



Research article

Single-step and multi-step methods for Caputo fractional-order differential equations with arbitrary kernels

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Abstract: We develop four numerical schemes to solve fractional differential equations involving the Caputo fractional derivative with arbitrary kernels. Firstly, we derive the four numerical schemes, namely, explicit product integration rectangular rule (forward Euler method), implicit product integration rectangular rule (backward Euler method), implicit product integration trapezoidal rule and Adam-type predictor-corrector method. In addition, the error estimation and stability for all four presented schemes are analyzed. To demonstrate the accuracy and effectiveness of the proposed methods, numerical examples are considered for various linear and nonlinear fractional differential equations with different kernels. The results show that these numerical schemes are feasible in application.

Keywords: fractional differential equations; numerical methods; product integration; single-step method; multi-step method

Mathematics Subject Classification: 26A33, 34A08, 65L05

1. Introduction

Fractional calculus has been as a mathematical theory of interest over three centuries. However, this theory was not initially applied to any real situation. The scientists and engineers in other fields commonly mentioned how the practical knowledge of fractional calculus has been used and how to operate it to the relevant studies. Fractional calculus and its application can be found in many fields such as physics [34, 35], neural networks [32], mechanics and dynamic systems [12, 38], biology [15, 40] and economics [37].

The effect of increasing attention on fractional order models has made the investigation and growth of numerical methods for nonlinear fractional integro-differential equations [5–11] and partial differential equations with time or space fractional derivatives [13, 31] become more active in the recent years. Moreover, the computational algorithm for the numerical solution of nonlinear fractional differential equations are found in [17–19, 23–25].

Researchers can obtain particular schemes for fractional differential equations based on the corresponding integral equation, for example, the product integration rules [39]. Nevertheless, the case of Adams predictor-corrector method [20] is different. It has been adjusted to create strong theoretical knowledge for numerical treatment of fractional differential equations. Some numerical schemes for fractional differential equations are developed directly from integral equations approaches, for example, product integration rules [39]. In addition, the Adams predictor-corrector approaches [20] is a completely different approach for the numerical solution of fractional differential equations.

Moreover, when the fractional order α converges to the nearest integer, the product integration rules and Adams predictor-corrector noticeably became the same methods. In fractional differential equations, the generalizations of the consistent approach for ordinary differential equations are considered as different methods.

The existence of different characteristics in approaches for fractional differential equations, which is established from the similar approach for ordinary differential equations, needs to be studied carefully. Particular distinctions are of theoretical concern, for instance, stability and convergence. However, these methods similarly indicate different types of computational nature, which have impact on the efficiency of the solution process. Therefore, our paper intends to explain various methods for fractional differential equations. A comparison of specific methods is also presented. The strengths and weaknesses will be investigated. Moreover, we compare the appropriateness among these methods.

Product integration rules and Adams predictor-corrector method are restricted to the analysis. These two approaches provide the least possible error constant [30], leading to a suitable balance between accuracy stability properties and computational complexity. This paper investigates when those strengths are rooted by the corresponding methods for fractional differential equations.

Additionally, we observe various definitions of fractional calculus [28, 29] including Riemann-Liouville, Caputo, Hilfer, Riesz, Erdelyi-Kober and Hadamard, among others. In particular, R. Almeida [2] suggests some generalizations of fractional operators of a function with arbitrary kernels involving a weight function Φ .

The Φ -Caputo fractional differential equations have gained more attention recently. Numerical schemes for solving the Φ -Caputo fractional differential equations are still under development. There are some research studies on the Φ -Caputo fractional derivative. For example, Almeida et al. [4] indicated that mathematical models based on Φ -Caputo fractional derivatives can be more adaptable. The Φ -Caputo fractional derivative has the ability to extract hidden parts of real-world situations. In 2019, Almeida et al. [3] demonstrated Φ -shifted Legendre polynomials to solve relaxation-oscillation equations with derivative of Φ -Caputo. In [16, 36], the monotone iteration of upper and lower solutions will be used to approximate the extremal solutions of Φ -Caputo fractional differential equations.

To the finest of our understanding, numerical schemes for nonlinear fractional differential equations

in the sense of Φ -Caputo derivative have never been investigated. On the basis of the above works on fractional derivative with arbitrary kernels, this paper investigates the numerical approaches for the solution of nonlinear fractional differential equations involving Φ -Caputo fractional derivative. In particular, the efficiency of numerical schemes, error and stability analysis are considered.

The paper is set as follows the next section, several preliminary knowledge of fractional derivatives and integrals are presented. The initial value problem involving Φ -Caputo fractional derivative is defined in Section 3. Moreover, we presented four numerical schemes, namely explicit product integration rectangular rule, implicit product integration rectangular rule, implicit product integration trapezoidal rule, and Adams predictor-corrector method to find numerical solutions of the fractional differential equations in sense of the Φ -Caputo fractional derivative. Next, the error estimation of the approximations and stability are obtained in Section 4. In Section 5, the simulation results including numerical convergence order are discussed for four test examples. The approximate solution and the error estimation for the test examples are presented through figures and tables, respectively. Further, a comparative study of these numerical schemes is performed.

2. Preliminaries

In this section, we will examine basic definitions and theorems, which will be used to declare and verify our essential results. Let Φ be a continuously differentiable function on $[t_0, T]$ such that $\Phi'(t) > 0$, for all $t \in [t_0, T]$.

Definition 2.1 ([14]). *The Gamma function is defined as*

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt, \quad \alpha > 0. \quad (2.1)$$

Definition 2.2 ([14]). *The Euler beta function is defined as*

$$B(\alpha, \beta) = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt, \quad \alpha, \beta > 0. \quad (2.2)$$

Definition 2.3 ([2]). *The Φ -Riemann-Liouville fractional integral of a real valued function y on $[t_0, T]$ is given by*

$$I_{t_0}^{\alpha, \Phi} y(t) := \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \Phi'(v) (\Phi(t) - \Phi(v))^{\alpha-1} y(v) dv, \quad \text{for } \alpha > 0. \quad (2.3)$$

Definition 2.4 ([2]). *The Φ -Riemann-Liouville fractional derivative of a real valued function y on $[t_0, T]$ is defined by*

$$D_{t_0}^{\alpha, \Phi} y(t) := \left(\frac{1}{\Phi'(t)} \frac{d}{dt} \right)^n I_{t_0}^{n-\alpha, \Phi} y(t), \quad (2.4)$$

where $n-1 < \alpha < n$ and $n \in \mathbb{N}$.

Definition 2.5 ([2]). *Let $n \in \mathbb{N}$, $\alpha \in (n-1, n)$ and $y, \Phi \in \mathcal{C}^n([t_0, T])$. The Φ -Caputo fractional derivative of y of order α is given by*

$${}^c D_{t_0}^{\alpha, \Phi} y(t) := I_{t_0}^{n-\alpha, \Phi} \left(\frac{1}{\Phi'(t)} \frac{d}{dt} \right)^n y(t). \quad (2.5)$$

Moreover, when $\alpha \in (0, 1)$, we have

$${}^C D_{t_0}^{\alpha, \Phi} y(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t (\Phi(t) - \Phi(v))^{-\alpha} y'(v) dv. \quad (2.6)$$

In particular, the Φ -Caputo fractional derivative is a generalization of the fractional derivatives as follows:

- the classical Caputo fractional derivative [29] when $\Phi(t) = t$,
- the Caputo-Erdélyi-Kober fractional derivative [33] when $\Phi(t) = t^\sigma$,
- the Caputo-Hadamard fractional derivative [21, 27] when $\Phi(t) = \ln(t)$.

Theorem 2.1 ([1]). If $y : [t_0, T] \rightarrow \mathbb{R}$ is a continuous function, then

$${}^C D_{t_0}^{\alpha, \Phi} I_{t_0}^{\alpha, \Phi} y(t) = y(t) - y(t_0).$$

3. Numerical schemes

In this work, we study the Φ -Caputo fractional derivative to differential equation as below:

$$\begin{cases} {}^C D_{t_0}^{\alpha, \Phi} y(t) = f(t, y(t)), & t \in [t_0, T], \\ y(t_0) = y_0, \end{cases} \quad (3.1)$$

where ${}^C D_{t_0}^{\alpha, \Phi}$ is the Φ -Caputo fractional derivative of order $\alpha \in (0, 1)$, $f : [t_0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a given continuous function and $y_0 \in \mathbb{R}$.

By Theorem 2.1, the solution of the problem (3.1) can be written in terms of the integral equation

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \Phi'(v) (\Phi(t) - \Phi(v))^{\alpha-1} f(v, y(v)) dv. \quad (3.2)$$

However, the Φ -Caputo fractional-order differential equation (3.1) is difficult to obtain the exact solution. In order to motivate the construction of our numerical methods, the concept of product integration rules [39] can be used to estimate the integral in (3.2) by the appropriate polynomials.

In order to numerically solve the integral equation (3.2), we consider the approximation solutions $y_n, n = 1, 2, \dots, N$. The uniform grid is divided as

$$0 \leq t_0 < t_1 < t_2 < \dots < t_{N-1} < t_N = T,$$

where $t_j = t_0 + jh$ and $h = \frac{T-t_0}{N}$ for $0 \leq j \leq N$.

Additionally, we assume the approximations $y_j \approx y(t_j)$ ($0 \leq j \leq n-1$) in the basic step. The piecewise decomposition of (3.2) is defined as

$$y_n = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \Phi'(v) (\Phi(t) - \Phi(v))^{\alpha-1} f(v, y(v)) dv, \quad t \geq t_n. \quad (3.3)$$

3.1. Explicit product integration rectangular rule (Ex. PI Rec.)

To obtain the approximation of $y_n \approx y(t_n)$, the function $f(\mathbf{v}, y(\mathbf{v}))$ in the integrand of (3.3) is approximated by the (explicit) forward Euler method, which is the constant values $f(t_j, y_j)$. The resulting methods is

$$\begin{aligned} y_n &= y_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \Phi'(\mathbf{v})(\Phi(t_n) - \Phi(\mathbf{v}))^{\alpha-1} f(t_j, y_j) d\mathbf{v} \\ &= y_0 + \frac{1}{\Gamma(\alpha+1)} \sum_{j=0}^{n-1} \left((\Phi(t_n) - \Phi(t_j))^{\alpha} - (\Phi(t_n) - \Phi(t_{j+1}))^{\alpha} \right) f(t_j, y_j) \\ &= y_0 + \frac{1}{\Gamma(\alpha+1)} \sum_{j=0}^{n-1} \left(w_{n,j}^{\alpha, \Phi} - w_{n,j+1}^{\alpha, \Phi} \right) f(t_j, y_j), \end{aligned} \quad (3.4)$$

where

$$w_{n,j}^{\alpha, \Phi} = (\Phi(t_n) - \Phi(t_j))^{\alpha}. \quad (3.5)$$

3.2. Implicit product integration rectangular rule (Im. PI Rec.)

We use a similar step of the explicit product integration rectangular rule to find the resulting method. However, the (implicit) backward Euler method is used to approximate, the function in the integrand of (3.3), which is

$$f(\mathbf{v}, y(\mathbf{v})) \approx f(t_{j+1}, y_{j+1}).$$

The resulting methods is

$$\begin{aligned} y_n &= y_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \Phi'(\mathbf{v})(\Phi(t_n) - \Phi(\mathbf{v}))^{\alpha-1} f(t_{j+1}, y_{j+1}) d\mathbf{v} \\ &= y_0 + \frac{1}{\Gamma(\alpha+1)} \sum_{j=1}^n \left((\Phi(t_n) - \Phi(t_{j-1}))^{\alpha} - (\Phi(t_n) - \Phi(t_j))^{\alpha} \right) f(t_j, y_j) \\ &= y_0 + \frac{1}{\Gamma(\alpha+1)} \sum_{j=1}^n \left(w_{n,j-1}^{\alpha, \Phi} - w_{n,j}^{\alpha, \Phi} \right) f(t_j, y_j), \end{aligned} \quad (3.6)$$

where $w_{n,j}^{\alpha, \Phi}$ is defined by (3.5).

