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Research article

Idempotent completion of right suspended categories

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Abstract: We show that the idempotent completion of a right suspended category has a natural structure of right suspended category and dually this is true for a left suspended category. This unifies and extends results of Balmer-Schlichting, Bühler and Liu-Sun for triangulated, exact and right triangulated categories, respectively.

Keywords: idempotent completion; suspended categories; triangulated categories; exact categories
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1. Introduction

Let $\mathcal{A}$ be an additive category. An idempotent morphism $e^2 = e\colon A \to A$ in $\mathcal{A}$ is said to be split if there are two morphisms $p\colon A \to B$ and $q\colon B \to A$ such that $e = qp$ and $pq = 1_B$. The category $\mathcal{A}$ is said to be idempotent complete if every idempotent morphism splits. Note that $\mathcal{A}$ is idempotent complete if and only if every idempotent morphism has a kernel if and only if every idempotent morphism has a cokernel, see [1]. Every additive category $\mathcal{A}$ can be embedded fully faithfully into an idempotent complete category $\widetilde{\mathcal{A}}$. Balmer and Schlichting [2] proved that the idempotent completion of a triangulated category is a triangulated category. Bühler showed that the idempotent completion of an exact category is an exact category. Liu and Sun [4] showed that the idempotent completion of a right triangulated category is again right triangulated.

Recently, suspended categories were introduced by Li in [3] as a simultaneous generalization of exact categories, triangulated categories and right triangulated categories. In this article, we will unify these conclusions stated above by showing that when $\mathcal{A}$ is a suspended category then the idempotent completion of $\mathcal{A}$ is also a suspended category.

2. Preliminaries

We first recall some notions and facts on the idempotent completion of additive categories.

Definition 2.1. [2, Definition 1.2] Let $\mathcal{A}$ be an additive category. The idempotent completion of $\mathcal{A}$ is denoted by $\widetilde{\mathcal{A}}$ which be defined as follows. The objects of $\widetilde{\mathcal{A}}$ are pairs (A, p), where A is an object of $\mathcal{A}$ and $p\colon A \to A$ is an idempotent morphism. A morphism in $\widetilde{\mathcal{A}}$ from (A, p) to (B, q) is a morphism $f\colon A \to B$ such that $qf = fp = f$. For any object (A, p) in $\widetilde{\mathcal{A}}$, the identity morphism $1_{(A,\ p)} = p$.

Remark 2.2. [1, Remark 6.3] Let $\mathcal{A}$ be an additive category and $\widetilde{\mathcal{A}}$ be an idempotent complete of $\mathcal{A}$. The biproduct in $\widetilde{\mathcal{A}}$ is defined as

$$(A, p) \oplus (B, q) = (A \oplus B, p \oplus q).$$

There exists a fully faithful additive functor $\ell_{\mathcal{A}}\colon \mathcal{A} \to \widetilde{\mathcal{A}}$ defined as follows. For an object A in $\mathcal{A}$, we have that $\ell_{\mathcal{A}}(A) = (A, 1_A)$ and for a morphism f in $\mathcal{A}$, we have that $\ell_{\mathcal{A}}(f) = f$. Since the functor $\ell_{\mathcal{A}}$ is fully faithful, we can view $\mathcal{A}$ as a full subcategory of $\widetilde{\mathcal{A}}$.

Proposition 2.3. [1, Proposition 6.10] *Let $\mathcal{A}$ be an additive category and $\mathcal{B}$ be an idempotent complete category. For every additive functor $\mathbb{F}\colon \mathcal{A} \to \mathcal{B}$, there exists a functor $\widetilde{\mathbb{F}}\colon \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ and a natural isomorphism $\phi\colon \mathbb{F} \Rightarrow \widetilde{\mathbb{F}}\ell_{\mathcal{A}}$.*

Now we recall the notion of suspended categories from [3].

Let $\mathcal{A}$ be an additive category and $\mathscr{X}$ be a full subcategory of $\mathcal{A}$. Recall that we say a morphism $f\colon A \to B$ in C is an $\mathscr{X}$*-monic* if

$$\mathrm{Hom}_{\mathcal{A}}(f, X)\colon \mathrm{Hom}_{\mathcal{A}}(B, X) \to \mathrm{Hom}_{\mathcal{A}}(A, X)$$

is an epimorphism for all $X \in \mathscr{X}$. Similarly, we say that f is a left $\mathscr{X}$-approximation of A if f is an $\mathscr{X}$-monic and $B \in \mathscr{X}$. The subcategory $\mathscr{X}$ is said to be covariantly finite in $\mathcal{A}$, if every object in $\mathcal{A}$ has a left $\mathscr{X}$-approximation. The notions of left $\mathscr{X}$-approximation and covariantly finite subcategories are also known as $\mathscr{X}$-preenvelope and preenveloping subcategories, respectively.

