and that our claim verified for $n-1$. That is,

$$x_{6n-8} = \frac{(-1)^{n-1} k^{3n-3}}{a^{2n-3}(a+2k)^{n-1}},\ x_{6n-7} = \frac{(-1)^{n-1} b f^{3n-3}}{(f-g)^{2n-2}(3f-g)^{n-1}},\ x_{6n-6} = \frac{(-1)^{n-1} c^{3n-2}}{d^{2n-2}(2c-d)^{n-1}},$$
$$x_{6n-5} = \frac{ek^{3n-2}}{(a-k)^{n-1}(a+k)^{2n-1}},\ x_{6n-4} = \frac{(-1)^{n-1} f^{3n-1}}{g^{n-1}(2f-g)^{2n-1}},\ x_{6n-3} = \frac{(-1)^{n-1} hc^{3n-1}}{(c-d)^{2n-1}(c+d)^n},$$

$$y_{6n-8} = \frac{(-1)^{n-1} c^{3n-3}}{d^{2n-3}(2c-d)^{n-1}},\ y_{6n-7} = \frac{ek^{3n-3}}{(a-k)^{n-1}(a+k)^{2n-2}},\ y_{6n-6} = \frac{(-1)^{n-1} f^{3n-2}}{g^{n-1}(2f-g)^{2n-2}},$$
$$y_{6n-5} = \frac{(-1)^{n-1} hc^{3n-2}}{(c-d)^{2n-2}(c+d)^n},\ y_{6n-4} = \frac{(-1)^{n-1} k^{3n-1}}{a^{2n-2}(a+2k)^n},\ y_{6n-3} = \frac{(-1)^{n-1} b f^{3n-1}}{(f-g)^{2n-1}(3f-g)^n},$$

and

$$z_{6n-8} = \frac{(-1)^{n-1} f^{3n-3}}{g^{n-2}(2f-g)^{2n-2}},\ z_{6n-7} = \frac{(-1)^{n-1} hc^{3n-3}}{(c-d)^{2n-2}(c+d)^{n-1}},\ z_{6n-6} = \frac{(-1)^{n-1} k^{3n-2}}{a^{2n-2}(a+2k)^{n-1}},$$
$$z_{6n-5} = \frac{(-1)^{n-1} b f^{3n-2}}{(f-g)^{2n-1}(3f-g)^{n-1}},\ z_{6n-4} = \frac{(-1)^n c^{3n-1}}{d^{2n-1}(2c-d)^{n-1}},\ z_{6n-3} = \frac{-ek^{3n-1}}{(a-k)^{n-1}(a+k)^{2n}}.$$

Now from Eq (2), it follows that

$$\begin{aligned}
x_{6n-2} &= \frac{y_{6n-4} z_{6n-3}}{z_{6n-3} + x_{6n-5}} \\
&= \frac{\left(\frac{(-1)^{n-1} k^{3n-1}}{a^{2n-2}(a+2k)^n}\right)\left(\frac{-ek^{3n-1}}{(a-k)^{n-1}(a+k)^{2n}}\right)}{\left(\frac{-ek^{3n-1}}{(a-k)^{n-1}(a+k)^{2n}}\right) + \left(\frac{ek^{3n-2}}{(a-k)^{n-1}(a+k)^{2n-1}}\right)} \\
&= \frac{\left(\frac{(-1)^n k^{3n}}{a^{2n-2}(a+2k)^n}\right)}{(-k+a+k)} = \frac{(-1)^n k^{3n}}{a^{2n-1}(a+2k)^n}, \\
y_{6n-2} &= \frac{z_{6n-4} x_{6n-3}}{x_{6n-3} + y_{6n-5}} = \frac{\left(\frac{(-1)^n c^{3n-1}}{d^{2n-1}(2c-d)^{n-1}}\right)\left(\frac{(-1)^{n-1} hc^{3n-1}}{(c-d)^{2n-1}(c+d)^n}\right)}{\left(\frac{(-1)^{n-1} hc^{3n-1}}{(c-d)^{2n-1}(c+d)^n}\right) + \left(\frac{(-1)^{n-1} hc^{3n-2}}{(c-d)^{2n-2}(c+d)^n}\right)} \\
&= \frac{\left(\frac{(-1)^n c^{3n}}{d^{2n-1}(2c-d)^{n-1}}\right)}{c+c-d} = \frac{(-1)^n c^{3n}}{d^{2n-1}(2c-d)^n}, \\
z_{6n-2} &= \frac{x_{6n-4} y_{6n-3}}{y_{6n-3} - z_{6n-5}} = \frac{\left(\frac{(-1)^{n-1} f^{3n-1}}{g^{n-1}(2f-g)^{2n-1}}\right)\left(\frac{(-1)^{n-1} b f^{3n-1}}{(f-g)^{2n-1}(3f-g)^n}\right)}{\left(\frac{(-1)^{n-1} b f^{3n-1}}{(f-g)^{2n-1}(3f-g)^n}\right) - \left(\frac{(-1)^{n-1} b f^{3n-2}}{(f-g)^{2n-1}(3f-g)^{n-1}}\right)} \\
&= \frac{\left(\frac{(-1)^{n-1} f^{3n}}{g^{n-1}(2f-g)^{2n-1}}\right)}{(f-3f+g)} = \frac{(-1)^n f^{3n}}{g^{n-1}(2f-g)^{2n}}.
\end{aligned}$$