3.3. Implicit product integration trapezoidal rule (Im. PI Trap.)

In this method, we replace the function in each subinterval of the integrand (3.3) by the first-degree polynomial interpolant

$$f(\mathbf{v}, y(\mathbf{v})) \approx f(t_{j+1}, y_{j+1}) + \frac{\mathbf{v} - t_{j+1}}{h} (f(t_{j+1}, y_{j+1}) - f(t_j, y_j)), \quad \mathbf{v} \in [t_j, t_{j+1}].$$

The implicit product integration trapezoidal rule is

$$\begin{aligned}
 y_n &= y_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \Phi'(v) (\Phi(t_n) - \Phi(v))^{\alpha-1} \left[f(t_{j+1}, y_{j+1}) \right. \\
 &\quad \left. + \left(\frac{v - t_{j+1}}{h} \right) \left(f(t_{j+1}, y_{j+1}) - f(t_j, y_j) \right) \right] dv \\
 &= y_0 + \frac{1}{\Gamma(\alpha + 1)} \left((\Phi(t_n) - \Phi(t_0))^\alpha - \frac{1}{h} I_{0,1}^{\alpha, \Phi} \right) f(t_0, y_0) \\
 &\quad + \sum_{j=1}^{n-1} \frac{1}{h\Gamma(\alpha + 1)} \left(I_{j-1,j}^{\alpha, \Phi} - I_{j,j+1}^{\alpha, \Phi} \right) f(t_j, y_j) \\
 &\quad + \frac{1}{h\Gamma(\alpha + 1)} \left(I_{n-1,n}^{\alpha, \Phi} \right) f(t_n, y_n) \\
 &= y_0 + \frac{1}{\Gamma(\alpha + 1)} u_{n,0}^{\alpha, \Phi} f(t_0, y_0) + \frac{1}{\Gamma(\alpha + 1)} \sum_{j=1}^n \left(u_{n,j}^{\alpha, \Phi} f(t_j, y_j) \right),
 \end{aligned} \tag{3.7}$$

with $I_{j,j+1}^{\alpha, \Phi} = \int_{t_j}^{t_{j+1}} (\Phi(t_n) - \Phi(v))^\alpha dv$, and

$$u_{n,j}^{\alpha, \Phi} = \begin{cases} w_{n,0}^{\alpha, \Phi} - \frac{1}{h} I_{0,1}^{\alpha, \Phi}, & \text{if } j = 0, \\ \frac{1}{h} \left(I_{j-1,j}^{\alpha, \Phi} - I_{j,j+1}^{\alpha, \Phi} \right), & \text{if } 1 \leq j \leq n-1, \\ \frac{1}{h} \left(I_{n-1,n}^{\alpha, \Phi} \right), & \text{if } j = n. \end{cases}$$

3.4. Adams predictor-corrector method (Adams PC)

According to (3.7), Newton's method is necessary for approximating y_n . However, we can predict the approximation y_n in (3.7) by using (3.4). The approximation in the predictor-step is called y_n^p . Furthermore, y_n^p can be used in (3.7). This step is called corrector-step. Overall, this numerical scheme is called Adams predictor-corrector method and is defined by

$$\begin{cases} y_n^p = y_0 + \frac{1}{\Gamma(\alpha + 1)} \sum_{j=0}^{n-1} \left(w_{n,j}^{\alpha, \Phi} - w_{n,j+1}^{\alpha, \Phi} \right) f(t_j, y_j), \\ y_n = y_0 + \frac{1}{\Gamma(\alpha + 1)} u_{n,0}^{\alpha, \Phi} f(t_0, y_0) + \frac{1}{\Gamma(\alpha + 1)} \sum_{j=1}^{n-1} \left(u_{n,j}^{\alpha, \Phi} f(t_j, y_j) \right) + \frac{1}{\Gamma(\alpha + 1)} u_{n,n}^{\alpha, \Phi} f(t_n, y_n^p), \end{cases} \tag{3.8}$$

where y_n^p is the resulting method at predictor-step and y_n is the resulting method at corrector-step.

In order to fulfill stability and error analysis, some lemmas are mentioned below.

Lemma 3.1. For $\alpha \in (0, 1)$ and $j = 1, \dots, n-1$, we have

$$w_{n,j}^{\alpha, \Phi} - w_{n,j+1}^{\alpha, \Phi} \leq C_{\alpha, \Phi} \left(w_{n,j}^{\alpha-1, \Phi} \cdot w_{j+1,j}^{1, \Phi} \right),$$

and

$$w_{n,j-1}^{\alpha,\Phi} - w_{n,j+1}^{\alpha,\Phi} \leq C_{\alpha,\Phi} \left(w_{n,j}^{\alpha-1,\Phi} \cdot w_{j+1,j}^{1,\Phi} \right).$$

Proof. Applying the mean value theorem, we can find $\xi \in (\Phi(t_n) - \Phi(t_{j+1}), \Phi(t_n) - \Phi(t_j))$ such that

$$\begin{aligned} w_{n,j}^{\alpha,\Phi} - w_{n,j+1}^{\alpha,\Phi} &= (\Phi(t_n) - \Phi(t_j))^\alpha - (\Phi(t_n) - \Phi(t_{j+1}))^\alpha \\ &= \alpha \xi^{\alpha-1} (\Phi(t_n) - \Phi(t_j) - \Phi(t_n) + \Phi(t_{j+1})) \\ &= \alpha \xi^{\alpha-1} (\Phi(t_{j+1}) - \Phi(t_j)) \\ &\leq \alpha (\Phi(t_n) - \Phi(t_{j+1}))^{\alpha-1} (\Phi(t_{j+1}) - \Phi(t_j)) \\ &= \alpha \frac{(\Phi(t_n) - \Phi(t_{j+1}))^{\alpha-1}}{(\Phi(t_n) - \Phi(t_j))^{\alpha-1}} (\Phi(t_n) - \Phi(t_j))^{\alpha-1} (\Phi(t_{j+1}) - \Phi(t_j)) \\ &= \alpha \left(\frac{\Phi(t_n) - \Phi(t_j)}{\Phi(t_n) - \Phi(t_{j+1})} \right)^{1-\alpha} (\Phi(t_n) - \Phi(t_j))^{\alpha-1} (\Phi(t_{j+1}) - \Phi(t_j)) \\ &= \alpha \left(1 + \frac{\Phi(t_{j+1}) - \Phi(t_j)}{\Phi(t_n) - \Phi(t_{j+1})} \right)^{1-\alpha} (\Phi(t_n) - \Phi(t_j))^{\alpha-1} (\Phi(t_{j+1}) - \Phi(t_j)) \\ &\leq C_{\alpha,\Phi} \left(w_{n,j}^{\alpha-1,\Phi} \cdot w_{j+1,j}^{1,\Phi} \right). \end{aligned}$$

In a similar way, the second inequality can be proved.

Lemma 3.2. If $\alpha \in (0, 1)$, $m \in \mathbb{N}$ and β be nonnegative, then

$$\sum_{j=1}^{m-1} (\Phi(t_m) - \Phi(t_j))^{\alpha-1} (\Phi(t_{j+1}) - \Phi(t_j)) (\Phi(t_j) - \Phi(t_0))^\beta \leq B(\alpha, \beta + 1) (\Phi(t_m) - \Phi(t_0))^{\alpha+\beta}.$$

Proof. Let $f(y) = (\Phi(t_m) - \Phi(t_0) - y)^{\alpha-1} y^\beta$ for $0 < y < \Phi(t_m) - \Phi(t_0)$.

Then, we have

$$\begin{aligned} f'(y) &= (1 - \alpha)y^\beta (\Phi(t_m) - \Phi(t_0) - y)^{\alpha-2} + \beta y^{\beta-1} (\Phi(t_m) - \Phi(t_0) - y)^{\alpha-1} \\ &= y^{\beta-1} (\Phi(t_m) - \Phi(t_0) - y)^{\alpha-2} [(1 - \alpha)y + \beta (\Phi(t_m) - \Phi(t_0) - y)] \\ &> 0. \end{aligned}$$

This implies that $f(y)$ is a monotone increasing function, and then

$$\begin{aligned} \sum_{j=1}^{m-1} (\Phi(t_m) - \Phi(t_j))^{\alpha-1} (\Phi(t_{j+1}) - \Phi(t_j)) (\Phi(t_j) - \Phi(t_0))^\beta \\ &= \sum_{j=1}^{m-1} f(\Phi(t_j) - \Phi(t_0)) (\Phi(t_{j+1}) - \Phi(t_j)) \\ &\leq \int_{t_0}^{\Phi(t_m) - \Phi(t_0)} f(y) dy \end{aligned}$$

$$\begin{aligned}
&= \int_{t_0}^{\Phi(t_m) - \Phi(t_0)} (\Phi(t_m) - \Phi(t_0) - y)^{\alpha-1} y^\beta dy \\
&= (\Phi(t_m) - \Phi(t_0))^{\alpha+\beta} \int_{t_0}^1 (1-v)^\beta v^{\alpha-1} dv \\
&= B(\alpha, \beta + 1) (\Phi(t_m) - \Phi(t_0))^{\alpha+\beta},
\end{aligned}$$

which completes the proof.

In order to verify the stability and error analysis of our numerical schemes, we present a modified Gronwall's inequality as follows.