Let $\mathcal{A}$ be an additive category with an additive endofunctor $\Sigma\colon \mathcal{A} \to \mathcal{A}$ and $\mathscr{X} \subseteq C$ be two full subcategories of $\mathcal{A}$. A right Σ-sequence $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ in $\mathcal{A}$ is called a right C-sequence if $C \in C$, g is a weak cokernel of f (i.e. the induced sequence $\mathrm{Hom}_{\mathcal{A}}(C, \mathcal{A}) \to \mathrm{Hom}_{\mathcal{A}}(B, \mathcal{A}) \to \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{A})$ is exact) and h is a weak cokernel of g.

Dually, a left Σ-sequence $\Sigma A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A$ is called a left C-sequence if $B \in C$, f is a weak kernel of g and g is a weak kernel of h.

Definition 2.4. [3, Definition 3.1] Let $\mathcal{A}$ be an additive category with an additive endofunctor $\Sigma\colon \mathcal{A} \to \mathcal{A}$ and $\mathscr{X} \subseteq C$ be two full subcategories of $\mathcal{A}$. A triple $(\mathcal{A}, R(C, \Sigma), \mathscr{X})$ is a right suspended category where $R(C, \Sigma)$ is a class of right C-sequences (whose elements are also called right C-triangles) if $R(C, \Sigma)$ is closed under isomorphisms and finite direct sums and the following conditions are satisfied:

(RS1) (a) For any $A \in C$, there exists a sequence $A \xrightarrow{i} X \longrightarrow U \longrightarrow \Sigma(A)$ in $R(C, \Sigma)$ where i is an $\mathscr{X}$-preenvelope such that for any morphism $f\colon A \longrightarrow B$ in C, there exists a sequence

$$A \xrightarrow{\binom{i}{f}} X \oplus B \longrightarrow N \longrightarrow \Sigma(A)$$

in $R(C, \Sigma)$.

(b) For any morphism $f\colon A \longrightarrow B$ in C, there exists a sequence

$$A \xrightarrow{\binom{1}{f}} A \oplus B \xrightarrow{(f\ -1)} B \xrightarrow{0} \Sigma(A)$$

in $R(C, \Sigma)$.

(RS2) For any commutative diagram of sequences in $R(C, \Sigma)$

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma(A) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Sigma(\alpha)} \\ A' & \xrightarrow{u} & X & \xrightarrow{v} & C' & \xrightarrow{w} & \Sigma(A') \end{array}$$

with $X \in \mathscr{X}$, if α factors through f, then γ factors through v.

(RS3) For each solid commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \vdots{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Sigma\alpha} \\ A' & \xrightarrow{u} & B' & \xrightarrow{v} & C' & \xrightarrow{w} & \Sigma A' \end{array}$$

with rows in $R(C, \Sigma)$, the dotted morphism exists which makes the whole diagram commutative.

(RS4) If any three sequences

$$A \xrightarrow{f} B \xrightarrow{l} D \xrightarrow{i} \Sigma(A)\ ,\ B \xrightarrow{g} C \xrightarrow{h} E \xrightarrow{j} \Sigma(B) \ \text{ and } \ A \xrightarrow{gf} C \xrightarrow{k} F \xrightarrow{m} \Sigma(A)$$

are in $R(C, \Sigma)$ and f, g are $\mathscr{X}$-monic, then there exists two morphisms $\alpha : D \longrightarrow F$ and $\beta : F \longrightarrow E$ of C, such that the diagram below is commutative:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{l} & D & \xrightarrow{i} & \Sigma A \\ \| & \circlearrowleft & \downarrow{\scriptstyle g} & & \vdots{\scriptstyle \alpha} & & \| \\ A & \xrightarrow[gf]{} & C & \xrightarrow[k]{} & F & \xrightarrow[m]{} & \Sigma A \\ & & {\scriptstyle h}\downarrow & & \vdots{\scriptstyle \beta} & & \\ & & E & = & E & & \\ & & {\scriptstyle j}\downarrow & & \downarrow & & \\ & & \Sigma B & \xrightarrow[\Sigma(l)]{} & \Sigma D & & \end{array}$$

where the third column from the left is in $R(C, \Sigma)$, with α is an $\mathscr{X}$-monic.

Dually, we can define the notion of a left suspended category.

Now we give some examples of right suspended categories from [3].

Example 2.5. (1) If $(\mathcal{A}, \Sigma, \Delta)$ is a right triangulated category, we take $\mathscr{X} = 0$, $C = \mathcal{A}$ and $R(\mathcal{A}, \Sigma) = \Delta$. Then the triple $(\mathcal{A}, R(\mathcal{A}, \Sigma) = \Delta, 0)$ is a right suspended category. We know that any triangulated category can be viewed as a right triangulated category. Hence any triangulated category can be viewed as a right suspended category.

