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*Research article*

# Semi-analytical and numerical computation of fractal-fractional sine-Gordon equation with non-singular kernels

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**Abstract:** In this article, we study the nonlinear sine-Gordon equation (sGE) under Mittag-Leffler and exponential decay type kernels in a fractal fractional sense. The Laplace Adomian decomposition method (LADM) is applied to investigate the sGE under the above-mentioned operators. The convergence analysis is provided for the proposed method. The results are validated by considering numerical examples with different initial conditions for both kernels and confirm the competence of the proposed technique. It is revealed that the obtained series solutions of the model with fractal fractional operators converge to the exact solutions. The numerical results converge to the particular exact solutions, proving the significance of LADM for nonlinear systems under fractal fractional derivatives. The absolute error analysis between the exact and obtained series solutions with both operators is shown in the tabulated form. The physical interpretations of the attained results with different fractal and fractional parameters are discussed in detail.

**Keywords:** sine-Gordon Equation; Laplace transform; ABC and C-F fractal-fractional derivatives
**Mathematics Subject Classification:** 26A33, 35Cxx, 35Qxx, 35R11, 41Axx

## 1. Introduction

Fractional calculus (FC) is the field of mathematics that investigate the extension or generalization of classical derivatives and integrals with fractional order. The idea was first explored in a document by Leibnitz to L-Hospital in 1695 [1–3]. In the late $20^{th}$ century, FC gained great importance due to its implementation in different areas of physics and engineering. FC offers numerous useful tools for solving a variety of differential and integral equations. Particularly, the field of fractional derivatives has shown a growing interest in time-fractional partial and ordinary differential equations. It has been discovered that when imperceptible generators of time evolution are used, fractional time derivatives develops. The significance of studying fractional differential equations improved many concepts such as stability, equilibrium and time evolution in long time bounds [4, 5]. The complexity of natural physical phenomena has stimulated the development of mathematical models. The fundamental calculus is particularly useful for studying problems that arise in several applied disciplines. However, several physical systems may be better explained using fractional calculus, because it is more appropriate to study fractional order and fractal dimension with long-term memory and chaotic behaviour [6].

The applications of fractional calculus have currently been extended to certain fields such as biology, physics, viscoelasticity, probability, transport theory, potential theory, electrochemistry, scattering theory, ground water theory, finance, wave propagation, and fluid mechanics [7–11]. It should be noted that several forms of fractional derivatives including Caputo, Capote-Fabrizio (C-F), Riemann-Liouville and Atangana Baleanu in Caputo's sense (ABC) have been defined to model the above physical phenomena [12]. In 2017, Atangana proposed the fractal fractional (FF) derivative involves two constraints, one of which signifies the fractal derivative while the other demonstrating the power-law kernel, the Mittag-Leffler or exponential decay type kernels [13]. This extended concept performs better than classical and fractional derivative approaches. It is because studying with fractal-fractional derivatives enables us to study at each fractional operator and fractal dimension simultaneously [14–16]. The idea of FF derivative is very suitable to study complex problems due to the fact that by dealing with FF derivatives, one can study the fractional operator as well as fractal dimensions at the same time [13]. Presently, this concept has extensively been considered for to investigate the dynamics of COVID-19 SIR models [17] and the fluid flow in fluid mechanics [18].

The analytical solutions of nonlinear FDE's are sometimes impossible to calculate or even just possible with suitable presumptions. Therefore, many methods have been proposed, such as the variational iteration technique, Fourier spectral methods, finite difference schemes, Haar wavelet numerical technique and the two-step Adams-Bashforth method [19, 20]. Similarly, the Adomian has presented a novel iterative scheme applied to solve approximate solutions for classical and non-integer integrable/non-integrable DE's [21–23]. The Adomian decomposition method (ADM) has also been extensively applied to study linear/nonlinear FDE's [23, 24].

Here, we consider one-dimensional sGE

$$\phi_{tt}(x,t) - \phi_{xx}(x,t) + \sin\phi(x,t) = 0. \tag{1.1}$$

Equation (1.1) is a real-valued, non-linear hyperbolic wave equation/partial differential equation including the operator of D'Alembert and the sine of the indefinite function known as the classical/integer order sine-Gordon equation. In the $19^{th}$ century, this equation with different solution techniques became recognized to study various differential geometry phenomena. In the 1970s, the equation got great importance when it was understood that it leads to solitons kink and anti-kink solutions. Eq (1.1) has various real applications to investigate localized modes in single and multi-stacked long Josephson junctions [25–31] and in modeling of ferromagnetism, physics of elementary particles and quantum optics [32]. We will investigate Eq (1.1) with both $C$-F and $A$BC fractional operators together with fractal dimension using Laplace transform with ADM [33] with

$$\phi(x,0) = A(x) \ \ and \ \ \phi_t(x,0) = B(x), \tag{1.2}$$

## 2. Preliminaries

**Definition 1.** *For $\phi(t) \in H(a,\ b)$ and $\alpha \in [0, 1)$, the exponential decay type kernel is given by [34]*

$$^{CF}D^{\alpha}\phi(t) = \frac{Q(\alpha)}{1-\alpha}\int_a^t \exp\left(\frac{-\alpha(t-s)}{1-s}\right)\phi'(s)\,ds.$$

