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Research article

On the linearized system of elasticity in the half-space

Nabil Kerdid*

Department of Mathematics and Statistics, College of Sciences, Imam Mohammad Ibn Saud Islamic University (IMSIU), PO-Box 90950, Riyadh 11623, KSA

* **Correspondence:** Email: nkerdid@gmail.com; Tel: +966508861145; Fax: +966112591040.

Abstract: The purpose of this paper is twofold. The first goal is to provide a simple and constructive proof of Korn inequalities in half-space with weighted norms. The proof leads to explicit values of the constants. The second objective is to use these inequalities to show that the linear elasticity system in half-space admits a coercive variational formulation. This formulation corresponds to the physical case in which the solution is evanescent at infinity.

Keywords: linear elasticity, unbounded domains; Korn's inequality; half-space; weighted spaces
Mathematics Subject Classification: 35E20, 35Q30, 76D07

1. Introduction

In this paper, we consider the linear elasticity equations in the half-space of $\mathbb{R}^n$ (see e.g., [1,2])

$$\begin{cases} -\displaystyle\sum_{j=1}^{n} \frac{\partial \sigma_{i,j}(\boldsymbol{u})}{\partial x_j} = f_i & \text{in } \mathbb{R}^n_+,\ i = 1, ..., n, \\ \boldsymbol{u} = \boldsymbol{g} & \text{at } x_n = 0, \end{cases} \tag{1.1}$$

where $\mathbb{R}^n_+ = \{\boldsymbol{x} = (x_1, ..., x_n) \in \mathbb{R}^n \ ; \ x_n > 0\}$ is the upper half-space of $\mathbb{R}^n$, $\boldsymbol{f} = (f_1, ..., f_n)$ is the body force and $\boldsymbol{g}$ is a prescribed displacement on the boundary $x_n = 0$. The stress tensor $(\sigma_{i,j})$ is related to the strain tensor $(\boldsymbol{\varepsilon}_{i,j})$ by the constitutive relation

$$\sigma_{i,j}(\boldsymbol{u}) = \lambda(\operatorname{div}\boldsymbol{u})\delta_{i,j} + 2\mu\,\boldsymbol{\varepsilon}_{i,j}(\boldsymbol{u}),\ 1 \le i, j \le n, \tag{1.2}$$

with

$$\boldsymbol{\varepsilon}_{i,j}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),\ 1 \le i, j \le n. \tag{1.3}$$

Here $\lambda > 0$ and $\mu > 0$ denote the Lamé coefficients.

Many difficulties arising in studying the system (1.1) are due to the unboundedness of the domain $\mathbb{R}^n_+$ and of its boundary $\{x_n = 0\}$. Moreover, system (1.1) must be completed with an asymptotic condition on $\boldsymbol{u}$ at large distances, that is when $|\boldsymbol{x}| \longrightarrow +\infty$. Qualitatively speaking, besides boundary conditions we shall further require that

$$\lim_{|\boldsymbol{x}|\longrightarrow+\infty} |\boldsymbol{u}(\boldsymbol{x})| = 0.$$

As in the case of a bounded domain (see, e.g., [1,2]), the study of the problem (1.1) is intimately related to Korn's inequalities

$$\|\frac{\boldsymbol{w}}{\langle x\rangle}\|_{L^2(\mathbb{R}^n_+)^n}, \quad \|\nabla \boldsymbol{w}\|_{L^2(\mathbb{R}^n_+)^{n^2}} \leq C\|\boldsymbol{\varepsilon}(\boldsymbol{w})\|_{L^2(\mathbb{R}^n_+)^{n^2}}.$$

In [4–6], generalizations of the Korn and Hardy inequalities were established for bounded domains and for various classes of unbounded domains and these inequalities were used to study the boundary value problem for the elasticity system under the assumption that the energy (Dirichlet) integral is finite. In [7–12], the uniqueness of the solution to the boundary value problems for the elasticity system and the biharmonic (polyharmonic) equation was investigated for a wide classe of unbounded domains with finite weighted energy (Dirichlet) integral and the dimension of the space of solutions was obtained.

In the approach we adopt here, we will propose a constructive proof of Korn's inequalities giving explicit values of the constants (see Proposition 3.1 below). For a non-constructive proof, the reader can refer to [1] and references therein. In particular, although we do not adopt this approach here, one could extend the proof based on the identity (see [1,3]):

$$\frac{\partial^2 w_i}{\partial x_k \partial x_j} = \frac{\partial(\boldsymbol{\varepsilon}(\boldsymbol{w}))_{i,k}}{\partial x_j} + \frac{\partial(\boldsymbol{\varepsilon}(\boldsymbol{w}))_{i,j}}{\partial x_k} - \frac{\partial(\boldsymbol{\varepsilon}(\boldsymbol{w}))_{j,k}}{\partial x_i} \text{ for } 1 \leq i, j, k \leq 3.$$

Concerning the system (1.1) we shall employ weighted function spaces to clarify what is going on in the remote regions of $\mathbb{R}^n$. Our approach is inspired from [13–22] where these spaces were used with success for studying the Poisson equation and the Stokes system in the whole space and in the half-space. Here, we prove on the one hand that the system (1.1) completed with asymptotic conditions is well posed in adequate function spaces. On the other hand, we give a regularity result by using a reflection principle which enables to use some isomorphism properties of the Laplace operator in the whole space. Explicit estimate on the solution are also given.

What remains of this paper is organized as follows. In Section 2, we give some basic definitions and notations concerning weighted Sobolev spaces. In Section 3, we state the main results and we outline proofs. The use of a weighted Korn inequality and a reflection principle are at the heart of our approach.

