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Research article

Fixed point results for nonlinear contractions of Perov type in abstract metric spaces with applications

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Abstract: In this paper, we present some common fixed point results for g-quasi-contractions of Perov type in cone b-metric spaces without the assumption of continuity. Besides, by constructing a non-expansive mapping from a real Banach algebra $\mathcal{A}$ to $\mathcal{B}(\mathcal{A})$, the space of all of its bounded linear operators, we explore the relationship between the results for the mappings of Perov type on cone metric (cone b-metric) spaces and that for the corresponding mappings on cone metric (cone b-metric) spaces over Banach algebras. As consequences, without the assumption of normality, we obtain common fixed point theorems for generalized g-quasi-contractions with the spectral radius $r(\lambda)$ of the g-quasi-contractive constant vector λ satisfying $r(\lambda) \in [0, \frac{1}{s})$ (where $s \geq 1$) in the setting of cone b-metric spaces over Banach algebras. In addition, we also get some fixed point theorems for nonlinear contractions of Perov type in the setting of cone normed spaces. The main results generalize, extend and unify several well-known comparable results in the literature. Finally, we apply our main results to some nonlinear equations.

Keywords: cone b-metric spaces (over Banach algebras); cone normed space; non-normal cones; nonlinear contractions of Perov type; fixed points
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1. Introduction

In 1905, the famous French mathematician Maurice Fréchet [1] introduced the concept of metric spaces. Metric spaces are a class of useful spaces since when coping with the practical or theoretical problems we often consider the distance of two objects discussed. For instance, a choice of a suitable definition of distance between images naturally leads to an environment in which many possible metrics can be considered simultaneously. Fortunately, cone metric spaces can play a crucial role because they lend themselves to this requirement. One specific instance of this is in the analysis of the structural similarity (SSIM) index of images. SSIM is used to improve the measuring of visual distortion between images. In both of these contexts the difference between two images is calculated using multiple criteria, which leads in a natural way to consider vector-valued distances. In 1934, Kurepa [2] introduced an abstract metric space, in which the metric takes values in an ordered vector space. The metric spaces with vector valued metrics are studied under various names.

In 1980, Rzepecki [3] introduced a generalized metric d_E on a set X in a way that $d_E : X \times X \to K$, replacing the set of real numbers with a Banach space E in the metric function where K is a normal cone in E with a partial order $\leq$. Seven years later, Lin [4] considered the notion of K-metric spaces by replacing the set of non-negative real numbers with a cone K in the metric function. Twenty years after Lin's work, Huang and Zhang [5] announced the notion of a cone metric space by replacing real numbers with an ordering Banach space, which is the same as either the definition of Rzepecki or of Lin. Huang and Zhang in 2007 called such spaces as cone metric spaces. Beg, Abbas, and Nazir [6] nad Beg, Azam, and Arshad [7]replaced the set of an ordered Banach space by a locally convex Hausdorff topological vector space in the definition of a cone metric and a generalized cone metric space. The connection between topological vector space valued cone metric spaces and standard metric spaces and the respective fixed point results were considered by several authors. There are a number of fixed point results concerning generalization of Banach contraction principle in the setting of metric spaces as well as all kinds of abstract spaces (see [8–58]).

In 1964, Perov [8] used used the concept of vector valued metric space, and obtained a Banach-like fixed point theorem on such a complete generalized metric space. After that, fixed point results of Perov type in vector valued metric spaces were studied by many other authors (see e.g., [9–15]). It is known that Perov theorem and related results have many applications in fixed point problems and differential functions and integral equations.

In 2007, Huang and Zhang [5] used the concept of cone metric spaces as a generalization of metric spaces. They have replaced the real numbers by an ordered Banach space. The authors also defined the convergence and completeness in cone metric spaces and proved some fixed point theorems for contractive type mappings in cone metric spaces. Later on, the existence of a fixed point or common fixed point on cone metric spaces was considered (see [16–21]). Recently, Hussain and Shah [22] introduced cone b-metric spaces, as a generalization of b-metric spaces and cone metric spaces, and established some important topological properties in such spaces. Following Hussain and Shah, Huang and Xu [23] obtained some interesting fixed point results for contractive mappings in cone b-metric spaces. Similar results can be seen in [24,25].

Let (X, d) be a complete metric space. Recall that a mapping $T : X \to X$ is called a quasi-contraction if, for some $k \in [0, 1)$ and for all $x, y \in X$, one has

$$d(Tx, Ty) \leqslant k \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Ćirić [26] introduced and studied quasi-contractions as one of the most general classes of contractive-type mappings. He proved the well-known theorem that any quasi-contraction T has a unique fixed point. Recently, scholars obtained various similar results on cone b-metric spaces (some authors call such spaces cone metric type spaces) and cone metric spaces. See, for instance, [19–21,27,28].

Recently, some authors investigated the problem of whether cone metric spaces are equivalent to metric spaces in terms of the existence of the fixed points of the mappings involved. They used to establish the equivalence between some fixed point results in metric and in (topological vector spaces valued) cone metric spaces (see [29–33]). Very recently, based on the concept of cone metric spaces, Liu and Xu [34] studied cone metric spaces with Banach algebras, replacing Banach spaces by Banach algebras as the underlying spaces of cone metric spaces. We may state that it is significant to study cone metric spaces with Banach algebras (which we would like to call cone metric spaces over Banach algebras in this paper). This is because there are examples to show that one is unable to conclude that the cone metric space (X, d) over a Banach algebra $\mathcal{A}$ discussed is equivalent to the metric space (X, d^*), where the metric d^* is defined by $d^* = \xi_e \circ d$. Here the nonlinear scalarization function $\xi_e : \mathcal{A} \to \mathbb{R}$ ($e \in \mathrm{int}P$) is defined by

$$\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - P\}.$$

See [34, Remark 2.3] for more details. In [35], the authors proved some fixed point theorems of quasi-contractions in cone metric spaces over Banach algebras, but the proof relied strongly on the assumption that the underlying cone is normal.

There are a number of generalization of Banach contraction principle. One such generalization is given by Perov [8]. Perov proved the following theorem (also see [9]).

Theorem 1.1. *Let (X, d) be a complete generalized metric space and $d : X \times X \to \mathbb{R}^k$, $f : X \to X$ and $A \in \mathcal{M}_k(\mathbb{R}^+)$ be a matrix convergent to zero, such that*

$$d(f(x), f(y)) \leq A \cdot d(x, y), \quad x, y \in X. \tag{1.1}$$

Then:

(i) f has a unique fixed point $x^* \in X$;

(ii) the sequence of successive approximations $x_n = f(x_{n-1}), n \in \mathbb{N}$, converges to x^* for all $x_0 \in X$;

(iii) $d(x_n, x^*) \leq A^n (I_k - A)^{-1} (d(x_0, x_1)), \quad n \in \mathbb{N}$;

(iv) if $g : X \to X$ satisfies the condition $d(f(x), g(y)) \leq c$ for all $x \in X$ and some $c \in \mathbb{R}^k$, then by considering the sequence $y_n = g^n(x_0), n \in \mathbb{N}$, one has

$$d(y_n, x^*) \leq (I_k - A)^{-1}(c) + A^n (I_k - A)^{-1} (d(x_0, x_1)), \quad n \in \mathbb{N}.$$

Obviously, in previous theorem (X, d) is actually a cone metric space over the normal solid cone $K = \{(x, y) : x \geq 0, y \geq 0\}$ in the Banach space $\mathbb{R}^k$.

There are two aspects to extend Banach contraction principle to the case of cone metric space. One is the case for the mappings of Perov type. A typical result for this case similar to Theorem 1.1 is indicated by the following Theorem 1.2 (see [11, Theorem 2.2]). The other is the case for the so-called generalized Lipschitz mappings in the setting of cone metric spaces over Banach algebras. A typical result for this case similar to Theorem 1.1 is described by the following Theorem 1.3 (see [34, Theorem 2.1]).

Theorem 1.2. *Let (X, d) be a complete cone metric space, $d : X \times X \to E$, $f : X \to X$ and $A \in \mathcal{B}(E)$,with the spectral radius $r(A)$ satisfying $r(A) < 1$ and $AP \subset P$ where P is the solid cone of the Banach space E, such that*

$$d(f(x), f(y)) \leq Ad(x, y), \quad x, y \in X. \tag{1.2}$$

Then:

(i) f has a unique fixed point $x^* \in X$;

(ii) the sequence of successive approximations $x_n = f(x_{n-1}), n \in \mathbb{N}$, converges to x^* for all $x_0 \in X$;

(iii) $d(x_n, x^*) \leq A^n (I - A)^{-1} (d(x_0, x_1)), \quad n \in \mathbb{N}$;

(iv) if $g : X \to X$ satisfies the condition$d(f(x), g(y)) \leq c$ for all $x \in X$ and some $c \in P$, then by considering the sequence $y_n = g^n(x_0), \quad n \in \mathbb{N}$, one has

$$d(y_n, x^*) \leq (I - A)^{-1}(c) + A^n (I - A)^{-1} (d(x_0, x_1)), \quad n \in \mathbb{N}.$$

Theorem 1.3. *Let (X, d) be a complete cone metric space over Banach algebra, $d : X \times X \to E$, $f : X \to X$ and $a \in P$ where P is the solid cone of the Banach space E, with $r(a) < 1$ such that*

$$d(f(x), f(y)) \leq ad(x, y), \quad x, y \in X. \tag{1.3}$$

Then:

(i) f has a unique fixed point $x^* \in X$;

(ii) the sequence of successive approximations $x_n = f(x_{n-1}), n \in \mathbb{N}$, converges to x^* for all $x_0 \in X$;

(iii) $d(x_n, x^*) \leq a^n (e - a)^{-1} (d(x_0, x_1)), \quad n \in \mathbb{N}$;

(iv) if $g : X \to X$ satisfies the condition $d(f(x), g(y)) \leq c$ for all $x \in X$ and some $c \in P$, then by considering the sequence $y_n = g^n(x_0), n \in \mathbb{N}$, one has

$$d(y_n, x^*) \leq (e - a)^{-1}(c) + a^n (e - a)^{-1} (d(x_0, x_1)), \quad n \in \mathbb{N}.$$

It is well known that Theorem 1.2 implies Theorem 1.1. Then a natural question arises, what is the relationship between Theorems 1.2 and 1.3? Does Theorem 1.2 also imply Theorem 1.3?