Lemma 3.3. Suppose $\alpha \in (0, 1)$, $t_0 \in (0, T)$, and

$$\tilde{b}_{j,m}^{\alpha,\Phi} = \begin{cases} (\Phi(t_m) - \Phi(t_j))^{\alpha-1} (\Phi(t_{j+1}) - \Phi(t_j)), & j = 1, 2, \dots, m-1, \\ 0, & j \geq n. \end{cases}$$

Let $\sum_{j=n}^{j=m} \tilde{b}_{j,m}^{\alpha,\Phi} |e_j| = 0$ for $1 \leq n < m$. Then, we have

$$|e_n| \leq C |\eta_0|, \quad n = 1, 2, \dots,$$

if

$$|e_m| \leq M \sum_{j=1}^{m-1} \tilde{b}_{j,m}^{\alpha,\Phi} |e_j| + |\eta_0|, \quad m = 1, 2, \dots, n,$$

where M and C are positive constants.

Proof. Let $c_{j,m}^{\alpha,\Phi} = M \tilde{b}_{j,m}^{\alpha,\Phi}$ be such that

$$|e_m| \leq |\eta_0| + \sum_{j=1}^{m-1} c_{j,m}^{\alpha,\Phi} |e_j|, \quad m = 1, 2, \dots, n. \quad (3.9)$$

From the inequality (3.9), we have $|e_1| \leq |\eta_0|$ to obtain

$$\begin{aligned}
|e_n| &\leq |\eta_0| + \sum_{j_1=1}^{n-1} c_{j_1,n}^{\alpha,\Phi} |e_{j_1}| \\
&\leq |\eta_0| + \sum_{j_1=1}^{n-1} c_{j_1,n}^{\alpha,\Phi} \left(|\eta_0| + \sum_{j_2=1}^{j_1-1} c_{j_2,j_1}^{\alpha,\Phi} |e_{j_2}| \right) \\
&= |\eta_0| + |\eta_0| \sum_{j_1=1}^{n-1} c_{j_1,n}^{\alpha,\Phi} + \sum_{j_1=1}^{n-1} c_{j_1,n}^{\alpha,\Phi} \sum_{j_2=1}^{j_1-1} c_{j_2,j_1}^{\alpha,\Phi} |e_{j_2}| \\
&\leq |\eta_0| + |\eta_0| \sum_{j_1=1}^{n-1} c_{j_1,n}^{\alpha,\Phi} + \sum_{j_1=1}^{n-1} c_{j_1,n}^{\alpha,\Phi} \sum_{j_2=1}^{j_1-1} c_{j_2,j_1}^{\alpha,\Phi} \left(|\eta_0| + \sum_{j_3=1}^{j_2-1} c_{j_3,j_2}^{\alpha,\Phi} |e_{j_3}| \right)
\end{aligned}$$

$$\begin{aligned}
&= |\eta_0| + |\eta_0| \sum_{j_1=1}^{n-1} c_{j_1, n}^{\alpha, \Phi} + |\eta_0| \sum_{j_1=1}^{n-1} c_{j_1, n}^{\alpha, \Phi} \sum_{j_2=1}^{j_1-1} c_{j_2, j_1}^{\alpha, \Phi} + \sum_{j_1=1}^{n-1} c_{j_1, n}^{\alpha, \Phi} \sum_{j_2=1}^{j_1-1} c_{j_2, j_1}^{\alpha, \Phi} \sum_{j_3=1}^{j_2-1} c_{j_3, j_2}^{\alpha, \Phi} |e_{j_3}| \\
&\vdots \\
&\leq |\eta_0| + |\eta_0| \sum_{j_1=1}^{n-1} c_{j_1, n}^{\alpha, \Phi} + |\eta_0| \sum_{j_2=1}^{n-1} c_{j_2, n}^{\alpha, \Phi} \sum_{j_1=1}^{j_2-1} c_{j_1, j_2}^{\alpha, \Phi} + |\eta_0| \sum_{j_3=1}^{n-1} c_{j_3, n}^{\alpha, \Phi} \sum_{j_2=1}^{j_3-1} c_{j_2, j_3}^{\alpha, \Phi} \sum_{j_1=1}^{j_2-1} c_{j_1, j_2}^{\alpha, \Phi} \\
&\quad + \cdots + |\eta_0| \sum_{j_{n-1}=1}^{n-1} c_{j_{n-1}, n}^{\alpha, \Phi} \sum_{j_{-2}=1}^{j_{n-1}-1} c_{j_{n-2}, j_{n-1}}^{\alpha, \Phi} \cdots \sum_{j_1=1}^{j_2-1} c_{j_1, j_2}^{\alpha, \Phi}. \tag{3.10}
\end{aligned}$$

According to Lemma 3.2, it yields

$$\begin{aligned}
&\sum_{j_q=1}^{j_s-1} c_{j_q, j_s}^{\alpha, \Phi} (\Phi(t_{j_q}) - \Phi(t_0))^\beta \\
&= M \sum_{j_q=1}^{j_s-1} (\Phi(t_{j_s}) - \Phi(t_{j_q}))^{\alpha-1} (\Phi(t_{j_q+1}) - \Phi(t_{j_q})) (\Phi(t_{j_q}) - \Phi(t_0))^\beta \\
&\leq MB(\alpha, \beta + 1) (\Phi(t_{j_s}) - \Phi(t_0))^{\alpha+\beta}, \quad \beta \geq 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
F_{q,n} &= |\eta_0| \sum_{j_q=1}^{n-1} c_{j_q, n}^{\alpha, \Phi} \sum_{j_{q-1}=1}^{j_q-1} c_{j_{q-1}, j_q}^{\alpha, \Phi} \cdots \sum_{j_1=1}^{j_2-1} c_{j_1, j_2}^{\alpha, \Phi} (\Phi(t_{j_1}) - \Phi(t_0))^0 \\
&\leq |\eta_0| \sum_{j_q=1}^{n-1} c_{j_q, n}^{\alpha, \Phi} \sum_{j_{q-1}=1}^{j_q-1} c_{j_{q-1}, j_q}^{\alpha, \Phi} \cdots \sum_{j_2=1}^{j_3-1} c_{j_2, j_3}^{\alpha, \Phi} (\Phi(t_{j_2}) - \Phi(t_0))^\alpha B(\alpha, 1) M \\
&\vdots \\
&\leq |\eta_0| M^q (\Phi(t_n) - \Phi(t_0))^{q\alpha} \prod_{j=0}^{q-1} B(\alpha, j\alpha + 1) \\
&= |\eta_0| M^q (\Phi(t_n) - \Phi(t_0))^{q\alpha} \frac{(\Gamma(\alpha))^q}{\Gamma(q\alpha + 1)} \\
&\leq |\eta_0| M^q (\Phi(T) - \Phi(t_0))^{q\alpha} \frac{(\Gamma(\alpha))^q}{\Gamma(q\alpha + 1)}.
\end{aligned}$$

Now, we want to show the following result:

$$\sum_{q=1}^{\infty} M^q (\Phi(T) - \Phi(t_0))^{q\alpha} \frac{(\Gamma(\alpha))^q}{\Gamma(q\alpha + 1)} =: \sum_{q=1}^{\infty} \rho_q < \infty.$$

In effect, we have

$$\begin{aligned}
\lim_{q \rightarrow \infty} \frac{\rho_{q+1}}{\rho_q} &= \frac{M^{q+1} (\Phi(T) - \Phi(t_0))^{(q+1)\alpha} \frac{(\Gamma(\alpha))^{q+1}}{\Gamma((q+1)\alpha + 1)}}{M^q (\Phi(T) - \Phi(t_0))^{q\alpha} \frac{(\Gamma(\alpha))^q}{\Gamma(q\alpha + 1)}} \\
&= M (\Phi(T) - \Phi(t_0))^\alpha \frac{\Gamma(\alpha) \Gamma(q\alpha + 1)}{\Gamma((q+1)\alpha + 1)}.
\end{aligned}$$

Applying the property of two gamma functions as follows:

$$\frac{\Gamma(z+m)}{\Gamma(z+n)} = z^{m-n} \left[1 + \mathcal{O}\left(\frac{1}{z}\right) \right], \quad |\arg(z+m)| < \pi \quad \text{as } |z| \rightarrow \infty.$$

Then, we have

$$\frac{\Gamma(q\alpha+1)}{\Gamma((q+1)\alpha+1)} = (q\alpha)^{-\alpha} \left[1 + \mathcal{O}\left(\frac{1}{q\alpha}\right) \right] \leq C(q\alpha)^{-\alpha} \quad \text{as } q \rightarrow \infty.$$

Therefore, we obtain

$$\lim_{q \rightarrow \infty} \frac{\Phi_{q+1}}{\Phi_q} = M(\Phi(T) - \Phi(t_0))^\alpha \Gamma(\alpha)(q\alpha)^{-\alpha} \left[1 + \mathcal{O}\left(\frac{1}{q\alpha}\right) \right] < 1,$$

which yields the convergence of $\sum_{q=1}^{\infty} \Phi_q$. Thus, the inequality of (3.10) is bounded, which means

$$|e_n| \leq |\eta_0| + |\eta_0| \sum_{q=1}^{n-1} \Phi_q = |\eta_0| \sum_{q=0}^{n-1} \Phi_r \leq C|\eta_0|.$$

The proof is completed.

Now, we outline the following assumptions on the nonlinearity f .

- (H₁) The function $f : [t_0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.
 (H₂) The function $f : [t_0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz continuous in the second variable, that is, there exists $L > 0$ such that

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$$

for all $t \in [t_0, T]$ and $y_1, y_2 \in \mathbb{R}$.

- (H₃) The function $f : [t_0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz condition in t and $y(t)$ with a Lipschitz constant K .

Lemma 3.4. Assume that (H₁) holds. Let $y(t)$ be the solution of problem (3.2) and $h > 0$ be sufficiently small. Then, there exists a constant C which is independent of h such that

$$|y(t+h) - y(t)| \leq Ch^\alpha, \quad t \in [t_0, T-h].$$

Proof. By the assumption (H₁), there is a positive constant M such that $|f(t, y(t))| \leq M$. From the integral equation (3.2), we get

$$y(t+h) - y(t) = \frac{1}{\Gamma(\alpha)} \left(\int_{t_0}^{t+h} \Phi'(\nu)(\Phi(t+h) - \Phi(\nu))^{\alpha-1} f(\nu, y(\nu)) d\nu - \int_{t_0}^t \Phi'(\nu)(\Phi(t) - \Phi(\nu))^{\alpha-1} f(\nu, y(\nu)) d\nu \right).$$