Also, we see from Eq (2) that

$$
\begin{aligned}
x_{6n-1} &= \frac{y_{6n-3}z_{6n-2}}{z_{6n-2}+x_{6n-4}} \\
&= \frac{\left(\frac{(-1)^{n-1}bf^{3n-1}}{(f-g)^{2n-1}(3f-g)^n}\right)\left(\frac{(-1)^n f^{3n}}{g^{n-1}(2f-g)^{2n}}\right)}{\left(\frac{(-1)^n f^{3n}}{g^{n-1}(2f-g)^{2n}}\right)+\left(\frac{(-1)^{n-1} f^{3n-1}}{g^{n-1}(2f-g)^{2n-1}}\right)} \\
&= \frac{\left(\frac{(-1)^n bf^{3n}}{(f-g)^{2n-1}(3f-g)^n}\right)}{(-f+2f-g)} = \frac{(-1)^n bf^{3n}}{(f-g)^{2n}(3f-g)^n}, \\
y_{6n-1} &= \frac{z_{6n-3}x_{6n-2}}{x_{6n-2}+y_{6n-4}} = \frac{\left(\frac{-ek^{3n-1}}{(a-k)^{n-1}(a+k)^{2n}}\right)\left(\frac{(-1)^n k^{3n}}{a^{2n-1}(a+2k)^n}\right)}{\left(\frac{(-1)^n k^{3n}}{a^{2n-1}(a+2k)^n}\right)+\left(\frac{(-1)^{n-1}k^{3n-1}}{a^{2n-2}(a+2k)^n}\right)} \\
&= \frac{\left(\frac{ek^{3n}}{(a-k)^{n-1}(a+k)^{2n}}\right)}{-k+a} = \frac{ek^{3n}}{(a-k)^n(a+k)^{2n}}, \\
z_{6n-1} &= \frac{x_{6n-3}y_{6n-2}}{y_{6n-2}-z_{6n-4}} = \frac{\left(\frac{(-1)^{n-1}hc^{3n-1}}{(c-d)^{2n-1}(c+d)^n}\right)\left(\frac{(-1)^n c^{3n}}{d^{2n-1}(2c-d)^n}\right)}{\left(\frac{(-1)^n c^{3n}}{d^{2n-1}(2c-d)^n}\right)-\left(\frac{(-1)^n c^{3n-1}}{d^{2n-1}(2c-d)^{n-1}}\right)} \\
&= \frac{\left(\frac{(-1)^{n-1}hc^{3n}}{(c-d)^{2n-1}(c+d)^n}\right)}{c-(2c-d)} = \frac{(-1)^n hc^{3n}}{(c-d)^{2n}(c+d)^n}.
\end{aligned}
$$

Also, we can prove the other relations.

Example 2. See below Figure 2 for system (2) with the initial conditions $x_{-2} = 11,\ x_{-1} = 5,\ x_0 = 13,\ y_{-2} = 6, y_{-1} = 7,\ y_0 = 3,\ z_{-2} = 14, z_{-1} = 9$ *and* $z_0 = 2$.

