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Research article

A novel formulation of the fuzzy hybrid transform for dealing nonlinear partial differential equations via fuzzy fractional derivative involving general order

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Abstract: The main objective of the investigation is to broaden the description of Caputo fractional derivatives (in short, CFDs) (of order $0 < \alpha < r$) considering all relevant permutations of entities involving t_1 equal to 1 and t_2 (the others) equal to 2 via fuzzifications. Under $g\mathcal{H}$-differentiability, we also construct fuzzy Elzaki transforms for CFDs for the generic fractional order $\alpha \in (r-1, r)$. Furthermore, a novel decomposition method for obtaining the solutions to nonlinear fuzzy fractional partial differential equations (PDEs) via the fuzzy Elzaki transform is constructed. The aforesaid scheme is a novel correlation of the fuzzy Elzaki transform and the Adomian decomposition method. In terms of CFD, several new results for the general fractional order are obtained via $g\mathcal{H}$-differentiability. By considering the triangular fuzzy numbers of a nonlinear fuzzy fractional PDE, the correctness and capabilities of the proposed algorithm are demonstrated. In the domain of fractional sense, the schematic representation and tabulated outcomes indicate that the algorithm technique is precise and straightforward. Subsequently, future directions and concluding remarks are acted upon with the most focused use of references.

Keywords: fuzzy set theory; Elzaki transform; Adomian decomposition method; nonlinear partial differential equation; Caputo fractional derivative
Mathematics Subject Classification: 46S40, 47H10, 54H25

1. Introduction

The idea of differential and integral calculus is essential for stronger and more comprehensive descriptions of natural reality. It aids in the modelling of the early evolution and forecasting the future of the respective manifestations. Furthermore, thanks to its capability to express more fascinating ramifications of heat flux [1–3], neural network [4], hydrodynamics [5], circuit theory [6], aquifers [7], chemical kinetics [8], epidemics [9–11], simulations [12], inequality theory [13–15] and henceforth. Numerous researchers have subsequently been drawn to the investigation of fractional calculus [16–22].

Fractional calculus is particularly effective at modelling processes or systems relying on hereditary patterns and legacy conceptions, and traditional calculus is a restricted component of fractional calculus. This approach seems to be as ancient as a classical notion, but it has just subsequently been applied to the detection of convoluted frameworks by numerous investigators, and it has been demonstrated by various researchers [23–25]. Fractional calculus has been advocated by a number of innovators [26–29]. Li et al. [30] contemplated a novel numerical approach to time-fractional parabolic equations with nonsmooth solutions. She et al. [31] developed a transformed method for solving the multi-term time-fractional diffusion problem. Qin et al. [32] presented a novel scheme to capture the initial dramatic evolutions of nonlinear sub-diffusion equations. Many scholars analyze simulations depicting viruses, bifurcation, chaos, control theory, image processing, quantum fluid flow, and several other related disciplines using the underlying concepts and properties of operators shown within the framework of fractional calculus [33–40].

Fuzzy set theory (FST) is a valuable tool for modelling unpredictable phenomena. As a result, fuzzy conceptions are often leveraged to describe a variety of natural phenomena. Fuzzy PDEs are an excellent means of modelling vagueness and misinterpretation in certain quantities for specified real-life scenarios, see [41–43]. In recent years, FPDEs have been exploited in a variety of disciplines, notably in control systems, knowledge-based systems, image processing, power engineering, industrial automation, robotics, consumer electronics, artificial intelligence/expert systems, management, and operations research.

Because of its relevance in a wide range of scientific disciplines, FST has a profound correlation with fractional calculus [44]. Kandel and Byatt [45] proposed fuzzy DEs in 1978, while Agarwal et al. [46] were the first to investigate fuzziness and the Riemann-Liouville (RL) differentiability concept via the Hukuhara-differentiability (HD) concept. FST and FC both use a variety of computational methodologies to gain a better understanding of dynamic structures while reducing the unpredictability of their computation. Identifying precise analytical solutions in the case of FPDEs is a complicated process.

Due to the model's intricacy, determining an analytical solution to PDEs is generally problematic. As a result, there is a developing trend of implementing mathematical approaches to get an exact solution. The Adomian decomposition method (ADM) is a prominent numerical approach that is widely used. Several researchers have employed different terminologies to address FPDEs. Nemati and Matinfar [47] constructed an implicit finite difference approach for resolving complex fuzzy PDEs. Also, to demonstrate the competence and acceptability of the synthesized trajectory, experimental investigations incorporating parabolic PDEs were provided. According to Allahviranloo and Kermani [48], an explicit numerical solution to the fuzzy hyperbolic and parabolic equations is provided. The validity and resilience of the proposed system were investigated in order to demonstrate that it is inherently robust. Arqub et al. [49] expounded the fuzzy FDE via the non-singular kernel considering the differential formulation of the Atangana-Baleanu operator. Authors [50] contemplated the numerical findings of fuzzy fractional initial value problems utilizing the non-singular kernel derivative operator.

Integral transforms are preferred by investigators when it pertains to identifying results for crucial difficulties. The Elzaki transformation [51], proposed by T. Elzaki in 2011, was used on the biological population model, the Fornberg–Whitham Model, and Fisher's models in [52–54].

The purpose of this study is to furnish a relatively effective Adomian decomposition approach [55] that can address complex partial fuzzy DEs by leveraging the fuzzy Elzaki transform. It can address the dynamics of partial fuzzy differential equations by utilizing the fuzzy Elzaki transform. A revolutionary computational concept is characterized by generating the result of a nonlinear fuzzy fractional PDE. To solve the nonlinear elements of the equation, the Adomian polynomial [56] methodology is implemented. The innovative decomposition approach is known as the "fuzzy Elzaki technique".

In this research, CFDs of order $\alpha \in (0, r)$ for a fuzzy-valued mapping by employing all conceivable configurations of objects with t_1 equal to 1 and t_2 (the others) equal to 2 are presented. Also, a new result in connection between Caputo's fractional derivative and the Elzaki transform via fuzzification is also presented. Taking into consideration $g\mathcal{H}$-differentiabilty for a new algorithm, the fuzzy Elzaki decomposition process, which is intended to generate the parameterized representation of fuzzy functions, is regarded as a promising technique for addressing fuzzy fractional nonlinear PDEs involving fuzzy initial requirements. The Elzaki transform, implemented here, is generally a modification of the Laplace and Sumudu transforms. The varying fractional order and uncertainty parameter, $\wp \in [0, 1]$, are utilized to reveal a demonstration case for the suggested approach. Both 2D and 3D models illustrate the test's superiority to previous approaches. As a result, the new revelation provides a couple of responses that are very identical to the earlier ones. We do, however, have the option of selecting the most suitable one. Ultimately, as part of our attempt to close remarks, we highlighted the information gathered during our investigation.

The following is a synopsis of the persisting sections with regard to introduction and implementation: Section 2 represents the fundamentals and essential details of fractional calculus and fuzzy set theory. Problem formulation, implementation, and execution were all used in Section 3. Section 4 utilized the Caputo fractional derivative formulation via fuzzification in generic order and some further results. Section 5 utilized numerical algorithms and mathematical debates with some tabulation and graphical results. Ultimately, Section 6 utilized concluding and future highlights.

2. Preliminaries

This section consists of some significant concepts and results from fractional calculus and FST. For more details, see [28, 57].

Here, there be a space of all continuous fuzzy-valued mappings $\mathbb{C}^F[\tilde{a}, \tilde{b}]$ on $[\tilde{a}, \tilde{b}]$. Moreover, the space of all Lebesgue integrable fuzzy-valued mappings on the bounded interval $[\tilde{a}, \tilde{b}] \subset \mathbb{R}$ are represented by $\mathbb{L}^F[\tilde{a}, \tilde{b}]$.

Definition 2.1. ([58]) A fuzzy number (FN) is a mapping $\mathbf{f} : \mathbb{R} \mapsto [0, 1]$, that fulfills the subsequent assumptions:

(i) $\mathbf{f}$ is upper semi-continuous on $\mathbb{R}$;

(ii) $\mathbf{f}(\mathbf{x}) = 0$ for some interval $[\tilde{c}, \tilde{d}]$;

(iii) For $\tilde{a}, \tilde{b} \in \mathbb{R}$ having $\tilde{c} \leq \tilde{a} \leq \tilde{b} \leq \tilde{d}$ such that $\mathbf{f}$ is increasing on $[\tilde{c}, \tilde{a}]$ and decreasing on $[\tilde{b}, \tilde{d}]$ and $\mathbf{f}(\mathbf{x}) = 1$ for every $\mathbf{x} \in [\tilde{a}, \tilde{b}]$;

(iv) $\mathbf{f}(\wp\mathbf{x} + (1-\wp)\mathbf{y}) \geq \min\{\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{y})\}$ for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}$, $\wp \in [0,1]$.

The set of all FNs is denoted by the letter E^1. If $\tilde{a} \in \mathbb{R}$, it can be regarded as a FN; $\tilde{\tilde{a}} = \chi_{\{\tilde{a}\}}$ is the characteristic function, and therefore $\mathbb{R} \subset E^1$.

Definition 2.2. ([59]) The $\wp$-level set of $\mathbf{f}$ is the crisp set $[\mathbf{f}]^{\wp}$, if $\wp \in [0,1]$ and $\mathbf{f} \in E^1$, then

$$[\mathbf{f}]^{\wp} = \{\mathbf{x} \in \mathbb{R} : \mathbf{f}(\mathbf{x}) \geq \wp\}. \tag{2.1}$$

Also, any $\wp$-level set is closed and bounded, signifies by $[\underline{\mathbf{f}}(\wp), \bar{\mathbf{f}}(\wp)]$, $\forall \wp \in [0,1]$, where $\underline{\mathbf{f}}, \bar{\mathbf{f}} : [0,1] \mapsto \mathbb{R}$ are the lower and upper bounds of $[\mathbf{f}]^{\wp}$, respectively.

Definition 2.3. ([59]) For each $\wp \in [0,1]$, a parameterize formulation of FN $\mathbf{f}$ is an ordered pair $\mathbf{f} = [\underline{\mathbf{f}}(\wp), \bar{\mathbf{f}}(\wp)]$ of mappings $\underline{\mathbf{f}}(\wp)$ and $\bar{\mathbf{f}}(\wp)$ that addresses the basic conditions:

(i) There be a bounded left continuous monotonic increasing mapping $\underline{\mathbf{f}}(\wp)$ on $[0,1]$;

(ii) There be a bounded left continuous monotonic decreasing $\bar{\mathbf{f}}(\wp)$ on $[0,1]$;

(iii) $\underline{\mathbf{f}}(\wp) \leq \bar{\mathbf{f}}(\wp)$.

Furthermore, the addition and scalar multiplication of FNs $\mathbf{f}_1 = [\underline{\mathbf{f}}_1(\wp), \bar{\mathbf{f}}_1(\wp)]$ and $\mathbf{f}_2 = [\underline{\mathbf{f}}_2(\wp), \bar{\mathbf{f}}_2(\wp)]$ are presented as follows:

$$\begin{aligned}
[\mathbf{f}_1 \oplus \mathbf{f}_2]^{\wp} &= [\mathbf{f}_1]^{\wp} + [\mathbf{f}_2]^{\wp} = [\underline{\mathbf{f}}_1(\wp) + \underline{\mathbf{f}}_2(\wp), \bar{\mathbf{f}}_1(\wp) + \bar{\mathbf{f}}_2(\wp)] \\
\textit{and} \;\; [k \odot \mathbf{f}]^{\wp} &= \begin{cases} [k\underline{\mathbf{f}}(\wp), k\bar{\mathbf{f}}(\wp)], & k > 0, \\ [k\bar{\mathbf{f}}(\wp), k\underline{\mathbf{f}}(\wp)], & k < 0. \end{cases}
\end{aligned} \tag{2.2}$$

As a distance between FNs, we employ the Hausdorff metric.

Definition 2.4. ([58]) Consider the two FNs $\mathbf{f}_1 = [\underline{\mathbf{f}}_1(\wp), \bar{\mathbf{f}}_1(\wp)]$ and $\mathbf{f}_2 = [\underline{\mathbf{f}}_2(\wp), \bar{\mathbf{f}}_2(\wp)]$ defined on E^1. Then the distance between two FNs is presented as follows:

$$d(\mathbf{f}_1, \mathbf{f}_2) = \sup_{\wp \in [0,1]} \max\{|\underline{\mathbf{f}}_1(\wp) - \underline{\mathbf{f}}_2(\wp)|, |\bar{\mathbf{f}}_1(\wp) - \bar{\mathbf{f}}_2(\wp)|\}. \tag{2.3}$$

Definition 2.5. ([60])A FN $\mathbf{f}$ has the following forms:

(i) If $\underline{\mathbf{f}}(1) \geq 0$, then $\mathbf{f}$ is positive;

(ii) If $\underline{\mathbf{f}}(1) > 0$, then $\mathbf{f}$ is strictly positive;

(iii) If $\bar{\mathbf{f}}(1) \leq 0$, then $\mathbf{f}$ is negative;

(iv) If $\underline{\mathbf{f}}(1) < 0$, then $\mathbf{f}$ is strictly negative.

The set of non-negative and non-positive FNs are indicated by E^+ and E^-, respectively.

Consider $\mathcal{D}$ be the set represents the domain of fuzzy-valued mappings $\mathbf{f}$. Define the mappings $\underline{\mathbf{f}}(.,.;\wp), \bar{\mathbf{f}}(.,.;\wp) : \mathcal{D} \mapsto \mathbb{R}$, $\forall \wp \in [0,1]$. These mappings are known to be the left and right $\wp$-level mappings of the map $\mathbf{f}$.

Definition 2.6. ([61]) A fuzzy valued mapping $\mathbf{f} : \mathcal{D} \mapsto E^1$ is known to be continuous at $(s_0, \xi_0) \in \mathcal{D}$ if for every $\epsilon > 0$ *exists* $\delta > 0$ such that $d(\mathbf{f}(s, \xi), \mathbf{f}(s_0, \xi_0)) < \epsilon$ whenever $|s - s_0| + |\xi - \xi_0| < \delta$. If $\mathbf{f}$ is continuous for each $(s_1, \xi_1) \in \mathcal{D}$, then $\mathbf{f}$ is said to be continuous on $\mathcal{D}$.

Definition 2.7. ([62]) Suppose $\mathbf{x}_1, \mathbf{x}_2 \in E^1$ and $\mathbf{y} \in E^1$ such that the subsequent satisfies:

(i) $\mathbf{x}_1 = \mathbf{x}_2 \oplus \mathbf{y}$

or

(ii) $\mathbf{y} = \mathbf{x}_1 \oplus (-1) \odot \mathbf{x}_2$.