**Definition 2.** *Suppose that $\phi(t)$ is a continuous function and $\beta \in (0, 1)$, then the exponential decay type kernel with order $(\alpha, \beta)$ for a function $\phi(t)$ is given by [13]*

$$\begin{aligned}
{}_0^{FFE}D_t^{\alpha,\beta}\phi(t) &= \frac{Q(\alpha)}{1-\alpha}\frac{d}{dt^{\beta}}\int_0^t \exp\left(\frac{-\alpha(t-s)}{1-s}\right)\phi(s)ds,\\
{}_0^{FFE}D_t^{\alpha,\beta}\phi(t) &= \frac{Q(\alpha)t^{\beta-1}}{\beta(1-\alpha)}\frac{d}{dt}\int_0^t \exp\left(\frac{-\alpha(t-s)}{1-s}\right)\phi(s)ds.
\end{aligned}$$

More generally, we can write as

$${}_0^{FFE}D_t^{\alpha,\beta,\gamma}\phi(t) = \frac{Q(\alpha)}{1-\alpha}\frac{d^{\gamma}}{dt^{\beta}}\int_0^t \exp\left(\frac{-\alpha(t-s)}{1-s}\right)\phi(s)ds, \quad 0<\alpha,\beta\le 1.$$

**Definition 3.** *Let $\phi(t) \in H(a,\ b)$, and $\alpha \in [0, 1]$, the Mittag-Leffler kernel is given by [35]*

$${}_a^{ABC}D_t^{\alpha}\phi(t) = \frac{A(\alpha)}{1-\alpha}\int_a^t E_{\alpha}\left(\frac{-\alpha(t-s)^{\alpha}}{1-\alpha}\right)\phi'(s)ds, \quad n-1<\alpha\le n.$$

**Definition 4.** *Let $\phi(t)$ fractal differentiable and a continuous function with order $\beta \in (0, 1)$, then the Mittag-Leffler kernel with $(\alpha, \beta)$ is given by [13]*

$$\begin{aligned}
{}_0^{FFMR}D_t^{\alpha,\beta}\phi(t) &= \frac{A(\alpha)}{1-\alpha}\frac{d}{dt^{\beta}}\int_0^t E_{\alpha}\left(\frac{-\alpha(t-s)^{\alpha}}{1-\alpha}\right)\phi(s)ds, \quad A(\alpha)=1-\alpha+\frac{\alpha}{\gamma(\alpha)}.\\
{}_0^{FFMR}D_t^{\alpha,\beta}\phi(t) &= \frac{A(\alpha)t^{\beta-1}}{\beta(1-\alpha)}\frac{d}{dt}\int_0^t E_{\alpha}\left(\frac{-\alpha(t-s)^{\alpha}}{1-\alpha}\right)\phi(s)ds.
\end{aligned}$$

*More generally, we can write*

$${}_0^{FFMR}D_t^{\alpha,\beta,\gamma}\phi(t) = \frac{A(\alpha)}{1-\alpha}\frac{d^{\gamma}}{dt^{\beta}}\int_0^t \exp\left(\frac{-\alpha(t-s)}{1-s}\right)\phi(s)ds, \quad 0<\alpha,\beta\le 1.$$

**Definition 5.** *The fractal-Laplace transform with order $\alpha$ for a continuous function $\phi$ in $(a,\ b)$ is given by [13, 36]*

$$^{F}\mathbb{L}_s^{\alpha}(\phi(t)) = \int_0^{\infty} t^{\alpha-1}e^{-st}\phi(t)dt.$$

**Definition 6.** *The Laplace transform for Caputo fractional derivative of order $\alpha$ for a function $\phi(t)$ is defined as*

$$\mathbb{L}\left\{{}_{0}^{C}D_t^{\alpha}\phi(t)\right\} = s^{\alpha}\phi(s) - \sum_{k=0}^{n-1}\left(s^{(\alpha-k-1)}\phi(0)^{k}\right), \quad n-1<\alpha\leq n,$$

*where*

$$\mathbb{L}\{\phi(t)\} = \int_0^{\infty}\phi(t)e^{-st}dt = \phi(s).$$

It should be noted that by putting $\beta$=1, one can obtain the usual Laplace transform.

## 3. Fractal fractional sine-Gordon equation with *C*-F derivative

Here, we consider fractal fractional sine-Gordon equation with exponential decay kernel as

$$^{FFE}D_t^{\alpha,\beta}\phi - \phi_{xx} + \sin\phi = 0, \;\; 0<\alpha\leq 1, \;\; 0<\beta\leq 1, \tag{3.1}$$

with

$$\phi(x,0) = A(x) \quad and \quad \phi_t(x,0) = B(x), \tag{3.2}$$

By re-grouping Eq (3.1), we obtain

$$^{CF}D_t^{\alpha}\phi \;\; = \;\; \beta t^{\beta-1}\{\phi_{xx} - \sin\phi\}. \tag{3.3}$$

Applying Laplace transform to Eq (3.3), gives

$$\mathbb{L}\left\{^{CF}D_t^{\alpha}\phi\right\} \;\; = \;\; \mathbb{L}\left\{\beta t^{\beta-1}\{\phi_{xx} - \mathbb{N}(\phi)\}\right\}. \tag{3.4}$$

$$\mathbb{L}\{\phi\} \;\; = \;\; \frac{A(x)}{s} + \frac{B(x)}{s^2} + \frac{(s+(1-s)\alpha)}{s^2}\mathbb{L}\left[\beta t^{\beta-1}\{\phi_{xx} - \mathbb{N}(\phi)\}\right]. \tag{3.5}$$