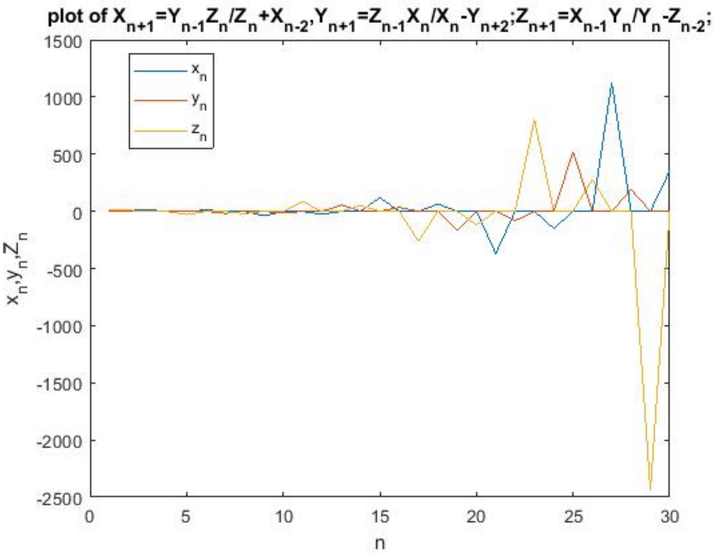

Figure 2. This figure shows the behavior of solutions of the systems of rational recursive sequence $x_{n+1} = \dfrac{y_{n-1}z_n}{z_n + x_{n-2}},\ y_{n+1} = \dfrac{z_{n-1}x_n}{x_n + y_{n-2}},\ z_{n+1} = \dfrac{x_{n-1}y_n}{y_n - z_{n-2}}$, when we take the initial conditions: $x_{-2} = 11,\ x_{-1} = 5,\ x_0 = 13,\ y_{-2} = 6, y_{-1} = 7,\ y_0 = 3,\ z_{-2} = 14, z_{-1} = 9$ *and* $z_0 = 2$. (See the figure we can conclude that all the solutions unboundedness solutions).

4. The system: $x_{n+1} = \frac{y_{n-1}z_n}{z_n + x_{n-2}},\ y_{n+1} = \frac{z_{n-1}x_n}{x_n - y_{n-2}},\ z_{n+1} = \frac{x_{n-1}y_n}{y_n + z_{n-2}}$

Here, we obtain the form of solutions of the system

$$x_{n+1} = \frac{y_{n-1}z_n}{z_n + x_{n-2}},\ y_{n+1} = \frac{z_{n-1}x_n}{x_n - y_{n-2}},\ z_{n+1} = \frac{x_{n-1}y_n}{y_n + z_{n-2}}, \tag{3}$$

where $n \in \mathbb{N}_0$ and the initial values are non-zero real numbers with $x_{-2} \neq \pm z_0, \neq 2z_0$, $z_{-2} \neq \pm y_0, \neq -2y_0$ and $y_{-2} \neq x_0, \neq 2x_0, \neq 3x_0$.

Theorem 3. *If* $\{x_n, y_n, z_n\}$ *are solutions of system (3) where* $x_{-2} = a,\ x_{-1} = b,\ x_0 = c,\ y_{-2} = d,\ y_{-1} = e,\ y_0 = f,\ z_{-2} = g,\ z_{-1} = h$ *and* $z_0 = k$. *Then for* $n = 0, 1, 2, ...,$

$$\begin{aligned} x_{6n-2} &= \frac{k^{3n}}{a^{2n-1}(a-2k)^n},\ x_{6n-1} = \frac{(-1)^n b f^{3n}}{(f-g)^n(f+g)^{2n}},\ x_{6n} = \frac{(-1)^n c^{3n+1}}{d^n(d-2c)^{2n}}, \\ x_{6n+1} &= \frac{(-1)^n e k^{3n+1}}{(a-k)^{2n}(a+k)^{n+1}},\ x_{6n+2} = \frac{(-1)^n f^{3n+2}}{g^{2n}(2f+g)^{n+1}},\ x_{6n+3} = \frac{(-1)^n h c^{3n+2}}{(c-d)^{2n+1}(3c-d)^{n+1}}, \end{aligned}$$

$$\begin{aligned} y_{6n-2} &= \frac{(-1)^n c^{3n}}{d^{n-1}(d-2c)^{2n}},\ y_{6n-1} = \frac{(-1)^n e k^{3n}}{(a-k)^{2n}(a+k)^n},\ y_{6n} = \frac{(-1)^n f^{3n+1}}{g^{2n}(2f+g)^n}, \\ y_{6n+1} &= \frac{(-1)^n h c^{3n+1}}{(c-d)^{2n+1}(3c-d)^n},\ y_{6n+2} = \frac{-k^{3n+2}}{a^{2n+1}(a-2k)^n},\ y_{6n+3} = \frac{(-1)^n b f^{3n+2}}{(f-g)^n(f+g)^{2n+2}}, \end{aligned}$$

and

$$\begin{aligned} z_{6n-2} &= \frac{(-1)^n f^{3n}}{g^{2n-1}(2f+g)^n},\ z_{6n-1} = \frac{(-1)^n h c^{3n}}{(c-d)^{2n}(3c-d)^n},\ z_{6n} = \frac{k^{3n+1}}{a^{2n}(a-2k)^n}, \\ z_{6n+1} &= \frac{(-1)^n b f^{3n+1}}{(f-g)^n(f+g)^{2n+1}},\ z_{6n+2} = \frac{(-1)^n c^{3n+2}}{d^n(2c-d)^{2n+1}},\ z_{6n+3} = \frac{(-1)^{n+1} e k^{3n+2}}{(a-k)^{2n+1}(a+k)^{n+1}}. \end{aligned}$$

Proof. As the proof of Theorem 2 and so will be left to the reader.