Then, $\mathbf{y}$ is known to be the generalized Hukuhara difference ($_{g\mathcal{H}}$-difference) of FNs $\mathbf{x}_1$ and $\mathbf{x}_2$ and is denoted by $\mathbf{x}_1 \ominus\, _{g\mathcal{H}}\mathbf{x}_2$.

Again, suppose $\mathbf{x}_1, \mathbf{x}_2 \in E^1$, then $\mathbf{x}_1 \ominus\, _{g\mathcal{H}}\mathbf{x}_2 = \mathbf{y} \Leftrightarrow$

(i) $\mathbf{y} = (\underline{\mathbf{x}_1}(\wp) - \underline{\mathbf{x}_2}(\wp), \bar{\mathbf{x}}_1(\wp) - \bar{\mathbf{x}}_2(\wp))$

or

(ii) $\mathbf{y} = (\bar{\mathbf{x}}_1(\wp) - \bar{\mathbf{x}}_2(\wp), \underline{\mathbf{x}_1}(\wp) - \underline{\mathbf{x}_2}(\wp))$.

The association regarding the $_{g\mathcal{H}}$-difference and the Housdroff distance is demonstrated by the following lemma.

Lemma 2.8. *([62]) For all* $\mathbf{f}_1, \mathbf{f}_2 \in E^1$, *then*

$$d(\mathbf{f}_1, \mathbf{f}_2) = \sup_{\wp \in [0,1]} \|[\mathbf{f}_1]^{\wp} \ominus\, _{g\mathcal{H}}[\mathbf{f}_2]^{\wp}\|, \tag{2.4}$$

where, for an interval $[\tilde{a}, \tilde{b}]$, *the norm is* $\|[\tilde{a}, \tilde{b}]\| = \max\{|\tilde{a}|, |\tilde{b}|\}$.

Definition 2.9. ([63]) Let $\mathbf{f} : \mathcal{D} \mapsto E^1$ and $(\mathbf{x}_0, \xi) \in \mathcal{D}$. A mapping $\mathbf{f}$ is known as the strongly strongly generalized Hukuhara differentiable on $(\mathbf{x}_0, \xi)$ ($_{g\mathcal{H}}$-differentiable for short) if there exists an element $\frac{\partial f_2(\mathbf{x}_0, \xi)}{\partial \mathbf{x}} \in E^1$, then the subsequent holds:

(i) The following $_{g\mathcal{H}}$-differences exist, if $\forall\ \epsilon > 0$ sufficiently small, then

$$\mathbf{f}(\mathbf{x}_0 + \epsilon, \xi) \ominus\, _{g\mathcal{H}}\mathbf{f}(\mathbf{x}_0, \xi), \quad \mathbf{f}(\mathbf{x}_0, \xi) \ominus\, _{g\mathcal{H}}\mathbf{f}(\mathbf{x}_0 + \epsilon, \xi),$$

the following limits hold as:

$$\lim_{\epsilon \mapsto 0} \frac{\mathbf{f}(\mathbf{x}_0 + \epsilon, \xi) \ominus\, _{g\mathcal{H}}\mathbf{f}(\mathbf{x}_0, \xi)}{\epsilon} = \lim_{\epsilon \mapsto 0} \frac{\mathbf{f}(\mathbf{x}_0, \xi) \ominus\, _{g\mathcal{H}}\mathbf{f}(\mathbf{x}_0 + \epsilon, \xi)}{\epsilon} = \frac{\partial \mathbf{f}(\mathbf{x}_0, \xi)}{\partial \mathbf{x}}. \tag{2.5}$$

(ii) The following $_{g\mathcal{H}}$-differences exist, if $\forall\ \epsilon > 0$ reasonably small, then

$$\mathbf{f}(\mathbf{x}_0, \xi) \ominus\, _{g\mathcal{H}}\mathbf{f}(\mathbf{x}_0 + \epsilon, \xi), \quad \mathbf{f}(\mathbf{x}_0 - \epsilon, \xi) \ominus\, _{g\mathcal{H}}\mathbf{f}(\mathbf{x}_0, \xi),$$

the following limits hold as:

$$\lim_{\epsilon \mapsto 0} \frac{\mathbf{f}(\mathbf{x}_0, \xi) \ominus\, _{g\mathcal{H}}\mathbf{f}(\mathbf{x}_0 + \epsilon, \xi)}{-\epsilon} = \lim_{\epsilon \mapsto 0} \frac{\mathbf{f}(\mathbf{x}_0 - \epsilon, \xi) \ominus\, _{g\mathcal{H}}\mathbf{f}(\mathbf{x}_0, \xi)}{-\epsilon} = \frac{\partial \mathbf{f}(\mathbf{x}_0, \xi)}{\partial \mathbf{x}}. \tag{2.6}$$

Lemma 2.10. *([64]) Suppose a continuous fuzzy-valued mapping* $\mathbf{f} : \mathcal{D} \mapsto E^1$ *and* $\mathbf{f}(\mathbf{x}, \xi) = [\underline{\mathbf{f}}(\mathbf{x}, \xi; \wp), \bar{\mathbf{f}}(\mathbf{x}, \xi; \wp)],\ \forall \wp \in [0, 1]$. *Then*

(i) *If* $\mathbf{f}(\mathbf{x}, \xi)$ *is (i)-differentiable for* $\mathbf{x}$ *under Definition 2.9(i), then we have the following:*

$$\frac{\partial \mathbf{f}(\mathbf{x}_0, \xi)}{\partial \mathbf{x}} = \Big(\frac{\partial \underline{\mathbf{f}}(\mathbf{x}_0, \xi)}{\partial \mathbf{x}}, \frac{\partial \bar{\mathbf{f}}(\mathbf{x}_0, \xi)}{\partial \mathbf{x}}\Big). \tag{2.7}$$

(ii) *If* $\mathbf{f}(\mathbf{x}, \xi)$ *is (ii)-differentiable for* $\mathbf{x}$ *under Definition 2.9(ii), then we have the following:*

$$\frac{\partial \mathbf{f}(\mathbf{x}_0, \xi)}{\partial \mathbf{x}} = \Big(\frac{\partial \bar{\mathbf{f}}(\mathbf{x}_0, \xi)}{\partial \mathbf{x}}, \frac{\partial \underline{\mathbf{f}}(\mathbf{x}_0, \xi)}{\partial \mathbf{x}}\Big). \tag{2.8}$$

Theorem 2.11. *([65]) Suppose* $\mathbf{f} : \mathbb{R}^+ \mapsto \mathbb{E}^{\Bbbk}$ *and* $\forall\ \wp \in [0, 1]$.
(i) *There be Riemann-integrable mappings* $\underline{\mathbf{f}}(\mathbf{x}; \xi; \wp)$ *and* $\bar{\mathbf{f}}(\mathbf{x}; \xi; \wp)$ *on* $[0, \tilde{b}]$ *for each* $\tilde{b} \geq 0$.
(ii) $\underline{\mathcal{M}}(\wp) > 0$ *and* $\bar{\mathcal{M}}(\wp) > 0$ *are the constants, then*

$$\int_0^{\tilde{b}} |\underline{\mathbf{f}}(\mathbf{x}; \xi; \wp)| d\mathbf{x} \leq \underline{\mathcal{M}}(\wp), \quad \int_0^{\tilde{b}} |\bar{\mathbf{f}}(\mathbf{x}; \xi; \wp)| d\mathbf{x} \leq \bar{\mathcal{M}}(\wp), \quad \forall\ \tilde{b} \geq 0.$$

Then, the mapping $\mathbf{f}$ *is improper fuzzy Riemann-integrable on* $[0, \infty)$ *and the subsequent satisfies:*

$$\mathcal{FR} \int_0^{\infty} \mathbf{f}(\mathbf{x}) d\mathbf{x} = \Big(\int_0^{\infty} \underline{\mathbf{f}}(\mathbf{x}; \wp) d\mathbf{x}, \int_0^{\infty} \bar{\mathbf{f}}(\mathbf{x}; \wp) d\mathbf{x}\Big). \tag{2.9}$$

Theorem 2.12. *([16]) Suppose there be a positive integer r and a continuous mapping* $\mathbf{D}^{r-1}\mathbf{f}$ *defined on* $\mathcal{J} = [0, \infty)$ *and a collection of piece wise continuous mappings* $\mathbb{C}$ *defined on* $\mathcal{J}' = (0, \infty)$ *is integrable on finite sub-interval of* $\mathcal{J} = [0, \infty)$ *and assume that* $\nu > 0$. *Then*
(i) *If* $\mathbf{D}^r\mathbf{f}$ *is in* $\mathbb{C}$, *then*

$$\mathbf{D}^{-\nu}\mathbf{f}(\mathbf{x}) = \mathbf{D}^{-\nu-r}[\mathbf{D}^r\mathbf{f}(\mathbf{x})] + X_r(\mathbf{x}, \nu).$$

(ii) *If there be a continuous mapping* $\mathbf{D}^r\mathbf{f}$ *on* $\mathcal{J}$, *then for* $\mathbf{x} > 0$

$$\mathbf{D}^r[\mathbf{D}^{-\nu}\mathbf{f}(\mathbf{x})] = \mathbf{D}^{-\nu}[\mathbf{D}^r\mathbf{f}(\mathbf{x})] + X_r(\mathbf{x}, \nu - r),$$

where

$$X_r(\mathbf{x}, \nu) = \sum_{\kappa=0}^{r-1} \frac{\mathbf{x}^{\nu+\kappa}}{\Gamma(\nu + \kappa + 1)} \mathbf{D}^{\kappa}\mathbf{f}(0).$$

3. Fuzzy Elzaki transform

Definition 3.1. ([51]) Suppose a continuous fuzzy-valued mapping $\mathbf{f} : \mathbb{R}^+ \mapsto E^1$ and for $\omega > 0$, there be an improper fuzzy Riemann-integrable mapping $\mathbf{f}(\xi) \odot \exp(-\xi/\omega)$ defined on $[0, \infty)$. Then we have

$$\mathcal{FR} \int_0^{\infty} \omega \mathbf{f}(\xi) \exp(-\xi/\omega) d\xi, \quad \omega \in (p_1, p_2),$$

which is known as the Fuzzy Elzaki transform and represented as

$$\mathcal{W}(\omega) = \mathbb{E}[\mathbf{f}(\xi)] = \mathcal{FR}\int_0^\infty \omega\mathbf{f}(\xi)\exp(-\xi/\omega)d\xi.$$

The parameterized version of fuzzy Elzaki transform:

$$\mathbb{E}[\mathbf{f}(\xi)] = \Big[\mathcal{E}[\underline{\mathbf{f}}(\xi;\wp)], \mathcal{E}[\bar{\mathbf{f}}(\xi;\wp)]\Big],$$

where

$$\mathcal{E}[\underline{\mathbf{f}}(\xi;\wp)] = \int_0^\infty \omega\underline{\mathbf{f}}(\xi;\wp)\exp(-\xi/\omega)d\xi,$$

$$\mathcal{E}[\bar{\mathbf{f}}(\xi;\wp)] = \int_0^\infty \omega\bar{\mathbf{f}}(\xi;\wp)\exp(-\xi/\omega)d\xi.$$

4. Fuzzy Elzaki transform of the fuzzy CFDs of orders $r-1<\alpha<r$

This section consists of CFDs of the general fractional order $0<\alpha<r$. Also, we obtain fuzzy Elzaki transform for CFD of the generic order $r-1<\alpha<r$ for fuzzy valued mapping $\mathbf{f}$ under $g\mathcal{H}$-differentiability.

For the sake of simplicity, for $0<\alpha<r$ and $\mathbf{f}(\mathbf{x}) \in \mathbb{C}^F[0,\tilde{b}]\bigcap\mathbb{L}^F[0,\tilde{b}]$, denoting

$$\mathcal{G}(\mathbf{x}) = \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\int_0^{\mathbf{x}} \frac{\mathbf{f}(\xi)d\xi}{(\mathbf{x}-\xi)^{1-\lceil\alpha\rceil+\alpha}} \ominus \sum_{\kappa=0}^{\lceil\alpha\rceil-\alpha} \frac{\mathbf{D}^\kappa\mathbf{f}(0)\mathbf{x}^{\lceil\alpha\rceil-\alpha+\kappa}}{\Gamma(1+\lceil\alpha\rceil-\alpha+\kappa)}. \tag{4.1}$$

Definition 4.1. Suppose $\mathbf{f}(\mathbf{x}) \in \mathbb{C}^F[0,\tilde{b}]\bigcap\mathbb{L}^F[0,\tilde{b}]$ and $\lceil\alpha\rceil$ and $\lfloor\alpha\rfloor$ indicates α values that have been rounded forward and descends to the closest integer value, respectively. It is clear that $\mathcal{G}(\mathbf{x})$ and the mappings $\mathcal{G}_{\jmath_1,\jmath_2,...,\jmath_\iota,1}$ and $\mathcal{G}_{\jmath_1,\jmath_2,...,\jmath_\iota,2}$ are stated as

$$\mathcal{G}_{\jmath_1,\jmath_2,...,\jmath_\iota,1}(\mathbf{x}_0) = \lim_{\epsilon\mapsto0^+}\frac{\mathcal{G}_{\jmath_1,\jmath_2,...,\jmath_\iota}(\mathbf{x}_0+\epsilon)\ominus\mathcal{G}_{\jmath_1,\jmath_2,...,\jmath_\iota}(\mathbf{x}_0)}{\epsilon} = \lim_{\epsilon\mapsto0^+}\frac{\mathcal{G}_{\jmath_1,\jmath_2,...,\jmath_\iota}(\mathbf{x}_0)\ominus\mathcal{G}_{\jmath_1,\jmath_2,...,\jmath_\iota}(\mathbf{x}_0-\epsilon)}{\epsilon}, \tag{4.2}$$

$$\mathcal{G}_{\jmath_1,\jmath_2,...,\jmath_\iota,2}(\mathbf{x}_0) = \lim_{\epsilon\mapsto0^+}\frac{\mathcal{G}_{\jmath_1,\jmath_2,...,\jmath_\iota}(\mathbf{x}_0)\ominus\mathcal{G}_{\jmath_1,\jmath_2,...,\jmath_\iota}(\mathbf{x}_0+\epsilon)}{-\epsilon} = \lim_{\epsilon\mapsto0^+}\frac{\mathcal{G}_{\jmath_1,\jmath_2,...,\jmath_\iota}(\mathbf{x}_0-\epsilon)\ominus\mathcal{G}_{\jmath_1,\jmath_2,...,\jmath_\iota}(\mathbf{x}_0)}{-\epsilon}, \tag{4.3}$$

for $\iota=0,1,2,...,r-2$ such that $\jmath_1,\jmath_2,...,\jmath_\iota$ are all possible arrangements of ι objects that represents the numbers in the following principal:

$$\iota P_{t_1t_2} = \frac{\iota!}{t_1!t_2!}, \quad t_1+t_2=\iota,$$

where t_1 of them equal 1 (means CD in the first version) and t_2 of them equal 2 (means CD in the second version). Also, $\jmath_1,\jmath_2,...,\jmath_0$.