Consider $\phi(x,t)$ in series form

$$\phi(x,t) = \sum_{n=0}^{\infty}\phi_n(x,t), \tag{3.6}$$

while the Taylor series expansion of $\mathbb{N}(\phi) = \sin\phi(x,t)$, gives

$$\mathbb{N}(\phi) \;\; = \;\; \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!}\cdots = \sum_{j=0}^{\infty}(-1)^{j+2}\frac{\phi^{2j+1}}{(2j+1)!}, \tag{3.7}$$

where the nonlinear term $\mathbb{N}(\phi)$ can be written as

$$\mathbb{N}(\phi) = \sum_{n=0}^{\infty}A_n, \tag{3.8}$$

where $A_n$ represents Adomian polynomial [37] described by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\mathbb{N}\left(\sum_{k=0}^{n}\lambda^k\phi_k\right)\right]_{\lambda=0}.$$

Now applying $\mathbb{L}^{-1}$ to Eq (3.5) together with Eqs (3.6), (3.8) and (3.2), we obtain

$$\sum_{n=0}^{\infty}\phi_n(x,t) = A(x) + tB(x) + \mathbb{L}^{-1}\left[\left(\frac{(s+(1-s)\alpha)}{s^2}\right)\mathbb{L}\left\{\beta t^{\beta-1}\left(\phi_{xx} - \sum_{n=0}^{\infty}A_n\right)\right\}\right]. \tag{3.9}$$

From Eq (3.9), we obtain

$$\begin{aligned}
\phi_0 &= A(x) + tB(x),\\
\phi_1 &= \mathbb{L}^{-1}\left[\left(\frac{(s+(1-s)\alpha)}{s^2}\right)\mathbb{L}\left\{\beta t^{\beta-1}(\phi_{0xx} - A_0)\right\}\right],\\
\phi_2 &= \mathbb{L}^{-1}\left[\left(\frac{(s+(1-s)\alpha)}{s^2}\right)\mathbb{L}\left\{\beta t^{\beta-1}(\phi_{1xx} - A_1)\right\}\right],\\
\phi_3 &= \mathbb{L}^{-1}\left[\left(\frac{(s+(1-s)\alpha)}{s^2}\right)\mathbb{L}\left\{\beta t^{\beta-1}(\phi_{2xx} - A_2)\right\}\right],\\
&\vdots
\end{aligned}$$

The obtained solution can be expressed as

$$\phi(x,t) = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \cdots. \tag{3.10}$$

## 4. Fractal fractional sine-Gordon equation with *A*BC derivative

Consider fractal fractional sine-Gordon with Mittag-Leffler kernel as

$${}^{FFM}D_t^{\alpha,\beta}\phi - \phi_{xx} + \sin\phi = 0, \quad 0 < \beta \le 1, \quad 1 < \alpha \le 2 \tag{4.1}$$

with

$$\phi(x,0) = A(x) \quad and \quad \phi_t(x,0) = B(x). \tag{4.2}$$

Using the Laplace transform on Eq (4.1)

$$\mathbb{L}\left\{{}^{CF}D_t^{\alpha}\phi\right\} = \mathbb{L}\left\{\beta t^{\beta-1}\left\{\phi_{xx} - \mathbb{N}(\phi)\right\}\right\}. \tag{4.3}$$

Expanding using Laplace transform, we obtain

$$\mathbb{L}\{\phi\} = \frac{A(x)}{s} + \frac{B(x)}{s^2} + \frac{(s+(1-s)\alpha)}{A(\alpha)s^2}\mathbb{L}\left[\beta t^{\beta-1}\left\{\phi_{xx} - \mathbb{N}(\phi)\right\}\right], \tag{4.4}$$

where $\alpha$ = fractional order, $\beta$ = fractal order and fractal order and $A(\alpha)$ is a scaling function such that $A(\alpha) = 1 - \alpha + \frac{\alpha}{\nu(\alpha)}$ *i.e.*, $A(1) = 1$. With the help of Eqs (3.6)–(3.8) and Eq (4.2), we obtain

$$\phi_0 = A(x) + tB(x),$$

$$\begin{aligned}
\phi_1 &= \mathbb{L}^{-1}\left[\left(1+\frac{\alpha}{s^{\alpha}}-\alpha\right)\mathbb{L}\left\{\beta t^{\beta-1}(\phi_{0xx}-A_0)\right\}\right],\\
\phi_2 &= \mathbb{L}^{-1}\left[\left(1+\frac{\alpha}{s^{\alpha}}-\alpha\right)\mathbb{L}\left\{\beta t^{\beta-1}(\phi_{1xx}-A_1)\right\}\right],\\
\phi_3 &= \mathbb{L}^{-1}\left[\left(1+\frac{\alpha}{s^{\alpha}}-\alpha\right)\mathbb{L}\left\{\beta t^{\beta-1}(\phi_{2xx}-A_2)\right\}\right],\\
&\vdots
\end{aligned}$$

The series solution can be written as

$$\phi = \sum_{n=0}^{\infty}\phi_n. \tag{4.5}$$

## 5. Convergence analysis

With the help of the following theorem, we investigate the convergence of the approximation series solutions derived using the suggested technique.

**Theorem 1.** *Let $A$ and $B$ are Banach spaces and $\mathbb{T}: A \to B$ is an operator such that $\forall a_1, a_1^* \in A$*

$$\|\mathbb{T}(a_1)-\mathbb{T}(a_1^*)\| \le k\|a_1-a_1^*\|,$$

*then $\mathbb{T}$ has a unique fixed point provide that $\mathbb{T}(a_1)=a_1$ [By Banach contraction theorem].*

*Let us consider the series in the form (see Eqs (3.10) and (4.5))*

$$\phi(x,t) = \sum_{n=0}^{\infty}\phi_n$$

*calculated by LADM then, we have*

$$\Phi(n) = \mathbb{T}(\Phi_{n-1}),\quad \Phi_{n-1} = \sum_{i=1}^{n-1} a_i,\quad n = 1,2,3,\cdots.$$

*Further consider that*

$$A_0 = a_0 \in B_r(a_1) \ \ where \ \ B_r(a_1) = \left\{a_1^* : \|a_1-a_1^*\| < r\right\},$$

*then*

*1)* $\Phi(n) \in B_r(a_1)$*,*
*2)* $\lim_{n\to\infty}\Phi(n) = a_1$.