Example 3. We put the initials $x_{-2} = 8,\ x_{-1} = 15,\ x_0 = 13,\ y_{-2} = 6,\ y_{-1} = 7,\ y_0 = 3,\ z_{-2} = 14,\ z_{-1} = 19$ *and* $z_0 = 2$, for the system (3), see Figure 3.

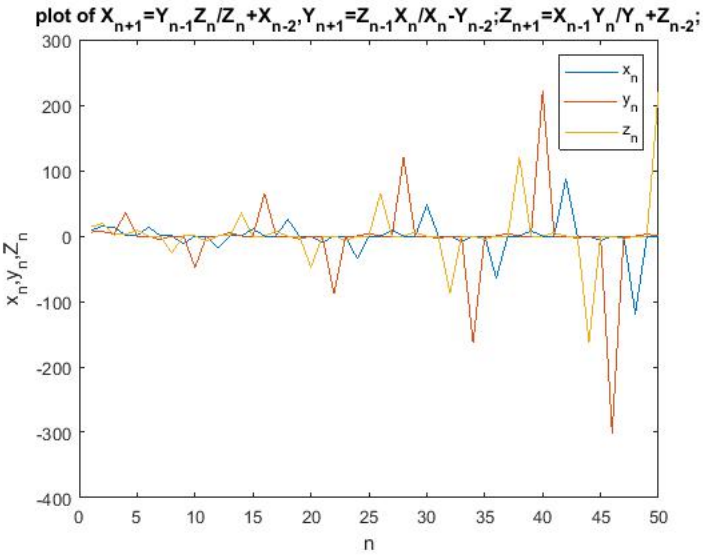

Figure 3. This figure shows the unstable of the solutions of the difference equations system (3) with the initial values $x_{-2} = 8,\ x_{-1} = 15,\ x_0 = 13,\ y_{-2} = 6,\ y_{-1} = 7,\ y_0 = 3,\ z_{-2} = 14,\ z_{-1} = 19$ *and* $z_0 = 2$.

The following systems can be treated similarly.

5. The system: $x_{n+1} = \frac{y_{n-1}z_n}{z_n - x_{n-2}},\ y_{n+1} = \frac{z_{n-1}x_n}{x_n + y_{n-2}},\ z_{n+1} = \frac{x_{n-1}y_n}{y_n + z_{n-2}}$

In this section, we deal with the solutions of the following system

$$x_{n+1} = \frac{y_{n-1}z_n}{z_n - x_{n-2}},\ y_{n+1} = \frac{z_{n-1}x_n}{x_n + y_{n-2}},\ z_{n+1} = \frac{x_{n-1}y_n}{y_n + z_{n-2}}, \tag{4}$$

where $n \in \mathbb{N}_0$ and the initial values are non-zero real with $x_{-2} \neq z_0, \neq 2z_0, \neq 3z_0,\ z_{-2} \neq \pm y_0, \neq 2y_0$ and $y_{-2} \neq \pm x_0, \neq -2x_0$.

Theorem 4. *The solutions of system (4) are given by*

$$\begin{aligned} x_{6n-2} &= \frac{(-1)^n k^{3n}}{a^{n-1}(a-2k)^{2n}},\ x_{6n-1} = \frac{(-1)^n bf^{3n}}{(f-g)^{2n}(f+g)^n},\ x_{6n} = \frac{(-1)^n c^{3n+1}}{d^{2n}(d+2c)^n}, \\ x_{6n+1} &= \frac{-ek^{3n+1}}{(a-k)^{2n+1}(a-3k)^n},\ x_{6n+2} = \frac{(-1)^{n+1} f^{3n+2}}{g^{2n+1}(2f-g)^n},\ x_{6n+3} = \frac{(-1)^{n+1} hc^{3n+2}}{(c-d)^n(c+d)^{2n+2}}, \end{aligned}$$

$$\begin{aligned} y_{6n-2} &= \frac{(-1)^n c^{3n}}{d^{2n-1}(d+2c)^n},\ y_{6n-1} = \frac{ek^{3n}}{(a-k)^{2n}(a-3k)^n},\ y_{6n} = \frac{(-1)^n f^{3n+1}}{g^{2n}(2f-g)^n}, \\ y_{6n+1} &= \frac{(-1)^n hc^{3n+1}}{(c+d)^{2n+1}(c-d)^n},\ y_{6n+2} = \frac{-k^{3n+2}}{a^n(a-2k)^{2n+1}},\ y_{6n+3} = \frac{(-1)^n bf^{3n+2}}{(f-g)^{2n+1}(f+g)^{n+1}}, \end{aligned}$$

and

$$z_{6n-2} = \frac{(-1)^n f^{3n}}{g^{2n-1}(2f-g)^n},\ z_{6n-1} = \frac{(-1)^n hc^{3n}}{(c+d)^{2n}(c-d)^n},\ z_{6n} = \frac{(-1)^n k^{3n+1}}{a^n(a-2k)^{2n}},$$