In the present paper we will first present some common fixed point results for g-quasi-contractions of Perov type in cone b-metric spaces without the assumption of continuity. Next, by constructing a non-expansive mapping from a real Banach algebra $\mathcal{A}$ to $\mathcal{B}(\mathcal{A})$, the space of all of its bounded linear operators, we explore the relationship between the results for the mappings of Perov type on cone metric (cone b-metric) spaces and that for the corresponding mappings on cone metric (cone b-metric) spaces over Banach algebras. As consequences, without the assumption of normality, we obtain common fixed point theorems for the generalized g-quasi-contractions with the spectral radius $r(\lambda)$ of the g-quasi-contractive constant vector λ satisfying $r(\lambda) \in [0, \frac{1}{s})$ (where $s \geq 1$) in the setting of cone b-metric spaces over Banach algebras. As a result, we obtain some fixed point results for quasi-contractions in cone b-metric spaces over Banach algebras, without the assumption that the underlying cone is normal. In addition, we also get some fixed point theorems for nonlinear contractions of Perov type in the setting of cone normed spaces. These results improve the main result of [10–12,35,40,49]. Finally, we apply our main results to a class of nonlinear equations.

2. Preliminaries

Let $\mathcal{A}$ always be a real Banach algebra. That is, $\mathcal{A}$ is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for all $x, y, z \in \mathcal{A}$, $\alpha \in \mathbb{R}$): (1) $(xy)z = x(yz)$; (2) $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$; (3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$; (4) $\|xy\| \leqslant \|x\|\,\|y\|$. Throughout this paper, we shall assume that a Banach algebra $\mathcal{A}$ has a unit (i.e., a multiplicative identity) e such that $ex = xe = x$ for all $x \in \mathcal{A}$. An element $x \in \mathcal{A}$ is said to be invertible if there is an inverse element $y \in \mathcal{A}$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}. For more details, we refer to [36].

Now let us recall the concepts of cone and partial ordering for a Banach algebra $\mathcal{A}$. A subset P of $\mathcal{A}$ is called a cone if (1) P is non-empty closed and $\{\theta, e\} \subset P$; (2) $\alpha P + \beta P \subset P$ for all non-negative real numbers α, β; (3) $P^2 = PP \subset P$; (4) $P \cap (-P) = \{\theta\}$, where θ denotes the null of the Banach algebra $\mathcal{A}$. For a given cone $P \subset \mathcal{A}$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \mathrm{int}P$, where $\mathrm{int}P$ denotes the interior of P. If E is a real Banach space, and $\mathcal{B}(E)$ denotes the space of all the bounded linear operators from E to E. For the given cone $P \subset E$, the partial ordering can be defined similarly.

The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in \mathcal{A}$,

$$\theta \preceq x \preceq y \Rightarrow \|x\| \leqslant M\|y\|.$$

The least positive number satisfying above is called the normal constant of P.

Proposition 2.1. Let $A \in \mathcal{B}(E)$. If the spectral radius $r(A)$ of A is less than 1, i.e.,

$$r(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} = \inf_{n\geqslant 1} \|A^n\|^{\frac{1}{n}} < 1,$$

then $I - A$ is invertible. Actually,

$$(I - A)^{-1} = \sum_{i=0}^{\infty} A^i.$$

Remark 2.1. Let $A, B \in \mathcal{B}(E)$. We say $A \preceq B$ (or $B \succeq A$) if $Ax \preceq Bx$ for any $x \in P$. $AP \subset P$ if and only if A is increasing if and only if $A \succeq O$. Here O denotes the null operator from E to E. So we sometime say that A is a positive operator if it is increasing.

Remark 2.2. Let $A, B, C \in \mathcal{B}(E)$. We have the following properties.

(i) $A \preceq B \iff B - A \succeq O$.

(ii) If $A \preceq B$, $B \preceq C$, then $A \preceq C$.

(iii) If $A \preceq B$, $C \succeq O$, then $CA \preceq CB$, where we define CA as the usual composite $C \circ A$ of A and C, i.e., $(CA)x = C(Ax)$ for any $x \in E$.

(iv) If $A \succeq O$, $B \succeq O$, then $A + B \succeq O$.

(v) If $A_i \succeq O$ $(i = 1, 2, ...)$, and $A = \sum_{i=1}^{\infty} A_i$, then $A \succeq O$.

(vi) If $r(A) < 1$, then $\|A^n\| \to 0$ $(n \to \infty)$.

Proposition 2.2. Let $A \in \mathcal{B}(E)$ with $A \succeq O$. If $r(A) < 1$, then the following assertions hold true.

(i) Suppose that for some $T \in \mathcal{B}(E)$, T is invertible with $T^{-1} \succeq O$ must imply $T \succeq O$, then for any integer $n \geq 1$, we have $A^n \preceq A \preceq I$;

(ii) For any $u \succ \theta$, we have $u \npreceq Au$. Moreover, we have $u \npreceq A^n u$ for any integer $n \geq 1$;

(iii) If $A \geq O$, then $(I-A)^{-1} \geq I \geq O$. In addition, we have $A^n \leq (I-A)^{-1}A^n \leq (I-A)^{-1}A$ for any integer $n \geq 1$.

Proof. (i) Since $r(A) < 1$, by Proposition 2.1, the element $I - A$ is invertible. Considering

$$I = (I-A)(I-A)^{-1} = (I-A)\sum_{i=0}^{\infty} A^i,$$

we have

$$A = (I-A)\sum_{i=1}^{\infty} A^i \leq (I-A)\sum_{i=0}^{\infty} A^i = (I-A)(I-A)^{-1} = I$$

which implies that, by induction on n,

$$A^n \leq A$$

for all $n \geq 1$ by induction. Therefore, the conclusion of (i) is true.

(ii) If it is not true, then there exists an element $u_0 \in E$ with $u_0 \succ \theta$ such that

$$u_0 \preceq Au_0.$$

Hence, it follows that

$$(I-A)u_0 \preceq \theta.$$

Then, multiplying the both sides with $(I-A)^{-1}$, it follows that $u_0 = \theta$, a contradiction.

(iii) It is obvious.

In the following, we always assume that P is a cone in Banach algebra $\mathcal{A}$ with $\text{int}P \neq \emptyset$ and $\preceq$ is the partial ordering with respect to P. For the sake of the main theorems, we need the following auxiliary results.

Lemma 2.1. ([36–42]) (1) If E is a real Banach space with a cone P and if $a \preceq \lambda a$ with $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.

(2) If E is a real Banach space with a solid cone P and $\|x_n\| \to 0$ $(n \to \infty)$, then for any $\theta \ll \epsilon$, there exists $N \in \mathbb{N}$ such that for any $n > N$, we have $x_n \ll \epsilon$.

Definition 2.1. ([27,34,35]) Let X be a nonempty set and $s \geq 1$ a given real number. A mapping $d: X \times X \to \mathcal{A}$ is said to be a cone b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $\theta \prec d(x,y)$ with $x \neq y$ and $d(x,y) = \theta$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,y) \preceq s[d(x,z) + d(z,y)]$.

The pair (X,d) is called a cone b-metric space over a Banach algebra $\mathcal{A}$.

If we replace Banach algebra $\mathcal{A}$ by Banach space E and take $s = 1$ in the above definition, then the corresponding (X,d) is called a cone metric space (see [14]).

Example 2.1. Denote by L_p $(0 < p < 1)$ the set of all real measurable functions $x(t)$ $(t \in [0,1])$ such that $\int_0^1 |x(t)|^p \mathrm{dt} < \infty$. Let $X = L_p$, $\mathcal{A} = \mathbb{R}^2$, $P = \{(x,y) \in \mathcal{A} \mid x, y \geq 0\} \subset \mathbb{R}^2$ and $d: X \times X \to \mathcal{A}$ such that

$$d(x,y) = \left(\alpha\left\{\int_0^1 |x(t)-y(t)|^p \mathrm{d}t\right\}^{\frac{1}{p}}, \beta\left\{\int_0^1 |x(t)-y(t)|^p \mathrm{d}t\right\}^{\frac{1}{p}}\right),$$

where $\alpha, \beta \geq 0$ are constants. Then (X, d) is a cone b-metric space over a Banach algebra $\mathcal{A}$ with the coefficient $s = 2^{\frac{1}{p}-1}$.

Example 2.2. Let $X = \mathbb{R}$, $\mathcal{A} = C^1_{\mathbb{R}}[0, 1]$ and $P = \{f \in \mathcal{A} : f \geq 0\}$. Define $d : X \times X \to \mathcal{A}$ by $d(x, y) = |x - y|^{1.5}\varphi(t)$ where $\varphi : [0, 1] \to \mathbb{R}$ is a function such that $\varphi(t) = \exp(t)$. It is easy to see that (X, d) is a cone b-metric space over a Banach algebra $\mathcal{A}$ with the coefficient $s = 2^{0.5}$, but it is not a cone metric space.

Definition 2.2. ([22,27,34,35]) Let (X, d) be a cone b-metric space over a Banach algebra $\mathcal{A}$, $x \in X$ and $\{x_n\}$ be a sequence in X.

(i) $\{x_n\}$ converges to x whenever for every $c \in E$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x(n \to \infty)$.