Then, we have

$$\begin{aligned}
& |y(t+h) - y(t)| \\
& \leq \frac{1}{\Gamma(\alpha)} \left| \int_{t_0}^t \Phi'(v) \left[(\Phi(t+h) - \Phi(v))^{\alpha-1} - (\Phi(t) - \Phi(v))^{\alpha-1} \right] f(v, y(v)) dv \right| \\
& \quad + \frac{1}{\Gamma(\alpha)} \left| \int_t^{t+h} \Phi'(v) (\Phi(t+h) - \Phi(v))^{\alpha-1} f(v, y(v)) dv \right| \\
& \leq \frac{M}{\Gamma(\alpha)} \int_{t_0}^t \Phi'(v) \left| (\Phi(t+h) - \Phi(v))^{\alpha-1} - (\Phi(t) - \Phi(v))^{\alpha-1} \right| dv \\
& \quad + \frac{M}{\Gamma(\alpha)} \int_t^{t+h} \Phi'(v) (\Phi(t+h) - \Phi(v))^{\alpha-1} dv \\
& = \frac{M}{\Gamma(\alpha)} \int_{t_0}^t \Phi'(v) \left[(\Phi(t) - \Phi(v))^{\alpha-1} - (\Phi(t+h) - \Phi(v))^{\alpha-1} \right] dv \\
& \quad + \frac{M}{\Gamma(\alpha+1)} (\Phi(t+h) - \Phi(t))^\alpha \\
& = \frac{M}{\Gamma(\alpha+1)} \left[(\Phi(t) - \Phi(t_0))^\alpha + (\Phi(t+h) - \Phi(t))^\alpha - (\Phi(t+h) - \Phi(t_0))^\alpha \right] \\
& \quad + \frac{M}{\Gamma(\alpha+1)} (\Phi(t+h) - \Phi(t))^\alpha \\
& \leq \frac{2M}{\Gamma(\alpha+1)} (\Phi(t+h) - \Phi(t))^\alpha \\
& \quad + \frac{M}{\Gamma(\alpha+1)} \left[(\Phi(t+h) - \Phi(t_0))^\alpha - (\Phi(t) - \Phi(t_0))^\alpha \right]. \tag{3.11}
\end{aligned}$$

In the following, we claim that

$$(\Phi(t+h) - \Phi(t))^\alpha \leq Ch^\alpha, \quad h \rightarrow 0.$$

In the fact that the inequality

$$b^\alpha - a^\alpha \leq (b-a)^\alpha, \quad 0 \leq a \leq b,$$

one immediately gets

$$\begin{aligned}
(\Phi(t+h) - \Phi(t_0))^\alpha - (\Phi(t) - \Phi(t_0))^\alpha & \leq (\Phi(t+h) - \Phi(t_0) - \Phi(t) + \Phi(t_0))^\alpha \\
& = (\Phi(t+h) - \Phi(t))^\alpha \leq Ch^\alpha. \tag{3.12}
\end{aligned}$$

Combining (3.11) and (3.12), we obtain

$$|y(t+h) - y(t)| \leq Ch^\alpha,$$

which completes the proof.

Lemma 3.5. Assume that (H_1) and (H_3) hold. Let $y(t)$ be the solution of problem (3.2) and $h > 0$ be sufficiently small. Then, we have

$$\left| \int_{t_0}^{t_n} \Phi'(v)(\Phi(t_n) - \Phi(v))^{\alpha-1} f(v, y(v)) dv - \frac{1}{\alpha} \sum_{j=0}^{n-1} (w_{n,j}^{\alpha, \Phi} - w_{n,j+1}^{\alpha, \Phi}) f(t_j, y_j) \right| \leq Ch^\alpha,$$

where $(w_{n,j}^{\alpha, \Phi} - w_{n,j+1}^{\alpha, \Phi})$ is given by (3.4).

Proof. By the assumption (H_3) and Lemma 3.4, which contributes to

$$\begin{aligned} & \left| \int_{t_0}^{t_n} \Phi'(v)(\Phi(t_n) - \Phi(v))^{\alpha-1} f(v, y(v)) dv - \frac{1}{\alpha} \sum_{j=0}^{n-1} (w_{n,j}^{\alpha, \Phi} - w_{n,j+1}^{\alpha, \Phi}) f(t_j, y_j) \right| \\ &= \sum_{j=0}^{n-1} \left| \int_{t_j}^{t_{j+1}} \Phi'(v)(\Phi(t_n) - \Phi(v))^{\alpha-1} [f(v, y(v)) - f(t_j, y(t_j))] dv \right| \\ &\leq \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \Phi'(v)(\Phi(t_n) - \Phi(v))^{\alpha-1} \\ &\quad (|f(v, y(v)) - f(v, y(t_j))| + |f(v, y(t_j)) - f(t_j, y(t_j))|) dv \\ &\leq L \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \Phi'(v)(\Phi(t_n) - \Phi(v))^{\alpha-1} [(v - t_j) + Ch^\alpha] dv \\ &\leq L(1+C)h^\alpha \int_{t_0}^{t_n} \Phi'(v)(\Phi(t_n) - \Phi(v))^{\alpha-1} dv \\ &= L(1+C)h^\alpha \frac{1}{\alpha} (\Phi(t_n) - \Phi(t_0))^\alpha \\ &\leq L(1+C)h^\alpha \frac{1}{\alpha} (\Phi(T) - \Phi(t_0))^\alpha \\ &\leq Ch^\alpha. \end{aligned}$$

Lemma 3.6. Assume that (H_1) and (H_3) hold. Let $y(t)$ be the solution of problem (3.2) and $h > 0$ be sufficiently small. Then, we have

$$\left| \int_{t_0}^{t_n} \Phi'(v)(\Phi(t_n) - \Phi(v))^{\alpha-1} f(v, y(v)) dv - \frac{1}{\alpha} \sum_{j=1}^n (w_{n,j}^{\alpha, \Phi} - w_{n,j+1}^{\alpha, \Phi}) f(t_j, y_j) \right| \leq Ch^\alpha,$$

where $(w_{n,j-1}^{\alpha, \Phi} - w_{n,j}^{\alpha, \Phi})$ is given by (3.6).

Proof. The proof follows the same argument as in Lemma 3.5.

Lemma 3.7. If $f(t) \in \mathcal{C}^1[t_0, T]$, then

$$\left| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_n} \Phi'(v)(\Phi(t_n) - \Phi(v))^{\alpha-1} f(v) ds - \frac{1}{\Gamma(\alpha+1)} \left(u_{n,0}^{\alpha, \Phi} f(t_0) + \sum_{j=1}^n u_{n,j}^{\alpha, \Phi} f(t_j) \right) \right|$$

$$\leq \frac{\|f'\|_{C[t_0, T]}}{\Gamma(\alpha + 1)} (\Phi(T) - \Phi(t_0))^\alpha h,$$

where $u_{n,0}^{\alpha, \Phi} f(t_0) + \sum_{j=1}^n u_{n,j}^{\alpha, \Phi} f(t_j)$ is given by (3.7).

Proof. Applying the mean value theorem, there exist $\xi_j, \eta_j \in (t_j, t_{j+1})$ for $j = 0, 1, \dots, n$ such that

$$\begin{aligned} & \left| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_n} (\Phi(t_n) - \Phi(v))^{\alpha-1} f(v) \Phi'(v) dv - \frac{1}{\Gamma(\alpha + 1)} \left(u_{n,0}^{\alpha, \Phi} f(t_0) + \sum_{j=1}^n u_{n,j}^{\alpha, \Phi} f(t_j) \right) \right| \\ &= \frac{1}{\Gamma(\alpha)} \left| \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (\Phi(t_n) - \Phi(v))^{\alpha-1} \left[f(v) - \frac{1}{2} (f(t_j) + f(t_{j+1})) \right] \Phi'(v) dv \right| \\ &= \frac{1}{\Gamma(\alpha)} \left| \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (\Phi(t_n) - \Phi(v))^{\alpha-1} \left[\frac{1}{2} (f(v) - (f(t_j))) + \frac{1}{2} (f(v) - f(t_{j+1})) \right] \Phi'(v) dv \right| \\ &\leq \frac{1}{2\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (\Phi(t_n) - \Phi(v))^{\alpha-1} |(s - t_j) f'(\xi_j) + (s - t_{j+1}) f'(\eta_j)| \Phi'(v) dv \\ &\leq \frac{h \|f'\|_{C[t_0, T]}}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (\Phi(t_n) - \Phi(v))^{\alpha-1} \Phi'(v) dv \\ &= \frac{h \|f'\|_{C[t_0, T]}}{\Gamma(\alpha + 1)} (\Phi(t_n) - \Phi(t_0))^\alpha \\ &\leq \frac{\|f'\|_{C[t_0, T]}}{\Gamma(\alpha + 1)} (\Phi(T) - \Phi(t_0))^\alpha h. \end{aligned}$$

This completes the proof.

Then, the stability analysis and error estimations of the explicit product integration rectangular rule (3.4), the implicit product integration rectangular rule (3.6), the implicit product integration trapezoidal rule (3.7), and the Adams predictor-corrector method (3.8) are investigated in the next section.

4. Error estimation of the approximation and stability analysis

Theorem 4.1. Assume that (H_3) holds. Let $y(t) \in \mathcal{C}[t_0, T]$ be the solution of problem (3.2) and $y_j (0 \leq j \leq n - 1)$ be the solution of the explicit product integration rectangular rule (3.4). Then, the error equation

$$|y(t_n) - y_n| \leq Ch^\alpha, \quad n = 1, 2, \dots, N \quad (4.1)$$

holds.

Proof. For each $n = 1, 2, \dots, N$, we let $e_{n-1} = y(t_{n-1}) - y_{n-1}$ and $e_0 = 0$. By the integral equation (3.2) and the fractional left rectangle scheme (3.4), we have the error equation as follows:

$$y(t_n) - y_n = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_n} \Phi'(v) (\Phi(t_n) - \Phi(v))^{\alpha-1} f(v, y(v)) dv \\ - \frac{1}{\Gamma(\alpha+1)} \sum_{j=0}^{n-1} \left(w_{n,j}^{\alpha,\Phi} - w_{n,j+1}^{\alpha,\Phi} \right) f(t_j, y_j).$$

By the assumption (H₃), and applying Lemmas 3.1 and 3.5, we obtain

$$|e_n| \leq \left| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_n} \Phi'(v) (\Phi(t_n) - \Phi(v))^{\alpha-1} f(v, y(v)) dv \right. \\ \left. - \frac{1}{\Gamma(\alpha+1)} \sum_{j=0}^{n-1} \left(w_{n,j}^{\alpha,\Phi} - w_{n,j+1}^{\alpha,\Phi} \right) f(t_j, y_j) \right| \\ + \frac{1}{\Gamma(\alpha+1)} \sum_{j=0}^{n-1} \left(w_{n,j}^{\alpha,\Phi} - w_{n,j+1}^{\alpha,\Phi} \right) |f(t_j, y(t_j)) - f(t_j, y_j)| \\ \leq Ch^\alpha + \frac{K}{\Gamma(\alpha+1)} \sum_{j=0}^{n-1} \left(w_{n,j}^{\alpha,\Phi} - w_{n,j+1}^{\alpha,\Phi} \right) |e_j|.$$

It follows that the inequality (4.1) is obtained by using Lemma 3.3.

Theorem 4.2. Assume that (H₃) holds. Let $y(t) \in \mathcal{C}[t_0, T]$ be the solution of problem (3.2) and y_j ($0 \leq j \leq n-1$) be the solution of the implicit product integration rectangular rule (3.6). Then, the error equation

$$|y(t_n) - y_n| \leq Ch^\alpha, \quad n = 1, 2, \dots, N \quad (4.2)$$

holds.