(2) Let $(\mathcal{A}, \mathcal{E})$ be an exact category and

$$R(\mathcal{A}, \Sigma = 0) = \{A \to B \to C \to 0 \mid A \rightarrowtail B \twoheadrightarrow C \in \mathcal{E}\}.$$

Then $(\mathcal{A}, R(\mathcal{A}, \Sigma = 0), \mathcal{A})$ is a right suspended category.

(3) Let $(\mathcal{A}, \mathcal{E})$ be an exact category with enough injectives. We denote by $\mathcal{I}$ the full subcategory of all injectives objects in $\mathcal{A}$. Then $(\mathcal{A}, R(\mathcal{A}, \Sigma = 0), \mathcal{I})$ is a right suspended category, where

$$R(\mathcal{A}, \Sigma = 0) = \{A \to B \to C \to 0 \mid A \rightarrowtail B \twoheadrightarrow C \in \mathcal{E}\}.$$

We collect some useful lemmas which can be used in the sequel.

Lemma 2.6. *Assume $(\mathcal{A}, R(C, \Sigma), \mathscr{X})$ be satisfies (RS1),(RS2),(RS3). If*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A) \ \ \text{and} \ \ A \xrightarrow{f} B \xrightarrow{g'} C' \xrightarrow{h'} \Sigma(A)$$

are in $R(C, \Sigma)$, then there exists an isomorphism $\gamma\colon C \longrightarrow C'$ which makes the following diagram commutative:

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\
\| & & \| & & \downarrow{\scriptstyle \gamma} & & \| \\
A & \xrightarrow[f]{} & B & \xrightarrow[g']{} & C' & \xrightarrow[h']{} & \Sigma A
\end{array}$$

Proof. It can be proved in a similar way as in [3, Lemma 3.2]

Lemma 2.7. *Let $(\mathcal{A}, R(C, \Sigma), X)$ be a right suspended category. Given a commutative diagram*

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & & & \downarrow{\scriptstyle \Sigma(p)} \\
A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & C & \xrightarrow[h]{} & \Sigma A
\end{array}$$

with rows in $R(C, \Sigma)$. If $p\colon A \to A$ and $q\colon B \to B$ are idempotent morphisms, then there exists an idempotent morphism $\alpha\colon C \to C$ such that the diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \Sigma(p)} \\
A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & C & \xrightarrow[h]{} & \Sigma A
\end{array}$$

commutes.

Proof. The proof is very similar to [2, Lemma 1.13], we omit it. □

3. Idempotent completion of right suspended categories

Let $(\mathcal{A}, R(\mathcal{C}, \Sigma), \mathscr{X})$ be a right suspended category. Then the additive endofunctor Σ of $\mathcal{A}$ induces the endofunctor $\widetilde{\Sigma}$ of idempotent completion $\widetilde{\mathcal{A}}$ given by $\widetilde{\Sigma}(A, e) = (\Sigma A, \Sigma e)$. Moreover, it is easy to see that there is a commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\Sigma} & \mathcal{A} \\ \downarrow{\scriptstyle \ell_{\mathcal{A}}} & & \downarrow{\scriptstyle \ell_{\mathcal{A}}} \\ \widetilde{\mathcal{A}} & \xrightarrow{\widetilde{\Sigma}} & \widetilde{\mathcal{A}} \end{array}$$

Clearly, $\ell_{\mathcal{A}}(\mathcal{C}) \subseteq \widetilde{\mathcal{C}}$, and $\ell_{\mathcal{A}}(\mathscr{X}) \subseteq \widetilde{\mathscr{X}}$.

We define a right $\widetilde{\Sigma}$-sequence in $\widetilde{\mathcal{A}}$,

$$A \xrightarrow{f_1} B \xrightarrow{f_2} C \xrightarrow{f_3} \widetilde{\Sigma}A \tag{Δ}$$

to be a right $\widetilde{\mathcal{C}}$-sequence in $R(\widetilde{\mathcal{C}}, \widetilde{\Sigma})$ if there is a right $\widetilde{\mathcal{C}}$-sequence in $R(\widetilde{\mathcal{C}}, \widetilde{\Sigma})$

$$A' \xrightarrow{f_1'} B' \xrightarrow{f_2'} C' \xrightarrow{f_3'} \widetilde{\Sigma}A' \tag{Δ'}$$

such that $\Delta \oplus \Delta'$ is isomorphic to a right $\mathcal{C}$-sequence in $R(\mathcal{C}, \Sigma)$ or equivalently, it is a direct summand of a right $\mathcal{C}$-sequence in $R(\mathcal{C}, \Sigma)$. It is easy to see that $R(\widetilde{\mathcal{C}}, \widetilde{\Sigma})$ is closed under isomorphisms and finite direct sums. For convenience, we usually write $\widetilde{\Sigma}$ as Σ.