(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(iii) (X, d) is a complete cone b-metric space over a Banach algebra $\mathcal{A}$ if every Cauchy sequence is convergent.

Lemma 2.2. ([22,27,34,35,39]) Let $\preceq$ be the partial ordering with respect to P, where P is the given cone P of the Banach algebra $\mathcal{A}$. The following properties are often used while dealing with cone b-metric spaces where the underlying cone is not necessarily normal.

(1) If $u \ll v$ and $v \preceq w$, then $u \ll w$.

(2) If $\theta \preceq u \ll c$ for each $c \in \text{int}P$, then $u = \theta$.

(3) If $a \preceq b + c$ for each $c \in \text{int}P$, then $a \preceq b$.

(4) If $c \in \text{int}P$ and $a_n \to \theta$, then there exists $n_0 \in \mathbb{N}$ such that $a_n \ll c$ for all $n > n_0$.

(5) Let (X, d) be a cone b-metric space over a Banach algebra $\mathcal{A}$, $x \in X$ and $\{x_n\}$ be a sequence in X. If $d(x_n, x) \preceq b_n$ and $b_n \to \theta$, then $x_n \to x$.

Lemma 2.3. ([22]) The limit of a convergent sequence in cone b-metric space is unique.

Definition 2.3. ([47]) Let X be a vector space over $\mathbb{R}$. Suppose the mapping $\|\cdot\|_P : X \to E$ satisfies

(N1) $\|x\|_P > 0$ for all $x \in X$;

(N2) $\|x\|_P = 0$ if and only if $x = 0$;

(N3) $\|x + y\|_P \leq \|x\|_P + \|y\|_P$ for all $x, y \in X$;

(N4) $\|kx\|_P = |k|\|x\|_P$ for all $k \in \mathbb{R}$.

Then $\|\cdot\|_P$ is called cone norm on X, and the pair $(X, \|\cdot\|_P)$ is called a cone normed space (CNS).

Definition 2.4. ([48]) Let $(X, \|\cdot\|_P)$ be a CNS, $x \in X$ and $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ converges to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $\|x_n - x\|_P \ll c$ for all $n \geq N$. It is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$;

(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number N, such that $\|x_n - x\|_P \ll c$ for all $n, m \geq N$;

(iii) $(X, \|\cdot\|_P)$ is a complete cone normed space if every Cauchy sequence is convergent. Complete cone normed spaces will be called cone Banach spaces.

Proposition 2.3. ([36]) If $a \in P$ and $b \notin P$, then $a \preceq b$ does not hold.

Remark 2.3. Note that each cone normed space is a cone metric space. Indeed, suppose $(X, \|\cdot\|_P)$ is a cone normed space. Let $d(x, y) = \|x - y\|_P$ for all $x, y \in X$. Then (X, d) is a cone metric space. Moreover, the following conclusions are true.

(i) If $x \in X$ and $\{x_n\}$ is a convergence sequence in $(X, \|\cdot\|_P)$, $\lim_{n\to\infty} x_n = x$, then $\{x_n\}$ is a convergence sequence in (X, d) and $\lim_{n\to\infty} x_n = x$.

(ii) If $x \in X$ and $\{x_n\}$ is a Cauchy sequence in $(X, \|\cdot\|_P)$, then $\{x_n\}$ is a Cauchy sequence in (X, d).

(iii) If $(X, \|\cdot\|_P)$ is a complete cone normed space, then (X, d) is a complete cone metric space.

Definition 2.5. ([16,24]) The mappings $f, g : X \to X$ are called weakly compatible, if for every $x \in X$ holds $fgx = gfx$ whenever $fx = gx$.

Definition 2.6. ([16,24,27]) Let f and g be self-maps of a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Lemma 2.4. ([16,24,27]) Let f and g be weakly compatible self-maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.

Lemma 2.5. Let $\mathcal{A}$ be a real Banach algebra and denote by $\mathcal{B}(\mathcal{A})$ the space of all of its bounded linear operators. Then there exists a mapping $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{A})$ satisfying the following

(i) for any $a \in \mathcal{A}$, there exists $\psi(a) \in \mathcal{B}(\mathcal{A})$ such that

$$\psi(a)(x) = ax, \quad x \in \mathcal{A}; \tag{2.1}$$

(ii) $r(\psi(a)) \leq r(a)$;

(iii) $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{A})$ is injective. Moreover, it is non-expansive, i.e.,

$$\|\psi(a) - \psi(b)\| \leq \|a - b\|, \quad a,\ b \in \mathcal{A}.$$

Proof. (i) For each $a \in \mathcal{A}$, there exists an operator $A : \mathcal{A} \to \mathcal{A}$ such that

$$Ax = ax, \quad x \in \mathcal{A}. \tag{2.2}$$

Firstly, we prove $A \in \mathcal{B}(\mathcal{A})$. In fact, for $x, y \in \mathcal{A}$, $k \in \mathbb{R}$, we see

$$A(x + y) = a(x + y) = ax + ay = Ax + Ay,$$

$$A(kx) = a(kx) = k(ax) = k(Ax)$$

and

$$\|Ax\| = \|ax\| \leq \|a\|\|x\|,$$

which imply $A \in \mathcal{B}(\mathcal{A})$ and $\|A\| \leq \|a\|$.

Next, we prove that for $a \in \mathcal{A}$, the operator A is unique. In fact, if there exists an operator $A_1 \in \mathcal{B}(\mathcal{A})$ such that $A_1x = ax$, $x \in \mathcal{A}$, then $A_1x = Ax$, $x \in \mathcal{A}$. So we have $(A_1 - A)x = \theta$ for any $x \in \mathcal{A}$. Thus, $A_1 - A = O$, i.e., $A_1 = A$.

Define $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{A})$ as $\psi(a) = A$. It is well defined and satisfies (2.1).

(ii) Let $\psi(a) = A$, as indicated above. By (2.1), for any $x \in \mathcal{A}$ we have

$$A^2x = A(Ax) = A(ax) = a^2x.$$

By induction on n, we have

$$A^nx = a^nx, \quad n \in \mathbb{N}.$$

Hence, we get

$$\|A^nx\| = \|a^nx\| \leq \|a^n\|\|x\|,$$

and so it deduces that $\|A^n\| \le \|a^n\|$, which implies that

$$r(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} \le \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = r(a).$$

(iii) For any $b \in \mathcal{A}$ with $b \neq a$, by (i), there exists $B \in \mathcal{B}(\mathcal{A})$ such that

$$Bx = bx, \quad x \in \mathcal{A}. \tag{2.3}$$

Since $b \neq a$, there is $x_0 \in \mathcal{A}$ with $x_0 \neq \theta$ such that $bx_0 \neq ax_0$. Thus by (2.2) and (2.3) we have $Bx_0 \neq Ax_0$, which shows that $B \neq A$. That is, ψ is injective. Now we prove that it is non-expansive. In fact, for any $a, b \in \mathcal{A}$, by (i), there exist $A, B \in \mathcal{B}(\mathcal{A})$ such that $\psi(a) = A, \psi(b) = B$ satisfying (2.2) and (2.3). Then, for any $x \in \mathcal{A}$, we have

$$\|(\psi(a) - \psi(b))(x)\| = \|\psi(a)(x) - \psi(b)(x)\| = \|ax - bx\| = \|(a - b)x\| \le \|a - b\|\|x\|,$$

so

$$\|\psi(a) - \psi(b)\| \le \|a - b\|, \quad a, b \in \mathcal{A}.$$

3. Fixed point results for g-quasi-contraction of Perov type in cone b-metric spaces

In this section, we give some common fixed point results for g-quasi-contractions of Perov type with the quasi-contractive constant operator A satisfying $r(A) \in [0, \frac{1}{s})$ in the setting of cone b-metric spaces without the assumption of normality or continuity. We recall the definition of g-quasi-contraction in cone b-metric spaces.

Definition 3.1. ([24]) Let (X, d) be a cone b-metric space with the coefficient $s \ge 1$. A mapping $f : X \to X$ is called a g-quasi-contraction where $g : X \to X$, $f(X) \subset g(X)$, if for some real number λ with $\lambda \in [0, 1/s)$ and for all $x, y \in X$, one has

$$d(fx, fy) \preceq \lambda u,$$

where

$$u \in C(g; x, y) = \{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\}.$$

Similarly, we also have the following definition of g-quasi-contraction of Perov type in cone b-metric spaces.

Definition 3.2. Let (X, d) be a cone b-metric space, with the coefficient $s \ge 1$ and $f, g : X \to X$. Then, f is called a g-quasi-contraction of Perov type if for some bounded linear operator $A \in \mathcal{B}(E)$, with $r(A) < \frac{1}{s}$ and for all $x, y \in X$, there exists

$$u \in C(g; x, y) = \{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\}$$

such that

$$d(fx, fy) \preceq A(u(x, y)),$$

where $r(A)$ denotes the spectral radius of A. Furthermore, if $g = I_x$ (the identity mapping from X to X), then f is called a quasi-contraction of Perov type.

Remark 3.1. Definition 3.2 extends the concept of g-quasi-contraction of Perov type in cone metric spaces to the one in the setting of cone b-metric spaces.

Theorem 3.1. Let (X, d) be a cone b-metric space over a Banach space E with the coefficient $s \geq 1$ and the underlying solid cone P. Let the mapping $f : X \to X$ be the g-quasi-contraction of Perov type with the g-quasi-contractive constant operator A satisfying $r(A) \in [0, \frac{1}{s})$ and $AP \subset P$. If $f(X) \subseteq g(X)$ and $g(X)$ or $f(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

We begin the proof of Theorem 3.1 with a useful lemma. For each $x_0 \in X$, set $gx_1 = fx_0$ and $gx_{n+1} = fx_n$. We will prove that $\{gx_n\}$ is a Cauchy sequence. First, we shall show the following lemmas. Note that for these lemmas, we suppose that all the conditions of Theorem 3.1 are satisfied.