$$z_{6n+1} = \frac{(-1)^n b f^{3n+1}}{(f-g)^{2n}(f+g)^{n+1}},\ z_{6n+2} = \frac{(-1)^n c^{3n+2}}{d^{2n}(2c+d)^{n+1}},\ z_{6n+3} = \frac{ek^{3n+2}}{(a-k)^{2n+1}(a-3k)^{n+1}},$$

where $x_{-2} = a,\ x_{-1} = b,\ x_0 = c,\ y_{-2} = d,\ y_{-1} = e,\ y_0 = f,\ z_{-2} = g,\ z_{-1} = h$ *and* $z_0 = k$.

Example 4. Figure 4 shows the behavior of the solution of system (4) with $x_{-2} = 18,\ x_{-1} = -15,\ x_0 = 3,\ y_{-2} = 6,\ y_{-1} = .7,\ y_0 = -3,\ z_{-2} = 4,\ z_{-1} = -9$ *and* $z_0 = 5$.

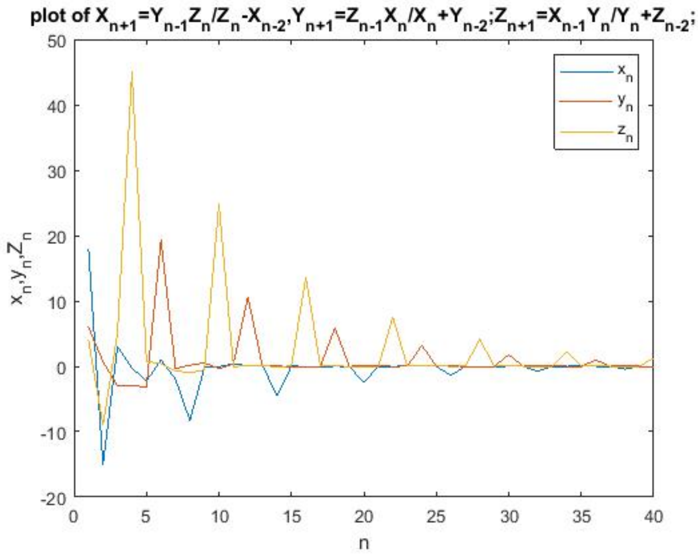

Figure 4. This figure shows the behavior of the system $x_{n+1} = \frac{y_{n-1}z_n}{z_n - x_{n-2}},\ y_{n+1} = \frac{z_{n-1}x_n}{x_n + y_{n-2}},\ z_{n+1} = \frac{x_{n-1}y_n}{y_n + z_{n-2}}$ with the initial conditions:- $x_{-2} = 18,\ x_{-1} = -15,\ x_0 = 3,\ y_{-2} = 6,\ y_{-1} = .7,\ y_0 = -3,\ z_{-2} = 4,\ z_{-1} = -9$ *and* $z_0 = 5. -0.6,\ x_{-1} = 0.2,\ x_0 = -5.$ (From the figure, we see that all solutions goes to zero).

6. The system: $x_{n+1} = \frac{y_{n-1}z_n}{z_n - x_{n-2}},\ y_{n+1} = \frac{z_{n-1}x_n}{x_n - y_{n-2}},\ z_{n+1} = \frac{x_{n-1}y_n}{y_n - z_{n-2}}$

In this section, we obtain the solutions of the difference system

$$x_{n+1} = \frac{y_{n-1}z_n}{z_n - x_{n-2}},\ y_{n+1} = \frac{z_{n-1}x_n}{x_n - y_{n-2}},\ z_{n+1} = \frac{x_{n-1}y_n}{y_n - z_{n-2}}, \tag{5}$$

where the initials are arbitrary non-zero real numbers with $x_{-2} \neq z_0$, $z_{-2} \neq y_0$ and $y_{-2} \neq x_0$.

Theorem 5. *If* $\{x_n, y_n, z_n\}$ *are solutions of system (5) where* $x_{-2} = a,\ x_{-1} = b,\ x_0 = c,\ y_{-2} = d,\ y_{-1} = e,\ y_0 = f,\ z_{-2} = g,\ z_{-1} = h$ *and* $z_0 = k$. *Then*

$$x_{6n-2} = \frac{k^{3n}}{a^{3n-1}},\ x_{6n-1} = \frac{bf^{3n}}{(f-g)^{3n}},\ x_{6n} = \frac{c^{3n+1}}{d^{3n}},$$
$$x_{6n+1} = \frac{ek^{3n+1}}{(k-a)^{3n+1}},\ x_{6n+2} = \frac{f^{3n+2}}{g^{3n+1}},\ x_{6n+3} = \frac{hc^{3n+2}}{(c-d)^{3n+2}},$$