Proof. The proof is essentially similar to the proof of Theorem 4.1.

Theorem 4.3. Assume that (H₁) and (H₃) hold. Let $y(t) \in \mathcal{C}[t_0, T]$ be the solution of problem (3.2) and y_j ($1 \leq j \leq N$) be the solution of the implicit product integration trapezoidal rule (3.7). Then, the error equation

$$|y(t_n) - y_n| \leq Ch^\alpha, \quad n = 1, \dots, N$$

holds.

Proof. The proof of this theorem follows the same technique as Theorem 4.1.

Theorem 4.4. Assume that (H₁) and (H₂) hold. Let $y(t) \in \mathcal{C}[t_0, T]$ be the solution of problem (3.2), $D_{t_0}^{\alpha,\Phi} y(t) \in \mathcal{C}[t_0, T]$, and y_j ($1 \leq j \leq N$) be the solution of the Adams predictor-corrector method (3.8). Then, the error equation

$$|y(t_n) - y_n| \leq Ch, \quad n = 1, \dots, N-1$$

holds.

Proof. For each $n = 0, 1, \dots, N-1$, we let $e_n = y(t_n) - y_n$ and $e_0 = 0$. By the integral equation (3.2) and the Adams predictor-corrector method (3.8), we obtain

$$y(t_n) - y_n = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_n} (\Phi(t_n) - \Phi(v))^{\alpha-1} f(v, y(v)) \Phi'(v) dv \\ - \frac{1}{\Gamma(\alpha+1)} \left(\sum_{j=0}^{n-1} u_{n,j}^{\alpha,\Phi} f(t_j, y_j) + u_{n,n}^{\alpha,\Phi} f(t_n, y_n^p) \right)$$

Then,

$$|e_n| = \left| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_n} (\Phi(t_n) - \Phi(v))^{\alpha-1} f(v, y(v)) \Phi'(v) dv \right. \\ \left. - \frac{1}{\Gamma(\alpha+1)} \left(\sum_{j=0}^{n-1} u_{n,j}^{\alpha,\Phi} f(t_j, y_j) + u_{n,n}^{\alpha,\Phi} f(t_n, y_n^p) \right) \right| \\ \leq \left| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_n} (\Phi(t_n) - \Phi(v))^{\alpha-1} f(v, y(v)) \Phi'(v) dv - \frac{1}{\Gamma(\alpha+1)} \sum_{j=0}^n u_{n,j}^{\alpha,\Phi} f(t_j, y(t_j)) \right| \\ + \frac{1}{\Gamma(\alpha+1)} \left| \sum_{j=0}^n u_{n,j}^{\alpha,\Phi} f(t_j, y(t_j)) - \left(\sum_{j=0}^{n-1} u_{n,j}^{\alpha,\Phi} f(t_j, y_j) + u_{n,n}^{\alpha,\Phi} f(t_n, y_n^p) \right) \right| \\ \leq \left| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_n} (\Phi(t_n) - \Phi(v))^{\alpha-1} f(v, y(v)) \Phi'(v) dv - \frac{1}{\Gamma(\alpha+1)} \sum_{j=0}^n u_{n,j}^{\alpha,\Phi} f(t_j, y(t_j)) \right| \\ + \frac{1}{\Gamma(\alpha+1)} \left| \sum_{j=0}^{n-1} u_{n,j}^{\alpha,\Phi} [f(t_j, y(t_j)) - f(t_j, y_j)] \right| \\ + \frac{1}{\Gamma(\alpha+1)} |u_{n,n}^{\alpha,\Phi} [f(t_n, y(t_n)) - f(t_n, y_n^p)]| \\ = I_1 + I_2 + I_3.$$

From Lemma 3.7, it follows that

$$I_1 \leq C_1 h.$$

By the assumption (H₂) of f , we obtain

$$I_2 = \frac{1}{\Gamma(\alpha+1)} \left| \sum_{j=0}^{n-1} u_{n,j}^{\alpha,\Phi} [f(t_j, y(t_j)) - f(t_j, y_j)] \right| \\ \leq \frac{1}{\Gamma(\alpha+1)} \sum_{j=0}^{n-1} u_{n,j}^{\alpha,\Phi} |f(t_j, y(t_j)) - f(t_j, y_j)| \\ \leq \frac{L}{\Gamma(\alpha+1)} \sum_{j=0}^{n-1} u_{n,j}^{\alpha,\Phi} |y(t_j) - y_j| \\ = \frac{L}{\Gamma(\alpha+1)} \sum_{j=0}^{n-1} u_{n,j}^{\alpha,\Phi} |e_j|.$$

As for $u_{n,n}^{\alpha,\Phi}$, one gets

$$u_{n,n}^{\alpha,\Phi} = \frac{1}{2} (\Phi(t_n) - \Phi(t_{n-1}))^\alpha \leq Ch^\alpha.$$

Since $f(t, y(t)) = D_{t_0}^{\alpha,\Phi} y(t)$ is continuous and bounded, it follows Lemma 3.7 that

$$\begin{aligned} I_3 &= |u_{n,n}^{\alpha,\Phi} [f(t_n, y(t_n)) - f(t_n, y_n^p)]| \\ &\leq Ch^\alpha |f(t_n, y(t_n)) - f(t_n, y_n^p)| \\ &\leq CLh^\alpha |y(t_n) - y_n^p| \\ &= CLh^\alpha \left| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_n} (\Phi(t_n) - \Phi(v))^{\alpha-1} \Phi'(v) f(v, y(v)) dv \right. \\ &\quad \left. - \frac{1}{\Gamma(\alpha+1)} \sum_{j=0}^{n-1} (w_{n,j}^{\alpha,\Phi} - w_{n,j+1}^{\alpha,\Phi}) f(t_j, y_j) \right| \\ &\leq CLh^\alpha \left| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_n} (\Phi(t_n) - \Phi(v))^{\alpha-1} \Phi'(v) D_{t_0}^{\alpha,\Phi} y(v) dv \right. \\ &\quad \left. - \frac{1}{\Gamma(\alpha+1)} \sum_{j=0}^{n-1} (w_{n,j}^{\alpha,\Phi} - w_{n,j+1}^{\alpha,\Phi}) D_{t_0}^{\alpha,\Phi} y(t_j) \right| \\ &\quad + CLh^\alpha \left| \frac{1}{\Gamma(\alpha+1)} \sum_{j=0}^{n-1} (w_{n,j}^{\alpha,\Phi} - w_{n,j+1}^{\alpha,\Phi}) D_{t_0}^{\alpha,\Phi} y(t_j) \right. \\ &\quad \left. - \frac{1}{\Gamma(\alpha+1)} \sum_{j=0}^{n-1} (w_{n,j}^{\alpha,\Phi} - w_{n,j+1}^{\alpha,\Phi}) f(t_j, y_j) \right| \\ &\leq CLh^\alpha \left| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_n} \Phi'(v) (\Phi(t_n) - \Phi(v))^{\alpha-1} D_{t_0}^{\alpha,\Phi} y(v) dv \right. \\ &\quad \left. - \frac{1}{\Gamma(\alpha+1)} \sum_{j=0}^{n-1} (w_{n,j}^{\alpha,\Phi} - w_{n,j+1}^{\alpha,\Phi}) D_{t_0}^{\alpha,\Phi} y(t_j) \right| \\ &\quad + \frac{CLh^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{n-1} (w_{n,j}^{\alpha,\Phi} - w_{n,j+1}^{\alpha,\Phi}) |f(t_j, y(t_j)) - f(t_j, y_j)| \\ &\leq Ch^{\alpha+1} + \frac{CL^2(T-t_0)^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{n-1} (w_{n,j}^{\alpha,\Phi} - w_{n,j+1}^{\alpha,\Phi}) |e_j|. \end{aligned}$$

According to Lemmas 3.1 and 3.3, we obtain

$$\begin{aligned} |e_n| &\leq I_1 + I_2 + I_3 \\ &\leq Ch + \frac{L}{\Gamma(\alpha+1)} \sum_{j=0}^{n-1} u_{n,j}^{\alpha,\Phi} |e_j| \\ &\quad + \frac{CL^2(T-a)^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{n-1} (w_{n,j}^{\alpha,\Phi} - w_{n,j+1}^{\alpha,\Phi}) |e_j| \\ &\leq \tilde{C}h, \end{aligned}$$

which implies the statement of Theorem 4.4.

Theorem 4.5. Assume that (H_1) and (H_2) are true. Let $y_j (j = 1, 2, \dots, n)$ be the solution to the explicit product integration rectangular rule (3.4) on the existed interval of its unique solution. Then the explicit product integration rectangular rule (3.4) is conditionally stable.

Proof. Suppose that y_0 and $y_j (j = 1, 2, \dots, n)$ have perturbations \tilde{y}_0 and \tilde{y}_j , respectively. From (3.4), it follows

$$y_n + \tilde{y}_n = y_0 + \tilde{y}_0 + \frac{1}{\Gamma(\alpha + 1)} \sum_{j=0}^{n-1} \left(w_{n,j}^{\alpha, \Phi} - w_{n,j+1}^{\alpha, \Phi} \right) f(t_j, y_j + \tilde{y}_j). \quad (4.3)$$

By the assumption (H_2) , and the equations (3.4) and (4.3), we get

$$\begin{aligned} |\tilde{y}_n| &= \left| \tilde{y}_{t_0} + \frac{1}{\Gamma(\alpha + 1)} \sum_{j=0}^{n-1} \left(w_{n,j}^{\alpha, \Phi} - w_{n,j+1}^{\alpha, \Phi} \right) [f(t_j, y_j + \tilde{y}_j) - f(t_j, y_j)] \right| \\ &\leq |\tilde{y}_{t_0}| + \frac{1}{\Gamma(\alpha + 1)} \sum_{j=0}^{n-1} \left(w_{n,j}^{\alpha, \Phi} - w_{n,j+1}^{\alpha, \Phi} \right) |f(t_j, y_j + \tilde{y}_j) - f(t_j, y_j)| \\ &\leq |\tilde{y}_{t_0}| + \frac{L}{\Gamma(\alpha + 1)} \sum_{j=0}^{n-1} \left(w_{n,j}^{\alpha, \Phi} - w_{n,j+1}^{\alpha, \Phi} \right) |\tilde{y}_j| \\ &\leq |\tilde{y}_{t_0}| + \frac{L \left(w_{n,0}^{\alpha, \Phi} - w_{n,1}^{\alpha, \Phi} \right)}{\Gamma(\alpha + 1)} |\tilde{y}_0| + \frac{L}{\Gamma(\alpha + 1)} \sum_{j=1}^{n-1} \left(w_{n,j}^{\alpha, \Phi} - w_{n,j+1}^{\alpha, \Phi} \right) |\tilde{y}_j| \\ &\leq \eta_0 + \frac{L}{\Gamma(\alpha + 1)} \sum_{j=1}^{n-1} \left(w_{n,j}^{\alpha, \Phi} - w_{n,j+1}^{\alpha, \Phi} \right) |\tilde{y}_j|. \end{aligned} \quad (4.4)$$

By Lemmas 3.1 and 3.3, it implies to

$$|\tilde{y}_n| \leq C\eta_0,$$

where $\eta_0 = \max_{0 \leq n \leq N-1} \left\{ |\tilde{y}_0| + \frac{L \left(w_{n,0}^{\alpha, \Phi} - w_{n,1}^{\alpha, \Phi} \right) |\tilde{y}_0|}{\Gamma(\alpha + 1)} \right\}$. The proof is completed.