Lemma 3.1. For any $N \geq 2$ and $1 \leq m \leq N - 1$, one has that

$$d(gx_N, gx_m) \preceq sA(I - sA)^{-1}d(gx_1, gx_0). \tag{3.1}$$

Proof. We prove Lemma 3.1 by induction. When $N = 2, m = 1$, since $f : X \to X$ is a g-quasi-contraction, there exists

$$u_1 \in C(g; x_1, x_0) = \{d(gx_1, gx_0), d(gx_1, gx_2), d(gx_0, gx_1), d(gx_1, gx_1), d(gx_0, gx_2)\}$$

such that

$$d(gx_2, gx_1) \preceq Au_1.$$

Hence, $u_1 = d(gx_1, gx_0)$ or $u_1 = d(gx_0, gx_2)$. (Note that it is obvious that $u_1 \neq d(gx_1, gx_2)$ since $d(gx_2, gx_1) \npreceq Ad(gx_1, gx_2)$ and $u_1 \neq d(gx_1, gx_1)$ since $d(gx_1, gx_2) \neq \theta$.)

When $u_1 = d(gx_1, gx_0)$, then we have

$$d(gx_2, gx_1) \preceq Ad(gx_0, gx_1) \preceq sAd(gx_0, gx_1) \preceq sA(I - sA)^{-1}d(gx_1, gx_0).$$

When $u_1 = d(gx_2, gx_0)$, then we have

$$d(gx_2, gx_1) \preceq Ad(gx_2, gx_0) \preceq sA[d(gx_2, gx_1) + d(gx_1, gx_0)].$$

So we get

$$(I - sA)d(gx_2, gx_1) \preceq sAd(gx_1, gx_0),$$

which implies that

$$d(gx_2, gx_1) \preceq sA(I - sA)^{-1}d(gx_1, gx_0).$$

Hence, (3.1) holds for $N = 2$ and $m = 1$.

Suppose that for some $N \geq 2$ and for any $2 \leq p \leq N$ and $1 \leq n < p$, one has

$$d(gx_p, gx_n) \preceq sA(I - sA)^{-1}d(gx_1, gx_0). \tag{3.2}$$

That is,

$$d(gx_p, gx_1) \preceq sA(I - sA)^{-1}d(gx_1, gx_0), \tag{3.2.1}$$

$$d(gx_p, gx_2) \preceq sA(I - sA)^{-1}d(gx_1, gx_0), \tag{3.2.2}$$

......

$$d(gx_p, gx_{p-1}) \preceq sA(I - sA)^{-1}d(gx_1, gx_0). \tag{3.2.p − 1}$$

Then, we need to prove that for $N + 1 \geq 2$ and any $1 \leq n < N + 1$, one has

$$d(gx_{N+1}, gx_n) \preceq sA(I - sA)^{-1}d(gx_1, gx_0). \tag{3.3}$$

That is,

$$d(gx_{N+1}, gx_1) \preceq sA(I - sA)^{-1}d(gx_1, gx_0), \tag{3.3.1}$$

$$d(gx_{N+1}, gx_2) \preceq sA(I - sA)^{-1}d(gx_1, gx_0), \tag{3.3.2}$$

......

$$d(gx_{N+1}, gx_{N-1}) \preceq sA(I - sA)^{-1}d(gx_1, gx_0), \tag{3.3.N − 1}$$

$$d(gx_{N+1}, gx_N) \preceq sA(I - sA)^{-1}d(gx_1, gx_0). \tag{3.3.N}$$

In fact, since $f : X \to X$ is a g-quasi-contraction, there exists

$$u_1 \in C(g; x_N, x_0) = \{d(gx_N, gx_0), d(gx_N, gx_{N+1}), d(gx_0, gx_1), d(gx_N, gx_1), d(gx_0, gx_{N+1})\}$$

such that

$$d(gx_{N+1}, gx_1) \preceq Au_1.$$

If $u_1 = d(gx_N, gx_1)$, then by (3.2.1) we have

$$d(gx_{N+1}, gx_1) \preceq sA^2(I - sA)^{-1}d(gx_1, gx_0) \preceq (sA)^2(I - sA)^{-1}d(gx_1, gx_0) \preceq sA(I - sA)^{-1}d(gx_1, gx_0).$$

If $u_1 = d(gx_0, gx_1)$, then we have

$$d(gx_{N+1}, gx_1) \preceq Ad(gx_1, gx_0) \preceq sAd(gx_1, gx_0) \preceq sA(I - sA)^{-1}d(gx_1, gx_0).$$

If $u_1 = d(gx_N, gx_0)$, then by (3.2.1) we have

$$\begin{aligned} d(gx_{N+1}, gx_1) &\preceq Ad(gx_N, gx_0) \preceq sA(d(gx_N, gx_1) + d(gx_1, gx_0)) \\ &\preceq sA(sA(I - sA)^{-1}d(gx_1, gx_0) + d(gx_1, gx_0)) \\ &= sA(sA(I - sA)^{-1} + e)d(gx_1, gx_0) \\ &= sA(I - sA)^{-1}d(gx_1, gx_0). \end{aligned}$$

If $u_1 = d(gx_0, gx_{N+1})$, then we have

$$d(gx_{N+1}, gx_1) \preceq Ad(gx_0, gx_{N+1}) \preceq sA(d(gx_0, gx_1) + d(gx_1, gx_{N+1})).$$

Hence, we see

$$(I - sA)d(gx_{N+1}, gx_1) \preceq sAd(gx_0, gx_1),$$

which implies that

$$d(gx_{N+1}, gx_1) \preceq (I - sA)^{-1}sAd(gx_0, gx_1).$$

Without loss of generality, suppose that $u_1 = d(gx_N, gx_{N+1})$. Since $f : X \to X$ is a g-quasi-contraction, there exists $u_2 \in C(g; x_{N-1}, x_N)$ such that

$$u_1 = d(gx_N, gx_{N+1}) \preceq Au_2,$$

where

$$C(g; x_{N-1}, x_N) = \{d(gx_{N-1}, gx_N), d(gx_{N-1}, gx_N), d(gx_N, gx_{N+1}), d(gx_{N-1}, gx_{N+1}), d(gx_N, gx_N)\}.$$

So, we get

$$d(gx_{N+1}, gx_1) \preceq Au_1 \preceq A^2u_2.$$

Similarly, it is easy to see that $u_2 \neq d(gx_N, gx_N)$ since $u_2 \neq \theta$ and $u_2 \neq d(gx_N, gx_{N+1})$ since $d(gx_N, gx_{N+1}) \npreceq A^2d(gx_N, gx_{N+1})$.

If $u_2 = d(gx_{N-1}, gx_N)$, then by the induction assumption (3.2) we have

$$\begin{aligned} d(gx_{N+1}, gx_1) &\preceq A^2u_2 \preceq sA^3(I - sA)^{-1}d(gx_1, gx_0) \\ &\preceq (sA)^3(I - sA)^{-1}d(gx_1, gx_0) \\ &\preceq sA(I - sA)^{-1}d(gx_1, gx_0). \end{aligned}$$

Without loss of generality, suppose that $u_2 = d(gx_{N-1}, gx_{N+1})$. There exists $u_3 \in C(g; x_{N-2}, x_N)$ such that

$$u_2 = d(gx_{N-1}, gx_{N+1}) \preceq Au_3,$$

where

$$C(g; x_{N-2}, x_N) = \{d(gx_{N-2}, gx_N), d(gx_{N-2}, gx_{N-1}), d(gx_N, gx_{N+1}), d(gx_{N-2}, gx_{N+1}), d(gx_N, gx_{N-1})\}.$$

In general, suppose that $u_{i-1} = d(gx_{N-i+2}, gx_{N+1})$. Since $f : X \to X$ is a g-quasi-contraction, by using similar arguments as above, there exists $u_i \in C(g; x_{N-i+1}, x_N)$ such that

$$u_{i-1} = d(gx_{N-i+2}, gx_{N+1}) \preceq Au_i,$$

for which we obtain

$$d(gx_{N+1}, gx_1) \preceq Au_1 \preceq A^2u_2 \preceq \cdots \preceq A^iu_i,$$

where

$$\begin{aligned} C(g; x_{N-i+1}, x_N) = \{&d(gx_{N-i+1}, gx_N), d(gx_{N-i+1}, gx_{N-i+2}), d(gx_N, gx_{N+1}), \\ &d(gx_{N-i+1}, gx_{N+1}), d(gx_N, gx_{N-i+2})\}. \end{aligned}$$

Similarly, it is easy to see that $u_i \neq d(gx_N, gx_{N+1})$. Namely, by Proposition 2.2(ii), we have

$$Au_1 \npreceq A^{i-1}(Au_1) = A^iu_1 = A^id(gx_N, gx_{N+1}),$$

So we know that if $u_i = d(gx_{N-i+1}, gx_N)$ or $u_i = d(gx_{N-i+1}, gx_{N-i+2})$ or $u_i = d(gx_N, gx_{N-i+2})$ then by the induction assumption (3.2) we have $u_i \preceq sA(I - sA)^{-1}d(gx_1, gx_0)$. Hence,

$$d(gx_{N+1}, gx_1) \preceq A^iu_i \preceq sA^{i+1}(I - sA)^{-1}d(gx_1, gx_0)$$

$$\leq (sA)^{i+1}(I - sA)^{-1}d(gx_1, gx_0)$$
$$\leq sA(I - sA)^{-1}d(gx_1, gx_0),$$

and (3.3.1) is true. Without loss of generality, suppose that $u_i = d(gx_{N-i+1}, gx_{N+1})$. Then by the similar arguments as above we have $u_i \leq Au_{i+1}$, where $u_{i+1} \in C(g; x_{N-i}, x_N)$. Hence, there is a sequence $\{u_n\}$ such that

$$d(gx_{N+1}, gx_1) \leq Au_1 \leq A^2u_2 \leq \cdots \leq A^{N-1}u_{N-1} \leq A^N u_N,$$

where

$$u_{N-1} = d(gx_2, gx_{N+1}) \leq Au_N$$

and

$$u_N \in C(g; x_1, x_N) = \{d(gx_1, gx_N), d(gx_1, gx_2), d(gx_N, gx_{N+1}), d(gx_N, gx_2), d(gx_1, gx_{N+1})\}.$$