$$y_{6n-2} = \frac{c^{3n}}{d^{3n-1}},\ y_{6n-1} = \frac{ek^{3n}}{(k-a)^{3n}},\ y_{6n} = \frac{f^{3n+1}}{g^{3n}},$$
$$y_{6n+1} = \frac{hc^{3n+1}}{(c-d)^{3n+1}},\ y_{6n+2} = \frac{k^{3n+2}}{a^{3n+1}},\ y_{6n+3} = \frac{bf^{3n+2}}{(f-g)^{3n+2}},$$

and

$$z_{6n-2} = \frac{f^{3n}}{g^{3n-1}},\ z_{6n-1} = \frac{hc^{3n}}{(c-d)^{3n}},\ z_{6n} = \frac{k^{3n+1}}{a^{3n}},$$
$$z_{6n+1} = \frac{bf^{3n+1}}{(f-g)^{3n+1}},\ z_{6n+2} = \frac{c^{3n+2}}{d^{3n+1}},\ z_{6n+3} = \frac{ek^{3n+2}}{(k-a)^{3n+2}}.$$

Example 5. Figure 5 shows the dynamics of the solution of system (5) with $x_{-2} = 18,\ x_{-1} = -15,\ x_0 = 3, y_{-2} = 6, y_{-1} = .7, y_0 = -3, z_{-2} = 4, z_{-1} = -9$ *and* $z_0 = 5$.

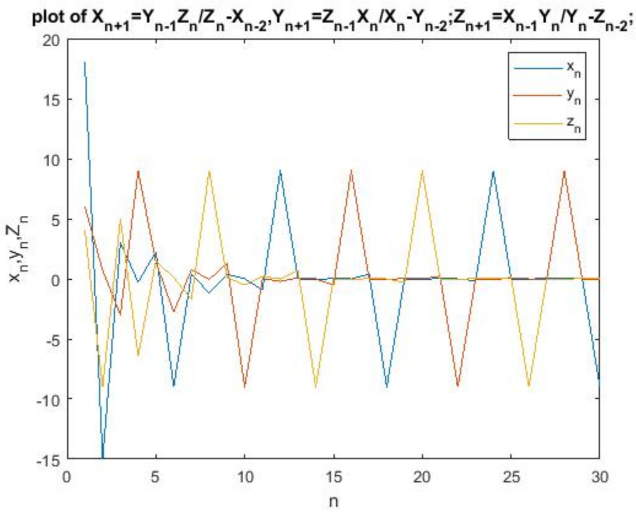

Figure 5. This figure shows the behavior of the system of nonlinear difference equations (5) with the initial conditions considered as follows:- $x_{-2} = 18,\ x_{-1} = -15,\ x_0 = 3,\ y_{-2} = 6,\ y_{-1} = .7,\ y_0 = -3,\ z_{-2} = 4,\ z_{-1} = -9$ *and* $z_0 = 5$.