In the same way, we obtain the stability results for the implicit rectangular and trapezoidal rules in Theorems 4.6 and 4.7, respectively. Hence, we omit the proof.

Theorem 4.6. Assume that (H_1) and (H_2) hold. Let $y_j (j = 1, 2, \dots, n)$ be the solution to the implicit product integration rectangular rule (3.6) on the existed interval of its unique solution. Then the implicit product integration rectangular rule (3.6) is conditionally stable.

Theorem 4.7. Assume that (H_1) and (H_2) hold. Let $y_j (j = 1, 2, \dots, n)$ be the solution to the implicit product integration trapezoidal rule (3.7) on the existed interval of its unique solution. Then the implicit product integration trapezoidal rule (3.7) is conditionally stable.

Next, we investigate the stability of the fractional predictor-corrector scheme (3.8).

Theorem 4.8. Assume that (H_1) and (H_2) hold. Let $y_j (j = 1, 2, \dots, n)$ be the solution to the Adams predictor-corrector method (3.8) on the existed interval of its unique solution. Then, the Adams predictor-corrector method (3.8) is conditionally stable.

Proof. Assume that $\tilde{y}_0, \tilde{y}_j (j = 1, 2, \dots, n)$, and $\tilde{y}_n^p (n = 1, \dots, N)$ are perturbation terms of y_0, y_j , and y_n^p , respectively. Then, we construct the perturbation equations as follows:

$$\begin{aligned} \tilde{y}_n^p &= \tilde{y}_0 + \frac{1}{\Gamma(\alpha + 1)} \sum_{j=0}^{n-1} \left(w_{n,j}^{\alpha,\Phi} - w_{n,j+1}^{\alpha,\Phi} \right) [f(t_j, y_j + \tilde{y}_j) - f(t_j, y_j)] \\ \tilde{y}_n &= \tilde{y}_0 + \frac{1}{\Gamma(\alpha + 1)} \left\{ \sum_{j=0}^{n-1} u_{n,j}^{\alpha,\Phi} [f(t_j, y_j + \tilde{y}_j) - f(t_j, y_j)] \right. \\ &\quad \left. + \frac{1}{2} (\Phi(t_n) - \Phi(t_{n-1}))^\alpha [f(t_n, y_n^p + \tilde{y}_n^p) - f(t_n, y_n^p)] \right\}. \end{aligned}$$

From the assumption (H_2) and the inequality (4.4), we get

$$\begin{aligned} |\tilde{y}_n| &= \left| \tilde{y}_0 + \frac{1}{\Gamma(\alpha + 1)} \left\{ \sum_{j=0}^{n-1} u_{n,j}^{\alpha,\Phi} [f(t_j, y_j + \tilde{y}_j) - f(t_j, y_j)] \right. \right. \\ &\quad \left. \left. + \frac{1}{2} (\Phi(t_n) - \Phi(t_{n-1}))^\alpha [f(t_n, y_n^p + \tilde{y}_n^p) - f(t_n, y_n^p)] \right\} \right| \\ &\leq |\tilde{y}_0| + \frac{1}{\Gamma(\alpha + 1)} \sum_{j=0}^{n-1} u_{n,j}^{\alpha,\Phi} |f(t_j, y_j + \tilde{y}_j) - f(t_j, y_j)| \\ &\quad + \frac{1}{2\Gamma(\alpha + 1)} (\Phi(t_n) - \Phi(t_{n-1}))^\alpha |f(t_n, y_n^p + \tilde{y}_n^p) - f(t_n, y_n^p)| \\ &\leq |\tilde{y}_0| + \frac{Lu_{0,n}^{\alpha,\Phi}}{\Gamma(\alpha + 1)} |\tilde{y}_0| + \frac{L}{\Gamma(\alpha + 1)} \sum_{j=1}^{n-1} u_{n,j}^{\alpha,\Phi} |\tilde{y}_j| \\ &\quad + \frac{L}{2\Gamma(\alpha + 1)} (\Phi(t_n) - \Phi(t_{n-1}))^\alpha |\tilde{y}_n^p| \\ &\leq |\tilde{y}_0| + \frac{Lu_{0,n}^{\alpha,\Phi}}{\Gamma(\alpha + 1)} |\tilde{y}_0| + \frac{L}{2\Gamma(\alpha + 1)} (\Phi(t_n) - \Phi(t_{n-1}))^\alpha \tilde{y}_0 \\ &\quad + \frac{L^2 (w_{n,0}^{\alpha,\Phi} - w_{n,1}^{\alpha,\Phi})}{2\Gamma(\alpha + 1)\Gamma(\alpha + 1)} (\Phi(t_n) - \Phi(t_{n-1}))^\alpha |\tilde{y}_0| \\ &\quad + \frac{L}{\Gamma(\alpha + 1)} \sum_{j=1}^{n-1} \left[u_{n,j}^{\alpha,\Phi} + \frac{L}{2\Gamma(\alpha + 1)} (\Phi(t_n) - \Phi(t_{n-1}))^\alpha (w_{n,j}^{\alpha,\Phi} - w_{n,j+1}^{\alpha,\Phi}) \right] |\tilde{y}_j| \\ &\leq \zeta_0 + \frac{L}{\Gamma(\alpha + 1)} \sum_{j=1}^{n-1} \left[u_{n,j}^{\alpha,\Phi} + \frac{L}{2\Gamma(\alpha + 1)} (\Phi(t_n) - \Phi(t_{n-1}))^\alpha (w_{n,j}^{\alpha,\Phi} - w_{n,j+1}^{\alpha,\Phi}) \right] |\tilde{y}_j|, \end{aligned}$$

where

$$\zeta_0 = \max_{0 \leq n \leq N-1} \left\{ |\tilde{y}_0| + \frac{L\mu_{0,n}^{\alpha,\Phi}}{\Gamma(\alpha+1)} |\tilde{y}_0| + \frac{L}{2\Gamma(\alpha+1)} (\Phi(t_n) - \Phi(t_{n-1}))^\alpha |\tilde{y}_0| + \frac{L^2 (w_{n,0}^{\alpha,\Phi} - w_{n,1}^{\alpha,\Phi})}{2\Gamma(\alpha+1)\Gamma(\alpha+1)} \cdot (\Phi(t_n) - \Phi(t_{n-1}))^\alpha |\tilde{y}_0| \right\}.$$

By Lemmas 3.1 and 3.3, it yields to

$$|\tilde{y}_n| \leq C\zeta_0.$$

This completes the proof.

5. Numerical examples

Motivated by [2], we assume that $J = [0, 1]$ and $\Phi \in \mathcal{C}(J)$ be a differentiable function such that $\Phi'(t) > 0, t \in J$ and $\Phi(J) = [0, 1]$. Moreover, we solve the numerical examples by using MATLAB software and investigate different choices of suitable functions Φ in the numerical examples as below

Example 5.1. Consider the fractional order initial value problem given by

$$\begin{cases} {}^C D_{t_0}^{\alpha,\Phi} y(t) + \frac{2}{\Gamma(3-\alpha)} y(t) = \frac{2}{\Gamma(3-\alpha)} ((\Phi(t))^{2-\alpha} + (\Phi(t))^2), & 0 < \alpha < 1, \\ y(0) = 0. \end{cases} \quad (5.1)$$

It is clearly seen that $y(t) = (\Phi(t))^2$ is the exact solution.

The exact solution and numerical solutions of (5.1) are plotted for different kernels Φ in Figure 1. Moreover, the numerical results are closed to the exact solution. Notice that the behaviors of the solutions are similar although we change the different kernels Φ . In Table 1, we display the maximum errors of four numerical schemes for (5.1) when $\Phi(t) = t$ and $\alpha = 0.8$. From the data given in Table 1, the accuracy of numerical solutions corresponds the Theorems 4.1–4.4 when h is sufficiently small. Tables 2 and 3 also present maximum errors for the kernels $\Phi(t) = \sin\left(\frac{t}{10}\right)$ and $\Phi(t) = \frac{t}{2}$, respectively. Overall, the implicit product integration trapezoidal rule and Adams predictor-corrector method have higher accuracy than the other method.

Table 1. Maximum errors for (5.1) with $\Phi(t) = t$ and $\alpha = 0.8$.

h	Ex. PI Rec.	Im. PI Rec.	Im. PI Trap.	Adams PC
2^{-4}	2.79E-02	2.64E-02	2.89E-04	1.50E-03
2^{-5}	1.38E-02	1.33E-02	7.73E-05	4.07E-04
2^{-6}	6.80E-03	6.70E-03	2.10E-05	1.13E-04
2^{-7}	3.40E-03	3.40E-03	5.66E-06	3.18E-05
2^{-8}	1.70E-03	1.70E-03	1.51E-06	9.00E-06
2^{-9}	8.49E-04	8.46E-04	4.02E-07	2.55E-06
2^{-10}	4.24E-04	4.23E-04	1.06E-07	7.26E-07

Table 2. Maximum errors for (5.1) with $\Phi(t) = \sin\left(\frac{t}{10}\right)$ and $\alpha = 0.8$.

h	Ex. PI Rec.	Im. PI Rec.	Im. PI Trap.	Adams PC
2^{-4}	5.38E-04	5.18E-04	4.31E-06	1.32E-05
2^{-5}	2.66E-04	2.62E-04	4.94E-06	4.44E-06
2^{-6}	1.32E-04	1.32E-04	5.98E-06	5.92E-06
2^{-7}	6.52E-05	6.68E-05	6.35E-06	6.47E-06
2^{-8}	3.35E-05	3.40E-05	6.51E-06	6.62E-06
2^{-9}	1.93E-05	1.76E-05	6.59E-06	6.66E-06
2^{-10}	1.27E-05	9.33E-06	6.63E-06	6.67E-06

Table 3. Maximum errors for (5.1) with $\Phi(t) = \frac{t}{2}$ and $\alpha = 0.8$.