Obviously, $u_N \neq d(gx_1, gx_{N+1})$ and $u_N \neq d(gx_N, gx_{N+1})$. On the contrary, if $u_N = d(gx_1, gx_{N+1})$, then $u_N \leq A^N u_N$, a contradiction. If $u_N = d(gx_N, gx_{N+1}) = u_1$, then we have

$$u_1 = d(gx_N, gx_{N+1}) \leq A^2u_2 \leq \cdots \leq A^{N-1}u_{N-1} \leq A^{N-1}u_1,$$

a contradiction. Hence, it follows that $u_N = d(gx_1, gx_N)$, $u_N = d(gx_1, gx_2)$ or $u_N = d(gx_N, gx_2)$. By the induction assumption (3.2), in any case, we have

$$u_N \leq sA(I - sA)^{-1}d(gx_1, gx_0). \tag{3.4}$$

Therefore, we get

$$\begin{aligned} d(gx_{N+1}, gx_1) &\leq Au_1 \leq A^2u_2 \leq \cdots \leq A^N u_N \\ &\leq A^N(I - sA)^{-1}sAd(gx_1, gx_0) \\ &\leq (sA)^{N+1}(I - sA)^{-1}d(gx_1, gx_0) \\ &\leq sA(I - sA)^{-1}d(gx_1, gx_0). \end{aligned} \tag{3.5}$$

That is to say, (3.3.1) is true. By (3.5), we have

$$u_1 \leq A^{N-1}sA(I - sA)^{-1}d(gx_1, gx_0).$$

Thus,

$$\begin{aligned} d(gx_N, gx_{N+1}) = u_1 &\leq A^{N-1}sA(I - sA)^{-1}d(gx_1, gx_0) \\ &\leq (sA)^N(I - sA)^{-1}d(gx_1, gx_0) \\ &\leq sA(I - sA)^{-1}d(gx_1, gx_0), \end{aligned}$$

which implies that (3.3.N) is true. Similarly, since

$$u_2 = d(gx_{N-1}, gx_{N+1}), \ldots, u_i = d(gx_{N-i+1}, gx_{N+1}), \ldots,$$

by (3.4) and (3.5) we get

$$u_i \preceq A^{N-i}u_N \preceq sA^{n-i+1}(I - sA)^{-1}d(gx_1, gx_0). \tag{3.6}$$

Hence, it follows from (3.6) that (3.3.2) and (3.3.$N-1$) are all true. That is, (3.3) is true. Therefore, we conclude that Lemma 3.1 holds true.

By Lemma 3.1, we immediately obtain the following result.

Lemma 3.2. For all $i, j \in \mathbb{N}$, one has

$$d(gx_i, gx_j) \preceq sA(I - sA)^{-1}d(gx_0, gx_1). \tag{3.7}$$

Now, we begin to prove Theorem 3.1. First, we need to show that $\{gx_n\}$ is a Cauchy sequence. For all $n > m$, there exists

$$\begin{aligned} v_1 \in C(g; x_{n-1}, x_{m-1}) = \{&d(gx_{n-1}, gx_{m-1}), d(gx_{n-1}, gx_n),\\ &d(gx_{m-1}, gx_m), d(gx_{n-1}, gx_m), d(gx_{m-1}, gx_n)\}\end{aligned}$$

such that

$$d(fx_{n-1}, fx_{m-1}) \preceq Av_1.$$

Using the g-quasi-contractive condition repeatedly, we easily show by induction that there must exist

$$v_k \in \{d(gx_i, gx_j) : 0 \leq i < j \leq n\}, \quad k = 2, 3, \ldots, m$$

such that

$$v_k \preceq Av_{k+1}, \quad k = 1, 2, \ldots, m-1. \tag{3.8}$$

For convenience, we write $v_m = d(gx_i, gx_j)$ for some $0 \leq i < j \leq n$.

By Lemma 3.2 we obtain

$$\begin{aligned} d(gx_n, gx_m) &= d(fx_{n-1}, fx_{m-1})\\ &\preceq Av_1 \preceq A^2 v_2 \preceq \cdots \preceq A^m v_m\\ &\preceq A^m d(gx_i, gx_j)\\ &\preceq sA^{m+1}(I - sA)^{-1}d(gx_1, gx_0).\end{aligned}$$

Since $r(A) < \frac{1}{s} \leq 1$, by Remark 2.2(vi), we have $sA^{m+1}(I - sA)^{-1}d(gx_1, gx_0) \to \theta$ as $m \to \infty$. Thus, it is easy to see that for any $c \in \mathrm{int}P$, there exists $n_0 \in \mathbb{N}$ such that for all $n > m > n_0$,

$$d(gx_n, gx_m) \preceq sA^{m+1}(I - sA)^{-1}d(gx_1, gx_0) \ll c.$$

So, $\{gx_n\}$ is a Cauchy sequence in $g(X)$. If $g(X) \subset X$ is complete, there exist $q \in g(X)$ and $p \in X$ such that $gx_n \to q$ as $n \to \infty$ and $gp = q$.

Now, we get

$$d(fx_n, fp) \preceq Av$$

where

$$v \in C(g; x_n, p) = \{d(gx_n, gp), d(gx_n, fx_n), d(gp, fp), d(gx_n, fp), d(fx_n, gp)\}.$$

Clearly at least one of the following five cases holds for infinitely many n.

(1) $d(fx_n, fp) \preceq Ad(gx_n, gp) \preceq sAd(gx_{n+1}, gp) + sAd(gx_{n+1}, gx_n)$;

(2) $d(fx_n, fp) \preceq Ad(gx_n, fx_n) = Ad(gx_n, gx_{n+1})$;

(3) $d(fx_n, fp) \preceq Ad(gp, fp) \preceq sAd(gx_{n+1}, gp) + sAd(gx_{n+1}, fp)$,
that is, $d(fx_n, fp) \preceq sA(I - sA)^{-1}d(gx_{n+1}, gp)$;

(4) $d(fx_n, fp) \preceq Ad(gx_n, fp) \preceq sAd(gx_{n+1}, fp) + sAd(gx_{n+1}, gx_n)$,
that is, $d(fx_n, fp) \preceq sA(I - sA)^{-1}d(gx_{n+1}, gx_n)$;

(5) $d(fx_n, fp) \preceq Ad(fx_n, gp) = Ad(gx_{n+1}, gp)$.

As $sA \preceq sA(I - sA)^{-1}$ (since $\theta \preceq sA$ and $r(sA) < 1$), we obtain that

$$d(gx_{n+1}, fp) \preceq sA(I - sA)^{-1}[d(gx_{n+1}, gx_n) + d(gx_{n+1}, q)].$$

Since $gx_n \to q$ as $n \to \infty$, we get that for any $c \in \text{int}P$, there exists $n_1 \in \mathbb{N}$ such that for all $n > n_1$, one has

$$d(gx_{n+1}, fp) \ll c.$$

By Lemmas 2.2 and 2.3, we have $gx_n \to fp$ as $n \to \infty$ and $q = fp$.

Now if w is another point such that $gu = fu = w$, hence $d(w, q) = d(fu, fp) \preceq Av$, where $r(A) \in [0, \frac{1}{s})$ and

$$v \in C(g; u, p) = \{d(gu, gp), d(gu, fu), d(gp, fp), d(gu, fp), d(fu, gp)\}.$$

It is obvious that $d(w, q) = \theta$, i.e., $w = q$. Therefore, q is the unique point of coincidence of f and g in X. Moreover, the mappings f and g are weakly compatible, by Lemma 2.4 we know that q is the unique common fixed point of f and g.

Similarly, if $f(X)$ is complete, the above conclusion is also established.

Corollary 3.1. Let (X, d) be a complete cone b-metric space over a Banach space E with the coefficient $s \geq 1$ and the underlying solid cone P. Let the mapping $f : X \to X$ be a quasi-contraction of Perov type with the quasi-contractive constant operator A satisfying $r(A) \in [0, \frac{1}{s})$ and $AP \subset P$. That is, there exists a constant operator $A \in \mathcal{B}(E)$ with $r(A) \in [0, \frac{1}{s})$ and $AP \subset P$ such that for all $x, y \in X$, one has

$$d(fx, fy) \preceq Au,$$

where

$$u \in C(I_X; x, y) = \{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}.$$

Then f has a unique fixed point in X. And for any $x_0 \in X$, the iterative sequence $\{f^n x_0\}$ converges to the fixed point.

Proof. Set $g = I_X$, the identity mapping from X to X. It is obvious to see that Theorem 3.1 yields Corollary 3.1.

Corollary 3.2. ([10]) Let E be a real Banach space with a solid cone P and (X, d) be a complete cone metric space. The mapping $f : X \to X$ is a quasi-contraction of Perov type. Then f has a unique fixed point in X. Moreover, for any $x_0 \in X$, $\{f^n x_0\}$ converges to the fixed point.

Remark 3.2. Theorem 3.1 generalizes [12, Corollary 3.5] and [10, Theorem 3.1].

Remark 3.3. Corollary 3.1 does not involve any assumptions about continuity of the mappings discussed. So Corollary 3.1 improves and generalizes Corollary 3.5 in [12].