Now, $\mathbf{f}(\mathbf{x})$ is the Caputo fractional type fuzzy differentiable mapping of order $0 < \alpha < r$, $\alpha \neq 1, 2, \ldots r-1$ at $\mathbf{x}_0 \in (0, \tilde{b})$ if $\exists$ an element $({}^c\mathbf{D}^\alpha \mathbf{f})(\mathbf{x}_0) \in \mathbb{C}^F$ such that $\forall \wp \in [0,1]$ and for $\epsilon > 0$ close to zero. Then

(i) If $J_{\lceil \alpha \rceil} = 1$, then

$$({}^c\mathbf{D}^\alpha \mathbf{f})(\mathbf{x}_0) = \lim_{\epsilon \mapsto 0^+} \frac{\mathcal{G}_{J_1, J_2, \ldots, J_{\lceil \alpha \rceil}}(\mathbf{x}_0 + \epsilon) \ominus \mathcal{G}_{J_1, J_2, \ldots, J_\iota}(\mathbf{x}_0)}{\epsilon} = \lim_{\epsilon \mapsto 0^+} \frac{\mathcal{G}_{J_1, J_2, \ldots, J_{\lceil \alpha \rceil}}(\mathbf{x}_0) \ominus \mathcal{G}_{J_1, J_2, \ldots, J_{\lceil \alpha \rceil}}(\mathbf{x}_0 - \epsilon)}{\epsilon}. \quad (4.4)$$

(ii) If $J_{\lceil \alpha \rceil} = 2$, then

$$({}^c\mathbf{D}^\alpha \mathbf{f})(\mathbf{x}_0) = \lim_{\epsilon \mapsto 0^+} \frac{\mathcal{G}_{J_1, J_2, \ldots, J_{\lceil \alpha \rceil}}(\mathbf{x}_0) \ominus \mathcal{G}_{J_1, J_2, \ldots, J_\iota}(\mathbf{x}_0 + \epsilon)}{-\epsilon} = \lim_{\epsilon \mapsto 0^+} \frac{\mathcal{G}_{J_1, J_2, \ldots, J_{\lceil \alpha \rceil}}(\mathbf{x}_0 - \epsilon) \ominus \mathcal{G}_{J_1, J_2, \ldots, J_{\lceil \alpha \rceil}}(\mathbf{x}_0)}{-\epsilon}, \quad (4.5)$$

for $\alpha \in (\kappa - 1, \kappa)$, $\kappa = 1, 2, \ldots, r$ such that $J_1, J_2, \ldots, J\lceil \alpha \rceil$ are all the suitable arrangements of $\lceil \alpha \rceil$ objects that have the following representation:

$$\lceil \alpha \rceil P_{t_1 t_2} = \frac{\lceil \alpha \rceil !}{t_1 ! t_2 !}, \; t_1 + t_2 = \lceil \alpha \rceil.$$

Theorem 4.2. *Suppose* $\mathbf{f}(\mathbf{x}) \in \mathbb{C}^F[0, \tilde{b}] \bigcap \mathbb{L}^F[0, \tilde{b}]$ *be a fuzzy-valued mapping such that* $\mathbf{f}(\mathbf{x}) = [\underline{\mathbf{f}}(\mathbf{x};\wp), \bar{\mathbf{f}}(\mathbf{x};\wp)]$ *for* $\wp \in [0,1]$, $\mathbf{x}_0 \in (0, \tilde{b})$ *and* $\mathcal{G}(\mathbf{x})$ *is stated in* (4.1).

Assume that $0 < \alpha < r$ *and* ℓ *is the number of repetitions of 2 among* $J_1, J_2, \ldots, J_{\lceil \alpha \rceil}$ *for* $\alpha \in (\kappa - 1, \kappa)$, $\kappa = 1, 2, \ldots, r$, *say*, $J_{\kappa_1}, J_{\kappa_2}, \ldots, J_{\kappa_\ell}$ *such that* $\kappa_1 < \kappa_2 < \ldots < \kappa_\ell$, *i.e.*, $J_{\kappa_1} = J_{\kappa_2} = \ldots = J_{\kappa_\ell} = 2$ *and* $0 \leq \ell \leq \lceil \alpha \rceil$. *Then we have the following*

If ℓ *is even number, then*

$$\left({}^c\mathbf{D}^\beta_{J_1, J_2, \ldots J_{\lceil \alpha \rceil}} \mathbf{f}\right)(\mathbf{x}_0) = \left[({}^c\mathbf{D}^\alpha \underline{\mathbf{f}})(\mathbf{x}_0;\wp), ({}^c\mathbf{D}^\alpha \bar{\mathbf{f}})(\mathbf{x}_0;\wp)\right]. \quad (4.6)$$

If ℓ *is odd number, then*

$$\left({}^c\mathbf{D}^\beta_{J_1, J_2, \ldots J_{\lceil \alpha \rceil}} \mathbf{f}\right)(\mathbf{x}_0) = \left[({}^c\mathbf{D}^\alpha \bar{\mathbf{f}})(\mathbf{x}_0;\wp), ({}^c\mathbf{D}^\alpha \underline{\mathbf{f}})(\mathbf{x}_0;\wp)\right], \quad (4.7)$$

where

$$({}^c\mathbf{D}^\alpha \underline{\mathbf{f}})(\mathbf{x}_0;\wp) = \left[\frac{1}{\Gamma(\lceil \alpha \rceil - \alpha)} \int_0^{\mathbf{x}} \frac{\mathbf{D}^{\lceil \alpha \rceil} \underline{\mathbf{f}}(\xi;\wp) d\xi}{(\mathbf{x} - \xi)^{1 - \lceil \alpha \rceil + \alpha}}\right]_{\mathbf{x} = \mathbf{x}_0},$$

$$({}^c\mathbf{D}^\alpha \bar{\mathbf{f}})(\mathbf{x}_0;\wp) = \left[\frac{1}{\Gamma(\lceil \alpha \rceil - \alpha)} \int_0^{\mathbf{x}} \frac{\mathbf{D}^{\lceil \alpha \rceil} \bar{\mathbf{f}}(\xi;\wp) d\xi}{(\mathbf{x} - \xi)^{1 - \lceil \alpha \rceil + \alpha}}\right]_{\mathbf{x} = \mathbf{x}_0}, \quad \mathbf{D}^\kappa \mathbf{f}(\xi) = \frac{d^\kappa \mathbf{f}(\xi)}{d\xi^\kappa}. \quad (4.8)$$

Proof. Let ℓ is an even number and then $\ell = 2s_1$, $s_1 \in \mathbb{N}$. Here, we have two assumptions as follows:

The first assumption is $\left({}^c\mathbf{D}^\beta_{J_1, \ldots, J_{\kappa_1}, \ldots, J_{\kappa_2}, \ldots, J_{\lceil \alpha \rceil}} \mathbf{f}\right)(\mathbf{x}_0)$ is the Caputo type fuzzy fractional differentiable mapping in the first form ($J_{\lceil \alpha \rceil} = 1$) and in view of (4.4) from Definition 4.1, we have

$$\begin{aligned} &\mathcal{G}_{J_1, \ldots, J_{\lceil \alpha \rceil}}(\mathbf{x}_0 + \epsilon) \ominus \mathcal{G}_{J_1, \ldots, J_{\lceil \alpha \rceil}}(\mathbf{x}_0) \\ =& \left[\underline{\mathcal{G}}_{J_1, \ldots, J_{\lceil \alpha \rceil}}(\mathbf{x}_0 + \epsilon;\wp) - \underline{\mathcal{G}}_{J_1, \ldots, J_{\lceil \alpha \rceil}}(\mathbf{x}_0;\wp), \bar{\mathcal{G}}_{J_1, \ldots, J_{\lceil \alpha \rceil}}(\mathbf{x}_0 + \epsilon;\wp) - \bar{\mathcal{G}}_{J_1, \ldots, J_{\lceil \alpha \rceil}}(\mathbf{x}_0;\wp)\right], \end{aligned}$$

$$\begin{aligned}&\mathcal{G}_{J1,\dots,J\lceil\alpha\rceil}(\mathbf{x}_0)\ominus\mathcal{G}_{J1,\dots,J\lceil\alpha\rceil}(\mathbf{x}_0-\epsilon)\\=&\Big[\underline{\mathcal{G}}_{J1,\dots,J\lceil\alpha\rceil}(\mathbf{x}_0;\wp)-\underline{\mathcal{G}}_{J1,\dots,J\lceil\alpha\rceil}(\mathbf{x}_0-\epsilon;\wp),\bar{\mathcal{G}}_{J1,\dots,J\lceil\alpha\rceil}(\mathbf{x}_0;\wp)-\bar{\mathcal{G}}_{J1,\dots,J\lceil\alpha\rceil}(\mathbf{x}_0-\epsilon;\wp)\Big].\end{aligned}\tag{4.9}$$

Conducting product on both sides by $1/\epsilon,\ \epsilon>0$, and then applying $\epsilon\mapsto^{+}$, yields

$$\Big({}^{\mathcal{RL}}\mathbf{D}^{\alpha}\mathbf{f}\Big)(\mathbf{x}_0)=\Big[\frac{d}{d\mathbf{x}}\underline{\mathcal{G}}_{J1,\dots,J\lceil\alpha\rceil}(\mathbf{x}_0;\wp),\frac{d}{d\mathbf{x}}\bar{\mathcal{G}}_{J1,\dots,J\lceil\alpha\rceil}(\mathbf{x}_0;\wp)\Big].\tag{4.10}$$

Thus, $\mathcal{G}_{J1,\dots,J_{\kappa_1-1}}$ is identical to the specified restrictions mentioned in (4.2) of Definition 4.1, then by employing (4.2) for (κ_1-1) *times*, we have that

$$\mathcal{G}_{J1,\dots,J_{\kappa_1}-1}(\mathbf{x}_0)=\Big[\underline{\mathcal{G}}^{(\kappa_1-1)}(\mathbf{x}_0;\wp),\bar{\mathcal{G}}^{(\kappa_1-1)}(\mathbf{x}_0;\wp)\Big].\tag{4.11}$$

Since $\mathcal{G}_{J1},..,J_{\kappa_1}(\mathbf{x}_0)$ is identical to the specified restrictions stated in (4.3) of Definition 4.1 then by employing (4.3), we have

$$\mathcal{G}_{J1,\dots,J_{\kappa_1}-1}(\mathbf{x}_0)=\Big[\bar{\mathcal{G}}^{(\kappa_1)}(\mathbf{x}_0;\wp),\underline{\mathcal{G}}^{(\kappa_1)}(\mathbf{x}_0;\wp)\Big].\tag{4.12}$$

Since $\mathcal{G}_{J1},..,J_{\kappa_2-1}(\mathbf{x}_0)$ is identical to the specified restrictions stated in (4.2) of Definition 4.1 then by employing (4.2), we have

$$\mathcal{G}_{J1,\dots,J_{\kappa_2-1}}(\mathbf{x}_0)=\Big[\bar{\mathcal{G}}^{(\kappa_2-1)}(\mathbf{x}_0;\wp),\underline{\mathcal{G}}^{(\kappa_2-1)}(\mathbf{x}_0;\wp)\Big].\tag{4.13}$$

Since $\mathcal{G}_{J1},..,J_{\kappa_2}(\mathbf{x}_0)$ is identical to the specified restrictions stated in (4.3) of Definition 4.1 then by employing (4.3), we have

$$\mathcal{G}_{J1,\dots,J_{\kappa_2}}(\mathbf{x}_0)=\Big[\underline{\mathcal{G}}^{(\kappa_2-1)}(\mathbf{x}_0;\wp),\bar{\mathcal{G}}^{(\kappa_2-1)}(\mathbf{x}_0;\wp)\Big].\tag{4.14}$$

On the other hand, from (4.14) we notice that we will have a similar equation, following the application of (4.2) and (4.3) for any even number of $J_{\kappa_1},J_{\kappa_2},...,J_{\kappa_m}$ of (4.14). Thus, for $\mathcal{G}_{J1,\dots,J2s_1}(\mathbf{x}_0)$, we have

$$\mathcal{G}_{J1,\dots,J_{\kappa_{2s_1}}}(\mathbf{x}_0)=\Big[\underline{\mathcal{G}}^{(\kappa_{2s_1})}(\mathbf{x}_0;\wp),\bar{\mathcal{G}}^{(\kappa_{2s_1})}(\mathbf{x}_0;\wp)\Big],\tag{4.15}$$

where $2s_1$ is even number.