*Proof.* To prove (1) and (2), we use mathematical induction.
1) Let the statement is true for $n=1$, then gives

$$\|A_1 - a_1\| = \|\mathbb{T}(A_0)-\mathbb{T}(a_1)\| \le k\|a_{10}-a_1\|,$$

which implies that it is also valid for $n-1$

$$\begin{aligned}
\|\Phi_{n-1} - a_1\| &\leq k^{n-1}\|a_{10} - a_1\|, \\
\|\Phi(n) - a_1\| &= \|\mathbb{T}(\Phi_{n-1}) - \mathbb{T}(a_1)\| \leq k\|\Phi_{n-1} - a_1\| \leq k^n\|A_0 - a_1\|, \\
\|\Phi(n) - a_1\| &= \|\mathbb{T}(\Phi_{n-1}) - \mathscr{T}(a_1)\| \leq k\|a_{10} - e\| \leq k^n\|a_{10} - a_1\|, \\
\|\Phi(n) - a_1\| &\leq k^n\|a_{10} - a_1\| \leq k^n r < r, \\
\Phi(n) &\in BQ_r(a_1).
\end{aligned}$$

2) Since we have $\|\Phi(n) - a_1\| \leq k^n\|a_{10} - a_1\|$, and $\lim_{n\to\infty} k^n = 0$, therefore

$$\lim_{n\to\infty}\|A - a_1\| = 0 \ \Rightarrow\ \lim_{n\to\infty}\Phi(n) = a_1.$$

Hence Theorem 1 is proved. □

*We can also demonstrate that*

$$\mathbb{G}(x,t) = \sum_{n=0}^{\infty}\mathbb{G}_n.$$

## 6. Numerical examples

To validate the proposed method, a specific examples with different subsidiary conditions of the governing equation is considered.

**Example 1.** *We consider Eq (1.1) with initial conditions*

$$\phi(x,0) = 0 \quad and \quad \phi_t(x,0) = 4\,\text{sech}(x). \tag{6.1}$$

*The exact solution of Eq (1.1) with initial conditions (6.1) can be obtained as*

$$\phi = 4\tanh^{-1}\left[t\,\text{sech}(x)\right]. \tag{6.2}$$

Here, we discuss two cases

**Case I**: Consider Eq (1.1) with exponential decay Kernel as

$${}^{FFE}D_t^{\alpha,\beta}\phi - \phi_{xx} + \sin\phi = 0, \quad 0 < \alpha,\ \beta \leq 1. \tag{6.3}$$

Proceeding with the techniques discussed in section 3 together with conditions (6.1), we obtain

$$\begin{aligned}
U_0 &= 4t\,\text{sech}(x), \\
U_1 &= \frac{8}{15}\beta t^{\beta+1}\text{sech}^3(x)\left[\frac{15\left(\alpha - 1 - \frac{\alpha t}{\beta+2}\right)}{\beta+1} - \frac{20t^2\left(\alpha - 1 - \frac{\alpha t}{\beta+4}\right)}{\beta+3} + \frac{16\,t^4\text{sech}^2(x)\left(\alpha - 1 - \frac{\alpha t}{\beta+6}\right)}{\beta+5}\right], \\
&\vdots
\end{aligned}$$

Finally, we can write

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t). \tag{6.4}$$

**Case II**: Consider Eq (1.1) with fractal-fractional Mittag-Leffler kernel as

$$ {}^{FFE}D_t^{\alpha,\beta}\phi - \phi_{xx} + \sin\phi = 0, \quad 1 < \alpha \le 2, \quad 0 < \beta \le 1, \tag{6.5}$$

Proceeding with the technique discussed in section 4 together with (6.1), we obtain

$$\begin{aligned} V_0 &= 4t\text{sech}(x), \\ V_1 &= \frac{8}{15}\beta t^{\beta+1}\text{sech}^3(x)\left[\frac{15\left(\alpha-1-\frac{\alpha t}{\beta+2}\right)}{\beta+1} - \frac{20t^2\left(\alpha-1-\frac{\alpha t}{\beta+4}\right)}{\beta+3} + \frac{16\,t^4\text{sech}^2(x)\left(\alpha-1-\frac{\alpha t}{\beta+6}\right)}{\beta+5}\right], \\ &\vdots \end{aligned}$$