2. Preliminaries

Throughout, given a point $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ of $\mathbb{R}^n$, we write

$$\boldsymbol{x'} = (x_1, ..., x_{n-1}), \ |x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2} \text{ and } \langle x\rangle = (1 + |\boldsymbol{x}|^2)^{1/2}.$$

Notice that for any $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{N}^n$ and $s \in \mathbb{R}$, one has

$$|D^\lambda \langle x\rangle^s| \leq C\langle x\rangle^{s-|\lambda|},$$

for a constant $C > 0$, depending only on λ and s. Here D^λ denotes the usual derivation operator $D^\lambda = \partial_1^{\lambda_1}\partial_2^{\lambda_2}\cdots\partial_n^{\lambda_n}$ and $|\lambda| = \lambda_1 + \cdots + \lambda_n$. The upper and lower half-spaces of $\mathbb{R}^n$ are

$$\mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n; x_n > 0\},\ \mathbb{R}^n_- = \{\boldsymbol{x} \in \mathbb{R}^n; x_n < 0\}.$$

Their common boundary is the hyperplane

$$\Sigma = \{\boldsymbol{x} \in \mathbb{R}^n\ ;\ x_n = 0\} \equiv \mathbb{R}^{n-1}.$$

Denote by $\mathbb{S}^n$ the unit sphere of $\mathbb{R}^n$ and set

$$\mathbb{S}^+_{n-1} = \{\boldsymbol{x} \in \mathbb{S}^n; x_n > 0\}.$$

By $L^2(\mathbb{R}^n_+)$ we mean the usual Lebesgue space of real measurable and square integrable functions over $\mathbb{R}^n_+$. We denote by $\|.\|_{L^2(\mathbb{R}^n_+)}$, the corresponding norm, that is

$$\|w\|_{L^2(\mathbb{R}^n_+)} = \left(\int_{\mathbb{R}^n_+} |w|^2 dx\right)^{1/2}.$$

The symbols $\langle .,.\rangle_{\mathbb{R}^n_+}$ and $\langle .,.\rangle_\Sigma$, or more simply $\langle .,.\rangle$ will be used to designate various duality pairing on $\mathbb{R}^n_+$ and on Σ.

We are now in position to introduce a family of weighted spaces we use here (see, e.g., Hanouzet [23], Giroire [13, 24] and Boulmezaoud [15, 16]). For each integer $m \geq 0$ and each real $\theta \in \mathbb{R}$, $W^m_\theta(\mathbb{R}^n_+)$ stands for the space of -class of- measurable functions on $\mathbb{R}^n_+$ satisfying

$$\forall |\lambda| \leq m,\ \int_{\mathbb{R}^n_+} (1 + |\boldsymbol{x}|^2)^{\theta - m + |\lambda|} |D^\lambda w|^2 dx < \infty.$$

Naturally, the corresponding norm is

$$\|w\|_{W^m_\theta(\mathbb{R}^n_+)} = (\sum_{0\leq|\lambda|\leq m} \int_{\mathbb{R}^n_+} (1 + |\boldsymbol{x}|^2)^{\theta - m + |\lambda|} |D^\lambda w|^2 dx)^{1/2}.$$

Endowed with this latter norm, $W^m_\theta(\mathbb{R}^n_+)$ is a Banach space. We obviously have the continuous embeddings

$$W^m_\theta(\mathbb{R}^n_+) \subset W^{m-1}_{\theta-1}(\mathbb{R}^n_+) \subset \cdots \subset W^0_{\theta-m}(\mathbb{R}^n_+).$$

Notice that the mapping $u \in W^m_\theta(\mathbb{R}^n_+) \to \langle x\rangle^s u \in W^m_{\theta-s}(\mathbb{R}^n_+)$, $s \in \mathbb{R}$, is an isomorphism. Moreover, the mapping $u \in W^m_\theta(\mathbb{R}^n_+) \to D^\lambda u \in W^{m-|\lambda|}_\theta(\mathbb{R}^n_+)$, $|\lambda| \leq m$, is continuous.

Set $\mathring{W}^m_\theta(\mathbb{R}^n_+)$ the closure of $\mathscr{D}(\mathbb{R}^n_+)$ in $W^m_\theta(\mathbb{R}^n_+)$ and set $W^{-m}_{-\theta}(\mathbb{R}^n_+)$ its dual. We can also consider the boundary space $W^{1/2}_0(\Sigma)$ composed of all distributions $z \in \mathscr{D}'(\Sigma) = \mathscr{D}'(\mathbb{R}^{n-1})$ such that $\langle x\rangle^{-1/2} z \in L^2(\Sigma)$ and

$$\int_0^\infty t^{-2} \int_\Sigma |z(\boldsymbol{x} + t\boldsymbol{e}_i) - z(\boldsymbol{x})|^2 dxdt < \infty \quad , \quad \forall i = 1, 2, ..., n-1.$$

This space is equipped with the norm

$$\|z\|_{W^{1/2}_0(\Sigma)} = \left(\int_\Sigma \frac{|z|^2}{\langle x\rangle} dx + \sum_{i=1}^{n-1} \int_{\mathbb{R}_+\times\Sigma} t^{-2} |z(\boldsymbol{x} + t\boldsymbol{e}_i) - z(\boldsymbol{x})|^2 dxdt\right)^{\frac{1}{2}}.$$