Consequently, since $\mathcal{G}_{J_1},..,J_{\lceil\alpha\rceil}(\mathbf{x}_0)$ is identical to the specified restrictions stated in (4.2) of Definition 4.1 then by employing (4.2) for $(\lceil\alpha\rceil-\kappa_{2s_1})$, we have

$$\mathcal{G}_{J1,\dots,J\lceil\alpha\rceil}(\mathbf{x}_0)=\Big[\underline{\mathcal{G}}^{(\lceil\alpha\rceil)}(\mathbf{x}_0;\wp),\bar{\mathcal{G}}^{(\lceil\alpha\rceil)}(\mathbf{x}_0;\wp)\Big],\tag{4.16}$$

then, we have

$$\begin{aligned}\underline{\mathcal{G}}_{J1,\dots,J\lceil\alpha\rceil}(\mathbf{x}_0;\wp)&=\underline{\mathcal{G}}^{(\lceil\alpha\rceil)}(\mathbf{x}_0;\wp),\\\bar{\mathcal{G}}_{J1,\dots,J\lceil\alpha\rceil}(\mathbf{x}_0;\wp)&=\bar{\mathcal{G}}^{(\lceil\alpha\rceil)}(\mathbf{x}_0;\wp).\end{aligned}\tag{4.17}$$

Plugging (4.17) and (4.10) gives the subsequent

$$({}^{c}\mathbf{D}^{\alpha}\mathbf{f})(\mathbf{x}_0)=[\mathbf{D}^{\lceil\beta\rceil}\underline{\mathcal{G}}(\mathbf{x}_0;\wp),\mathbf{D}^{\lceil\beta\rceil}\bar{\mathcal{G}}(\mathbf{x}_0;\wp)],\quad\mathbf{D}=d/d\mathbf{x}.\tag{4.18}$$

Thus,

$$({}^c\mathbf{D}^{\alpha}\mathbf{f})(\mathbf{x}_0) = \Big[\mathbf{D}^{\lceil\beta\rceil}\Big(\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\int_0^{\mathbf{x}}\frac{\underline{\mathbf{f}}(\xi;\wp)d\xi}{(\mathbf{x}-\xi)^{1-\lceil\alpha\rceil+\alpha}} - \sum_{\kappa=0}^{\lceil\alpha\rceil-\alpha}\frac{\mathbf{D}^{\kappa}\underline{\mathbf{f}}(0)\mathbf{x}^{\lceil\alpha\rceil-\alpha+\kappa}}{\Gamma(1+\lceil\alpha\rceil-\alpha+\kappa)}\Big)\Big|_{\mathbf{x}=\mathbf{x}_0},$$

$$\mathbf{D}^{\lceil\beta\rceil}\Big(\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\int_0^{\mathbf{x}}\frac{\bar{\mathbf{f}}(\xi;\wp)d\xi}{(\mathbf{x}-\xi)^{1-\lceil\alpha\rceil+\alpha}} - \sum_{\kappa=0}^{\lceil\alpha\rceil-\alpha}\frac{\mathbf{D}^{\kappa}\bar{\mathbf{f}}(0)\mathbf{x}^{\lceil\alpha\rceil-\alpha+\kappa}}{\Gamma(1+\lceil\alpha\rceil-\alpha+\kappa)}\Big)\Big|_{\mathbf{x}=\mathbf{x}_0}\Big]. \quad (4.19)$$

Utilizing the fact of (4.1) we have

$$({}^c\mathbf{D}^{\alpha}\mathbf{f})(\mathbf{x}_0) = \Big[\mathbf{D}^{\lceil\alpha\rceil}(\mathbf{D}^{-(\lceil\alpha\rceil-\alpha)}\underline{\mathbf{f}})(\mathbf{x}_0;\wp) - \Big(\sum_{\kappa=0}^{\lceil\alpha\rceil}\frac{\mathbf{D}^{\kappa}\underline{\mathbf{f}}(0;\wp)\mathbf{D}^{\lceil\alpha\rceil}\mathbf{x}^{\lceil\alpha\rceil-\alpha+\kappa}}{\Gamma(1+\lceil\alpha\rceil-\alpha+\kappa)}\Big)\Big|_{\mathbf{x}=\mathbf{x}_0},$$

$$\mathbf{D}^{\lceil\alpha\rceil}(\mathbf{D}^{-(\lceil\alpha\rceil-\alpha)}\bar{\mathbf{f}})(\mathbf{x}_0;\wp) - \Big(\sum_{\kappa=0}^{\lceil\alpha\rceil}\frac{\mathbf{D}^{\kappa}\bar{\mathbf{f}}(0;\wp)\mathbf{D}^{\lceil\alpha\rceil}\mathbf{x}^{\lceil\alpha\rceil-\alpha+\kappa}}{\Gamma(1+\lceil\alpha\rceil-\alpha+\kappa)}\Big)\Big|_{\mathbf{x}=\mathbf{x}_0}\Big], \quad (4.20)$$

where $(\mathbf{D}^{-(\lceil\alpha\rceil-\alpha)}\underline{\mathbf{f}})(\mathbf{x}_0;\wp)$ and $(\mathbf{D}^{-(\lceil\alpha\rceil-\alpha)}\bar{\mathbf{f}})(\mathbf{x}_0;\wp)$ are the $\mathcal{RL}$ fractional integrals of the mappings $\underline{\mathbf{f}}(\mathbf{x}_0;\wp)$ and $\bar{\mathbf{f}}(\mathbf{x}_0;\wp)$ at $\mathbf{x}=\mathbf{x}_0$, respectively. By the use of continuity of $\mathbf{D}^r\mathbf{f}$ having $r=\lceil\alpha\rceil,\ \nu=\lceil\alpha\rceil-\alpha$ and by the virtue of Theorem 2.12, $\mathbf{D}^r\mathbf{x}^{\ell}=\frac{\Gamma(\ell+1)}{\Gamma(\ell+1-r)}\mathbf{x}^{\ell-r}$, we have

$$({}^c\mathbf{D}^{\alpha}\mathbf{f})(\mathbf{x}_0) = \Big[\mathbf{D}^{-(\lceil\alpha\rceil-\alpha)}(\mathbf{D}^{\lceil\alpha\rceil}\underline{\mathbf{f}}(\mathbf{x}_0;\wp)) + \underline{Q}(\mathbf{x}_0,-\alpha) - \sum_{\kappa=0}^{\lceil\alpha\rceil}\frac{\mathbf{D}^{\kappa}\underline{\mathbf{f}}(0;\wp)\mathbf{x}^{\kappa-\alpha}}{\Gamma(1-\alpha+\kappa)}\Big|_{\mathbf{x}=\mathbf{x}_0},$$

$$\mathbf{D}^{-(\lceil\alpha\rceil-\alpha)}(\mathbf{D}^{\lceil\alpha\rceil}\bar{\mathbf{f}}(\mathbf{x}_0;\wp)) + \bar{Q}(\mathbf{x}_0,-\alpha) - \sum_{\kappa=0}^{\lceil\alpha\rceil}\frac{\mathbf{D}^{\kappa}\bar{\mathbf{f}}(0;\wp)\mathbf{x}^{\kappa-\alpha}}{\Gamma(1-\alpha+\kappa)}\Big|_{\mathbf{x}=\mathbf{x}_0}\Big]. \quad (4.21)$$

Thus,

$$\begin{aligned}({}^c\mathbf{D}^{\alpha}\mathbf{f})(\mathbf{x}_0) &= \Big[\mathbf{D}^{-(\lceil\alpha\rceil-\alpha)}(\mathbf{D}^{\lceil\alpha\rceil}\underline{\mathbf{f}}(\mathbf{x}_0;\wp)), \mathbf{D}^{-(\lceil\alpha\rceil-\alpha)}(\mathbf{D}^{\lceil\alpha\rceil}\bar{\mathbf{f}})(\mathbf{x}_0;\wp)\Big]\\ &= [({}^c\mathbf{D}^{\alpha}\underline{\mathbf{f}})(\mathbf{x}_0;\wp), ({}^c\mathbf{D}^{\alpha}\bar{\mathbf{f}})(\mathbf{x}_0;\wp)]. \end{aligned} \quad (4.22)$$

If ℓ is an odd, solution is similar as we did before. □

Theorem 4.3. *Assume that there be a fuzzy-valued mapping* $\mathbf{f}(\mathbf{x}) \in \mathbb{C}^F[0,\infty)\bigcap\mathbb{L}^F[0,\infty)$ *such that* $\mathbf{f}(\mathbf{x}) = [\underline{\mathbf{f}}(\mathbf{x};\wp), \bar{\mathbf{f}}(\mathbf{x},\wp)]$ *for* $\wp\in[0,1]$. *Also, let* $r-1<\alpha<r$ *and* ℓ *is the quantity replicated of two amongest* $\jmath_1,\jmath_2,\jmath_3,...,\jmath_r$ *say* $\jmath_{\kappa_1},\jmath_{\kappa_2},\jmath_{\kappa_3},...,\jmath_{\kappa_\ell}$ *such that* $\kappa_1<\kappa_2<...<\kappa_m$; *i.e.,* $\jmath_{\kappa_1},\jmath_{\kappa_2},\jmath_{\kappa_3},...,\jmath_{\kappa_\ell}=2$ *and* $0\le\ell\le r$. *Then,*

(1) If ℓ *is even number, then*

$$\mathbb{E}\Big[\Big({}^c\mathbf{D}^{\alpha}_{\jmath_1,\jmath_2,...,\jmath_r}\mathbf{f}\Big)(\mathbf{x})\Big] = \omega^{-\alpha}\mathbb{E}[\mathbf{f}(\mathbf{x})] \ominus \omega^{2-\alpha}\mathbf{f}(0) \otimes \sum_{\kappa=1}^{r-1}\omega^{2-\alpha+\kappa}\mathbf{f}^{(\kappa)}(0), \quad (4.23)$$

then

$$\otimes = \begin{cases}\ominus, & if\ such\ quantity\ is\ replication\ of\ two\ amongest\ \jmath_1,\jmath_2,...\jmath_{r-(\kappa+1)}\ \text{is an even number},\\ -, & if\ such\ quantity\ is\ replication\ of\ two\ amongest \jmath_1,\jmath_2,...\jmath_{r-(\kappa+1)}\ \text{is an odd number}.\end{cases} \quad (4.24)$$

(2) If ℓ is odd number, we have

$$\mathbb{E}\Big[\big({}^{c}\mathbf{D}^{\alpha}_{J_1,J_2,\ldots,J_r}\mathbf{f}\big)(\mathbf{x})\Big] = -\omega^{2-\alpha}\mathbf{f}(0) \ominus (-\omega^{-\alpha})\mathbb{E}[\mathbf{f}(\mathbf{x})] \otimes \sum_{\kappa=0}^{r-1} \omega^{2-\alpha+\kappa}\mathbf{f}(0), \tag{4.25}$$

$$\otimes = \begin{cases} \ominus, \ if \ such \ quantity \ is \ replication \ of \ two \ amongest \ J_1, J_2, \ldots J_{r-(\kappa+1)} \ is \ an \ odd \ number, \\ -, \ if \ such \ quantity \ is \ replication \ of \ two \ amongest \ J_1, J_2, \ldots J_{r-(\kappa+1)} \ is \ an \ even \ number. \end{cases} \tag{4.26}$$

Proof. Considering $({}^{c}\mathbf{D}^{\alpha}_{J_1,J_2,\ldots,J_r}\mathbf{f})(\mathbf{x})$, that can be expressed as $({}^{c}\mathbf{D}^{\alpha}_{J_1,\ldots,J_{\kappa_1},\ldots,J_{\kappa_2},\ldots,J_{\kappa_\ell},\ldots,J_r}\mathbf{f})(\mathbf{x})$. Also, assume that ℓ is an odd number, then by means of Theorem 4.2 and $r-1<\alpha<r$, we have

$$({}^{c}\mathbf{D}^{\alpha}_{J_1,J_2,\ldots,J_r}\mathbf{f})(\mathbf{x}) = [({}^{c}\mathbf{D}^{\alpha}\bar{\mathbf{f}})(\mathbf{x};\wp), ({}^{c}\mathbf{D}^{\alpha}\underline{\mathbf{f}})(\mathbf{x};\wp)]. \tag{4.27}$$

Thus, we have

$$\begin{aligned} \underline{({}^{c}\mathbf{D}^{\alpha}\mathbf{f})}(\mathbf{x};\wp) &= ({}^{c}\mathbf{D}^{\alpha}\bar{\mathbf{f}})(\mathbf{x};\wp), \\ \overline{({}^{c}\mathbf{D}^{\alpha}\mathbf{f})}(\mathbf{x};\wp) &= ({}^{c}\mathbf{D}^{\alpha}\underline{\mathbf{f}})(\mathbf{x};\wp). \end{aligned} \tag{4.28}$$

Using the fact of (4.28), we have

$$\begin{aligned} \mathbb{E}\Big[({}^{c}\mathbf{D}^{\alpha}_{J_1,J_2,\ldots,J_r}\mathbf{f})(\mathbf{x})\Big] &= \mathbb{E}\Big[\underline{({}^{c}\mathbf{D}^{\alpha}\mathbf{f})}(\mathbf{x};\wp), \overline{({}^{c}\mathbf{D}^{\alpha}\mathbf{f})}(\mathbf{x};\wp)\Big] \\ &= \Big[\mathcal{E}[({}^{c}\mathbf{D}^{\alpha}\bar{\mathbf{f}})(\mathbf{x};\wp)], \mathcal{E}[({}^{c}\mathbf{D}^{\alpha}\underline{\mathbf{f}})(\mathbf{x};\wp)]\Big]. \end{aligned} \tag{4.29}$$

In view of Elzaki transform of the Caputo fractional derivative of order α ([66]), we have

$$\begin{aligned} \mathcal{E}[({}^{c}\mathbf{D}^{\alpha}\underline{\mathbf{f}})(\mathbf{x};\wp)] &= \omega^{-\alpha}\mathcal{E}[\underline{\mathbf{f}}(\mathbf{x};\wp)] - \sum_{\kappa=0}^{r-1}\omega^{2-\alpha+\kappa}\underline{\mathbf{f}}^{(\kappa)}(0;\wp) \\ &= \omega^{-\alpha}\mathcal{E}[\underline{\mathbf{f}}(\mathbf{x};\wp)] - \omega^{2-\alpha}\underline{\mathbf{f}}(0;\wp) - \sum_{\kappa=1}^{r-1}\omega^{2-\alpha+\kappa}\underline{\mathbf{f}}^{(\kappa)}(0;\wp). \end{aligned} \tag{4.30}$$

The aforementioned expression can be represented as

$$\begin{aligned} \mathcal{E}[({}^{c}\mathbf{D}^{\alpha}\underline{\mathbf{f}})(\mathbf{x};\wp)] =& \omega^{-\alpha}\mathcal{E}[\underline{\mathbf{f}}(\mathbf{x};\wp)] - \omega^{2-\alpha}\underline{\mathbf{f}}(0;\wp) \\ & - \sum_{\kappa=1}^{\kappa_1-1}\omega^{2-\alpha+\kappa}\underline{\mathbf{f}}^{(\kappa)}(0;\wp) - \sum_{\kappa=\kappa_1}^{\kappa_2-1}\omega^{2-\alpha+\kappa}\underline{\mathbf{f}}^{(\kappa)}(0;\wp) - \ldots \\ & - \sum_{\kappa=\kappa_{\ell-1}}^{\kappa_\ell-1}\omega^{2-\alpha+\kappa}\underline{\mathbf{f}}^{(\kappa)}(0;\wp) - \sum_{\kappa=\kappa_\ell}^{r-1}\omega^{2-\alpha+\kappa}\underline{\mathbf{f}}(0;\wp). \end{aligned} \tag{4.31}$$

Repeating the same process, we can write

$$\begin{aligned}\mathcal{E}[({}^c\mathbf{D}^{\alpha}\bar{\mathbf{f}})(\mathbf{x};\wp)] =&\omega^{-\alpha}\mathcal{E}[\bar{\mathbf{f}}(\mathbf{x};\wp)] - \omega^{2-\alpha}\bar{\mathbf{f}}(0;\wp)\\ &-\sum_{\kappa=1}^{\kappa_1-1}\omega^{2-\alpha+\kappa}\bar{\mathbf{f}}(0;\wp) - \sum_{\kappa=\kappa_1}^{\kappa_2-1}\omega^{2-\alpha+\kappa}\bar{\mathbf{f}}(0;\wp) - ...\\ &d - \sum_{\kappa=\kappa_{\ell-1}}^{\kappa_\ell-1}\omega^{2-\alpha+\kappa}\underline{\mathbf{f}}^{(\kappa)}(0;\wp) - \sum_{\kappa=\kappa_\ell}^{r-1}\omega^{2-\alpha+\kappa}\underline{\mathbf{f}}^{(\kappa)})(0;\wp).\end{aligned} \tag{4.32}$$

Even though $J_{\kappa_1} = J_{\kappa_2} = ... = J_{\kappa_\ell} = 2$ and ℓ is an odd number, then we have the subsequent forms

$$\begin{aligned}&\underline{\mathbf{f}}^{(\kappa)}(0;\wp) = \underline{\mathbf{f}^{(\kappa)}}(0;\wp),\\ &\bar{\mathbf{f}}^{(\kappa)}(0;\wp) = \underline{\mathbf{f}^{(\kappa)}}(0;\wp), \quad \forall\, \kappa \in [1, \kappa_1 - 1],\\ &\underline{\mathbf{f}}^{(\kappa)}(0;\wp) = \overline{\mathbf{f}^{(\kappa)}}(0;\wp)\\ &\bar{\mathbf{f}}^{(\kappa)}(0;\wp) = \underline{\mathbf{f}^{(\kappa)}}(0;\wp), \quad \forall\, \kappa \in [\kappa_1, \kappa_2 - 1],\\ &\vdots\\ &\underline{\mathbf{f}}^{(\kappa)}(0;\wp) = \underline{\mathbf{f}^{(\kappa)}}(0;\wp),\\ &\mathbf{f}^{\bar{(\kappa)}}(0;\wp) = \overline{\mathbf{f}^{(\kappa)}}(0;\wp), \quad \forall\, \kappa \in [\kappa_{\ell-1}, \kappa_\ell - 1],\\ &\underline{\mathbf{f}}^{(\kappa)}(0;\wp) = \overline{\mathbf{f}^{(\kappa)}}(0;\wp),\\ &\bar{\mathbf{f}}^{(\kappa)}(0;\wp) = \underline{\mathbf{f}^{(\kappa)}}(0;\wp), \quad \forall\, \kappa \in [\kappa_\ell, r - 1].\end{aligned} \tag{4.33}$$

When ℓ is odd number and utilizing Theorem 4.2, we get the aforementioned equations.