Similar to the above result, we can write

$$V(x,t) = \sum_{n=0}^{\infty} V_n(x,t). \tag{6.6}$$

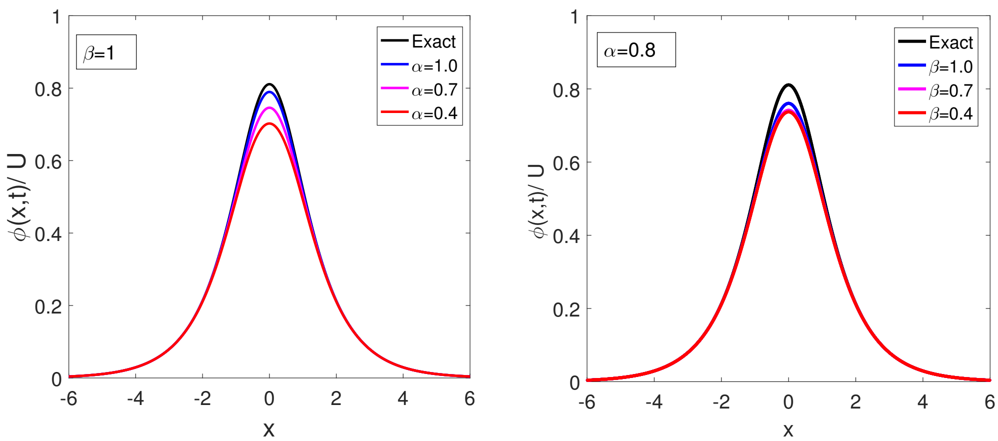


**Figure 1.** The comparison between Eq (6.2) versus Eq (6.4) for different values of fractional and fractal variables.

## 7. Discussion

In Figure 1, the left upper panel signify the estimate amongst the exact (6.2) and approximate *C-F* solution (6.4) for $\beta = 1$ a with numerous values of $\alpha$ as shown in figure for $t = 0.2$. One can see that, for different fractional order the amplitude of soliton wave solution is does not altered. This shows that fractional order does not affect qualitatively the solitary wave solution of the system. The right panel of Figure 1, signifies the fractal dimension effect of the obtained *C-F* solution (6.4) for $\alpha = 0.8$ with different values of fractal variable $\beta$ and time $t = 0.3$. Similar to the left panel the fractal dimension qualitatively does not affect the solution of the system. In Figure 2, the left panel shows the contrast between the exact versus ABC solution (6.6) for $\beta = 1$ with altered values of $\alpha$ . The right panel displays the influence of fractal dimension on considered equation for the fractional order $\alpha$. From the figures, a good agreement is obtained between the exact and the solution with exponential decay kernel with different fractal and fractional orders. Figure 3 represent the surface plots for both *C*-F and ABC solutions with $\alpha = \beta = 1$.

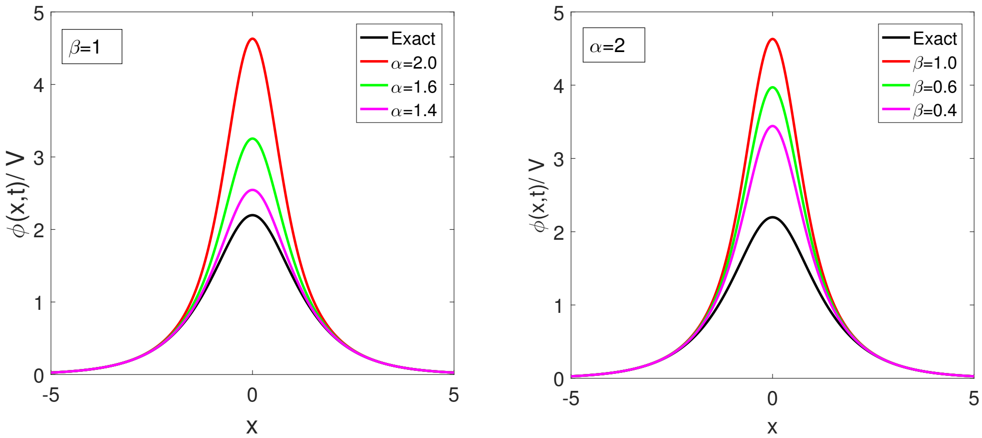


**Figure 2.** The comparison between exact solution (6.2) versus ABC solution (6.6) for different values of fractional and fractal variables.

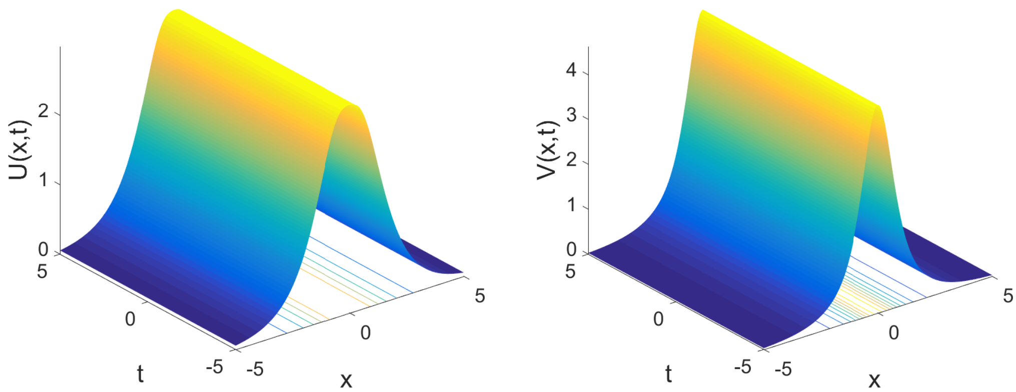


**Figure 3.** The left panel shows the surface plot of Eq (6.4), while the panel shows surface plot for solution (6.6).

**Example 2. case I:** *Consider Eq (6.3) with initial conditions*

$$\phi(x,0) = 0 \quad and \quad \phi_t(x,0) = \frac{8}{e^x + e^{-x}}, \tag{7.1}$$