For later use, we also need the space

$$W_0^{3/2}(\Sigma) = \{z \in W^1_{-1/2}(\Sigma),\ \ \frac{\partial z}{\partial x_i} \in W_0^{1/2}(\Sigma),\ \ \text{for } i \leq n-1\},$$

endowed with the norm

$$\|z\|_{W_0^{3/2}(\Sigma)} = (\|z\|^2_{W^1_{-1/2}(\Sigma)} + \sum_{i=1}^{n-1} \|\frac{\partial z}{\partial x_i}\|^2_{W_0^{1/2}(\Sigma)})^{1/2}.$$

Lastly, consider the space

$$W_1^{3/2}(\Sigma) = \{z \in \mathscr{D}'(\Sigma)\ ;\ \langle x\rangle z \in W_0^{3/2}(\Sigma)\}$$

endowed with the norm

$$\|u\|_{W_1^{3/2}(\Sigma)} = \|\langle x\rangle u\|_{W_0^{3/2}(\Sigma)}.$$

Hanouzet [23] proved that the trace map

$$\gamma_0 \quad : \quad \mathscr{D}(\overline{\mathbb{R}^n_+}) \to \mathscr{D}(\Sigma),$$

defined by

$$\gamma_0(u)(\boldsymbol{x}) = u(\boldsymbol{x}', 0)$$

can be extended by continuity to a linear continuous mapping (still denoted by γ_0) from $W_0^1(\mathbb{R}^n_+)$ into $W_0^{1/2}(\Sigma)$. Moreover, γ_0 is onto and ker $\gamma_0 = \mathring{W}_0^1(\mathbb{R}^n_+)$. Similary, the map

$$\gamma_1 \quad : \quad \mathscr{D}(\overline{\mathbb{R}^n_+}) \to \mathscr{D}(\Sigma) \times \mathscr{D}(\Sigma),$$

defined as

$$\gamma_1(u)(\boldsymbol{x}) = (u(\boldsymbol{x}', 0), \frac{\partial u}{\partial x_n}(\boldsymbol{x}', 0))$$

can also be extended to a linear continuous operator from $W_1^2(\mathbb{R}^n_+)$ onto $W_1^{3/2}(\Sigma) \times W_1^{1/2}(\Sigma)$.
It is worth noting that the definition of $W^s_\theta(\mathbb{R}^n)$ can be extended to all real numbers s (see [23]). This extension is unnecessary here, except when $s = 1/2$ or $s = 3/2$.

3. The main result

Before dealing with system (1.1), we state the following result concerning the Korn's type inequalities in the half-space, but with explicitly given constants.

Proposition 3.1. *Assume that $n \geq 3$. The following inequalities hold for each $\boldsymbol{w} \in \mathring{W}_0^1(\mathbb{R}^n_+)^n$:*

$$\|\frac{\boldsymbol{w}}{\langle x\rangle}\|_{L^2(\mathbb{R}^n_+)^n} \leq \frac{2\sqrt{2}}{n-2}\|\boldsymbol{\varepsilon}(\boldsymbol{w})\|_{L^2(\mathbb{R}^n_+)^{n^2}}, \tag{3.1}$$

$$\|\nabla \boldsymbol{w}\|_{L^2(\mathbb{R}^n_+)^{n^2}} \leq \sqrt{2}\|\boldsymbol{\varepsilon}(\boldsymbol{w})\|_{L^2(\mathbb{R}^n_+)^{n^2}}, \tag{3.2}$$

$$\|\operatorname{div} \boldsymbol{w}\|_{L^2(\mathbb{R}^n_+)} \leq \|\boldsymbol{\varepsilon}(\boldsymbol{w})\|_{L^2(\mathbb{R}^n_+)^{n^2}}. \tag{3.3}$$

In particular, the semi-norm $\boldsymbol{w} \in \mathring{W}_0^1(\mathbb{R}^n_+)^n \longrightarrow \|\boldsymbol{\varepsilon}(\boldsymbol{w})\|_{L^2(\mathbb{R}^n_+)^{n^2}}$ is a norm on $\mathring{W}_0^1(\mathbb{R}^n_+)^n$ equivalent to the norm $\|.\|_{W_0^1(\mathbb{R}^n_+)^n}$.

Remark 3.2. *Inequalities (3.2) and (3.3) remain valid when $n = 1$ or $n = 2$. On the other hand, when $n = 1$, inequality* (3.1) *is no longer valid because the constant vector functions belong to $\mathring{W}_0^1(\mathbb{R}_+)$.*

Proof. Let $\boldsymbol{\varphi} \in \mathscr{D}(\mathbb{R}^n_+)^n$. By Green's formula, we have

$$\begin{aligned}\sum_{k,j=1}^{n}\int_{\mathbb{R}^n}\frac{\partial\varphi_j}{\partial x_k}\varphi_j\frac{x_k}{\langle x\rangle^2}dx &= -\frac{1}{2}\sum_{k,j=1}^{n}\int_{\mathbb{R}^n}\frac{\varphi_j^2}{\langle x\rangle^2}dx+\sum_{k,j=1}^{n}\int_{\mathbb{R}^n}\frac{x_k^2\varphi_j^2}{\langle x\rangle^4}dx\\ &= -\frac{n}{2}\int_{\mathbb{R}^n}\frac{|\boldsymbol{\varphi}|^2}{\langle x\rangle^2}dx+\int_{\mathbb{R}^n}\frac{|x|^2|\boldsymbol{\varphi}|^2}{\langle x\rangle^4}dx\\ &\le \frac{2-n}{2}\int_{\mathbb{R}^n}\frac{|\boldsymbol{\varphi}|^2}{\langle x\rangle^2}dx.\end{aligned}$$