In view of (4.32), (4.31) and (4.29) reduce to

$$\mathbb{E}\Big[\big({}^c\mathbf{D}^{\alpha}_{J_1,J_2,...,J_r}\mathbf{f}\big)(\mathbf{x})\Big] = -\omega^{2-\alpha}\mathbf{f}(0) \ominus (-\omega^{-\alpha})\mathbb{E}[\mathbf{f}(\mathbf{x})] \otimes \sum_{\kappa=1}^{r-1}\omega^{2-\alpha+\kappa}f^{(\kappa)}(0;\wp). \tag{4.34}$$

where $\otimes$ defined in (4.26).

Adopting the same way, we can prove ℓ to be even number on parallel lines. □

Corollary 1. *Assume that* $\mathbf{f}(\mathbf{x}) \in \mathbb{C}^F[0,\infty) \bigcap \mathbb{L}^\infty[0,\infty)$. *Also, let* $\alpha \in (2,3)$. *Then we obtain the following:*

If $({}^c\mathbf{D}^{\alpha}_{1,1}\mathbf{f})(\mathbf{x})$ *is* ${}^c[(i)-\alpha]$*-differentiable fuzzy-valued mapping, then*

$$\mathbb{E}\Big[\big({}^c\mathbf{D}^{\alpha}_{1,1,1}\mathbf{f}\big)(\mathbf{x})\Big] = \omega^{-\alpha}\mathbb{E}[\mathbf{f}(\mathbf{x})] \ominus \omega^{-\alpha+2}\mathbf{f}(0) \ominus \omega^{-\alpha+3}\mathbf{f}'(0) \ominus \omega^{-\alpha+4}\mathbf{f}''(0).$$

If $({}^c\mathbf{D}^{\alpha}_{1,1}\mathbf{f})(\mathbf{x})$ *is* ${}^c[(ii)-\alpha]$*-differentiable fuzzy-valued mapping, then*

$$\mathbb{E}\Big[\big({}^c\mathbf{D}^{\alpha}_{1,1,1}\mathbf{f}\big)(\mathbf{x})\Big] = -\omega^{-\alpha+2}\mathbf{f}(0) \ominus (-\omega^{-\alpha})\mathbb{E}[\mathbf{f}(\mathbf{x})] - \omega^{-\alpha+3}\mathbf{f}'(0) - \omega^{-\alpha+4}\mathbf{f}''(0).$$

If $({}^c\mathbf{D}^{\alpha}_{1,2}\mathbf{f})(\mathbf{x})$ *is* ${}^c[(i)-\alpha]$*-differentiable fuzzy-valued mapping, then*

$$\mathbb{E}\Big[\big({}^c\mathbf{D}^{\alpha}_{1,2,1}\mathbf{f}\big)(\mathbf{x})\Big] = -\omega^{-\alpha+2}\mathbf{f}(0) \ominus (-\omega^{-\alpha})\mathbb{E}[\mathbf{f}(\mathbf{x})] - \omega^{-\alpha+3}\mathbf{f}'(0) \ominus \omega^{-\alpha+4}\mathbf{f}''(0).$$

If $({}^{c}\mathbf{D}^{\alpha}_{1,2}\mathbf{f})(\mathbf{x})$ *is* ${}^{c}[(ii)-\alpha]$*-differentiable fuzzy-valued mapping, then*

$$\mathbb{E}\Big[\Big({}^{c}\mathbf{D}^{\alpha}_{1,2,2}\mathbf{f}\Big)(\mathbf{x})\Big] = \omega^{-\alpha}\mathbb{E}[\mathbf{f}(\mathbf{x})] \ominus \omega^{-\alpha+2}\mathbf{f}(0) \ominus \omega^{-\alpha+3}\mathbf{f}'(0) - \omega^{-\alpha+4}\mathbf{f}''(0).$$

If $({}^{c}\mathbf{D}^{\alpha}_{2,1}\mathbf{f})(\mathbf{x})$ *is* ${}^{c}[(i)-\alpha]$*-differentiable fuzzy-valued mapping, then*

$$\mathbb{E}\Big[\Big({}^{c}\mathbf{D}^{\alpha}_{2,1,1}\mathbf{f}\Big)(\mathbf{x})\Big] = -\omega^{-\alpha+2}\mathbf{f}(0) \ominus (-\omega)^{-\alpha}\mathbb{E}[\mathbf{f}(\mathbf{x})] \ominus \omega^{-\alpha+3}\mathbf{f}'(0) \ominus \omega^{-\alpha+4}\mathbf{f}''(0).$$

If $({}^{c}\mathbf{D}^{\alpha}_{2,1}\mathbf{f})(\mathbf{x})$ *is* ${}^{c}[(ii)-\alpha]$*-differentiable fuzzy-valued mapping, then*

$$\mathbb{E}\Big[\Big({}^{c}\mathbf{D}^{\alpha}_{2,1,2}\mathbf{f}\Big)(\mathbf{x})\Big] = \omega^{-\alpha}\mathbb{E}[\mathbf{f}(\mathbf{x})] \ominus \omega^{-\alpha+2}\mathbf{f}(0) - \omega^{-\alpha+3}\mathbf{f}'(0) - \omega^{-\alpha+4}\mathbf{f}''(0).$$

If $({}^{c}\mathbf{D}^{\alpha}_{2,2}\mathbf{f})(\mathbf{x})$ *is* ${}^{c}[(i)-\alpha]$*-differentiable fuzzy-valued mapping, then*

$$\mathbb{E}\Big[\Big({}^{c}\mathbf{D}^{\alpha}_{2,2,1}\mathbf{f}\Big)(\mathbf{x})\Big] = \omega^{-\alpha}\mathbb{E}[\mathbf{f}(\mathbf{x})] \ominus \omega^{-\alpha+2}\mathbf{f}(0) - \omega^{-\alpha+3}\mathbf{f}'(0) \ominus \omega^{-\alpha+4}\mathbf{f}''(0).$$

If $({}^{c}\mathbf{D}^{\alpha}_{2,2}\mathbf{f})(\mathbf{x})$ *is* ${}^{c}[(ii)-\alpha]$*-differentiable fuzzy-valued mapping, then*

$$\mathbb{E}\Big[\Big({}^{c}\mathbf{D}^{\alpha}_{2,2,2}\mathbf{f}\Big)(\mathbf{x})\Big] = -\omega^{-\alpha+2}\mathbf{f}(0) \ominus (-\omega)^{-\alpha}\mathbb{E}[\mathbf{f}(\mathbf{x})] \ominus \omega^{-\alpha+3}\mathbf{f}'(0) - \omega^{-\alpha+4}\mathbf{f}''(0).$$

5. Fuzzy Elzaki decomposition method for finding solution of nonlinear fuzzy partial differential equation

In this note, we coupled the fuzzy Elzaki transform and the ADM for obtaining the solution of NFPDE. The generic form of NFPDE is presented as follows:

$$\sum_{\iota=0}^{p} c_\iota \odot \mathbf{D}^{\alpha}_{\xi}\mathbf{f}(\mathbf{x},\xi) \oplus \sum_{j=1}^{q} c_j \odot \frac{\partial^j \mathbf{f}(\mathbf{x},\xi)}{\partial \mathbf{x}^j} \oplus \sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2} c_{\eta\sigma} \odot \frac{\partial^\eta \mathbf{f}(\mathbf{x},\xi)}{\partial \mathbf{x}^\eta} \odot \frac{\partial^\sigma \mathbf{f}(\mathbf{x},\xi)}{\partial \mathbf{x}^\sigma} = \mathbf{g}(\mathbf{x},\xi), \tag{5.1}$$

subject to initial conditions

$$\frac{\partial^\iota \mathbf{f}(\mathbf{x},0)}{\partial \xi^\iota} = \psi_\iota(\mathbf{x}),\ \iota = 0,1,...,p-1, \tag{5.2}$$

where $\mathbf{f},\mathbf{g} : [0,\tilde{b}] \times [0,\tilde{d}] \mapsto E^1, \psi_\iota : [0,\tilde{b}] \mapsto E^1$ are continuous fuzzy mappings and $c_\iota,\ \iota = 1,2,...,p,\ c_j,\ j = 1,2,...,q,\ c_{\eta\sigma}, \eta = 0,1,2, \sigma = 0,1,2$, are non-negative constants.

Implementing the fuzzy Elzaki transform on (5.1), yields

$$\sum_{\iota=0}^{p} c_\iota \odot \mathbb{E}[\mathbf{D}^{\alpha}_{\xi}\mathbf{f}(\mathbf{x},\xi)] \oplus \sum_{j=1}^{q} c_j \odot \mathbb{E}\Big[\frac{\partial^j \mathbf{f}(\mathbf{x},\xi)}{\partial \mathbf{x}^j}\Big] \oplus \sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2} c_{\eta\sigma} \odot \mathbb{E}\Big[\frac{\partial^\eta \mathbf{f}(\mathbf{x},\xi)}{\partial \mathbf{x}^\eta}\Big] \odot \mathbb{E}\Big[\frac{\partial^\sigma \mathbf{f}(\mathbf{x},\xi)}{\partial \mathbf{x}^\sigma}\Big] = \mathbb{E}[\mathbf{g}(\mathbf{x},\xi)]. \tag{5.3}$$

Consider $\frac{\partial^\eta \mathbf{f}(\mathbf{x},\xi)}{\partial \xi^\eta}$, $\eta = 0,1,2$ be a positive fuzzy-valued mappings.

Then, the parametric version of (5.3) is as follows:

$$\sum_{\iota=0}^{p} c_\iota \mathcal{E}[\mathbf{D}_{\xi}^{\alpha}\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)] + \sum_{j=1}^{q} c_j \mathcal{E}\Big[\frac{\partial^j \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^j}\Big] + \sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2} c_{\eta\sigma}\mathcal{E}\Big[\frac{\partial^\eta \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\sigma}\Big] = \mathcal{E}[\underline{\mathbf{g}}(\mathbf{x},\xi;\wp)], \tag{5.4}$$

and

$$\sum_{\iota=0}^{p} c_\iota \mathcal{E}[\mathbf{D}_{\xi}^{\alpha}\bar{\mathbf{f}}(\mathbf{x},\xi;\wp)] + \sum_{j=1}^{q} c_j \mathcal{E}\Big[\frac{\partial^j \bar{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^j}\Big] + \sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2} c_{\eta\sigma}\mathcal{E}\Big[\frac{\partial^\eta \bar{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \bar{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\sigma}\Big] = \mathcal{E}[\bar{\mathbf{g}}(\mathbf{x},\xi;\wp)]. \tag{5.5}$$

Case I. Consider the mapping $\mathbf{f}(\mathbf{x},\xi;\wp)$ be $[(i)-\alpha]$-differentiable of the *qth*-order with respect to $\mathbf{x}$. In view of (5.4), then from (4.31) and (4.32) and IC, we have

$$\begin{aligned}\frac{1}{\omega^\alpha}\sum_{\iota=0}^{p} c_\iota \mathcal{E}[\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)] =& \mathcal{E}[\mathbf{g}(\mathbf{x},\xi;\wp)] + \sum_{\iota=1}^{p}\omega^2\underline{\psi}_0(\mathbf{x};\wp) - \sum_{j=1}^{q} c_j \mathcal{E}\Big[\frac{\partial^j \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^j}\Big]\\ &- \sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2} c_{\eta\sigma}\mathcal{E}\Big[\frac{\partial^\eta \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\sigma}\Big].\end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{E}[\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)] =& \Big(\sum_{\iota=0}^{p}\frac{c_\iota}{\omega^\alpha}\Big)^{-1}\Big[\mathcal{E}[\underline{\mathbf{g}}(\mathbf{x},\xi;\wp)] + \sum_{\iota=1}^{p}\omega^2\underline{\psi}_0(\mathbf{x};\wp) - \sum_{j=1}^{q} c_j \mathcal{E}\Big[\frac{\partial^j \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^j}\Big]\\ &- \sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2} c_{\eta\sigma}\mathcal{E}\Big[\frac{\partial^\eta \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\sigma}\Big]\Big].\end{aligned}$$

Now, employing the inverse Elzaki fuzzy transform to the aforementioned formulation, gives

$$\begin{aligned}\underline{\mathbf{f}}(\mathbf{x},\xi;\wp) =& \mathcal{E}^{-1}\Big[\Big(\sum_{\iota=0}^{p}\frac{c_\iota}{\omega^\alpha}\Big)^{-1}\Big(\mathcal{E}[\underline{\mathbf{g}}(\mathbf{x},\xi;\wp)] + \sum_{\iota=1}^{p}\omega^2\underline{\psi}_0(\mathbf{x};\wp)\Big)\Big]\\ &- \mathcal{E}^{-1}\Big[\Big(\sum_{\iota=0}^{p}\frac{c_\iota}{\omega^\alpha}\Big)^{-1}\Big(\sum_{j=1}^{q} c_j \mathcal{E}\Big[\frac{\partial^j \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^j}\Big]\\ &+ \sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2} c_{\eta\sigma}\mathcal{E}\Big[\frac{\partial^\eta \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\sigma}\Big)\Big]\Big].\end{aligned} \tag{5.6}$$