7. Conclusions

This paper discussed the expression's form and boundedness of some systems of rational third order difference equations. In Section 2, we studied the qualitative behavior of system $x_{n+1} = \frac{y_{n-1}z_n}{z_n + x_{n-2}},\ y_{n+1} = \frac{z_{n-1}x_n}{x_n + y_{n-2}},\ z_{n+1} = \frac{x_{n-1}y_n}{y_n + z_{n-2}}$, first we have got the form of the solutions of this system, studied the boundedness and gave numerical example and drew it by using Matlab. In Section 3, we have got the solution's of the system $x_{n+1} = \frac{y_{n-1}z_n}{z_n + x_{n-2}},\ y_{n+1} = \frac{z_{n-1}x_n}{x_n + y_{n-2}},\ z_{n+1} = \frac{x_{n-1}y_n}{y_n - z_{n-2}}$, and take a numerical example. In Sections 4–6, we obtained the solution of the following systems respectively, $x_{n+1} = \frac{y_{n-1}z_n}{z_n + x_{n-2}},\ y_{n+1} = \frac{z_{n-1}x_n}{x_n - y_{n-2}},\ z_{n+1} = \frac{x_{n-1}y_n}{y_n + z_{n-2}},\ x_{n+1} = \frac{y_{n-1}z_n}{z_n - x_{n-2}},\ y_{n+1} = \frac{z_{n-1}x_n}{x_n + y_{n-2}},\ z_{n+1} = \frac{x_{n-1}y_n}{y_n + z_{n-2}}$, and $x_{n+1} = \frac{y_{n-1}z_n}{z_n - x_{n-2}},\ y_{n+1} = \frac{z_{n-1}x_n}{x_n - y_{n-2}},\ z_{n+1} = \frac{x_{n-1}y_n}{y_n - z_{n-2}}$. Also, in each case we take a numerical example to illustrates the results.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. R. Abo-Zeid, On the solutions of a fourth order difference equation, *Univ. J. Math. Appl.*, **4** (2021), 76–81. https://doi.org/10.1017/S0040298221000693
2. Y. Akrour, N. Touafek, Y. Halim, On a system of difference equations of third order solved in closed form, *J. Innov. Appl. Math. Comput. Sci.*, **1** (2021), 1–15. https://doi.org/10.48550/arXiv.1910.14365
3. M. B. Almatrafi, Analysis of solutions of some discrete systems of rational difference equations, *J. Comput. Anal. Appl.*, **29** (2021), 355–368.
4. A. M. Alotaibi, M. A. El-Moneam, On the dynamics of the nonlinear rational difference equation $x_{n+1} = \frac{\alpha x_{n-m}+\delta x_n}{\beta+\gamma x_{n-k}x_{n-l}(x_{n-k}+x_{n-l})}$, *AIMS Math.*, **7** (2022), 7374–7384. https://doi.org/10.3934/math.2022411
5. N. Battaloglu, C. Cinar, I. Yalçınkaya, The dynamics of the difference equation, *ARS Combinatoria*, **97** (2010), 281–288.
6. C. Cinar, I. Yalcinkaya, R. Karatas, On the positive solutions of the difference equation system $x_{n+1} = m/y_n,\ y_{n+1} = py_n/x_{n-1}y_{n-1}$, *J. Inst. Math. Comp. Sci.*, **18** (2005), 135–136.
7. C. Cinar, I. Yalçinkaya, On the positive solutions of the difference equation system $x_{n+1} = 1/z_n,\ y_{n+1} = y_n/x_{n-1}y_{n-1},\ z_{n+1} = 1/x_{n-1}$, *J. Inst. Math. Comp. Sci.*, **18** (2005), 91–93.
8. S. E. Das, M. Bayram, On a system of rational difference equations, *World Appl. Sci. J.*, **10** (2010), 1306–1312.
9. E. M. Elabbasy, H. El-Metwally, E. M. Elsayed, On the solutions of a class of difference equations systems, *Demonstr. Math.*, **41** (2008), 109–122. https://doi.org/10.1515/dema-2008-0111
10. E. M. Elsayed, Solution and attractivity for a rational recursive sequence, *Discrete Dyn. Nat. Soc.*, **2011** (2011). https://doi.org/10.1155/2011/982309
11. E. M. Elsayed, Solutions of rational difference system of order two, *Math. Comput. Model.*, **55** (2012), 378–384. https://doi.org/10.1016/j.mcm.2011.08.012
12. E. M. Elsayed, M. M. El-Dessoky, A. Alotaibi, On the solutions of a general system of difference equations, *Discrete Dyn. Nat. Soc.*, **2012** (2012). https://doi.org/10.1155/2012/892571
13. E. M. Elsayed, A. Alshareef, Qualitative behavior of a system of second order difference equations, *Eur. J. Math. Appl.*, **1** (2021), 1–11.

14. E. M. Elsayed, B. S. Alofi, A. Q. Khan, Qualitative behavior of solutions of tenth-order recursive sequence equation, *Math. Probl. Eng.*, **2022** (2022). https://doi.org/10.1155/2022/5242325

15. E. M. Elsayed, F. Alzahrani, Periodicity and solutions of some rational difference equations systems, *J. Appl. Anal. Comput.*, **9** (2019), 2358–2380. https://doi.org/10.11948/20190100

16. T. F. Ibrahim, A. Q. Khan, A. Ibrahim, Qualitative behavior of a nonlinear generalized recursive sequence with delay, *Math. Probl. Eng.*, **2021** (2021). https://doi.org/10.1155/2021/6162320

17. T. F. Ibrahim, A. Q. Khan, Forms of solutions for some two-dimensional systems of rational partial recursion equations, *Math. Probl. Eng.*, **2021** (2021). https://doi.org/10.1155/2021/9966197

18. M. Kara, Y. Yazlik, On a solvable three-dimensional system of difference equations, *Filomat*, **34** (2020), 1167–1186. https://doi.org/10.2298/FIL2004167K

19. K. Y. Liu, Z. J. Zhao, X. R. Li, P. Li, More on three-dimensional systems of rational difference equations, *Discrete Dyn. Nat. Soc.*, **2011** (2011). https://doi.org/10.1155/2011/178483

20. A. Khaliq, M. Shoaib, Dynamics of three-dimensional system of second order rational difference equations, *Electron. J. Math. Anal. Appl.*, **9** (2021), 308–319.