It follows that for $n \ge 3$

$$\begin{aligned}\frac{n-2}{2}\int_{\mathbb{R}^n}\frac{|\boldsymbol{\varphi}|^2}{\langle x\rangle^2}dx &\le \sum_{k,j=1}^{n}\int_{\mathbb{R}^n}|\frac{\partial\varphi_j}{\partial x_k}\varphi_j\frac{x_k}{\langle x\rangle^2}|dx\\ &\le \sum_{k,j=1}^{n}\left(\int_{\mathbb{R}^n}(\frac{\partial\varphi_j}{\partial x_k})^2dx\right)^{1/2}\left(\int_{\mathbb{R}^n}\frac{x_k^2\varphi_j^2}{\langle x\rangle^4}dx\right)^{1/2}\\ &\le \left(\sum_{k,j=1}^{n}\int_{\mathbb{R}^n}(\frac{\partial\varphi_j}{\partial x_k})^2dx\right)^{1/2}\left(\sum_{k,j=1}^{n}\int_{\mathbb{R}^n}\frac{x_k^2\varphi_j^2}{\langle x\rangle^4}dx\right)^{1/2}\\ &= \left(\int_{\mathbb{R}^n}|\nabla\boldsymbol{\varphi}|^2dx\right)^{1/2}\left(\int_{\mathbb{R}^n}\frac{|\boldsymbol{\varphi}|^2}{\langle x\rangle^2}dx\right)^{1/2}.\end{aligned}$$

We get the following Hardy's type inequality

$$\int_{\mathbb{R}^n}\frac{|\boldsymbol{\varphi}|^2}{\langle x\rangle^2}dx \le \frac{4}{(n-2)^2}\int_{\mathbb{R}^n}|\nabla\boldsymbol{\varphi}|^2dx. \tag{3.4}$$

We still denote by $\boldsymbol{\varphi} \in \mathscr{D}(\mathbb{R}^n)^n$ its extension by zero to the whole space. Let $\widehat{\boldsymbol{\varphi}}$ be the Fourier transform of $\boldsymbol{\varphi}$ defined by the usual formula

$$\boldsymbol{\varphi}(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{-i\boldsymbol{\xi}.\boldsymbol{x}}\boldsymbol{\varphi}(\boldsymbol{x})dx, \quad \text{for } \boldsymbol{\xi} \in \mathbb{R}^n.$$

By Plancherel theorem, we can write

$$\begin{aligned}\int_{\mathbb{R}^n}|\boldsymbol{\varepsilon}(\boldsymbol{\varphi})|^2dx &= \sum_{k,j=1}^{n}\int_{\mathbb{R}^n}(\frac{\partial\varphi_k}{\partial x_j}+\frac{\partial\varphi_j}{\partial x_k})^2dx\\ &= \frac{1}{4}\sum_{k,j=1}^{n}\int_{\mathbb{R}^n}|\widehat{\frac{\partial\varphi_k}{\partial x_j}}+\widehat{\frac{\partial\varphi_j}{\partial x_k}}|^2d\xi\\ &= \frac{1}{4}\sum_{k,j=1}^{n}\int_{\mathbb{R}^n}|i\xi_j\widehat{\varphi}_k+i\xi_k\widehat{\varphi}_j|^2d\xi\\ &= \frac{1}{4}\sum_{k,j=1}^{n}\int_{\mathbb{R}^n}\left(\xi_j^2|\widehat{\varphi}_k|^2+\xi_k^2|\widehat{\varphi}_j|^2+\xi_j\xi_k(\widehat{\varphi}_j\overline{\widehat{\varphi}_k}+\widehat{\varphi}_k\overline{\widehat{\varphi}_j})\right)d\xi\\ &= \frac{1}{2}\sum_{k,j=1}^{n}\int_{\mathbb{R}^n}|\xi_j\widehat{\varphi}_k|^2dx+\frac{1}{2}\sum_{k,j=1}^{n}\int_{\mathbb{R}^n}\xi_j\widehat{\varphi}_j\overline{\xi_k\widehat{\varphi}_k}d\xi\\ &= \frac{1}{2}\int_{\mathbb{R}^n}|\nabla\boldsymbol{\varphi}|^2dx+\frac{1}{2}\int_{\mathbb{R}^n}|\operatorname{div}\boldsymbol{\varphi}|^2dx.\end{aligned}$$

Notice in addition that

$$\int_{\mathbb{R}^n} |\mathrm{div}\, \boldsymbol{\varphi}|^2 dx = \int_{\mathbb{R}^n} |\boldsymbol{\xi}.\widehat{\boldsymbol{\varphi}}|^2 d\xi \leq \int_{\mathbb{R}^n} |\boldsymbol{\xi}|^2 |\widehat{\boldsymbol{\varphi}}|^2 d\xi = \int_{\mathbb{R}^n} |\nabla \boldsymbol{\varphi}|^2 dx.$$

Thus,

$$\int_{\mathbb{R}^n} |\boldsymbol{\varepsilon}(\boldsymbol{\varphi})|^2 dx \geq \int_{\mathbb{R}^n} |\mathrm{div}\, \boldsymbol{\varphi}|^2 dx.$$

It follows that (3.2) and (3.3) are valid for functions in $\mathscr{D}(\mathbb{R}^n)^n$. Since $\mathscr{D}(\mathbb{R}^n_+)^n$ is dense in $\mathring{W}^1_0(\mathbb{R}^n_+)^n$, these inequalities are also valid for any $w \in \mathring{W}^1_0(\mathbb{R}^n)^n$. Inequality (3.1) follows from (3.2) and (3.4). □