In view of the Adomian decomposition technique, this approach has infinite series solution for the subsequent unknown mappings:

$$\underline{\mathbf{f}}(\mathbf{x},\xi;\wp) = \sum_{r=0}^{\infty}\underline{\mathbf{f}}_r(\mathbf{x},\xi;\wp). \tag{5.7}$$

The non-linearity is dealt by an infinite series of the Adomian polynomials $\underline{A}_r^{\eta\sigma},\ \eta = 0,1,2,\ \sigma = 0,1,2$ has the subsequent representation:

$$\frac{\partial^\eta \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\sigma} = \sum_{r=0}^{\infty} \underline{\mathcal{A}}_r^{\eta\sigma}, \tag{5.8}$$

where

$$\underline{\mathcal{A}}_r^{\eta\sigma} = \begin{cases} \frac{\partial^\eta \underline{\mathbf{f}}_0}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}_0}{\partial \mathbf{x}^\sigma}, & r = 0, \\ \frac{\partial^\eta \underline{\mathbf{f}}_0}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}_1}{\partial \mathbf{x}^\sigma} + \frac{\partial^\eta \underline{\mathbf{f}}_1}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}_0}{\partial \mathbf{x}^\sigma}, & r = 1, \\ \frac{\partial^\eta \underline{\mathbf{f}}_0}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}_2}{\partial \mathbf{x}^\sigma} + \frac{\partial^\eta \underline{\mathbf{f}}_1}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}_1}{\partial \mathbf{x}^\sigma} + \frac{\partial^\eta \underline{\mathbf{f}}_2}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}_0}{\partial \mathbf{x}^\sigma}, & r = 2, \\ \vdots \end{cases} \tag{5.9}$$

Inserting (5.8) and (5.9) in (5.7) refers to the following equation:

$$\begin{aligned} \sum_{r=0}^{\infty} \underline{\mathbf{f}}_r(\mathbf{x},\xi;\wp) =& \mathcal{E}^{-1}\Bigg[\Big(\sum_{\iota=0}^{p}\frac{c_\iota}{\omega^\alpha}\Big)^{-1}\Big(\mathcal{E}[\underline{\mathbf{g}}(\mathbf{x},\xi;\wp)] + \sum_{\iota=1}^{p}\omega^2\underline{\psi}_0(\mathbf{x};\wp)\Big)\Bigg] \\ &- \mathcal{E}^{-1}\Bigg[\Big(\sum_{\iota=0}^{p}\frac{c_\iota}{\omega^\alpha}\Big)^{-1}\Big(\sum_{j=1}^{q}c_j\mathcal{E}\Big[\sum_{r=0}^{\infty}\frac{\partial^j\underline{\mathbf{f}}_r(\mathbf{x};\wp)}{\partial \mathbf{x}^j}\Big] + \sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2}c_{\eta\sigma}\mathcal{E}\Big[\sum_{r=0}^{\infty}\underline{\mathcal{A}}_r^{\eta\sigma}\Big]\Big)\Bigg]. \end{aligned} \tag{5.10}$$

The recursive terms of Elzaki decomposition method can be computed for $r \geq 0$ as follows:

$$\begin{aligned} \underline{\mathbf{f}}_0(\mathbf{x},\xi;\wp) =& \mathcal{E}^{-1}\Bigg[\Big(\sum_{\iota=0}^{p}\frac{c_\iota}{\omega^\alpha}\Big)^{-1}\Big(\mathcal{E}[\underline{\mathbf{g}}(\mathbf{x},\xi;\wp)] + \sum_{\iota=1}^{p}\omega^2\underline{\psi}_0(\mathbf{x},\xi;\wp)\Big)\Bigg], \\ \underline{\mathbf{f}}_{r+1}(\mathbf{x},\xi;\wp) =& -\mathcal{E}^{-1}\Bigg[\Big(\sum_{\iota=0}^{p}\frac{c_\iota}{\omega^\alpha}\Big)^{-1}\Big(\sum_{j=1}^{q}c_j\mathcal{E}\Big[\sum_{r=0}^{\infty}\frac{\partial^j\underline{\mathbf{f}}_r(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^j}\Big] + \sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2}c_{\eta\sigma}\mathcal{E}\Big[\sum_{r=0}^{\infty}\underline{\mathcal{A}}_r^{\eta\sigma}\Big]\Big)\Bigg]. \end{aligned} \tag{5.11}$$

Case II. Suppose the mapping $\mathbf{f}(\mathbf{x},\xi;\wp)$ be $[(i)-\alpha]$-differentiable of the *qth* order in regard to $\mathbf{x}$ and $[(ii)-\alpha]$ differentiable of the $2pth$ order in regard to ξ. Then, the parametric version of (5.3) has the following representation:

$$\begin{aligned} &\sum_{\iota=0}^{p}c_{2\iota}\mathcal{E}[\mathbf{D}_\xi^{2\alpha}\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)] + \sum_{\iota=1}^{p}c_{2\iota-1}\mathcal{E}[\mathbf{D}_\xi^{\alpha}\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)] \\ &\quad + \sum_{j=1}^{q}c_j\mathcal{E}\Big[\frac{\partial^j\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^j}\Big] + \sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2}c_{\eta\sigma}\mathcal{E}\Big[\frac{\partial^\eta\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\sigma}\Big] = \mathcal{E}[\underline{\mathbf{g}}(\mathbf{x},\xi;\wp)], \end{aligned}$$

and

$$\begin{aligned} &\sum_{\iota=0}^{p}c_{2\iota}\mathcal{E}[\mathbf{D}_\xi^{2\alpha}\bar{\mathbf{f}}(\mathbf{x},\xi;\wp)] + \sum_{\iota=1}^{p}c_{2\iota-1}\mathcal{E}[\mathbf{D}_\xi^{\alpha}\bar{\mathbf{f}}(\mathbf{x},\xi;\wp)] \\ &\quad + \sum_{j=1}^{q}c_j\mathcal{E}\Big[\frac{\partial^j\bar{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^j}\Big] + \sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2}c_{\eta\sigma}\mathcal{E}\Big[\frac{\partial^\eta\bar{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma\bar{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial \mathbf{x}^\sigma}\Big] = \mathcal{E}[\bar{\mathbf{g}}(\mathbf{x},\xi;\wp)]. \end{aligned}$$

Utilizing the fact of Theorem 4.3 and ICs, we have

$$\begin{aligned}&\mathcal{B}\mathcal{E}[\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)]+\mathcal{C}\mathcal{E}[\mathbf{D}_{\xi}^{\alpha}\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)]\\ =&\mathcal{E}[\bar{\mathbf{g}}(\mathbf{x},\xi;\wp)+\mathcal{F}_1(\mathbf{x};\wp)]-\sum_{j=1}^{q}c_j\mathcal{E}\Big[\frac{\partial^j\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial\mathbf{x}^j}\Big]\\ &-\sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2}c_{\eta\sigma}\mathcal{E}\Big[\frac{\partial^{\eta}\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial\mathbf{x}^{\eta}}\frac{\partial^{\sigma}\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial\mathbf{x}^{\sigma}}\Big],\end{aligned}\tag{5.12}$$

and

$$\begin{aligned}&\mathcal{B}\mathcal{E}[\bar{\mathbf{f}}(\mathbf{x},\xi;\wp)]+\mathcal{C}\mathcal{E}[\bar{\mathbf{f}}(\mathbf{x},\xi;\wp)]\\ =&\mathcal{E}[\bar{\mathbf{g}}(\mathbf{x},\xi;\wp)+\mathcal{F}_2(\mathbf{x};\wp)]\\ &-\sum_{j=1}^{q}c_j\mathcal{E}\Big[\frac{\partial^j\bar{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial\mathbf{x}^j}\Big]-\sum_{\eta=0}^{2}\sum_{\sigma=\eta}^{2}c_{\eta\sigma}\mathcal{E}\Big[\frac{\partial^{\eta}\bar{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial\mathbf{x}^{\eta}}\frac{\partial^{\sigma}\bar{\mathbf{f}}(\mathbf{x},\xi;\wp)}{\partial\mathbf{x}^{\sigma}}\Big],\end{aligned}\tag{5.13}$$

where $\mathcal{B}=\sum_{\iota=0}^{p}c_{2\iota}\omega^{\alpha},\ \ \mathcal{C}=\sum_{\iota=1}^{p}c_{2\iota-1}\omega^{2-\alpha}$,

$$\mathcal{F}_1(\mathbf{x};\wp)=\sum_{\iota=0}^{p}c_{2\iota}\Big(\omega^{2-2\alpha}\underline{\psi}_0(\mathbf{x};\wp)+\omega^{3-2\alpha}\bar{\psi}_1(\mathbf{x};\wp)\Big)+\sum_{\iota=0}^{p}c_{2\iota-1}\Big(\omega^{3-2\alpha}\bar{\psi}_0(\mathbf{x};\wp)+\omega^{2-2\alpha}\underline{\psi}_0(\mathbf{x};\wp)\Big),$$

and

$$\mathcal{F}_2(\mathbf{x};\wp)=\sum_{\iota=0}^{p}c_{2\iota}\Big(\omega^{2-2\alpha}\bar{\psi}_0(\mathbf{x};\wp)+\omega^{3-2\alpha}\underline{\psi}_1(\mathbf{x};\wp)\Big)+\sum_{\iota=0}^{p}c_{2\iota-1}\Big(\omega^{3-2\alpha}\underline{\psi}_0(\mathbf{x};\wp)+\omega^{2-2\alpha}\bar{\psi}_0(\mathbf{x};\wp)\Big).$$

For the aforementioned Eqs (5.12) and (5.13), we obtain $\mathcal{E}[\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)]$ and $\mathcal{E}[\bar{\mathbf{f}}(\mathbf{x},\xi;\wp)]$ similar to **Case I,** we find the the general solution $\mathbf{f}(\mathbf{x};\wp)=\Big[\underline{\mathbf{f}}(\mathbf{x},\xi;\wp),\bar{\mathbf{f}}(\mathbf{x},\xi;\wp)\Big]$.

Example 5.1. Consider the fuzzy fractional partial differential equation as follows:

$$\mathbf{D}_{\xi}^{2\alpha}\mathbf{f}(\mathbf{x},\xi)\oplus\frac{\partial\mathbf{f}(\mathbf{x},\xi)}{\partial\mathbf{x}}\odot\frac{\partial\mathbf{f}^2(\mathbf{x},\xi)}{\partial\mathbf{x}^2}=g_3(\mathbf{x},\xi),\ \mathbf{x}\geq 0,\ \xi>0,\tag{5.14}$$

subject to ICs

$$\mathbf{f}(\mathbf{x},0)=\Big(\frac{\mathbf{x}^2}{2}\wp,\frac{\mathbf{x}^2}{2}(2-\wp)\Big),\ \ \mathbf{f}'_{\xi}(\mathbf{x},0)=(0,0),\ \mathbf{x}>0,\tag{5.15}$$

and $g_3(\mathbf{x},\xi)=\Big(\wp+\mathbf{x}\wp^2,2-\wp+\mathbf{x}(2-\wp)^2\Big)$.

In order to find solution of (5.14), we have the following three cases.

Case I. If $\mathbf{f}(\mathbf{x},\xi)$ is $[(i)-\alpha]$-differentiable.

Employing the Elzaki transform on (5.14), then we have

$$\frac{1}{\omega^{2\alpha}}\mathcal{E}[\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)]-\omega^{2-2\alpha}\underline{\mathbf{f}}(\mathbf{x},0;\wp)=\mathcal{E}\Big[g_3(\mathbf{x},\xi)-\frac{\partial\mathbf{f}(\mathbf{x},\xi)}{\partial\mathbf{x}}\frac{\partial\mathbf{f}^2(\mathbf{x},\xi)}{\partial\mathbf{x}^2}\Big],$$

or equivalently, we have

$$\mathcal{E}[\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)] - \omega^2\underline{\mathbf{f}}(\mathbf{x},0;\wp) = \omega^{2\alpha}\mathcal{E}\Big[g_3(\mathbf{x},\xi) - \frac{\partial \mathbf{f}(\mathbf{x},\xi)}{\partial \mathbf{x}}\frac{\partial \mathbf{f}^2(\mathbf{x},\xi)}{\partial \mathbf{x}^2}\Big].$$

Further, implementing the inverse fuzzy Elzaki transform, we have

$$\underline{\mathbf{f}}(\mathbf{x},\xi;\wp) = \mathcal{E}^{-1}\Big[\omega^2\underline{\mathbf{f}}(\mathbf{x},0;\wp) + \omega^{2\alpha}\mathcal{E}\Big[g_3(\mathbf{x},\xi) - \frac{\partial \mathbf{f}(\mathbf{x},\xi)}{\partial \mathbf{x}}\frac{\partial \mathbf{f}^2(\mathbf{x},\xi)}{\partial \mathbf{x}^2}\Big]\Big].$$

Also, applying the scheme described in Section 4, we have

$$\sum_{r=0}^{\infty}\underline{\mathbf{f}}_r(\mathbf{x},\xi;\wp) = \mathcal{E}^{-1}\Big[\omega^2\underline{\mathbf{f}}(\mathbf{x},0;\wp) + \omega^{2\alpha}\mathcal{E}[g_3(\mathbf{x},\xi)] - \omega^{2\alpha}\mathcal{E}[\sum_{r=0}^{\infty}\underline{\mathcal{A}}_r]\Big]. \tag{5.16}$$

Utilizing the iterative procedure defined in (5.11), we have

$$\underline{\mathbf{f}}_0(\mathbf{x},\xi;\wp) = \mathcal{E}^{-1}\Big[\omega^2\underline{\mathbf{f}}(\mathbf{x},0;\wp) + \omega^{2\alpha}\mathcal{E}[g_3(\mathbf{x},\xi)]\Big] = \wp\frac{\mathbf{x}^2}{2} + (\wp + \mathbf{x}\wp^2)\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}. \tag{5.17}$$

Also,

$$\underline{\mathbf{f}}_{r+1}(\mathbf{x},\xi;\wp) = \mathcal{E}^{-1}\Big[\omega^{2\alpha}\mathcal{E}[\sum_{r=0}^{\infty}\underline{\mathcal{A}}_r]\Big]. \tag{5.18}$$

Utilizing the first few Adomian polynomials mentioned in (5.9), we have

$$\begin{aligned}
\underline{\mathbf{f}}(\mathbf{x},\xi;\wp) &= \mathcal{E}^{-1}\Big[\omega^{2\alpha}[\underline{\mathcal{A}_0}]\Big] \\
&= -\wp^2\mathbf{x}\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)} - \wp^3\frac{\xi^{4\alpha}}{\Gamma(4\alpha+1)}, \\
\underline{\mathbf{f}}_2(\mathbf{x},\xi;\wp) &= \mathcal{E}^{-1}\Big[\omega^{2\alpha}[\underline{\mathcal{A}_1}]\Big] = -\wp^3\frac{\xi^{4\alpha}}{\Gamma(4\alpha+1)}, \\
\underline{\mathbf{f}}_3(\mathbf{x},\xi;\wp) &= 0, \\
&\vdots
\end{aligned} \tag{5.19}$$

In a similar way we obtained the upper solutions as follows:

$$\begin{aligned}
\bar{\mathbf{f}}_0(\mathbf{x},\xi;\wp) &= \frac{\mathbf{x}^2}{2}(2-\wp) + (2-\wp+\mathbf{x}(2-\wp)^2)\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}, \\
\bar{\mathbf{f}}(\mathbf{x},\xi;\wp) &= -\mathbf{x}(2-\wp)^2\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)} - (2-\wp)^3\frac{\xi^{4\alpha}}{\Gamma(4\alpha+1)}, \\
\bar{\mathbf{f}}_2(\mathbf{x},\xi;\wp) &= -(2-\wp)^3\frac{\xi^{4\alpha}}{\Gamma(4\alpha+1)}, \\
\bar{\mathbf{f}}_3(\mathbf{x},\xi;\wp) &= 0, \\
&\vdots
\end{aligned} \tag{5.20}$$

The series form solution of Example 5.1 is presented as follows:

$$\mathbf{f}(\mathbf{x},\xi)=\left(\left(\frac{\mathbf{x}^2}{2}+\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}\right)\wp,\left(\frac{\mathbf{x}^2}{2}+\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}\right)(2-\wp)\right). \tag{5.21}$$

The numerical solution to the fuzzy fractional nonlinear PDE is presented in this section. Incorporating all of the data in regard to the numerous parameters involved in the related equation is a monumental task. Uncertain responses subject to Caputo fractional order derivatives have been considered, as previously said.