*and exact solution*

$$\phi \;=\; 4\arctan\left(\frac{2t}{e^x + e^{-x}}\right). \tag{7.2}$$

The series solution of Eq (6.3) with initial conditions (7.1) is obtained in the form

$$\begin{aligned}
\phi_0 &= \frac{8t}{e^{-x} + e^x}, \\
\phi_1 &= \frac{64\beta e^{3x} t^{\beta+1}}{3\left(e^{2x}+1\right)^3}\left(\frac{3(\alpha-1)}{\beta+1} - \frac{3\alpha t}{(\beta+1)(\beta+2)} + \frac{4t^2(\alpha\, t + (\beta+4)(1-\alpha))}{(\beta+3)(\beta+4)}\right)
\end{aligned}$$

$$-\frac{64\beta e^{3x}t^{\beta+1}}{15\left(e^{2x}+1\right)^5}\left(\frac{64t^4e^{2x}(\alpha\, t+(\beta+6)(1-\alpha))}{(\beta+5)(\beta+6)}\right),$$

$$\vdots$$

The final series can be written as

$$\phi(x,t)=\sum_{n=0}^{\infty}\phi_n(x,t). \tag{7.3}$$

**Case II:** Consider Eq (6.5) with initial conditions

$$\phi(x,0)=0 \quad and \quad \phi_t(x,0)=\frac{8}{e^{-x}+e^x}. \tag{7.4}$$

The series solution of Eq (6.5) with initial conditions (7.4) is

$$\begin{aligned}
\phi_0 &= \frac{8t}{e^{-x}+e^x},\\
\phi_1 &= \frac{64\beta e^{3x}t^{\beta}}{15\left(e^{2x}+1\right)^5}\left(2e^{2x}\left((\alpha-1)\left(5\left(3-4t^2\right)\cosh(2x)+32t^4-20t^2+15\right)-\frac{32\alpha\Gamma(\beta+5)t^{\alpha+4}}{\Gamma(\alpha+\beta+5)}\right)\right)\\
&\quad -\frac{64\beta e^{3x}t^{\beta}}{15\left(e^{2x}+1\right)^5}\left(\frac{5\alpha\left(e^{2x}+1\right)^2\Gamma(\beta+1)t^{\alpha}\left(3(\alpha+\beta+1)(\alpha+\beta+2)-4(\beta+1)(\beta+2)t^2\right)}{\Gamma(\alpha+\beta+3)}\right),\\
&\vdots
\end{aligned}$$

The final approximate solutions can be expressed in the form

$$\phi(x,t)=\sum_{n=0}^{\infty}\phi_n(x,t). \tag{7.5}$$

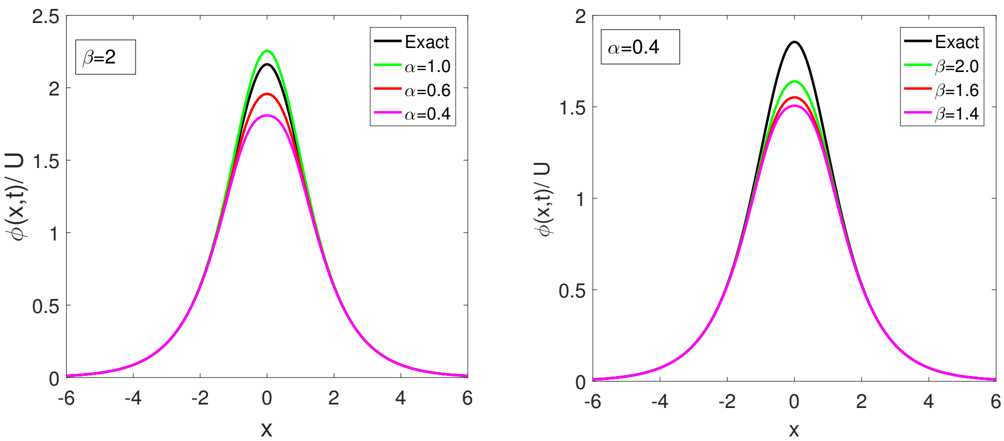


**Figure 4.** The comparison between exact solution (7.2) versus CF solution (7.3) for different values of fractional and fractal variables.

The left panel in Figure 4 represents comparison between exact solution (7.2) and approximate $C$-F solution (7.3) for $\beta = 1$ with different values of $\alpha$ as shown in the figure. The right panel represent the $C$-F approximate solution (7.3) for $\alpha = 1$ for different values of fractal variable $\beta$. Similarly, The left panel in Figure 5 shows the comparison of exact versus ABC approximate solution (7.5) for $\beta = 1$ and different values of $\alpha$, while, the right panel represents the effect of fractal dimension on considered model for $\alpha = 1.3$ and different values of $\beta$. The surface plots for$C$-F and ABC solutions are given in Figure 6.

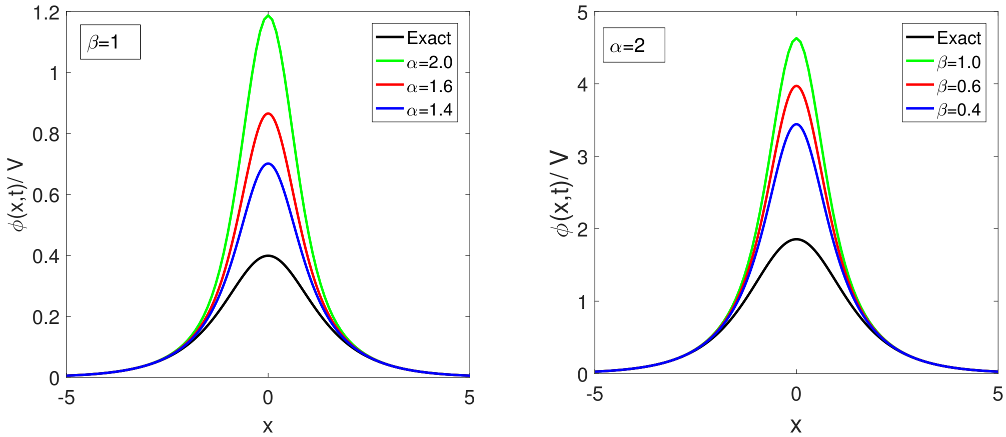


**Figure 5.** The comparison between an exact solution (7.2) against ABC solution(7.5) for different values of fractional and fractal variables.