Lemma 3.1. *Let $(\mathcal{A}, R(\mathcal{C}, \Sigma), \mathscr{X} = 0)$ be a right suspended category. A sequence*

$$A \oplus A' \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & x' \end{pmatrix}} B \oplus B' \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & y' \end{pmatrix}} C \oplus C' \xrightarrow{\begin{pmatrix} z & 0 \\ 0 & z' \end{pmatrix}} \Sigma(A \oplus A')$$

is a right $\mathcal{C}$-sequence in $R(\mathcal{C}, \Sigma)$ if and only if both two sequences

$$A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{z} \Sigma(A) \text{ and } A' \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{z'} \Sigma(A')$$

are right $\mathcal{C}$-sequences in $R(\mathcal{C}, \Sigma)$.

Proof. Since $R(\mathcal{C}, \Sigma)$ is closed under finite direct sums, it is enough to show the necessity. By axiom (RS1), there are two right $\mathcal{C}$-sequences in $R(\mathcal{C}, \Sigma)$

$$A \xrightarrow{x} B \xrightarrow{a} C_1 \xrightarrow{b} \Sigma A,$$
$$A' \xrightarrow{x'} B' \xrightarrow{a'} C_1' \xrightarrow{b'} \Sigma A'.$$

By axiom (RS3), there exists a commutative diagram

$$\begin{array}{ccccccc} A \oplus A' & \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & x' \end{pmatrix}} & B \oplus B' & \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & y' \end{pmatrix}} & C \oplus C' & \xrightarrow{\begin{pmatrix} z & 0 \\ 0 & z' \end{pmatrix}} & \Sigma A \oplus A' \\ \downarrow{\scriptstyle (1\ 0)} & & \downarrow{\scriptstyle (1\ 0)} & & \downarrow{\scriptstyle (f\ g)} & & \downarrow{\scriptstyle (1\ 0)} \\ A & \xrightarrow{x} & B & \xrightarrow{a} & C_1 & \xrightarrow{b} & \Sigma A \end{array}$$

Thus, we have $fy = a$ and $bf = z$. Similarly, one can find a morphism $f' : C' \to C_1'$ such that $f'y' = a'$ and and $b'f' = z'$. Hence, we have the following commutative diagram

$$\begin{array}{ccccccc}
A \oplus A' & \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & x' \end{smallmatrix}\right)} & B \oplus B' & \xrightarrow{\left(\begin{smallmatrix} y & 0 \\ 0 & y' \end{smallmatrix}\right)} & C \oplus C' & \xrightarrow{\left(\begin{smallmatrix} z & 0 \\ 0 & z' \end{smallmatrix}\right)} & \Sigma A \oplus A' \\
\| & & \| & & \downarrow{\scriptstyle \left(\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right)} & & \| \\
A \oplus A' & \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & x' \end{smallmatrix}\right)} & B \oplus B' & \xrightarrow{\left(\begin{smallmatrix} a & 0 \\ 0 & a' \end{smallmatrix}\right)} & C_1 \oplus C_1' & \xrightarrow{\left(\begin{smallmatrix} b & 0 \\ 0 & b' \end{smallmatrix}\right)} & \Sigma A
\end{array}$$

By Lemma 2.6, we know that $\left(\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right)$ is an isomorphism. It follows that f and f' are isomorphisms. It is easy to see that there exists a commutative diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{x} & B & \xrightarrow{y} & C & \xrightarrow{z} & \Sigma A \\
\| & & \| & & \downarrow{\scriptstyle f} & & \downarrow \\
A & \xrightarrow{x} & B & \xrightarrow{a} & C_1 & \xrightarrow{b} & \Sigma A
\end{array}$$

where the second row lies in $R(C, \Sigma)$. It follows that $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{z} \Sigma A$ lies in $R(C, \Sigma)$. Similarly, we can show that $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{z'} \Sigma A'$ lies in $R(C, \Sigma)$. $\square$

Now we state and prove our main result in this article.

Theorem 3.2. *Let* Σ *be an endofunctor when restricted to* C, $(\mathcal{A}, R(C, \Sigma), \mathscr{X} = 0)$ *be a right suspended category. Then the triple* $(\widetilde{\mathcal{A}}, R(\widetilde{C}, \widetilde{\Sigma}), \widetilde{\mathscr{X}} = 0)$ *is a right suspended category.*

Proof. We will check the axioms of suspended categories.