• Table 1 represents the obtained findings with $\mathbf{x} = 0.4$ and $\xi = 0.7$. Table 1 also comprises the outcomes of a Georgieva and Pavlova [67]. As a consequence, the findings acquired by fuzzy Elzaki decomposition method are the same if $\alpha = 1$, as those reported by a Georgieva and Pavlova [67].

• Figure 1 demonstrates the three-dimensional illustration of the lower and upper estimates for different uncertainties $\wp \in [0, 1]$.

• Figure 2 shows the fuzzy responses for different fractional orders.

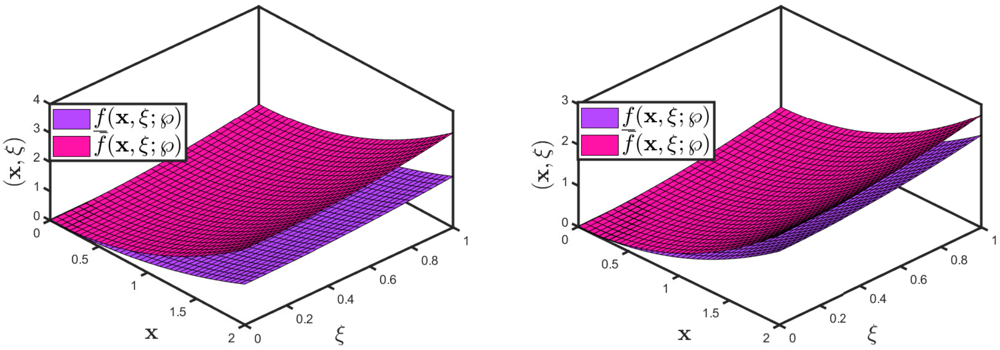
• Figure 3 illustrates the fuzzy responses for different uncertainty parameters.

• The aforementioned representations illustrate that all graphs are substantially identical in their perspectives and have good agreement with one another, especially when integer-order derivatives are taken into account.

Finally, this generic approach for dealing with nonlinear PDEs is more accurate and powerful than the method applied by [67]. Our findings for the fuzzy Elzaki decomposition method, helpful for fuzzy initial value problems, demonstrate the consistency and strength of the offered solutions.

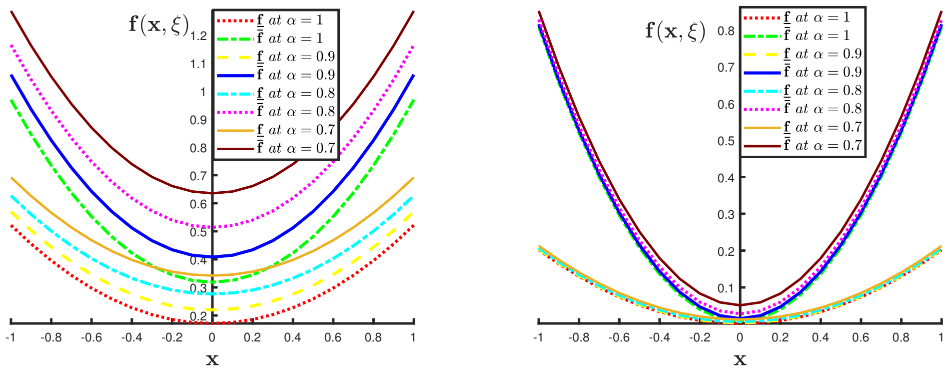
Table 1. Lower and upper solutions of **Case I** of Example 5.1 for various fractional orders in comparison with the solution derived by [67].

$\wp$	$\underline{\mathbf{f}}(\alpha=0.7)$	$\bar{\mathbf{f}}(\alpha=0.7)$	$\underline{\mathbf{f}}(\alpha=1)$	$\bar{\mathbf{f}}(\alpha=1)$	$\underline{\mathbf{f}}$ [67]	$\bar{\mathbf{f}}$ [67]
0.1	1.9420×10^{-2}	3.6899×10^{-1}	9.0000×10^{-3}	1.7100×10^{-1}	9.0000×10^{-3}	1.7100×10^{-1}
0.2	3.8841×10^{-2}	3.4957×10^{-1}	1.8000×10^{-2}	1.6200×10^{-1}	1.8000×10^{-2}	1.6200×10^{-1}
0.3	5.8262×10^{-2}	3.30152×10^{-1}	2.7000×10^{-2}	1.5300×10^{-1}	2.7000×10^{-2}	1.5300×10^{-1}
0.4	7.7682×10^{-2}	3.1073×10^{-1}	3.6000×10^{-2}	1.4400×10^{-1}	3.6000×10^{-2}	1.4400×10^{-1}
0.5	9.7103×10^{-2}	2.9131×10^{-1}	4.5000×10^{-2}	1.3500×10^{-1}	4.5000×10^{-2}	1.3500×10^{-1}
0.6	1.1652×10^{-2}	2.71890×10^{-1}	5.4000×10^{-2}	1.2600×10^{-1}	5.4000×10^{-2}	1.2600×10^{-1}
0.7	1.3594×10^{-2}	2.5246×10^{-1}	6.3000×10^{-2}	1.1700×10^{-1}	6.3000×10^{-2}	1.1700×10^{-1}
0.8	1.5536×10^{-2}	2.3304×10^{-1}	7.2000×10^{-2}	1.0800×10^{-1}	7.2000×10^{-2}	1.0800×10^{-1}
0.9	1.7478×10^{-2}	2.1362×10^{-1}	8.1000×10^{-2}	9.9000×10^{-2}	8.1000×10^{-2}	9.9000×10^{-2}
1.0	1.9420×10^{-1}	1.9420×10^{-1}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}



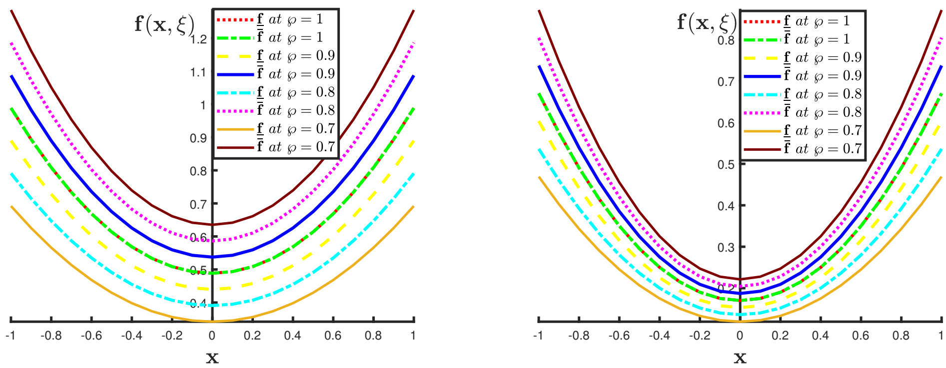
(a) **(b)**

Figure 1. Three-dimensional fuzzy responses of Example 5.1 for **Case I** at (a) $\wp = 0.7$, (b) $\wp = 0.9$ with fractional order $\alpha = 1$.



(a) **(b)**

Figure 2. Two-dimensional fuzzy responses of Example 5.1 for **Case I** at (a) $\wp = 0.7$ *and* $\xi = 0.7$, (b) $\wp = 0.4$ *and* $\xi = 0.1$ with varing fractional orders.



(a) **(b)**

Figure 3. Two-dimensional fuzzy responses of Example 5.1 for **Case I** at (a) $\alpha = 0.7$ *and* $\xi = 0.7$, (b) $\alpha = 0.4$ *and* $\xi = 0.1$ with varing uncertainity parameters $\wp \in [0, 1]$.

Case II. If $\mathbf{f}(\mathbf{x},\xi)$ is $[(ii)-\alpha]$-differentiable, taking into account (5.12) and (5.13), we find

$$\begin{aligned}\frac{1}{\omega^{2\alpha}}\mathcal{E}[\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)] &= \omega^{2-2\alpha}\underline{\mathbf{f}}(\mathbf{x},0;\wp)+\mathcal{E}[\underline{g_3}(\mathbf{x},\xi)]-\mathcal{E}\Big[\frac{\partial\underline{\mathbf{f}}(\mathbf{x},\xi)}{\partial\mathbf{x}}\frac{\partial\underline{\mathbf{f}}^2(\mathbf{x},\xi)}{\partial\mathbf{x}^2}\Big],\\ \frac{1}{\omega^{2\alpha}}\mathcal{E}[\bar{\mathbf{f}}(\mathbf{x},\xi;\wp)] &= \omega^{2-2\alpha}\bar{\mathbf{f}}(\mathbf{x},0;\wp)+\mathcal{E}[\bar{g}_3(\mathbf{x},\xi)]-\mathcal{E}\Big[\frac{\partial\bar{\mathbf{f}}(\mathbf{x},\xi)}{\partial\mathbf{x}}\frac{\partial\bar{\mathbf{f}}^2(\mathbf{x},\xi)}{\partial\mathbf{x}^2}\Big].\end{aligned} \tag{5.22}$$

Employing the inverse fuzzy Elzaki transform to the aforementioned equations and incorporation of Elzaki decomposition technique, we find the solution on same lines as we did in **Case I**.

Case III. If $\mathbf{f}(\mathbf{x},\xi)$ is $[(i)-\alpha]$-differentiable and $\mathbf{f}'(\mathbf{x},\xi)$ is $[(ii)-\alpha]$-differentiable, then

$$\mathbb{E}(\mathbf{f}'(\mathbf{x},\xi)) = [\mathcal{E}(\underline{\mathbf{f}}'(\mathbf{x},\xi;\wp)),\mathcal{E}(\bar{\mathbf{f}}'(\mathbf{x},\xi;\wp))] \tag{5.23}$$

and

$$\mathbb{E}(\mathbf{f}''(\mathbf{x},\xi)) = [\mathcal{E}(\bar{\mathbf{f}}''(\mathbf{x},\xi;\wp)),\mathcal{E}(\underline{\mathbf{f}}''(\mathbf{x},\xi;\wp))]. \tag{5.24}$$

In view of (5.11) and Theorem 4.3 with IC, we follow the iterative process:

Employing the Elzaki transform on (5.14), then we have

$$\frac{1}{\omega^{2\alpha}}\mathcal{E}[\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)]-\omega^{2-2\alpha}\underline{\mathbf{f}}(\mathbf{x},0;\wp)=\mathcal{E}\Big[\bar{g}_3(\mathbf{x},\xi)-\frac{\partial\bar{\mathbf{f}}(\mathbf{x},\xi)}{\partial\mathbf{x}}\frac{\partial\bar{\mathbf{f}}^2(\mathbf{x},\xi)}{\partial\mathbf{x}^2}\Big].$$

or equivalently, we have

$$\mathcal{E}[\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)]-\omega^{2}\underline{\mathbf{f}}(\mathbf{x},0;\wp)=\omega^{2\alpha}\mathcal{E}\Big[\bar{g}_3(\mathbf{x},\xi)-\frac{\partial\bar{\mathbf{f}}(\mathbf{x},\xi)}{\partial\mathbf{x}}\frac{\partial\bar{\mathbf{f}}^2(\mathbf{x},\xi)}{\partial\mathbf{x}^2}\Big].$$

Further, implementing the inverse fuzzy Elzaki transform, we have

$$\underline{\mathbf{f}}(\mathbf{x},\xi;\wp)=\mathcal{E}^{-1}\Big[\omega^{2}\underline{\mathbf{f}}(\mathbf{x},0;\wp)+\omega^{2\alpha}\mathcal{E}\Big[\bar{g}_3(\mathbf{x},\xi)-\frac{\partial\bar{\mathbf{f}}(\mathbf{x},\xi)}{\partial\mathbf{x}}\frac{\partial\bar{\mathbf{f}}^2(\mathbf{x},\xi)}{\partial\mathbf{x}^2}\Big]\Big].$$

Also, applying the scheme described in Section 4, we have

$$\sum_{r=0}^{\infty}\underline{\mathbf{f}}_r(\mathbf{x},\xi;\wp)=\mathcal{E}^{-1}\Big[\omega^{2}\underline{\mathbf{f}}(\mathbf{x},0;\wp)+\omega^{2\alpha}\mathcal{E}[\bar{g}_3(\mathbf{x},\xi)]-\omega^{2\alpha}\mathcal{E}[\sum_{r=0}^{\infty}\bar{\mathcal{A}}_r]\Big].$$

Utilizing the iterative procedure defined in (5.11), we have

$$\begin{aligned}\underline{\mathbf{f}}_0(\mathbf{x},\xi;\wp) &= \mathcal{E}^{-1}\Big[\omega^{2}\underline{\mathbf{f}}(\mathbf{x},0;\wp)+\omega^{2\alpha}\mathcal{E}[\bar{g}_3(\mathbf{x},\xi)]\Big]\\ &= \wp\frac{\mathbf{x}^2}{2}+[(2-\wp)+\mathbf{x}(2-\wp)^2]\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}.\end{aligned}$$