h	Ex. PI Rec.	Im. PI Rec.	Im. PI Trap.	Adams PC
2^{-4}	9.40E-03	8.90E-03	7.73E-05	4.06E-04
2^{-5}	4.70E-03	4.50E-03	2.10E-05	1.13E-04
2^{-6}	2.30E-03	2.30E-03	5.66E-06	3.16E-05
2^{-7}	1.20E-03	1.10E-03	1.51E-06	8.91E-06
2^{-8}	5.75E-04	5.71E-04	4.02E-07	2.52E-06
2^{-9}	2.87E-04	2.86E-04	1.06E-07	7.16E-07
2^{-10}	1.43E-04	1.43E-04	2.78E-08	2.04E-07

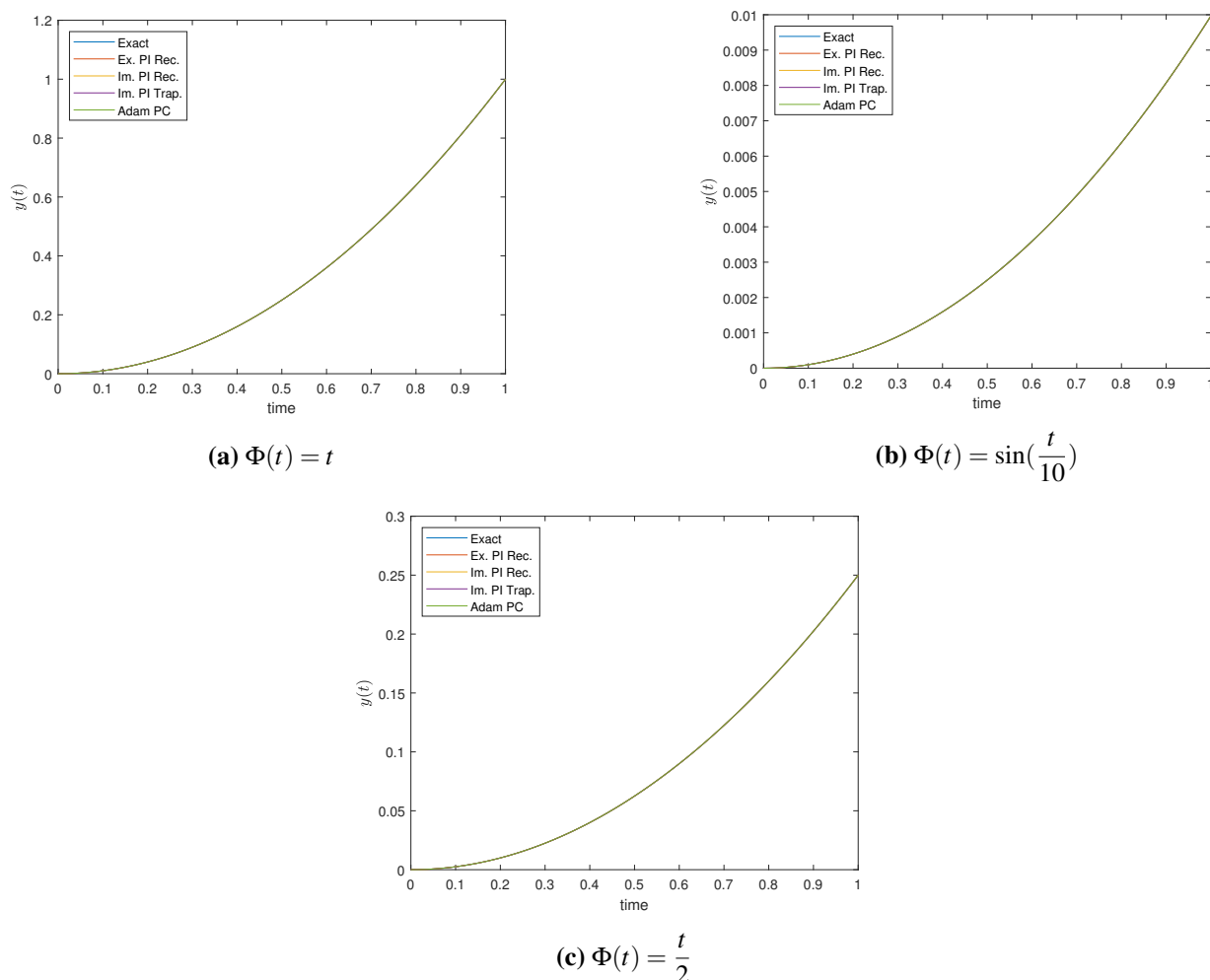


Figure 1. Testing between exact and numerical solutions of (5.1) for different Φ with $h = 2^{-10}$ and $\alpha = 0.8$.

Example 5.2. Consider the nonlinear Φ -Caputo fractional differential equations given by

$$\begin{cases} {}^c D_{t_0}^{\alpha, \Phi} y(t) = \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)} (\Phi(t) - \Phi(t_0))^\alpha - \frac{2}{\Gamma(3 - \alpha)} (\Phi(t) - \Phi(t_0))^{2-\alpha} \\ \quad + ((\Phi(t) - \Phi(t_0))^{2\alpha} - (\Phi(t) - \Phi(t_0))^2)^4 - y^4(t), \\ y(0) = 0. \end{cases} \quad (5.2)$$

From Figure 2, the numerical and exact solutions of (5.2) are plotted for kernels Φ . Moreover, the numerical solutions are close to the exact solution. However, the behavior of the solutions is changed when the functions of Φ are change.

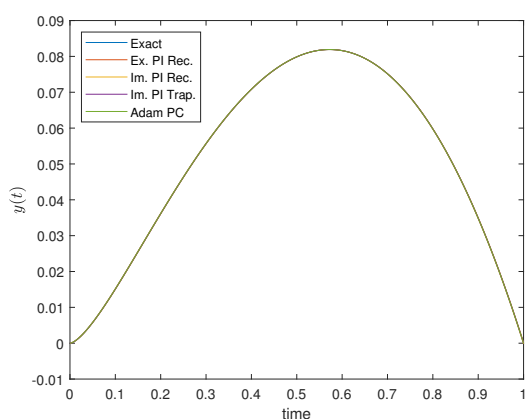
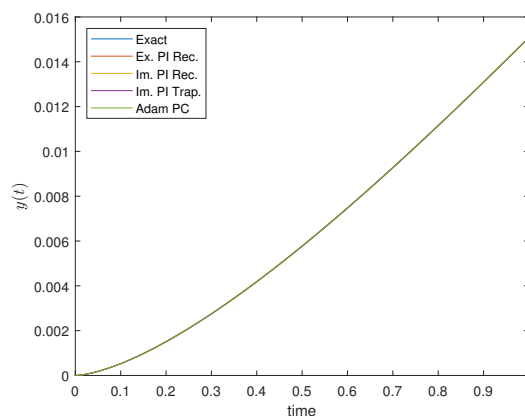
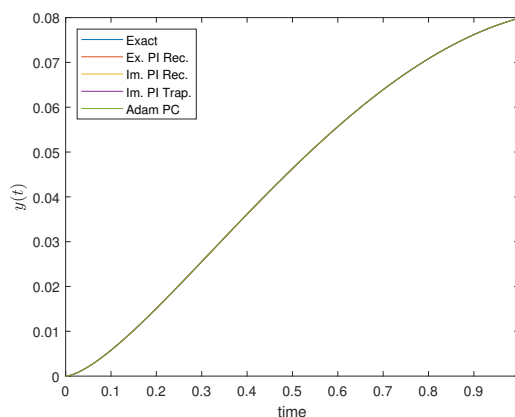
(a) $\Phi(t) = t$ (b) $\Phi(t) = \sin\left(\frac{t}{10}\right)$ (c) $\Phi(t) = \frac{t}{2}$

Figure 2. Testing between exact and numerical solutions of (5.2) for different Φ with $h = 2^{-10}$ and $\alpha = 0.8$.

When $\alpha = 0.8$ and $\Phi(t) = t$, the maximum errors of four numerical schemes of (5.2) are presented in Table 4. The accuracy of numerical solutions also corresponds to the Theorems 4.1–4.4 when the step size h is getting small. Next, the maximum errors of (5.2) with the different kernels $\Phi(t) = \sin\left(\frac{t}{10}\right)$ and $\Phi(t) = \frac{t}{2}$, respectively, for different values of h are showed in Tables 5 and 6. Therefore, the implicit product integration trapezoidal rule has high accuracy than the other method. However, the implicit product integration rectangular rule gives higher accuracy than the other method when $\Phi(t) = \sin\left(\frac{t}{10}\right)$. Particularly, the four numerical schemes can be reduced to the numerical schemes in [22] when $\Phi(t) = t$. Furthermore, if $\Phi(t) = \ln(t)$, the numerical schemes are agreed in the example of [26]. It can be seen that $y(t) = (\Phi(t) - \Phi(t_0))^{2\alpha} - (\Phi(t) - \Phi(t_0))^2$.

Table 4. Maximum errors for (5.2) with $\Phi(t) = t$ and $\alpha = 0.8$.

h	Ex. PI Rec.	Im. PI Rec.	Im. PI Trap.	Adams PC
2^{-4}	1.17E-02	1.32E-02	1.30E-03	1.30E-03
2^{-5}	6.00E-03	6.40E-03	4.11E-04	4.11E-04
2^{-6}	3.10E-03	3.20E-03	1.29E-04	1.29E-04
2^{-7}	1.50E-03	1.60E-03	4.09E-05	4.09E-05
2^{-8}	7.77E-04	7.85E-04	1.31E-05	1.31E-05
2^{-9}	3.90E-04	3.92E-04	4.22E-06	4.22E-06
2^{-10}	1.95E-04	1.96E-04	1.37E-06	1.37E-06

Table 5. Maximum errors for (5.2) with $\Phi(t) = \sin\left(\frac{t}{10}\right)$ and $\alpha = 0.8$.

h	Ex. PI Rec.	Im. PI Rec.	Im. PI Trap.	Adams PC
2^{-4}	6.61E-04	5.99E-04	3.65E-05	3.91E-05
2^{-5}	3.24E-04	3.06E-04	1.99E-05	2.14E-05
2^{-6}	1.63E-04	1.55E-04	1.52E-05	1.61E-05
2^{-7}	8.64E-05	7.80E-05	1.40E-05	1.45E-05
2^{-8}	4.94E-05	3.91E-05	1.38E-05	1.41E-05
2^{-9}	3.14E-05	1.96E-05	1.38E-05	1.39E-05
2^{-10}	2.26E-05	9.82E-06	1.38E-05	1.39E-05

Table 6. Maximum errors for (5.2) with $\Phi(t) = \frac{t}{2}$ and $\alpha = 0.8$.

h	Ex. PI Rec.	Im. PI Rec.	Im. PI Trap.	Adams PC
2^{-4}	3.80E-03	2.90E-03	4.11E-04	4.11E-04
2^{-5}	1.80E-03	1.50E-03	1.29E-04	1.29E-04
2^{-6}	8.72E-04	7.92E-04	4.09E-05	4.09E-05
2^{-7}	4.28E-04	4.04E-04	1.31E-05	1.31E-05
2^{-8}	2.11E-04	2.04E-04	4.22E-06	4.22E-06
2^{-9}	1.05E-04	1.03E-04	1.37E-06	1.37E-06
2^{-10}	5.22E-05	5.17E-05	4.45E-07	4.45E-07

6. Conclusions and future work

Four numerical schemes namely explicit product integration rectangular rule, implicit product integration rectangular rule, implicit product integration trapezoidal rule, and Adams method are extended to solve the nonlinear Φ -Caputo fractional differential problems. We also analyze the error estimation and stability for those numerical schemes. When the exact solutions are known as in Examples 5.1 and 5.2, the implicit product integration trapezoidal rule and Adams predictor-corrector method can perform comparatively better than the explicit product integration rectangular and the implicit product integration rectangular rules, where the convergence order depends on the step size h .