The main result is the following:

Theorem 3.3. *Suppose that $n \geq 3$. Then, for all $\boldsymbol{f} \in W^{-1}_0(\mathbb{R}^n_+)^n$ and all $\boldsymbol{g} \in W^{1/2}_0(\Sigma)^n$, the system* (1.1) *has one and only one solution $\boldsymbol{u} \in W^1_0(\mathbb{R}^n_+)^n$ and*

$$\|\boldsymbol{u}\|_{W^1_0(\mathbb{R}^n_+)^n} \leq C_1\{\|\boldsymbol{f}\|_{W^{-1}_0(\mathbb{R}^n_+)^n} + \|\boldsymbol{g}\|_{W^{1/2}_0(\Sigma)^n}\}, \tag{3.5}$$

where $C_1 > 0$ is a constant depending only on n. Moreover

- *we have*

$$\lim_{|\boldsymbol{x}|\longrightarrow+\infty} |\boldsymbol{x}|^{(n-2)/2} \|\boldsymbol{u}(|\boldsymbol{x}|.)\|^2_{L^2(\mathbb{S}^+_{n-1})} = 0. \tag{3.6}$$

- *if $\boldsymbol{f} \in W^0_1(\mathbb{R}^n_+)^n$ and $\boldsymbol{g} \in W^{3/2}_1(\Sigma)^n$, then $\boldsymbol{u} \in W^2_1(\mathbb{R}^n_+)^n$ and*

$$\|\boldsymbol{u}\|_{W^2_1(\mathbb{R}^n_+)^n} \leq C_2\{\|\boldsymbol{f}\|_{W^0_1(\mathbb{R}^n_+)^n} + \|\boldsymbol{g}\|_{W^{3/2}_1(\Sigma)^n}\}, \tag{3.7}$$

  where $C_2 > 0$ is a constant depending only on n.
- *if $\boldsymbol{f} \in W^0_1(\mathbb{R}^n_+)^n$ and $\boldsymbol{g} = \boldsymbol{0}$, then the following estimates hold*

$$\|\langle x\rangle^{-1}\boldsymbol{u}\|_{L^2(\mathbb{R}^n_+)^n} \leq \frac{4}{(n-2)^2\mu}\|\langle x\rangle \boldsymbol{f}\|_{L^2(\mathbb{R}^n_+)^n}, \tag{3.8}$$

$$\|\nabla \boldsymbol{u}\|_{L^2(\mathbb{R}^n_+)^{n^2}} \leq \frac{2}{(n-2)\mu}\|\langle x\rangle \boldsymbol{f}\|_{L^2(\mathbb{R}^n_+)^n}, \tag{3.9}$$

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{L^2(\mathbb{R}^n_+)} \leq \frac{\sqrt{2}}{(n-2)\mu}\|\langle x\rangle \boldsymbol{f}\|_{L^2(\mathbb{R}^n_+)^n}, \tag{3.10}$$

$$\|\mathrm{div}\, \boldsymbol{u}\|_{L^2(\mathbb{R}^n_+)} \leq \frac{2\sqrt{2}}{(n-2)\sqrt{\mu(\lambda+2\mu)}}\|\langle x\rangle \boldsymbol{f}\|_{L^2(\mathbb{R}^n_+)^n}. \tag{3.11}$$

Proof. The proof is composed of several steps and is based on some intermediate results. Before detailing these steps, let us observe that there exists a function $\boldsymbol{w}_0 \in W^1_0(\mathbb{R}^n_+)^n$ such that

$$\gamma_0 \boldsymbol{w}_0 = g, \text{ and } \|\boldsymbol{w}_0\|_{W^1_0((\mathbb{R}^n_+)^n} \leq C_0 \|\boldsymbol{g}\|_{W^{1/2}_0(\Sigma)^n},$$

with $C_0 > 0$ a constant not depending on $\boldsymbol{w}_0$ (see [23]). Set

$$\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{w}_0 \in \mathring{W}^1_0(\mathbb{R}^n_+)^n, \; \tilde{\boldsymbol{f}} = \boldsymbol{f} + \mathrm{div}\, \sigma(\boldsymbol{w}_0) \in W^{-1}_0(\mathbb{R}^n_+)^n.$$

The displacement $\boldsymbol{u} \in W_0^1(\mathbb{R}_+^n)^n$ is solution of (1.1) if and only if $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{w}_0 \in \mathring{W}_0^1(\mathbb{R}_+^n)^n$ is solution of the homogeneous problem

$$\begin{cases} -\displaystyle\sum_{j=1}^{n} \frac{\partial \sigma_{i,j}(\boldsymbol{v})}{\partial x_j} & = & \tilde{f}_i \text{ in } \mathbb{R}_+^n,\ i = 1, ..., n, \\ \boldsymbol{v} & = & 0 \text{ at } x_n = 0. \end{cases} \tag{3.12}$$