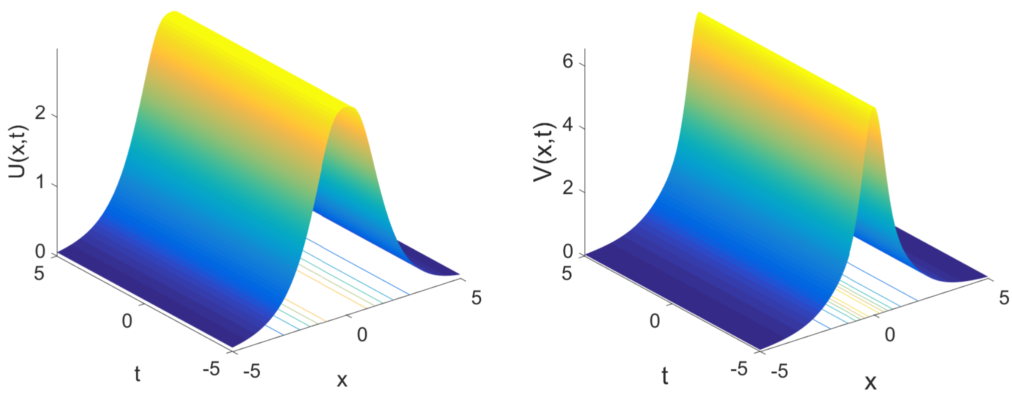


**Figure 6.** The left panel shows surface plot for (7.3), while the right panel shows surface plot for (7.5).

## 8. Error analysis

In Figure 7, the left panel represent the error between exact solution (6.2) and approximate solution (6.4) for $\alpha = \beta = 1$. The right panel contains error between the exact solution (6.2) and approximate ABC solution (6.6) for $\alpha = 2$ *and* $\beta = 1$ shown in Table 1. One can easily observe the comparison of error amongst $C$-F and ABC approximate solutions.

The absolute error estimate between an exact (7.2) and an approximate $C$-F solution (7.3) is given in the left panel of Figure 8, by considering $\alpha = \beta = 1$. Similarly, the right panel shows the absolute error between an exact solution (6.2) and an approximate ABC solution (7.5) for $\alpha = 2, \beta = 1$. One can easily observed from Table 2 that, the absolute error decreases on both sides of the domain between CF and ABC solutions and the solitary wave solution maintains its shape.

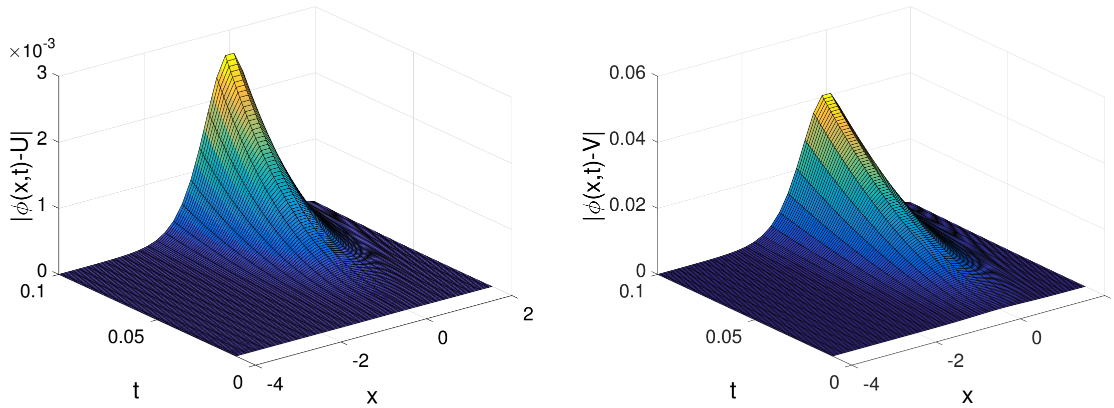


**Figure 7.** The surface plot for error estimate for approximate solutions (6.4) versus (6.6).

**Table 1.** Comparison between an exact and approximate solutions for $\alpha = \beta = 1$ for $C$-F solutions, $\alpha = 2$ *and* $\beta = 1$ for ABC solution, for Example1.

| (x,t) | $\phi(x,t)$ | $U(x,t)$ | $V(x,t)$ | $\mid \phi - U\mid$ | $\mid \phi - V\mid$ |
|---|---|---|---|---|---|
| (-8,0.2) | $5.3674\times10^{-04}$ | $5.3674\times10^{-04}$ | $5.3674\times10^{-04}$ | $6.3914\times10^{-12}$ | $4.4788\times10^{-10}$ |
| (-4,0.2) | 0.0239 | 0.0239 | 0.294 | $1.0392\times10^{-6}$ | $7.2822\times10^{-5}$ |
| (0,0.2) | 0.8109 | 0.7895 | 2.2964 | 0.0214 | 1.4855 |
| (4,0.2) | 0.0293 | 0.0293 | 0.294 | $1.0392\times10^{-6}$ | $7.2822\times10^{-5}$ |
| (8,0.2) | $5.3674\times10^{-04}$ | $5.3674\times10^{-04}$ | $5.3674\times10^{-4}$ | $6.3914\times10^{-12}$ | $4.4788\times10^{-10}$ |
| (-8,0.4) | 0.0011 | 0.0011 | 0.0011 | $4.9894\times10^{-11}$ | $6.8625\times10^{-10}$ |
| (-4,0.4) | 0.0586 | 0.0586 | 0.0587 | $8.1128\times10^{-6}$ | $1.1158\times10^{-4}$ |
| (0,0.4) | 1.6946 | 1.5198 | 4.0443 | 0.1748 | 2.23497 |
| (4,0.4) | 0.0586 | 0.0586 | 0.0587 | $8.1128\times10^{-6}$ | $1.1158\times10^{-4}$ |
| (8,0.4) | 0.0011 | 0.0011 | 0.0011 | $4.9894\times10^{-11}$ | $6.8625\times10^{-10}$ |