Remark 3.4. From the proof of Theorem 3.1, we note that the technique of induction appearing in Theorem 3.1 is somewhat different from that in [12, Corollary 3.5], and also different from that in [10, Theorem 3.1], which is more interesting and easily to understood.

The following corollary is the Jungck's result in the setting of cone b-metric spaces.

Corollary 3.3. Let (X, d) be a cone b-metric space over a Banach space E with the coefficient $s \geq 1$ and the underlying solid cone P. Let the mappings $f,\ g : X \to X$ satisfy the condition that for $A \in \mathcal{B}(E)$ with $r(A) \in [0, \frac{1}{s})$ and $AP \subset P$, and for every $x, y \in X$ holds $d(fx, fy) \preceq Ad(gx, gy)$. If $g(X) \subset f(X)$ and $g(X)$ or $f(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

The next result is the Banach contraction principle for the mappings of Perov type in the setting of cone b-metric spaces.

Corollary 3.4. Let (X, d) be a cone b-metric space over a Banach algebra $\mathcal{A}$ with the coefficient $s \geq 1$ and the underlying solid cone P. Let the mapping $f : X \to X$ satisfy the condition that for $A \in \mathcal{B}(E)$ with $r(A) \in [0, \frac{1}{s})$ and $AP \subset P$, and for every $x, y \in X$ holds $d(fx, fy) \preceq Ad(x, y)$. If $f(X)$ is a complete subspace of X, then f has a unique point in X.

4. Applications to fixed point theory in the setting of cone b-metric spaces over Banach algebras

In this section, we use the main results obtained in the last section to give some common fixed point results for generalized g-quasi-contractions with the quasi-contractive constant vector satisfying $r(\lambda) \in [0, 1/s)$ in the setting of cone b-metric spaces over Banach algebras without the assumption of normality. The following definition is needed.

Definition 4.1. Let $(X,\ d)$ be a cone b-metric space with the coefficient $s \geq 1$ over a Banach algebra $\mathcal{A}$. A mapping $f : X \to X$ is called a generalized g-quasi-contraction where $g : X \to X$, $f(X) \subset g(X)$, if for some $\lambda \in P$ with $r(\lambda) \in [0, 1/s)$ and for all $x,\ y \in X$, one has

$$d(fx, fy) \preceq \lambda u,$$

where

$$u \in C(g; x, y) = \{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\}.$$

Moreover, if $g = I_X$ (the identity mapping from X to X), then the mapping f is called a quasi-contraction.

Theorem 4.1. Let (X, d) be a cone b-metric space over a Banach algebra $\mathcal{A}$ with the coefficient $s \geq 1$ and the underlying solid cone P. Let the mapping $f : X \to X$ be the g-quasi-contraction with the g-quasi-contractive constant vector satisfying $r(\lambda) \in [0, 1/s)$. If $f(X) \subseteq g(X)$ and $g(X)$ or $f(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Since $f : X \to X$ be the g-quasi-contraction with the g-quasi-contractive constant vector satisfying $r(\lambda) \in [0, 1/s)$, by Definition 4.1, there exists some $\lambda \in P$ with $r(\lambda) \in [0, 1/s)$ such that for all $x,\ y \in X$, one has

$$d(fx, fy) \preceq \lambda u, \tag{4.1}$$

where

$$u \in C(g; x, y) = \{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\}. \tag{4.2}$$

By Lemma 2.5, there exists $\psi(\lambda)$ (denoted by B) $\in \mathcal{B}(\mathcal{A})$ with $r(\psi(\lambda)) \leq r(\lambda)$ such that

$$B(x) = \lambda x, \quad x \in \mathcal{A}. \tag{4.3}$$

Thus, there exists $B \in \mathcal{B}(\mathcal{A})$ with $r(B) \leq r(\lambda) \leq \frac{1}{s}$ such that

$$d(fx, fy) \leq Bu, \tag{4.4}$$

where u is as indicated in (4.2). Hence, by Theorem 3.1 we know that all the conclusions of Theorem 4.1 hold.

Corollary 4.1. Let $(X,\ d)$ be a complete cone b-metric space over a Banach algebra $\mathcal{A}$ and let P be the underlying cone with $k \in P$. If the mapping $T : X \to X$ is a quasi-contraction, then T has a unique fixed point in X and for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Proof. Set $g = I_X$. It is obvious to see that Theorem 4.1 yields Corollary 4.1.

Remark 4.1. Corollary 4.1 does not need to require the assumption of normality of the cone P. So, Corollary 4.1 improves and generalizes Theorem 9 in [35].

Remark 4.2. Taking $E = \mathbb{R}$, $P = [0, +\infty)$, $\lambda \in [0, 1/s)$ in Theorem 4.1, we get Das-Naik's result from [43]; if $g = I_X$ we get Ćirić's result, both in the setting of b-metric spaces.

The following corollary is the Jungck's result in the setting of cone b-metric spaces.

Corollary 4.2. Let (X, d) be a cone b-metric space over a Banach algebra $\mathcal{A}$ with the coefficient $s \geq 1$ and the underlying solid cone P. Let the mappings $f,\ g : X \to X$ satisfy the condition that for $\lambda \in P$ with $r(\lambda) \in [0, 1/s)$ and for every $x, y \in X$ holds $d(fx, fy) \leq \lambda d(gx, gy)$. If $g(X) \subset f(X)$ and $g(X)$ or $f(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

The next result is the Banach contraction principle in the setting of cone b-metric spaces.

Corollary 4.3. ([40]) Let (X, d) be a cone b-metric space over a Banach algebra $\mathcal{A}$ with the coefficient $s \geq 1$ and the underlying solid cone P. Let the mapping $f : X \to X$ satisfy the condition that $d(fx, fy) \leq \lambda d(x, y)$ for $\lambda \in P$ with $r(\lambda) \in [0, 1/s)$ and every $x, y \in X$ (namely, f is a generalized Lipschitz contraction). If $f(X)$ is a complete subspace of X, then f has a unique point in X.

Remark 4.3. Corollary 4.3 generalizes Theorem 3.1 in [40].

5. Applications to fixed point theory in the setting of cone normed spaces

In this section, we use the fixed point results for nonlinear contractions of Perov type in cone metric space to obtain some fixed point results for nonlinear contractions of Perov type in the setting of cone normed spaces without the assumption of normality.

Definition 5.1. Let (X, d) be a cone metric space over a Banach space E with a solid cone P and $(X, \|\cdot\|_P)$ be a cone normed space. The mapping $f : X \to X$ is called a quasi-contraction of Perov type if for some bounded linear operator $A \in \mathcal{B}(E)$ with $r(A) < 1$ and $A(P) \subset P$ we have

$$d(fx, fy) \leq A\left(u\left(x, y\right)\right),$$

where

$$u(x, y) \in \{\|x - y\|_P, \|x - fx\|_P, \|y - fy\|_P, \|x - fy\|_P, \|y - fx\|_P\}, \tag{5.1}$$

for arbitrary $x, y \in X$.

Lemma 5.1. ([11]) Let E be a real Banach space with a solid cone P and (X, d) be a complete cone metric space. There exist bounded linear operators $A, B \in \mathcal{B}(E)$ with $r(A + B) < 1$ and $A(P) \subset P$, $B(P) \subset P$. For any $x, y \in X$, the mapping $f : X \to X$ satisfies:

$$d(fx, fy) \preceq Ad(x, y) + Bd(x, fy).$$

Then f has a unique fixed point in F. Moreover, for any $x_0 \in X$, $\{f^n x_0\}$ converges to the fixed point.
Theorem 5.1. Let E be a Banach space with a solid cone P, $(X, \|\cdot\|_P)$ be a complete cone normed space and F be a closed subset of $(X, \|\cdot\|_P)$. Let $f : F \to F$ satisfy the quasi-contraction of Perov type condition: there exists a bounded linear operator $A \in \mathcal{B}(E)$ with $r(A) < 1$ and $A(P) \subset P$ such that for any $x, y \in X$,

$$d(fx, fy) \preceq A(u(x, y)),$$

where $u(x, y)$ satisfies (5.1). Then f has a unique fixed point in F. Moreover, for any $x_0 \in F$, $\{f^n x_0\}$ converges to the fixed point.
Proof. For any $x, y \in X$, let $d(x, y) = \|x - y\|_P$, then (X, d) and (F, d) are cone metric spaces. By (5.1), for any $x, y \in F$, we have

$$d(fx, fy) \preceq A(u(x, y)),$$

where

$$u(x, y) \in \{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}.$$

Since F is a closed subset of $(X, \|\cdot\|_P)$, it follows that (F, d) is a complete cone metric space. In fact, if $\{x_n\}$ is a Cauchy sequence of (F, d) and $F \subset X$, then $\{x_n\}$ is a Cauchy sequence of (X, d). By the completeness of $(X, \|\cdot\|_P)$, (X, d) is complete. Thus, $\{x_n\}$ is a convergence sequence in (X, d). Suppose $\lim_{n\to\infty} x_n = x$, that is, $\lim_{n\to\infty} d(x_n, x) = \theta$, then $\|x_n - x\|_P = \theta$. Note that F is a closed subset of $(X, \|\cdot\|_P)$, so $x \in F$, which implies (F, d) is complete. By Corollary 3.2, the conclusion is true.
Theorem 5.2. Let E be a Banach space with a solid cone P, $(X, \|\cdot\|_P)$ be a complete cone normed space and F be a closed subset of $(X, \|\cdot\|_P)$. If $f : F \to F$ satisfies: there exist bounded linear operators $A, B \in \mathcal{B}(E)$, with $r(A + B) < 1$ and $A(P) \subset P, B(P) \subset P$ such that for any $x, y \in F$

$$d(fx, fy) \preceq A\|x - y\|_P + B\|x - fy\|_P, \tag{5.2}$$

then f has a unique fixed point in F. Moreover, for any $x_0 \in F$, $\{f^n x_0\}$ converges to the fixed point.
Proof. For any $x, y \in X$, let $d(x, y) = \|x - y\|_P$. Then (X, d) and (F, d) are cone metric spaces. By (5.2), for any $x, y \in F$, we have

$$d(fx, fy) \preceq Ad(x, y) + Bd(x, fy).$$

As in the proof of Theorem 5.1, Lemma 5.1 gives the conclusion.