Moreover, these schemes are investigated for the linear and nonlinear Φ -Caputo fractional differential equations. Further studies on numerical methods for fractional differential equations based on Φ -Caputo derivative could be investigated for various types of fractional differential equations such as the fractional differential equations with delay and a nonlocal term, integro-differential equations, and higher order fractional differential equations.

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Conflict of interest

The authors declare no conflict of interest in this paper.

References

1. O. P. Agrawal, Some generalized fractional calculus operators and their applications in integral equations, *Fract. Calc. Appl. Anal.*, **15** (2012), 700–711. <https://doi.org/10.2478/s13540-012-0047-7>
2. R. Almeida, A caputo fractional derivative of a function with respect to another function, *Commun. Nonlinear Sci.*, **44** (2017), 460–481. <https://doi.org/10.1016/j.cnsns.2016.09.006>
3. R. Almeida, M. Jleli, B. Samet, A numerical study of fractional relaxation-oscillation equations involving ψ -caputo fractional derivative, *RACSAM*, **113** (2019), 1873–1891. <https://doi.org/10.1007/s13398-018-0590-0>
4. R. Almeida, A. B. Malinowska, M. T. T. Monteiro, Fractional differential equations with a caputo derivative with respect to a kernel function and their applications, *Math. Method. Appl. Sci.*, **41** (2018), 336–352. <https://doi.org/10.1002/mma.4617>
5. H. Alrabaiah, I. Ahmad, R. Amin, K. Shah, A numerical method for fractional variable order pantograph differential equations based on haar wavelet, *Eng. Comput.*, **38** (2022), 2655–2668. <https://doi.org/10.1007/s00366-020-01227-0>
6. R. Amin, H. Ahmad, K. Shah, M. B. Hafeez, W. Sumelka, Theoretical and computational analysis of nonlinear fractional integro-differential equations via collocation method, *Chaos Soliton. Fract.*, **151** (2021), 111252. <https://doi.org/10.1016/j.chaos.2021.111252>
7. R. Amin, B. Alshahrani, M. Mahmoud, A. H. Abdel-Aty, K. Shah, W. Deebani, Haar wavelet method for solution of distributed order time-fractional differential equations, *Alex. Eng. J.*, **60** (2021), 3295–3303. <https://doi.org/10.1016/j.aej.2021.01.039>
8. R. Amin, N. Senu, M. B. Hafeez, N. I. Arshad, A. Ahmadian, S. Salahshour, et al., A computational algorithm for the numerical solution of nonlinear fractional integral equations, *Fractals*, **30** (2021), 2240030. <https://doi.org/10.1142/S0218348X22400308>

9. R. Amin, K. Shah, H. Ahmad, A. H. Ganie, A. H. Abdel-Aty, T. Botmart, Haar wavelet method for solution of variable order linear fractional integro-differential equations, *AIMS Mathematics*, **7** (2022), 5431–5443. <https://doi.org/10.3934/math.2022301>
10. R. Amin, K. Shah, M. Asif, I. Khan, A computational algorithm for the numerical solution of fractional order delay differential equations, *Appl. Math. Comput.*, **402** (2021), 125863. <https://doi.org/10.1016/j.amc.2020.125863>
11. R. Amin, K. Shah, M. Asif, I. Khan, F. Ullah, An efficient algorithm for numerical solution of fractional integro-differential equations via haar wavelet, *J. Comput. Appl. Math.*, **381** (2021), 113028. <https://doi.org/10.1016/j.cam.2020.113028>
12. D. Baleanu, G. C. Wu, Y. R. Bai, F. L. Chen, Stability analysis of caputo-like discrete fractional systems, *Commun. Nonlinear Sci.*, **48** (2017), 520–530. <https://doi.org/10.1016/j.cnsns.2017.01.002>
13. K. Burrage, N. Hale, D. Kay, An efficient implicit fem scheme for fractional-in-space reaction-diffusion equations, *SIAM J. Sci. Comput.*, **34** (2012), A2145–A2172. <https://doi.org/10.1137/110847007>
14. M. A. Chaudhry, A. Qadir, M. Rafique, S. Zubair, Extension of euler’s beta function, *J. Comput. Appl. Math.*, **78** (1997), 19–32. [https://doi.org/10.1016/S0377-0427\(96\)00102-1](https://doi.org/10.1016/S0377-0427(96)00102-1)
15. W. Chen, J. Zhang, J. Zhang, A variable-order time-fractional derivative model for chloride ions sub-diffusion in concrete structures, *Fract. Calc. Appl. Anal.*, **16** (2013), 76–92. <https://doi.org/10.2478/s13540-013-0006-y>
16. C. Derbazi, Z. Baitiche, M. Benchohra, A. Cabada, Initial value problem for nonlinear fractional differential equations with ψ -caputo derivative via monotone iterative technique, *Axioms*, **9** (2020), 57. <https://doi.org/10.3390/axioms9020057>
17. K. Diethelm, An investigation of some nonclassical methods for the numerical approximation of caputo-type fractional derivatives, *Numer. Algor.*, **47** (2008), 361–390. <https://doi.org/10.1007/s11075-008-9193-8>
18. K. Diethelm, *The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type*, Springer Science & Business Media, 2010.
19. K. Diethelm, An efficient parallel algorithm for the numerical solution of fractional differential equations, *Fract. Calc. Appl. Anal.*, **14** (2011), 475–490. <https://doi.org/10.2478/s13540-011-0029-1>
20. K. Diethelm, N. J. Ford, A. D. Freed, A predictor-corrector approach for the numerical solution of fractional differential equations, *Nonlinear Dynam.*, **29** (2002), 3–22. <https://doi.org/10.1023/A:1016592219341>
21. Y. Y. Gambo, F. Jarad, D. Baleanu, T. Abdeljawad, On caputo modification of the hadamard fractional derivatives, *Adv. Differ. Equ.*, **2014** (2014), 10. <https://doi.org/10.1186/1687-1847-2014-10>
22. R. Garrappa, Numerical solution of fractional differential equations: A survey and a software tutorial, *Mathematics*, **6** (2018), 16. <https://doi.org/10.3390/math6020016>

23. R. Garrappa, On linear stability of predictor-corrector algorithms for fractional differential equations, *Int. J. Comput. Math.*, **87** (2010), 2281–2290. <https://doi.org/10.1080/00207160802624331>
24. R. Garrappa, M. Popolizio, On accurate product integration rules for linear fractional differential equations, *J. Comput. Appl. Math.*, **235** (2011), 1085–1097. <https://doi.org/10.1016/j.cam.2010.07.008>
25. R. Garrappa, M. Popolizio, Evaluation of generalized mittag-leffler functions on the real line, *Adv. Comput. Math.*, **39** (2013), 205–225. <https://doi.org/10.1007/s10444-012-9274-z>
26. M. Gohar, C. Li, Z. Li, Finite difference methods for caputo-hadamard fractional differential equations, *Mediterr. J. Math.*, **17** (2020), 194. <https://doi.org/10.1007/s00009-020-01605-4>
27. F. Jarad, T. Abdeljawad, D. Baleanu, Caputo-type modification of the hadamard fractional derivatives, *Adv. Differ. Equ.*, **2012** (2012), 142. <https://doi.org/10.1186/1687-1847-2012-142>
28. A. A. Kilbas, H. M. Srivastava, J. J. Trujill, *Theory and applications of fractional differential equations*, Isevier, 2006.
29. A. A. Kilbas, O. Marichev, S. Samko, *Fractional integrals and derivatives: Theory and applications*, Switzerland: Gordon and Breach, 1993.
30. J. D. Lambert, *Numerical methods for ordinary differential systems: The initial value problem*, John Wiley & Sons, 1991.
31. C. Li, A. Chen, Numerical methods for fractional partial differential equations, *Int. J. Comput. Math.*, **95** (2018), 1048–1099. <https://doi.org/10.1080/00207160.2017.1343941>
32. H. L. Li, C. Hu, L. Zhang, H. Jiang, J. Cao, Complete and finite-time synchronization of fractional-order fuzzy neural networks via nonlinear feedback control, *Fuzzy Set. Syst.*, **443** (2022), 50–69. <https://doi.org/10.1016/j.fss.2021.11.004>
33. Y. Luchko, J. Trujillo, Caputo-type modification of the erdélyi-kober fractional derivative, *Fract. Calc. Appl. Anal.*, **10** (2007), 249–267.
34. R. Metzler, J. H. Jeon, A. G. Cherstvy, E. Barkai, Anomalous diffusion models and their properties: non-stationarity, non-ergodicity, and ageing at the centenary of single particle tracking, *Phys. Chem. Chem. Phys.*, **16** (2014), 24128–24164. <https://doi.org/10.1039/C4CP03465A>
35. J. F. Reverey, J. H. Jeon, H. Bao, M. Leippe, R. Metzler, C. Selhuber-Unkel, Superdiffusion dominates intracellular particle motion in the supercrowded cytoplasm of pathogenic *acanthamoeba castellanii*, *Sci. Rep.*, **5** (2015), 11690. <https://doi.org/10.1038/srep11690>
36. A. Suechoei, P. S. Ngiamsunthorn, Extremal solutions of φ -caputo fractional evolution equations involving integral kernels, *AIMS Mathematics*, **6** (2021), 4734–4757. <https://doi.org/10.3934/math.2021278>
37. V. E. Tarasov, V. V. Tarasova, Long and short memory in economics: Fractional-order difference and differentiation, 2016, arXiv: 1612.07903V3.
38. G. C. Wu, D. Baleanu, H. P. Xie, F. L. Chen, Chaos synchronization of fractional chaotic maps based on the stability condition, *Physica A*, **460** (2016), 374–383. <https://doi.org/10.1016/j.physa.2016.05.045>

-
39. A. Young, Approximate product-integration, *Proc. Roy. Soc. A*, **224** (1954), 552–561. <https://doi.org/10.1098/rspa.1954.0179>
40. B. Yu, X. Jiang, C. Wang, Numerical algorithms to estimate relaxation parameters and caputo fractional derivative for a fractional thermal wave model in spherical composite medium, *Appl. Math. Comput.*, **274** (2016), 106–118. <https://doi.org/10.1016/j.amc.2015.10.081>



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