Using the density of $\mathscr{D}(\mathbb{R}_+^n)^n$ in $\mathring{W}_0^1(\mathbb{R}^n)^n$ we obtain easily the weak formulation of system (3.12): Find $\boldsymbol{v} \in \mathring{W}_0^1(\mathbb{R}_+^n)^n$ such that, for all $\boldsymbol{w} \in \mathring{W}_0^1(\mathbb{R}_+^n)^n$

$$\lambda \int_{\mathbb{R}_+^n} (\operatorname{div} \boldsymbol{v}).(\operatorname{div} \boldsymbol{w}) dx + 2\mu \int_{\mathbb{R}_+^n} \boldsymbol{\varepsilon}(\boldsymbol{v}).\boldsymbol{\varepsilon}(\boldsymbol{w}) dx = \langle \tilde{\boldsymbol{f}}, \boldsymbol{w} \rangle_{W_0^{-1}(\mathbb{R}_+^n)^n, \mathring{W}_0^1(\mathbb{R}_+^n)^n}. \tag{3.13}$$

According to Proposition 3.1, the bilinear form on the left-hand side of the weak formulation (3.13) is coercive on $\mathring{W}_0^1(\mathbb{R}_+^n)^n$. It is also continuous and so is the linear form on the right hand side. By Lax-Milgram lemma, we conclude that (3.13) has a unique solution $\boldsymbol{v} \in \mathring{W}_0^1(\mathbb{R}_+^n)^n$. This ends the proof of existence and uniqueness of a solution to (1.1). Since $\boldsymbol{u} \in W_0^1(\mathbb{R}_+^n)^n$, we also deduce that $\lim_{|\boldsymbol{x}| \longrightarrow +\infty} |\boldsymbol{x}|^{(n-2)/2} \|\boldsymbol{u}(|\boldsymbol{x}|.)\|^2_{L^2(\mathbb{S}^+_{n-1})} = 0$ (see, e.g., [24]).

Assume now that $\boldsymbol{f} \in W_1^0(\mathbb{R}_+^n)^n$ and $\boldsymbol{g} \in W_1^{3/2}(\Sigma)^n$. According to [23], then we can choose the function $\boldsymbol{w}_0$ here above in $W_1^2(\mathbb{R}_+^n)^n$ and such that it depends continuously on $\boldsymbol{g}$. We shall first prove that the difference $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{w}_0$ satisfies

$$\boldsymbol{v} \in W_1^2(\mathbb{R}_+^n),$$

by means of a reflection argument. Set

$$\boldsymbol{U}(\boldsymbol{x'}, x_n) = \begin{cases} (v_1(\boldsymbol{x'}, x_n), \cdots, v_n(\boldsymbol{x'}, x_n)) \text{ if } \boldsymbol{x} \in \mathbb{R}_+^n, \\ (v_1(\boldsymbol{x'}, -x_n), \cdots, v_{n-1}(\boldsymbol{x'}, -x_n), -v_n(\boldsymbol{x'}, -x_n)) \text{ if } \boldsymbol{x} \in \mathbb{R}_-^n. \end{cases}$$

Since $v = 0$ on Σ, we easily deduce that $U \in W_0^1(\mathbb{R}^n)^n$. Moreover, we have for $x \in \mathbb{R}_-^n$

$$(\operatorname{div} \boldsymbol{U})(\boldsymbol{x}) = (\operatorname{div} \boldsymbol{v})(\boldsymbol{x'}, -x_n),$$

and

$$\boldsymbol{\varepsilon}_{i,j}(\boldsymbol{U})(\boldsymbol{x}) = \boldsymbol{\varepsilon}_{i,j}(\boldsymbol{v})(\boldsymbol{x'}, -x_n) \quad \text{ for } 1 \leq i, j \leq n-1,$$

$$\boldsymbol{\varepsilon}_{i,n}(\boldsymbol{U})(\boldsymbol{x}) = -\boldsymbol{\varepsilon}_{i,n}(\boldsymbol{v})(\boldsymbol{x'}, -x_n) \quad \text{ for } 1 \leq i \leq n-1,$$

$$\boldsymbol{\varepsilon}_{n,n}(\boldsymbol{U})(\boldsymbol{x}) = \boldsymbol{\varepsilon}_{n,n}(\boldsymbol{v})(\boldsymbol{x'}, -x_n).$$

Hence, we get in the sense of distributions

$$-\sum_{j=1}^{n} \frac{\partial \sigma_{i,j}(\boldsymbol{U})}{\partial x_j} = \boldsymbol{F}_i \quad \text{in } \mathbb{R}^n,\ i = 1, ..., n, \tag{3.14}$$

where

$$\boldsymbol{F}(\boldsymbol{x'}, x_n) = \begin{cases} (\tilde{f}_1(\boldsymbol{x'}, x_n), \cdots, \tilde{f}_n(\boldsymbol{x'}, x_n)) \quad \text{if } \boldsymbol{x'} \in \mathbb{R}_+^n, \\ (\tilde{f}_1(\boldsymbol{x'}, -x_n), \cdots, \tilde{f}_{n-1}(\boldsymbol{x'}, -x_n), -\tilde{f}_n(\boldsymbol{x'}, -x_n)) \quad \text{if } \boldsymbol{x'} \in \mathbb{R}_-^n. \end{cases}$$

Since $\tilde{\boldsymbol{f}} \in W_1^0(\mathbb{R}_+^n)^n$, we can easily deduce that $\boldsymbol{F} \in W_1^0(\mathbb{R}_+^n)^n$. In addition, we know that

$$\sum_{j=1}^{n} \frac{\partial \sigma_{i,j}(\boldsymbol{U})}{\partial x_j} = (\lambda + \mu)\frac{\partial}{\partial x_i}(\operatorname{div} \boldsymbol{U}) + \mu \Delta U_i,\ i = 1, ..., n.$$

Hence

$$-(\lambda + \mu)\nabla(\operatorname{div} \boldsymbol{U}) - \mu\Delta U = F \text{ in } \mathbb{R}^n. \tag{3.15}$$