(RS1) (a) Let A be an arbitrary object in $\widetilde{C}$. Then there is A' in $\widetilde{C}$ such that $A \oplus A' \in C$ actually ,if $A = (N, e)$ take $A' = (N, id_N - e)$ we have $A \oplus A' \cong \ell_{\mathcal{A}}(N))$. Note that $A \oplus A' \xrightarrow{0} 0 \longrightarrow 0 \longrightarrow \Sigma(A \oplus A')$ is a right C-sequence in $R(C, \Sigma)$. It is clear that 0 is an $\mathscr{X}$-preenvelope. By the definition of right $\widetilde{C}$-sequences in $\widetilde{\mathcal{A}}$, we obtain $A \xrightarrow{0} 0 \longrightarrow 0 \longrightarrow \Sigma(A)$ in $R(\widetilde{C}, \widetilde{\Sigma})$ with 0 is an $\widetilde{\mathscr{X}}$-preenvelope.

For any morphism $f\colon A \to B$ in $\widetilde{C}$, there exists two objects $A', B' \in \widetilde{C}$ such that $A \oplus A', B \oplus B' \in C$. For the morphism $A \oplus A' \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right)} B \oplus B'$ in C, by axiom (RS1)(a), there exists a right C-sequence

$$A \oplus A' \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right)} B \oplus B' \xrightarrow{a_1} N \xrightarrow{a_2} \Sigma(A \oplus A') \tag{3.1}$$

in $R(C, \Sigma)$. By Lemma 2.7, there exists an idempotent morphism $p = p^2\colon N \to N$ which makes the following diagram commutative:

$$\begin{array}{ccccccc}
A \oplus A' & \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right)} & B \oplus B' & \xrightarrow{a_1} & N & \xrightarrow{a_2} & \Sigma(A \oplus A') \\
\downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)} \\
A \oplus A' & \xrightarrow[\left(\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right)]{} & B \oplus B' & \xrightarrow[a_1]{} & N & \xrightarrow[a_2]{} & \Sigma(A \oplus A')
\end{array}$$

Therefore, the sequence $A \xrightarrow{f} B \xrightarrow{pa_1} (N, p) \xrightarrow{a_2 p} \Sigma(A)$ is in $R(\widetilde{C}, \widetilde{\Sigma})$.

(b) For each morphism $f : A \to B$ in $\widetilde{C}$, there are two objects $A', B' \in \widetilde{C}$ such that $A \oplus A', B \oplus B' \in C$. For the morphism $A \oplus A' \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right)} B \oplus B'$ in C, by axiom (RS1)(b), there is a right C-sequence in $R(C, \Sigma)$

$$A \oplus A' \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ f & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)} A \oplus B \oplus A' \oplus B' \xrightarrow{\left(\begin{smallmatrix} f & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{smallmatrix}\right)} B \oplus B' \xrightarrow{0} \Sigma(A \oplus A')$$

which guarantees

$$A \xrightarrow{\left(\begin{smallmatrix} 1 \\ f \end{smallmatrix}\right)} A \oplus B \xrightarrow{(f\ -1)} B \xrightarrow{0} \Sigma(A)$$

is a right $\widetilde{C}$-sequence in $R(\widetilde{C}, \widetilde{\Sigma})$.

(RS2) For any two right $\widetilde{C}$-sequences

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A) , \tag{3.2}$$

$$A' \xrightarrow{0} 0 \xrightarrow{0} C' \xrightarrow{n} \Sigma(A') \tag{3.3}$$

lies in $R(\widetilde{C}, \widetilde{\Sigma})$. For any commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\Sigma(\alpha)} \\ A' & \xrightarrow[0]{} & 0 & \xrightarrow[0]{} & C' & \xrightarrow[n]{} & \Sigma A' \end{array}$$

with α factors through f. Next we will prove $\gamma = 0$, thus we are done.

By the definition of right $\widetilde{C}$-sequences, there are two right $\widetilde{C}$-sequences

$$U \xrightarrow{f'} V \xrightarrow{g'} W \xrightarrow{h'} \Sigma(U) , \tag{3.4}$$

$$U' \xrightarrow{l'} V' \xrightarrow{m'} W' \xrightarrow{n'} \Sigma(U') \tag{3.5}$$

lie in $R(\widetilde{C}, \widetilde{\Sigma})$. Taking the direct sum of right $\widetilde{C}$-sequences (3.2) and (3.4), we get a right $\widetilde{C}$-sequence

$$A \oplus U \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right)} B \oplus V \xrightarrow{\left(\begin{smallmatrix} g & 0 \\ 0 & g' \end{smallmatrix}\right)} C \oplus W' \xrightarrow{\left(\begin{smallmatrix} h & 0 \\ 0 & h' \end{smallmatrix}\right)} \Sigma(A \oplus U) \tag{3.6}$$

in $R(\widetilde{C}, \widetilde{\Sigma})$ such that (3.6) is isomorphic to a right C-sequence in $R(C, \Sigma)$.