Also,

$$\underline{\mathbf{f}}_{r+1}(\mathbf{x},\xi;\wp)=\mathcal{E}^{-1}\Big[\omega^{2\alpha}\mathcal{E}[\sum_{r=0}^{\infty}\bar{\mathcal{A}}_r]\Big].$$

Utilizing the first few Adomian polynomials as follows:

$$\underline{\mathcal{A}}_r = \begin{cases} \frac{\partial^\eta \underline{\mathbf{f}}_0}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}_0}{\partial \mathbf{x}^\sigma}, & r=0,\\ \frac{\partial^\eta \underline{\mathbf{f}}_0}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}_1}{\partial \mathbf{x}^\sigma}+\frac{\partial^\eta \underline{\mathbf{f}}_1}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}_0}{\partial \mathbf{x}^\sigma}, & r=1,\\ \frac{\partial^\eta \underline{\mathbf{f}}_0}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}_2}{\partial \mathbf{x}^\sigma}+\frac{\partial^\eta \underline{\mathbf{f}}_1}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}_1}{\partial \mathbf{x}^\sigma}+\frac{\partial^\eta \underline{\mathbf{f}}_2}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \underline{\mathbf{f}}_0}{\partial \mathbf{x}^\sigma}, & r=2,\\ \vdots \end{cases} \qquad \bar{\mathcal{A}}_r = \begin{cases} \frac{\partial^\eta \bar{\mathbf{f}}_0}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \bar{\mathbf{f}}_0}{\partial \mathbf{x}^\sigma}, & r=0,\\ \frac{\partial^\eta \bar{\mathbf{f}}_0}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \bar{\mathbf{f}}_1}{\partial \mathbf{x}^\sigma}+\frac{\partial^\eta \bar{\mathbf{f}}_1}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \bar{\mathbf{f}}_0}{\partial \mathbf{x}^\sigma}, & r=1,\\ \frac{\partial^\eta \bar{\mathbf{f}}_0}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \bar{\mathbf{f}}_2}{\partial \mathbf{x}^\sigma}+\frac{\partial^\eta \bar{\mathbf{f}}_1}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \bar{\mathbf{f}}_1}{\partial \mathbf{x}^\sigma}+\frac{\partial^\eta \bar{\mathbf{f}}_2}{\partial \mathbf{x}^\eta}\frac{\partial^\sigma \bar{\mathbf{f}}_0}{\partial \mathbf{x}^\sigma}, & r=2,\\ \vdots \end{cases}$$

$$\begin{aligned} \underline{\mathbf{f}}(\mathbf{x},\xi;\wp) &= \mathcal{E}^{-1}\Big[\omega^{2\alpha}[\bar{\mathcal{A}}_0]\Big]\\ &= -\wp^2(2-\wp)\frac{\xi^{4\alpha}}{\Gamma(4\alpha+1)} - \mathbf{x}(2-\wp)^2\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)},\\ \underline{\mathbf{f}}_2(\mathbf{x},\xi;\wp) &= \mathcal{E}^{-1}\Big[\omega^{2\alpha}[\bar{\mathcal{A}}_1]\Big] = \wp^2(2-\wp)\frac{\xi^{4\alpha}}{\Gamma(4\alpha+1)},\\ \underline{\mathbf{f}}_3(\mathbf{x},\xi;\wp) &= 0,\\ &\vdots \end{aligned}$$

In a similar way we obtained the upper solutions as follows:

$$\begin{aligned} \bar{\mathbf{f}}_0(\mathbf{x},\xi;\wp) &= \frac{\mathbf{x}^2}{2}(2-\wp) + (2-\wp+\mathbf{x}(2-\wp)^2)\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)},\\ \bar{\mathbf{f}}(\mathbf{x},\xi;\wp) &= -\mathbf{x}(2-\wp)^2\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)} - (2-\wp)^3\frac{\xi^{4\alpha}}{\Gamma(4\alpha+1)},\\ \bar{\mathbf{f}}_2(\mathbf{x},\xi;\wp) &= -(2-\wp)^3\frac{\xi^{4\alpha}}{\Gamma(4\alpha+1)},\\ \bar{\mathbf{f}}_3(\mathbf{x},\xi;\wp) &= 0,\\ &\vdots \end{aligned}$$

The series form solution of Example 5.1 is presented as follows:

$$\mathbf{f}(\mathbf{x},\xi) = \left(\left(\frac{\mathbf{x}^2}{2}\wp + (2-\wp)\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}\right), \left(\frac{\mathbf{x}^2}{2}(2-\wp) + \wp\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}\right)\right).$$

The results show that perfect fractional order precision and uncertainty for fuzzy numerical solutions of the function $\mathbf{f}(\mathbf{x},\xi)$ are highly correlated to stuffing time and the fractional order used, whereas additional precision solutions can be obtained by using more redundancy and iterative development.

• Table 2 represents the obtained findings with $\mathbf{x} = 0.4$ and $\xi = 0.7$. Table 2 also comprises the outcomes of a Georgieva and Pavlova [67]. As a consequence, the findings acquired by fuzzy Elzaki decomposition method are the same if $\alpha = 1$, as those reported by a Georgieva and Pavlova [67].

• Figure 4 demonstrates the three-dimensional illustration of the lower and upper estimates for different uncertainties $\wp \in [0, 1]$.

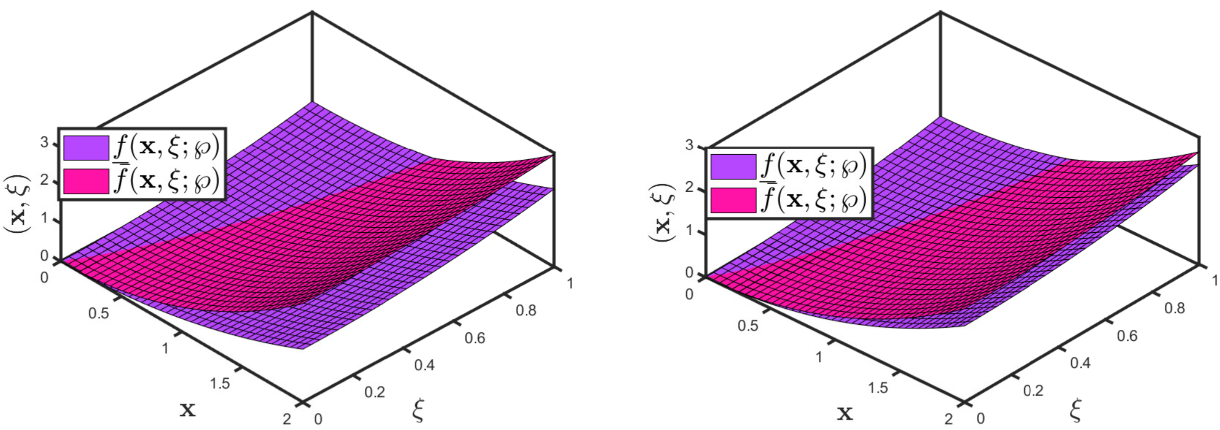
• Figure 5 shows the fuzzy responses for different fractional orders. Figure 6 illustrates the fuzzy responses for different uncertainty parameters.

• The aforementioned representations illustrate that all graphs are substantially identical in their perspectives and have good agreement with one another, especially when integer-order derivatives are taken into account.

Finally, this generic approach for dealing with nonlinear PDEs is more accurate and powerful than the method applied by [67]. Our findings for the fuzzy Elzaki decomposition method, helpful for fuzzy initial value problems, demonstrate the consistency and strength of the offered solutions.

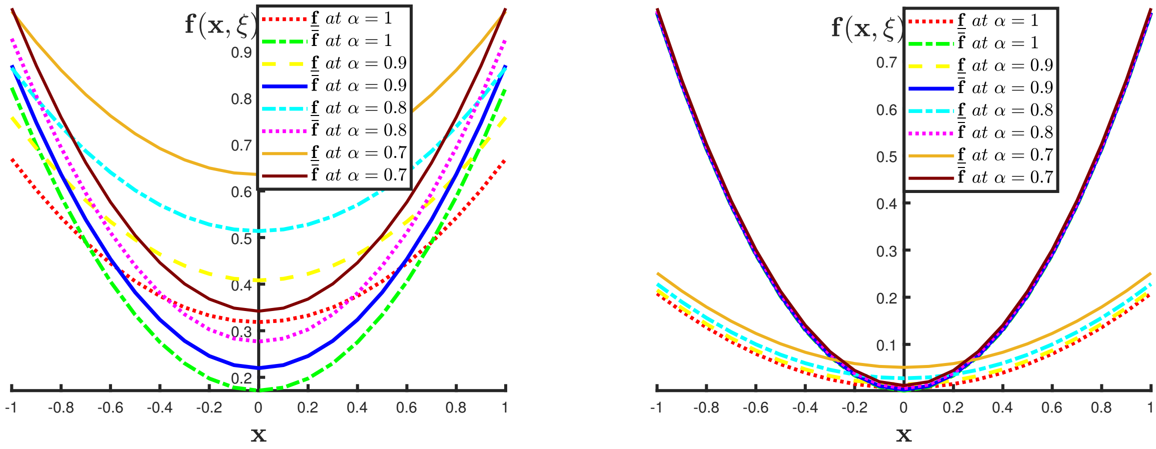
Table 2. Lower and upper solutions of **Case II** of Example 5.1 for various fractional orders in comparison with the solution derived by [67].

$\wp$	$\underline{\mathbf{f}}(\alpha = 0.7)$	$\bar{\mathbf{f}}(\alpha = 0.7)$	$\underline{\mathbf{f}}(\alpha = 1)$	$\bar{\mathbf{f}}(\alpha = 1)$	$\underline{\mathbf{f}}$ [67]	$\bar{\mathbf{f}}$ [67]
0.1	2.8799×10^{-1}	1.0042×10^{-1}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}
0.2	2.7757×10^{-1}	1.1084×10^{-1}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}
0.3	2.6715×10^{-1}	1.2126×10^{-1}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}
0.4	2.5673×10^{-1}	1.3168×10^{-1}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}
0.5	2.4631×10^{-1}	1.4210×10^{-1}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}
0.6	2.3589×10^{-1}	1.5252×10^{-1}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}
0.7	2.2546×10^{-1}	1.6294×10^{-1}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-1}
0.8	2.1504×10^{-1}	1.7336×10^{-1}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-1}
0.9	2.0462×10^{-1}	1.8378×10^{-1}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}
1.0	1.9420×10^{-1}	1.9420×10^{-1}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}	9.0000×10^{-2}



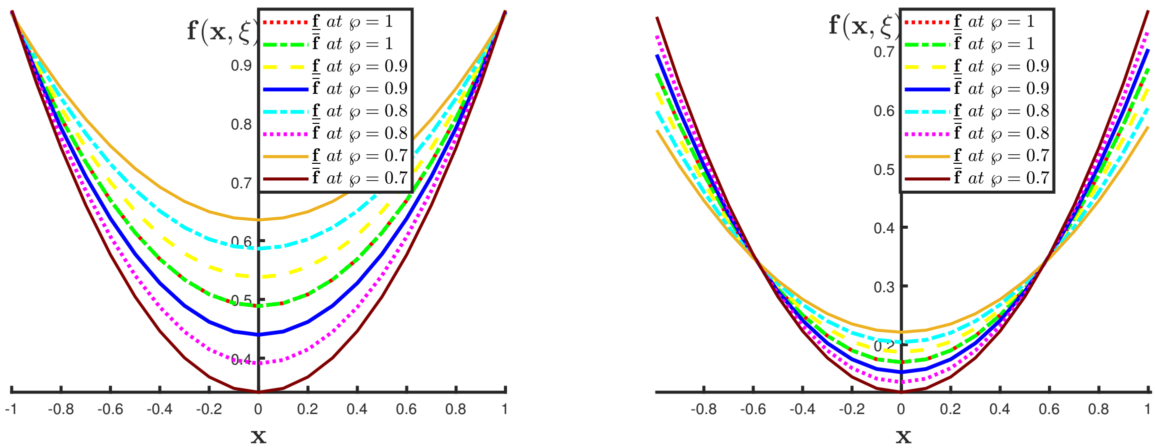
(a) **(b)**

Figure 4. Three-dimensional fuzzy responses of Example 5.1 for **Case II** at (a) $\wp = 0.7$, (b) $\wp = 0.9$ with fractional order $\alpha = 1$.



(a) (b)

Figure 5. Two-dimensional fuzzy responses of Example 5.1 for **Case II** at (a) $\wp = 0.7$ *and* $\xi = 0.7$, (b) $\wp = 0.4$ *and* $\xi = 0.1$ with varing fractional orders.



(a) (b)

Figure 6. Two-dimensional fuzzy responses of Example 5.1 for **Case II** at (a) $\alpha = 0.7$ *and* $\xi = 0.7$, (b) $\alpha = 0.4$ *and* $\xi = 0.1$ with varing uncertainity parameters $\wp \in [0, 1]$.

6. Conclusions

In this investigation, the fuzzy Caputo fractional problem formalism, homogenized fuzzy initial condition, partial differential equation, exemplification of fuzzy Caputo fractional derivative and numerical solutions under $g\mathcal{H}$ are the main significations of the following subordinate part.

• The generic formulation of fuzzy CFDs pertaining to the generic order of $0 < \alpha < r$ is derived by combining all conceivable groupings of items such that t_1 equals 1 and t_2 (the others) equals 2 and utilized for the first time.

• The generic formulas for CFDs regarding the order $\alpha \in (r-1, r)$ are generated under the $g\mathcal{H}$-

difference.

• Under $\mathcal{H}$-differentiabilty, a semi-analytical approach for finding the solution of nonlinear fuzzy fractional PDE has been applied. Besides that, this methodology offers a series of solutions as an analytical expression is its significant aspect.

• A test problem is solved to demonstrate the proposed approach. The simulation results can solve nonlinear partial fuzzy differential equations in a flexible and efficient manner, whilst, frame of numerical programming is natural and the computations are very swift in terms of fractional orders and uncertainty parameters $\wp \in [0, 1]$.

• The results of the projected methodology are more general and fractional in nature than the results provided by [67].

• For futuristic research, a similar method can be applied to Fitzhugh-Nagumo-Huxley by formulating the Henstock integrals (fuzzy integrals in the Lebesgue notion) at infinite intervals [68, 69]. Furthermore, one can explore the implementation of this strategy for relatively intricate challenges, such as the spectral problem [70] and maximum likelihood estimation [71].

Conflict of interest

The authors declare that they have no competing interests.

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