By symmetry, we immediately have the next assertion.
Theorem 5.3. Let E be a Banach space with a solid cone P, $(X, \|\cdot\|_P)$ be a complete cone normed space and F be a closed subset of $(X, \|\cdot\|_P)$. If $f : F \to F$ satisfies: there exist bounded linear operators $A, B \in \mathcal{B}(E)$, with $r(A + B) < 1$ such that for any $x, y \in F$

$$d(fx, fy) \preceq A\|x - y\|_P + B\|y - fx\|_P,$$

then f has a unique fixed point in F. Moreover, for any $x_0 \in F$, $\{f^n x_0\}$ converges to the fixed point.

Corollary 5.1. Let E be a Banach space with a solid cone P, $(X, \|\cdot\|_P)$ be a complete cone normed space and F be a closed subset of $(X, \|\cdot\|_P)$. If $f : F \to F$ satisfies: there exists a bounded linear operator $A \in \mathcal{B}(E)$ with $r(A) < 1$ such that for any $x, y \in F$, $d(fx, fy) \preceq A\|x-y\|_P$, then f has a unique fixed point in F. Moreover, for any $x_0 \in F$, $\{f^n x_0\}$ converges to the fixed point.

Proof. By Proposition 2.3, the operator A satisfies $A(P) \subset P$. Otherwise, there exists $a \in P$ with $Aa \in P$. Take $u, v \in X$ such that $a = \|u - v\|_P$. Then $A(\|u - v\|_P) = Aa \notin P$. However, $\|fu - fv\|_P \in P$, then $d(fx, fy) \preceq A\|x - y\|_P$ does not hold by Proposition 2.3, which is a contradiction. Thus, the conclusion is true.

By Theorem 5.1, we obtain the following fixed point theorems in cone normed spaces.

Corollary 5.2. Let E be a Banach space with a solid cone P, $(X, \|\cdot\|_P)$ be a complete cone normed space and F be a closed subset of $(X, \|\cdot\|_P)$. If $f : F \to F$ satisfies: there exists a bounded linear operator $A \in \mathcal{B}(E)$ with $r(A) < 1$ such that for any $x, y \in F$, $d(fx, fy) \preceq A\|x - fy\|_P$, then f has a unique fixed point in F. Moreover, for any $x_0 \in F$, $\{f^n x_0\}$ converges to the fixed point.

Corollary 5.3. Let E be a Banach space with a solid cone P, $(X, \|\cdot\|_P)$ be a complete cone normed space and F be a closed subset of $(X, \|\cdot\|_P)$. If $f : F \to F$ satisfies: there exists a bounded linear operator $A \in \mathcal{B}(E)$ with $r(A) < 1$ such that for any $x, y \in F$, $d(fx, fy) \preceq A\|fx - y\|_P$, then f has a unique fixed point in F. Moreover, for any $x_0 \in F$, $\{f^n x_0\}$ converges to the fixed point.

Corollary 5.4. Let E be a Banach space with a solid cone P, $(X, \|\cdot\|_P)$ be a complete cone normed space and F be a closed subset of $(X, \|\cdot\|_P)$. If $f : F \to F$ satisfies: there exists a bounded linear operator $A \in \mathcal{B}(E)$ with $r(A) < 1$ such that for any $x, y \in F$, $d(fx, fy) \preceq A\|x - fx\|_P$, then f has a unique fixed point in F. Moreover, for any $x_0 \in F$, $\{f^n x_0\}$ converges to the fixed point.

Corollary 5.5. Let E be a Banach space with a solid cone P, $(X, \|\cdot\|_P)$ be a complete cone normed space and F be a closed subset of $(X, \|\cdot\|_P)$. If $f : F \to F$ satisfies: there exists a bounded linear operator $A \in \mathcal{B}(E)$ with $r(A) < 1$ such that for any $x, y \in F$, $d(fx, fy) \preceq A\|y - fy\|_P$, then f has a unique fixed point in F. Moreover, for any $x_0 \in F$, $\{f^n x_0\}$ converges to the fixed point.

Remark 5.1. Compared with [49, Theorem 1.3], we do not require the solid cone is normal in our theorems. Therefore, Theorem 5.1 and Corollary 5.1 improve and generalize [49, Theorem 1.3] in the solid cone. In addition, Theorem 5.1 and the corollaries are obtained by applying Perov-type fixed point theory in cone metric spaces. The techniques and methods in the proof are different from [49] and the related literatures. The main results of this paper are beneficial supplements to [20].

6. Applications to nonlinear equations

We present some examples to show that the main results obtained in the previous sections have meaningful applications in nonlinear equations.

Example 6.1. Let $\mathcal{A} = C^1_{\mathbb{R}}[0, 1]$ denote the set of all real-valued functions on [0,1] which also have continuous derivatives on [0,1] and define a norm on $\mathcal{A}$ by $\|x\| = \|x\|_\infty + \|x'\|_\infty$ for $x \in \mathcal{A}$. Define multiplication in $\mathcal{A}$ just as the pointwise multiplication. Then $\mathcal{A}$ is a real Banach algebra with the unit $e = 1$ ($e(t) = 1$ for all $t \in [0, 1]$). The set $P = \{x \in \mathcal{A} : x(t) \geq 0 \text{ for all } t \in [0, 1]\}$ is a cone in $\mathcal{A}$. Moreover P is not normal (see [33]).

Let $X = \{0, 1, 3\}$. Define $d : X \times X \to \mathcal{A}$ by $d(0, 1)(t) = d(1, 0)(t) = \exp(t)$, $d(0, 3)(t) = d(3, 0)(t) = 9\exp(t)$, $d(3, 1)(t) = d(1, 3)(t) = 4\exp(t)$ and $d(x, x)(t) = \theta$ for all $t \in [0, 1]$ and $x \in X$. It is clear that (X, d) is a solid cone b-metric space over Banach algebra $\mathcal{A}$ with $s = \frac{9}{5}$ without normality.

Further, let $f : X \to X$ be a mapping defined with $f(0) = f(1) = 1$ and $f(3) = 0$ and $\lambda \in P$ defined by $\lambda(t) = \frac{11}{38}t + \frac{1}{4}$. By the careful calculations, one can get that all conditions of Theorem 4.1 for $g = I_X$ are fulfilled. The point $x = 1$ is the unique fixed point of f.

Example 6.2. Let $X = C^1_{\mathbb{R}}[0,1]$ and $\mathcal{A} = C^1_{\mathbb{R}}[0,1]$. Consider the following nonlinear integral equation

$$\int_0^1 F(t, f(s))\,\mathrm{d}s = f(t), \tag{6.1}$$

where F satisfies:

(a) $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function,

(b) the partial derivative F_y of F with respect to y exists and $|F_y(x,y)| \le L$ for some $L \in [0, \frac{1}{2})$.

Theorem 6.1. The Eq (6.1) has a unique solution in $C^1_{\mathbb{R}}[0,1]$.

Proof. Let $P = \{x \in C^1_{\mathbb{R}}[0,1] \mid x = x(t) \ge 0, t \in [0,1]\}$. Then P is a non-normal cone of the real Banach algebra $\mathcal{A}$ with the operations as

$$(x+y)(t) = x(t) + y(t),$$
$$(cx)(t) = cx(t),$$
$$(xy)(t) = x(t)x(t),$$

for all $x = x(t), y = y(t) \in \mathcal{A}$ and $c \in \mathbb{R}$. Moreover, $\mathcal{A}$ owns the unit element $e = 1$. Define a norm on $\mathcal{A}$ by $\|f\| = \|f\|_\infty + \|f'\|_\infty$ for $f \in \mathcal{A}$ where $\|f\|_\infty = \max_{0\le t\le 1}|f(t)|$ and let T be a self map of X defined by $Tf(t) = \int_0^1 F(t, f(s))\,\mathrm{d}s$.

It is easy shown that (X,d) is a complete cone b-metric space (where $s = 2^p$ $(p > 1)$) over Banach algebra $\mathcal{A}$ where the cone b-metric is defined by $d(x,y) = \exp(t)\,\|(x-y)^p\|_\infty$ but not a cone metric space. In fact, for $x, y, z \in X$, set $u = x - z, v = z - y$, so $x - y = u + v$. From the inequality

$$(a+b)^p \le (2\max\{a,b\})^p \le 2^p(a^p + b^p), \quad a, b > 0,$$

we have

$$|x-y|^p = |u+v|^p \le (|u| + |v|)^p \le 2^p(|x-z|^p + |z-y|^p),$$
$$|x-y|^p \exp(t) \le 2^p(|x-z|^p \exp(t) + |z-y|^p \exp(t)),$$

which implies that

$$d(x,y) \le s[d(x,z) + d(z,y)]$$

where $s = 2^p > 1$. Then, we can check that $T : X \to X$ is a generalized Lipschitz contraction with the generalized Lipschitz coefficient L^p satisfying the spectral radius $r(L^p) \le \|L^p\| = L^p < \frac{1}{2^p} = \frac{1}{s}$. In fact, we see

$$\begin{aligned} d(Tf, Tg) &= \|(Tf - Tg)^p\|_\infty \exp(t) \\ &= \exp(t) \max_{0\le x\le 1} \left| \int_0^1 (F(x, f(t)) - F(x, g(t)))\mathrm{d}t \right|^p \\ &\le \exp(t) \max_{0\le x\le 1} \left(\int_0^1 \left|F(x, f(t)) - F(x, g(t))\right|\mathrm{d}t \right)^p \\ &\le \exp(t) \left(\int_0^1 L\left|f(t) - g(t)\right|\mathrm{d}t \right)^p \end{aligned}$$

$$\begin{aligned} &\leq \exp(t)\int_0^1 (L\big|f(t)-g(t)\big|)^p \mathrm{d}t \\ &\leq L^p \exp(t)\max_{0\leq t\leq 1}\big|f(t)-g(t)\big|^p \\ &= L^p d(f,g). \end{aligned}$$

Thus by Corollary 4.3, the integral equation (6.1) has a unique continuous solution in $C^1_{\mathbb{R}}\,[0,1]$.