21. A. Khelifa, Y. Halim, M. Berkal, Solutions of a system of two higher-order difference equations in terms of Lucas sequence, *Univ. J. Math. Appl.*, **2** (2019), 202–211. https://doi.org/10.32323/ujma.610399

22. A. Khelifa, Y. Halim, Global behavior of P-dimensional difference equations system, *Electron. Res. Arch.*, **29** (2021), 3121–3139. https://doi.org/10.3934/era.2021029

23. A. S. Kurbanli, C. Cinar, I. Yalçınkaya, On the behavior of positive solutions of the system of rational difference equations, *Math. Comput. Model.*, **53** (2011), 1261–1267. https://doi.org/10.1016/j.mcm.2010.12.009

24. A. S. Kurbanli, On the behavior of solutions of the system of rational difference equations, *Adv. Differ. Equ.,* **2011** (2011), 40. https://doi.org/10.1186/1687-1847-2011-40

25. A. S. Kurbanli, On the behavior of solutions of the system of rational difference equations: $x_{n+1} = x_{n-1}/x_{n-1}y_n - 1,\ y_{n+1} = y_{n-1}/y_{n-1}x_n - 1,\ z_{n+1} = z_{n-1}/z_{n-1}y_n - 1$, *Discrete Dyn. Nat. Soc.*, **2011** (2011). https://doi.org/10.1186/1687-1847-2011-40

26. A. Kurbanli, C. Cinar, M. Erdoğan, On the behavior of solutions of the system of rational difference equations $x_{n+1} = \dfrac{x_{n-1}}{x_{n-1}y_n - 1},\ y_{n+1} = \dfrac{y_{n-1}}{y_{n-1}x_n - 1},\ z_{n+1} = \dfrac{x_n}{z_{n-1}y_n}$, *Appl. Math.*, **2** (2011), 1031–1038.

27. B. Oğul, D. Şimşek, On the recursive sequence $x_{n+1} = x_{n-14}/1 + x_{n-2}x_{n-5}x_{n-8}x_{n-11}$, *MANAS J. Eng.*, **8** (2020), 155–163. https://hdl.handle.net/20.500.13091/1672

28. A. Y. Ozban, On the system of rational difference equations $x_{n+1} = a/y_{n-3},\ y_{n+1} = by_{n-3}/x_{n-q}y_{n-q}$, *Appl. Math. Comput.*, **188** (2007), 833–837. https://doi.org/10.1016/j.amc.2006.10.034

29. D. Tollu, İ. Yalçınkaya, H. Ahmad, S. Yao, A detailed study on a solvable system related to the linear fractional difference equation, *Math. Biosci. Eng.*, **18** (2021), 5392–5408. https://doi.org/10.3934/mbe.2021273

30. N. Touafek, E. M. Elsayed, On the solutions of systems of rational difference equations, *Math. Comput. Model.*, **55** (2012), 1987–1997. https://doi.org/10.1016/j.mcm.2011.11.058

31. N. Touafek, D. Tollu, Y. Akrour, On a general homogeneous three-dimensional system of difference equations, *Electron. Res. Arch.*, **29** (2021), 2841–2876. https://doi.org/10.3934/era.2021017
32. I. Yalcinkaya, On the global asymptotic stability of a second-order system of difference equations, *Discrete Dyn. Nat. Soc.*, **2008** (2008). https://doi.org/10.1155/2008/860152
33. I. Yalçınkaya, On the global asymptotic behavior of a system of two nonlinear difference equations, *ARS Combinatoria*, **95** (2010), 151–159. https://doi.org/10.1016/j.ygeno.2009.12.003
34. I. Yalçınkaya, C. Cinar, D. Simsek, Global asymptotic stability of a system of difference equations, *Appl. Anal.*, **87** (2008), 689–699. https://doi.org/10.1097/PHM.0b013e31817e4b84
35. I. Yalcinkaya, C. Cinar, Global asymptotic stability of two nonlinear difference equations, *Fasciculi Math.*, **43** (2010), 171–180.
36. I. Yalcinkaya, C. Cinar, M. Atalay, On the solutions of systems of difference equations, *Adv. Differ. Equ.*, **2008** (2008). https://doi.org/10.1155/2008/143943
37. X. Yang, Y. Liu, S. Bai, On the system of high order rational difference equations $x_n = a/y_{n-p}$, $y_n = by_{n-p}/x_{n-q}y_{n-q}$, *Appl. Math. Comput.*, **171** (2005), 853–856. https://doi.org/10.1016/j.amc.2005.01.092
38. Y. Yazlik, D. T. Tollu, N. Taskara, On the behavior of solutions for some systems of difference equations, *J. Comput. Anal. Appl.*, **18** (2015), 166–178.
39. Y. Zhang, X. Yang, G. M. Megson, D. J. Evans, On the system of rational difference equations, *Appl. Math. Comput.*, **176** (2006), 403–408. https://doi.org/10.1016/j.amc.2005.09.039

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