Similarly, taking the direct sum of right $\widetilde{C}$-sequences (3.3) and (3.5), we get a right $\widetilde{C}$-sequence

$$A' \oplus U' \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 0 & l' \end{smallmatrix}\right)} 0 \oplus V' \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 0 & m' \end{smallmatrix}\right)} C' \oplus W' \xrightarrow{\left(\begin{smallmatrix} n & 0 \\ 0 & n' \end{smallmatrix}\right)} \Sigma(A' \oplus U') \tag{3.7}$$

in $R(\widetilde{C},\widetilde{\Sigma})$ such that (3.7) is isomorphic to a right C-sequence in $R(C,\Sigma)$. Thus we have a commutative diagram in $R(C,\Sigma)$

$$\begin{array}{ccccccc} A\oplus U & \xrightarrow{\left(\begin{smallmatrix} f & 0\\ 0 & f'\end{smallmatrix}\right)} & B\oplus V & \xrightarrow{\left(\begin{smallmatrix} g & 0\\ 0 & g'\end{smallmatrix}\right)} & C\oplus W & \xrightarrow{\left(\begin{smallmatrix} h & 0\\ 0 & h'\end{smallmatrix}\right)} & \Sigma(A\oplus U)\\ \downarrow{\scriptstyle\left(\begin{smallmatrix} \alpha & 0\\ 0 & 0\end{smallmatrix}\right)} & & \downarrow{\scriptstyle\left(\begin{smallmatrix} 0 & 0\\ 0 & 0\end{smallmatrix}\right)} & & \downarrow{\scriptstyle\left(\begin{smallmatrix} \gamma & 0\\ 0 & 0\end{smallmatrix}\right)} & & \downarrow{\scriptstyle\left(\begin{smallmatrix} \Sigma(\alpha) & 0\\ 0 & 0\end{smallmatrix}\right)}\\ A'\oplus U' & \xrightarrow[\left(\begin{smallmatrix} 0 & 0\\ 0 & l'\end{smallmatrix}\right)]{} & 0\oplus V' & \xrightarrow[\left(\begin{smallmatrix} 0 & 0\\ 0 & m'\end{smallmatrix}\right)]{} & C'\oplus W' & \xrightarrow[\left(\begin{smallmatrix} n & 0\\ 0 & n'\end{smallmatrix}\right)]{} & \Sigma(A'\oplus U'). \end{array}$$

Note that $\left(\begin{smallmatrix} \alpha & 0\\ 0 & 0\end{smallmatrix}\right)$ factors through $\left(\begin{smallmatrix} f & 0\\ 0 & f'\end{smallmatrix}\right)$ since α factors through f, hence $\left(\begin{smallmatrix} \gamma & 0\\ 0 & 0\end{smallmatrix}\right)$ factors through $\left(\begin{smallmatrix} 0 & 0\\ 0 & m'\end{smallmatrix}\right)$. In particular, we have

$$\begin{pmatrix} \gamma & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix} 0 & 0\\ 0 & m'\end{pmatrix}\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a'_{22}\end{pmatrix}$$

which implies $\gamma = 0$.

(RS3) For any two right $\widetilde{C}$-sequences

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A) \;\text{ and }\; X \xrightarrow{x} Y \xrightarrow{y} Z \xrightarrow{z} \Sigma(X)$$

in $R(\widetilde{C},\widetilde{\Sigma})$, the diagram below with the leftmost square is commutative

$$\begin{array}{cccccccc} \Delta & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} \Sigma A\\ \downarrow{\scriptstyle(\alpha,\beta)} & & \downarrow{\scriptstyle\alpha} & \circlearrowright & \downarrow{\scriptstyle\beta} & & & \downarrow{\scriptstyle\gamma}\\ \Gamma & & X & \xrightarrow[x]{} & Y & \xrightarrow[y]{} & Z & \xrightarrow[z]{} \Sigma(X) \end{array}$$

Next we will prove that there exists a morphism $\gamma\colon C \longrightarrow Z$ which makes the whole diagram commutative in $\widetilde{\mathscr{A}}$. By the definition of right $\widetilde{C}$-sequences, there exists right C-sequence Δ',Γ' and morphisms $i:\Delta \longrightarrow \Delta', p:\Delta' \longrightarrow \Delta, j:\Gamma \longrightarrow \Gamma', q:\Gamma' \longrightarrow \Gamma$, such that $pi = 1_\Delta, qj = 1_\Gamma$, which induce a morphism $j\circ(\alpha,\beta)\circ p:\Delta' \longrightarrow \Gamma'$ in $\mathscr{A}$, since Δ' and Γ' are right C-sequence in $R(C,\Sigma)$. According to axiom (RS3), we have a right C-sequence map $u:\Delta' \longrightarrow \Gamma'$, which induces a right C-sequence morphism $q\circ u\circ i:\Delta \longrightarrow \Gamma$ extending (α,β) in $R(\widetilde{C},\widetilde{\Sigma})$.