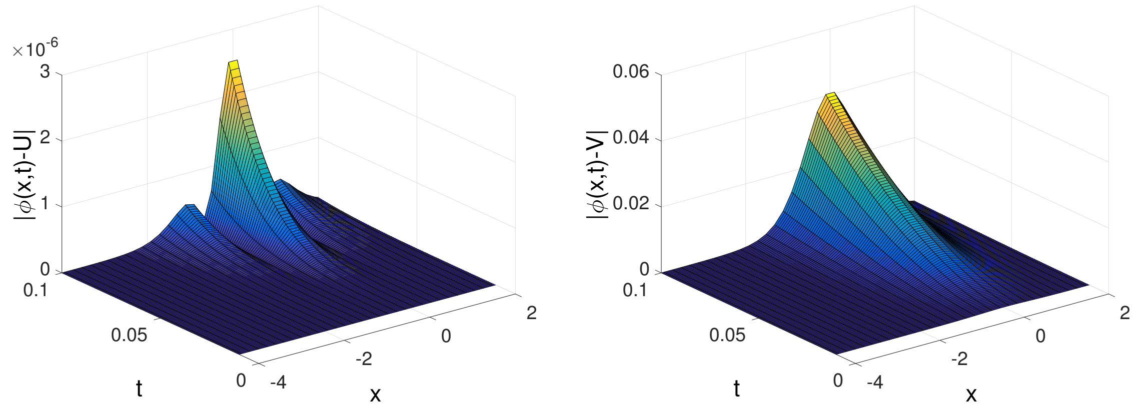


**Figure 8.** The surface plot of the absolute error for solution obtained (7.3) and (7.5).

**Table 2.** The comparison between an exact and approximate solutions for $\alpha = \beta = 1$ for $C$-F solutions and $\alpha = 2$ *and* $\beta = 1$ for ABC solutions for Example 2.

| (x,t) | $\phi(x,t)$ | $U(x,t)$ | $V(x,t)$ | $\lvert \phi - U \rvert$ | $\lvert \phi - V \rvert$ |
|---|---|---|---|---|---|
| (-8,0.6) | 0.0016 | 0.0016 | 0.0016 | $7.4454\times10^{-11}$ | $6.0491\times10^{-10}$ |
| (-4,0.2) | 0.0879 | 0.0879 | 0.0880 | $1.2166\times10^{-5}$ | $9.8396\times10^{-5}$ |
| (0,0.2) | 2.4000 | 2.1478 | 5.0551 | 0.2522 | 2.6551 |
| (4,0.2) | 0.0879 | 0.0879 | 0.0880 | $1.2106\times10^{-5}$ | $9.8396\times10^{-5}$ |
| (8,0.2) | 0.0016 | 0.0016 | 0.0016 | $7.4454\times10^{-11}$ | $6.0491\times10^{-10}$ |
| (-8,0.4) | $8.0511\times10^{-4}$ | $8.0511\times10^{-4}$ | $8.0511\times10^{-4}$ | $1.0481\times10^{-11}$ | $6.1688\times10^{-10}$ |
| (-4,0.4) | 0.0439 | 0.0439 | 0.0440 | $1.7041\times10^{-6}$ | $1.0030\times10^{-4}$ |
| (0,0.4) | 1.2000 | 1.1653 | 3.2632 | 0.0347 | 2.0622 |
| (4,0.4) | 0.0439 | 0.0439 | 0.0440 | $1.7041\times10^{-6}$ | $1.0030\times10^{-4}$ |
| (8,0.4) | $8.0511\times10^{-4}$ | $8.0511\times10^{-4}$ | $8.0511\times10^{-4}$ | $1.0481\times10^{-11}$ | $6.1688\times10^{-10}$ |

## 9. Conclusions

We have investigated the time fractional non-integrable sine-Gordon equation with Mittag Leffler and exponential decay type kernels in FF sense. The series solution of the sGE is obtained using ADM with Laplace transform. The main advantage of LAMD is that calculating only two iterations, comparatively precise results of sG solitary wave solution is obtained with fractal fractional operators. The suggested approach has the key advantage of calculating series solutions to the considered problem without any perturbation, discretization, or measuring complicated polynomials. For applications, we have considered a numerical example with both kernels and different subsidiary conditions. The numerical results are presented and compared with analytical solutions for different fractional variables and fractal dimensions. One can see that good agreements are obtained for solitary wave solutions of the model. It is revealed that, by bearing in mind the solitonic solution in the form of sech and exponential functions, kink type solitary wave solutions of sGE is obtained supposed examples for both non-singular fractional operators with fractional dimensions. It is well known that sGE is evidently the most famous demonstration of a kink-bearing nonlinear PDE, and therefore a perfect system for considerate kink and anti-kink occurrences. The absolute error estimate is calculated, where, for comparatively small time (less than one), the error minimizes between the approximate and exact solutions.

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**Conflict of interest**

It is declared that all the authors have no conflict of interest regarding this manuscript.

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