Applying the divergence operator to both sides gives

$$-(\lambda + 2\mu)\Delta(\operatorname{div} \boldsymbol{U}) = \operatorname{div} \boldsymbol{F} \text{ in } \mathbb{R}^n. \tag{3.16}$$

We know from [13] (see also [24]) that for $n \neq 2$ and $m \in \mathbb{Z}$, the Laplacian Δ defines an isomorphism from $W_m^{m+1}(\mathbb{R}^n)$ into $W_m^{m-1}(\mathbb{R}^n)$. So, since $\boldsymbol{F} \in W_1^0(\mathbb{R}^n)^n$, we deduce that there exists a unique $\boldsymbol{H} \in W_1^2(\mathbb{R}^n)^n$ such that

$$\Delta \boldsymbol{H} = \boldsymbol{F}.$$

Set $G = -(\lambda + 2\mu)^{-1}\operatorname{div} \boldsymbol{H} \in W_1^1(\mathbb{R}^n) \hookrightarrow W_0^0(\mathbb{R}^n)$. Then,

$$-\Delta G = (\lambda + 2\mu)^{-1}\operatorname{div} \boldsymbol{F}.$$

Let $\psi = G - \operatorname{div} \boldsymbol{U}$. Then, $\psi \in W_0^0(\mathbb{R}^n)$ and $\Delta\psi = 0$. Since elements of $W_0^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ are tempered distributions, we deduce that ψ is a polynomial function of $L^2(\mathbb{R}^n)$. That is $\psi = 0$ and $\operatorname{div} \boldsymbol{U} = G \in W_1^1(\mathbb{R}^n)$. On the other hand, from (3.15) we have

$$-\mu\Delta U = \boldsymbol{F} + (\lambda + \mu)\nabla(\operatorname{div} \boldsymbol{U}).$$

Since $\operatorname{div} \boldsymbol{U} \in W_1^1(\mathbb{R}^n)$, we deduce that $\nabla(\operatorname{div} \boldsymbol{U}) \in W_1^0(\mathbb{R}^n)^n$ and $\Delta \boldsymbol{U} \in W_1^0(\mathbb{R}^n)^n$. Then,

$$\boldsymbol{U} \in W_1^2(\mathbb{R}^n)^n.$$

Therefore, $\boldsymbol{u} \in W_1^2(\mathbb{R}_+^n)^n$. Moreover, if $\boldsymbol{g} = \boldsymbol{0}$, by (3.13) we get

$$\lambda\|\operatorname{div} \boldsymbol{u}\|^2_{L^2(\mathbb{R}_+^n)} + 2\mu\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|^2_{L^2(\mathbb{R}_+^n)^{n^2}} \leq \|\langle x\rangle \boldsymbol{f}\|_{L^2(\mathbb{R}_+^n)^n}\|\frac{\boldsymbol{u}}{\langle x\rangle}\|_{L^2(\mathbb{R}_+^n)^n}.$$

Combining with (3.1)–(3.4), gives (3.8)–(3.11). □

Remark 3.4. *(3.6) means the following*

$$|\boldsymbol{u}| = o(\frac{1}{|x|^{(n-2)/2}}) \textit{ when } |\boldsymbol{x}| \longrightarrow +\infty.$$

Remark 3.5. *(Extension to the two-dimensional case)*

Theorem 3.3 can be extended to the case $n = 2$. In the latter case, instead of $W_0^1(\mathbb{R}_+^n)$, one can consider the space $W_{0,\log}^1(\mathbb{R}_+^2)$ of all the (generalized) functions $\boldsymbol{v} \in \mathscr{D}'(\mathbb{R}_+^2)^2$ satisfying

$$\int_{\mathbb{R}_+^2} \frac{|\boldsymbol{v}|^2}{\langle x\rangle^2(\log(2 + |\boldsymbol{x}|^2))^2}dx < \infty,\ \int_{\mathbb{R}_+^2} |\nabla \boldsymbol{v}|^2 dx < \infty.$$

As in the proof of Proposition 3.1, the following Hardy inequality can be proved: For all $\boldsymbol{v} \in W^1_{0,\log}(\mathbb{R}^2_+)^2$:

$$\int_{\mathbb{R}^2_+} \frac{|\boldsymbol{v}|^2}{(|\boldsymbol{x}|^2+2)(\log(2+|\boldsymbol{x}|^2))^2} dx \leq C_1 \int_{\mathbb{R}^2_+} |\nabla \boldsymbol{v}|^2 dx, \tag{3.17}$$

with $C_1 > 0$ (see, e.g., [13, 14, 16]). Combining with (3.2) gives a two dimensional weighted Korn's inequality of the form

$$\int_{\mathbb{R}^2_+} \frac{|\boldsymbol{v}|^2}{(|\boldsymbol{x}|^2+2)(\log(2+|\boldsymbol{x}|^2))^2} dx \leq C_2 \int_{\mathbb{R}^2_+} |\boldsymbol{\varepsilon}(\boldsymbol{v})|^2 dx. \tag{3.18}$$

With these inequalities and a trace theorem on $x_n = 0$, Theorem 3.3 can be easily extended to the two dimensional case.