(RS4) For any three right $\widetilde{C}$-sequence $A \xrightarrow{f} B \xrightarrow{l} D \xrightarrow{i} \Sigma(A)$, $B \xrightarrow{g} C \xrightarrow{h} E \xrightarrow{j} \Sigma(B)$ and $A \xrightarrow{gf} C \xrightarrow{k} E \xrightarrow{m} \Sigma(A)$ are in $R(\widetilde{C},\widetilde{\Sigma})$,with f,g are $\mathscr{X}$-monics in $\widetilde{C}$. For the morphism $f:A \longrightarrow B$ in $\widetilde{C}$, there exists A',B' in $\widetilde{C}$, such that $A\oplus A', B\oplus B'$ in C, Clearly

$$A \xrightarrow{f} B \xrightarrow{l} D \xrightarrow{i} \Sigma(A)\,, \tag{3.8}$$

$$A' \longrightarrow 0 \longrightarrow \Sigma(A') \longrightarrow \Sigma(A') \tag{3.9}$$

and

$$0 \longrightarrow B' \longrightarrow B' \longrightarrow 0 \tag{3.10}$$

are right $\widetilde{C}$-sequences in $R(\widetilde{C}, \widetilde{\Sigma})$. Take the direct sum of right triangles (3.8)–(3.10), we get the following right $\widetilde{C}$-sequence:

$$A \oplus A' \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}} B \oplus B' \xrightarrow{\begin{pmatrix} l & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}} D \oplus B' \oplus \Sigma(A') \xrightarrow{\begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \Sigma(A \oplus A') \tag{3.11}$$

By the proof of (RS1), we know that any morphism in C can be embedded into a right C-sequence, since the morphism $A \oplus A' \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}} B \oplus B'$ in C, therefore,it can be extended to a right C-sequence (3.1). By Lemma 2.6, (3.11) is isomorphic to (3.1) in $R(C, \Sigma)$.
Similarly, the following right $\widetilde{C}$-sequence

$$B \oplus B' \xrightarrow{\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}} C \oplus C' \xrightarrow{\begin{pmatrix} h & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}} E \oplus C' \oplus \Sigma(B') \xrightarrow{\begin{pmatrix} j & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \Sigma(B \oplus B') \tag{3.12}$$

is isomorphic to a right C-sequence in $R(C, \Sigma)$. Since the morphism $gf\colon A \to C$ in $\widetilde{C}$, similar to above, the following right $\widetilde{C}$-sequence

$$A \oplus A' \xrightarrow{\begin{pmatrix} gf & 0 \\ 0 & 0 \end{pmatrix}} C \oplus C' \xrightarrow{\begin{pmatrix} k & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}} F \oplus C' \oplus \Sigma(A') \xrightarrow{\begin{pmatrix} m & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \Sigma(A \oplus A') \tag{3.13}$$

is isomorphic to a right C-sequence in $R(C, \Sigma)$.

By axiom (RS4), we can get the following commutative diagram in $\mathscr{A}$:

$$\begin{array}{ccccccc}
A \oplus A' & \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}} & B \oplus B' & \xrightarrow{\begin{pmatrix} l & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}} & D \oplus B' \oplus \Sigma(A') & \xrightarrow{\begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} & \Sigma(A \oplus A') \\
\Vert & & \downarrow{\scriptsize\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}} & & \downarrow{h_1} & & \Vert \\
A \oplus A' & \xrightarrow{\begin{pmatrix} gf & 0 \\ 0 & 0 \end{pmatrix}} & C \oplus C' & \xrightarrow{\begin{pmatrix} k & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}} & F \oplus C' \oplus \Sigma(A') & \xrightarrow{\begin{pmatrix} m & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} & \Sigma(A \oplus A') \\
 & & \downarrow{\scriptsize\begin{pmatrix} h & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}} & & \downarrow{h_2} & & \\
 & & E \oplus C' \oplus \Sigma(B') & = & E \oplus C' \oplus \Sigma(B') & & \\
 & & \downarrow{\scriptsize\begin{pmatrix} j & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} & & \downarrow{\scriptsize\begin{pmatrix} \Sigma(l) \circ j & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} & & \\
 & & \Sigma(B \oplus B') & \xrightarrow[\begin{pmatrix} \Sigma(l) & 0 \\ 0 & 1 \end{pmatrix}]{} & \Sigma D \oplus \Sigma(B') \oplus \Sigma^2(A) & &
\end{array}$$

where the third column is a right C-sequence in $R(C, \Sigma)$ and h_1 is an $\mathscr{X}$-monic.
We write

$$h_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \; h_2 = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}.$$

According to the above commutative diagram, we have

$$\begin{pmatrix} k & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} l & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} m & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} h & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

$$\begin{pmatrix} \Sigma(l) \circ j & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = 0.$$