Remark 6.1. Compared with [44, Theorem 3.1], Example 6.2 shows that the unique solution of the integral equation (6.1) is not only continuous but differentiable in [0,1] under the same conditions, while [44, Theorem 3.1] does only show that it is continuous. In addition, the techniques of Example 6.2 are new and interesting since Example 6.2 is discussed in the setting of the cone b-metric space over a Banach algebra and does not require the normality of the underlying cone, while [44, Theorem 3.1] is proved in the setting of a cone metric space and relies strongly on the normality of the underlying cone.

Example 6.3. Let $X = \mathbb{R}^2$, $f = f(s,t) : X \to \mathbb{R}$, $g = g(s,t) : X \to \mathbb{R}$. Consider the following group of nonlinear coupled equations

$$\text{(I)}\begin{cases} f(x,\ y) = x, \\ g(x,\ y) = y - px, \end{cases}$$

where $p \geq 0$. Suppose that there exists $0 < k < 1$ such that

$$\left|\frac{\partial f}{\partial s}\right| \leq k,\ \left|\frac{\partial g}{\partial t}\right| \leq k$$

for all $(s,t) \in X$.

Theorem 6.2. The coupled equations in (I) have a unique common solution in X.

Proof. Let $\mathcal{A} = \mathbb{R}^2$ with the norm defined as $\|(x,y)\| = |x| + |y|$ for each $(x,y) \in \mathcal{A}$. Then $\mathcal{A}$ is a Banach algebra with the operations as

$$(x_1,y_1) + (x_2,y_2) = (x_1 + x_2, y_1 + y_2),$$

$$c(x_1,y_1) = (cx_1, cy_1),$$

$$(x_1,y_1)(x_2,y_2) = (x_1x_2, x_1y_2 + x_2y_1),$$

for all $(x_1,y_1),(x_2,y_2) \in \mathcal{A}$ and $c \in \mathbb{R}$. Moreover, $\mathcal{A}$ owns the unit element $e = (1,0)$.

Let $P = \{(x,y) \in \mathbb{R}^2 \mid x,y \geq 0\}$. Then P is a cone of $\mathcal{A}$.

Let $X = \mathbb{R}^2$ and the metric $d : X \times X \to \mathcal{A}$ be defined by

$$d\left((x_1,y_1),\ (x_2,y_2)\right) = (|x_1 - x_2|, |y_1 - y_2|) \in P.$$

Then (X,d) is a complete cone b-metric space over a Banach algebra $\mathcal{A}$ with the coefficient $s = 1$.

Now define mapping $T : X \to X$ by

$$T(x,y) = (f(x,y), g(x,y) + px)\,. \tag{6.2}$$

From Lagrange mean value theorem, we have

$$d(T(x_1,y_1),T(x_2,y_2)) \;=\; d\big(\,(f(x_1,y_1),g(x_1,y_1)+px_1)\,,(f(x_2,y_2),g(x_2,y_2)+px_2)\,\big)$$

$$
\begin{aligned}
&= \left(\left|f(x_1,y_1)-f(x_2,y_2)\right|, \left|g(x_1,y_1)-g(x_2,y_2)+p(x_1-x_2)\right|\right)\\
&\le (k|x_1-x_2|, k|y_1-y_2|+p|x_1-x_2|)\\
&= (k,p)\left(\left|x_1-x_2\right|, \left|y_1-y_2\right|\right)\\
&= (k,p)\, d((x_1,y_1),(x_2,y_2)),
\end{aligned}
$$

and

$$\|(k,p)^n\|^{\frac{1}{n}} = \left\|\left(k^n, pnk^{n-1}\right)\right\|^{\frac{1}{n}} = (k^n + pnk^{n-1})^{\frac{1}{n}} \to k < 1 \quad (n \to \infty),$$

which implies $r((k,p)) < \frac{1}{s}$. Then by Corollary 4.3, T has a unique fixed point in X.

The following example is a direct result of Theorem 6.2.

Example 6.4. Consider the following group of nonlinear coupled equations

$$\text{(II)}\begin{cases} \log(m+x) = x, \\ \arctan(n+y) = y - px, \end{cases}$$

where $p \ge 0, m > 1$ and $n \ge \sqrt{m-1}$. The coupled equations in (II) have a unique common solution.

In fact, set $f(t,s) = \log(m+s)$, $g(t,s) = \arctan(n+y)$. Then all the conditions of Theorem 6.2 are satisfied. Thus it follows from Theorem 6.2 that the coupled equations (II) have a unique positive common solution.

Remark 6.2. In Example 6.3, the mapping $T(x,y)$ described by (6.2) is related to the famous Poincaré mapping

$$T(x,y) = (f(x,y), y + g(x,y))$$

which is useful in Poincaré fixed point theorem and Poincaré geometry theorem (see, for instance, [45, 46]).

Remark 6.3. In Example 6.3, if $p > 1$, then $\|(k,p)\| = k + p > 1$, so by the arguments in Theorem 6.2, T is not a contraction in the Euclidean metric on X. Hence, one is unable to directly use Banach contraction principle to show T has a unique fixed point in X.

Example 6.5. Suppose $E = (E, \|\cdot\|)$ is a real Banach space, $T > 0, 0 \le \alpha < 1, I = [0,T], F \subset C[I,E]$, while $C[I,E]$ is the set of all continuous abstract functions in interval I. Consider Volterra type integral equation

$$x(t) = (fx)(t) = \frac{1}{t^\alpha}\int_0^t h(s,t,x(s))\mathrm{d}s, \quad t \in [0,T]. \tag{6.3}$$

Theorem 6.3. Suppose $h : [0,T]\times[0,T]\times F \to F$ is continuous and satisfies

$$\|h(t,s,u) - h(t,s,v)\| \le M\|u - v\|,$$

where $MT^{1-\alpha} < 1$. Then the integral equation (6.3) has a unique solution in F.

Proof. By (6.3), we have

$$(fu)(t) - (fv)(t) = \frac{1}{t^\alpha}\int_0^t (h(s,t,u(s)) - h(s,t,v(s)))\,\mathrm{d}s, \quad t \in [0,T].$$

Let $0 = T_0 < T_1 < \cdots < T_{k-1} < T_k = T$ be a division of interval $I = [0,T]$. Denote

$$I_s = [T_{s-1}, T_s], \quad s = 1, \ldots, k\ (k > 1, k \in \mathbb{N}).$$

For any $u \in C[I, E]$, define

$$\|u\|_P = (\max_{\sigma \in I_1} \|u(\sigma)\|, \ldots, \max_{\sigma \in I_k} \|u(\sigma)\|),$$

then $\|u\|_P$ is a complete norm on the linear space $C[I, E]$. Denote

$$\|fu - fv\|_P = ((fu - fv)_1, \ldots, (fu - fv)_k) \in \mathbb{R}^k,$$

then

$$\begin{aligned}
(fu - fv)_1 &= \max_{t \in I_1} \|fu - fv\| \\
&= \max_{t \in I_1} \frac{1}{t^\alpha} \left\| \int_0^t [h(s, t, u(s)) - h(s, t, v(s))] \mathrm{d}s \right\| \\
&\le \max_{t \in I_1} \frac{1}{t^\alpha} \int_0^t \|h(s, t, u(s)) - h(s, t, v(s))\| \mathrm{d}s \\
&\le \max_{t \in I_1} \frac{1}{t^\alpha} \int_0^t \|u(s) - v(s)\| \mathrm{d}s \\
&\le \max_{t \in I_1} \frac{1}{t^\alpha} \int_0^t \left(\sup_{s \in I_1} \|u(s) - v(s)\| \right) \mathrm{d}s \\
&\le \max_{t \in I_1} \frac{1}{t^\alpha} \int_0^t M(u - v)_1 \mathrm{d}s \\
&\le \max_{t \in I_1} M t^{1-\alpha} (u - v)_1 \\
&\le M T^{1-\alpha} (u - v)_1.
\end{aligned}$$

In a similar analysis, we obtain $(fu - fv)_i \le MT^{1-\alpha}(u - v)_i$, $i = 1, \ldots, k$. Thus, $\|fu - fv\|_P \le A(\|u - v\|_P)$, while

$$A = \begin{pmatrix} MT^{1-\alpha} & & & \\ & MT^{1-\alpha} & & \\ & & \ddots & \\ & & & MT^{1-\alpha} \end{pmatrix}.$$

As $MT^{1-\alpha} < 1$, we see $r(A) < 1$. Taking $u(x, y) = \|x - y\|_P$, we observe that the conclusion is true by Theorem 5.1.

7. Conclusions

In this manuscript, we introduced g-quasi-contractions of Perov type in cone b-metric spaces and established fixed point results for such kind of nonlinear contractions. Moreover, we use the main results to obtain some theoretical results, such as the common fixed point results for generalized g-quasi-contractions in cone b-metric spaces over Banach algebra as well as fixed point results for nonlinear contractions of Perov type in cone normed spaces without the assumption of normality. Further, we provided several applications to nonlinear equations that elaborated on the usability of our results.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

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