4. Conclusions

Korn's inequalities play a prominent role in the variational analysis of partial differential equations arising in linear elasticity theory. The Korn inequalities are essential in establishing existence, uniqueness, and stability of the solution. Boundary value problems for the elasticity system in bounded domains were quite well studied in the literature. Generalizations of Korn's inequalities for a wide class of unbounded domains were also established and used for the analysis of the main boundary value problems for the elasticity system. This article considers the boundary value problem for the elasticity system in the uper half-space of $\mathbb{R}^n$ with a Dirichlet boundary condition. We provide a simple and constructive proof of Korn inequalities and obtained explicit values of the constants. These inequalities were used to establish the coerciveness of the variational formulation of the linear elasticity system in half-space. Then, a regularity result is proved when the data is sufficiently smooth and explicit estimates on the solution are given.

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Conflict of interest

The author declares no conflict of interest.

References

1. P. G. Ciarlet, *Mathematical elasticity: Studies in mathematics and its applications*, Amsterdam: North-Holland Publishing Co., 1988.
2. G. Fichera, *Existence theorems in elasticity*, Berlin: Springer-Verlag, 1974.
3. G. Duvaut, J. L. Lions, *Les inéquations en mécanique et en physique*, Paris: Duvaut, 1972.

4. V. A. Kondratiev, O. A. Oleinik, Hardy and Korn inequalities for a class of unbounded domains and their applications in elasticity theory, *Dokl. Math.*, **41** (1990), 527–531.
5. V. A. Kondratiev, O. A. Oleinik, Boundary-value problems for the system of elasticity theory in unbounded domains. Korn's inequalities, *Russian Math. Surv.,* **43** (1998), 65–119. https://doi.org/10.1070/RM1988v043n05ABEH001945
6. V. A. Kondratiev, O. A. Oleinik, Hardy's and Korn's inequality and their application, *Rend. Mat. Appl. Ser. VII*, **10** (1990), 641–666.
7. O. A. Matevosyan, On solutions of boundary value problems for a system in the theory of elasticity and for the biharmonic equation in a half-space, *Differ. Equations*, **34** (1998), 803–808.
8. O. A. Matevosyan, Solutions of exterior boundary-value problems for the elasticity system in weighted spaces, *Sb. Math.*, **192** (2001), 1763–1798. https://doi.org/10.1070/SM2001v192n12ABEH000615
9. H. A. Matevossian, On solutions of mixed boundary-value problems for the elasticity system in unbounded domains, *Izvestiya Math.*, **67** (2003), 895–929. https://doi.org/10.1070/IM2003v067n05ABEH000451
10. H. A. Matevossian, On the biharmonic Steklov problem in weighted spaces, *Russ. J. Math. Phys.*, **24** (2017), 134–138. https://doi.org/10.1134/S1061920817010125
11. H. A. Matevossian, On the polyharmonic Neumann problem in weighted spaces, *Complex Var. Elliptic*, **64** (2019), 1–7. https://doi.org/10.1080/17476933.2017.1409740
12. H. A. Matevossian, Asymptotics and uniqueness of solutions of the elasticity system with the mixed Dirichlet–Robin boundary conditions, *MDPI Math.*, **8** (2020), 2241. https://doi.org/10.3390/math8122241
13. J. Giroire, *Etude de quelques problèmes aux limites extérieurs et résolution par équations intégrales*, Thèse de Doctorat d'Etat, Université Pierre et Marie Curie, Paris, 1987.
14. T. Z. Boulmezaoud, Espaces de Sobolev avec poids pour l'équation de Laplace dans le demi-espace, *C. R. Acad. Sci. Paris Sér. I Math.*, **328** (1999), 221–226.
15. T. Z. Boulmezaoud, On the Stokes system and on the biharmonic equation in the half-space: An approach via weighted Sobolev spaces, *Math. Method. Appl. Sci.*, **25** (2002), 373–398. https://doi.org/10.1002/mma.296
16. T. Z. Boulmezaoud, On the Laplace operator and on the vector potential problems in the half-space: An approach using weighted spaces, *Math. Method. Appl. Sci.*, **26** (2003), 633–669. https://doi.org/10.1002/mma.369
17. T. Z. Boulmezaoud, Vorticity-vector potential formulations of the Stokes equations in the half-space, *Math. Method. Appl. Sci.*, **28** (2005), 903–915. https://doi.org/10.1002/mma.596
18. N. Kerdid, A mixed formulation of the Stokes equations with slip conditions in exterior domains and the half-space, *Hiroshima Math. J.*, **48** (2018), 119–131. https://doi.org/10.32917/hmj/1533088823
19. V. Girault, J. Giroire, A. Sequeira, Formulation variationnelle en fonction courant-tourbillon du problème de Stokes extérieur dans des espaces de Sobolev à poids, *C. R. Acad. Sci. Paris Sér. I. Math.*, **313** (1991), 499–502.

20. V. Girault, A. Sequeira, A well-posed problem for the exterior Stokes equations in two and three dimensions, *Arch. Ration. Mech. An.*, **313** (1991), 499–502. https://doi.org/10.1007/BF00376137

21. V. Girault, The divergence, curl and Stokes operators in exterior domains of R^3, *Pitman Res. Notes Math. Ser.*, **291** (1993), 34–77.

22. V. Girault, The Stokes problem and vector potential operator in three-dimensional exterior domains: An approach in weighted Sobolev spaces, *Differ. Integral Equ.*, **7** (1994), 535–570.

23. B. Hanouzet, Espaces de Sobolev avec poids application au problème de Dirichlet dans un demi espace, *Rend. Semin. Mat. Univ. Pad.*, **46** (1971), 227–272.

24. F. Alliot, *Etude des équations stationnaires de Stokes et Navier-Stokes dans des domaines extérieurs*, PhD. Thesis, Ecole Nationale des Ponts et Chaussées, France, 1998.