Hence

$$h_1 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & 0 & 1 \end{pmatrix},\ h_2 = \begin{pmatrix} b_{11} & 0 & b_{13} \\ b_{21} & 1 & b_{23} \\ 0 & 0 & 0 \end{pmatrix}.$$

According to $h_2 \circ h_1 = 0$, we have

$$\begin{pmatrix} b_{11} & 0 & b_{13} \\ b_{21} & 1 & b_{23} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b_{11}a_{11} & 0 & b_{11}a_{13} + b_{13} \\ b_{21}a_{11} + a_{21} & 0 & b_{21}a_{13} + a_{23} + b_{23} \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

Thus we obtain

$$b_{21}a_{11} + a_{21} = 0,$$
$$b_{11}a_{11} = 0,$$
$$b_{21}a_{13} + a_{23} + b_{23} = 0,$$
$$b_{11}a_{13} + b_{13} = 0.$$

For the object $F \oplus C' \oplus \Sigma(A')$, there are morphisms $u, v\colon F \oplus C' \oplus \Sigma(A') \longrightarrow F \oplus C' \oplus \Sigma(A')$ where

$$u = \begin{pmatrix} 1 & 0 & -a_{13} \\ b_{21} & 1 & b_{23} \\ 0 & 0 & 1 \end{pmatrix}, v = \begin{pmatrix} 1 & 0 & a_{13} \\ -b_{21} & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

such that u and v are inverse of each other. Therefore we can get a commutative diagram as follows:

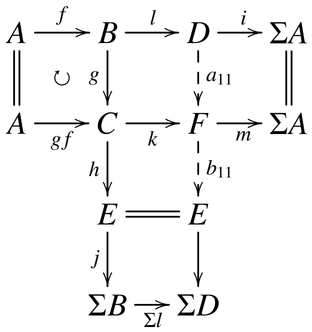
$$\begin{array}{ccccccc}
D \oplus B' \oplus \Sigma(A') & \xrightarrow{h_1} & F \oplus C' \oplus \Sigma(A') & \xrightarrow{h_2} & E \oplus C' \oplus \Sigma(B') & \xrightarrow{\left(\begin{smallmatrix} \Sigma(l)\circ j & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)} & \Sigma(D) \oplus \Sigma(B') \oplus \Sigma^2(A') \\
\| & & \downarrow u & & \| & & \| \\
D \oplus B' \oplus \Sigma(A') & \xrightarrow{u\circ h_1} & F \oplus C' \oplus \Sigma(A') & \xrightarrow{h_2\circ v} & E \oplus C' \oplus \Sigma(B') & \xrightarrow{\left(\begin{smallmatrix} \Sigma(l)\circ j & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)} & \Sigma(D) \oplus \Sigma(B') \oplus \Sigma^2(A')
\end{array}$$

Note that

$$uh_1 = \begin{pmatrix} 1 & 0 & -a_{13} \\ b_{21} & 1 & b_{23} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$h_2 v = \begin{pmatrix} b_{11} & 0 & b_{13} \\ b_{21} & 1 & b_{23} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & a_{13} \\ -b_{21} & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we obtain the right $\widetilde{C}$-sequence $D \xrightarrow{a_{11}} F \xrightarrow{b_{11}} E \xrightarrow{\Sigma(l)\circ j} \Sigma(D)$ in $R(\widetilde{C}, \widetilde{\Sigma})$.
Therefore, we can get the following commutative diagram in $\widetilde{\mathscr{A}}$:



where a_{11} is an $\widetilde{\mathscr{X}}$-monic.

This completes the proof. □

Remark 3.3. In Theorem 3.2, when $\mathscr{A} = C$ is a triangulated category, it is just Theorem 1.5 in [2]; when $\mathscr{A} = C$ is an exact category, it is just Proposition 6.13 in [1]; when $\mathscr{A} = C$ is a right triangulated category, it is just Theorem 2.14 in [4].

4. Conclusions

In this article, we show that the idempotent completion of a right suspended category has a natural structure of right suspended category and dually this is true for a left suspended category.

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Conflict of interest

The author declares no conflict of interests.

References

1. T. Bühler, Exact categories, *Expo. Math.*, **28** (2020), 1–69. https://doi.org/10.1016/0021-8693(81)90214-3

2. P. Balmer, M. Schlichting, Idempotent completion of triangulated categories, *J. Algebra*, **236** (2001), 819–834. https://doi.org/10.1006/jabr.2000.8529

3. Z. Li, Homotopy theory in additive categories with suspensions, *Commun. Algebra*, **49** (2021), 5137–5170. https://doi.org/10.1080/00927872.2021.1938102

4. J. Liu, L. Sun, Idempotent completion of pretriangulated categories, *Czechoslovak Math. J.*, **64** (2014), 477–494. https://doi.org/10.1007/s10587-014-0114-9