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Research article

On distance signless Laplacian eigenvalues of zero divisor graph of commutative rings

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Abstract: For a simple connected graph G of order n, the distance signless Laplacian matrix is defined by $D^Q(G) = D(G) + Tr(G)$, where $D(G)$ and $Tr(G)$ is the distance matrix and the diagonal matrix of vertex transmission degrees, respectively. The zero divisor graph $\Gamma(R)$ of a finite commutative ring R is a simple graph, whose vertex set is the set of non-zero zero divisors of R and two vertices $v, w \in \Gamma(R)$ are edge connected whenever $vw = wv = 0$. In this article, we find the D^Q-eigenvalues of zero divisor graph of the ring $\mathbb{Z}_n$ for general value $n = p_1^{l_1} p_2^{l_2}$, where $p_1 < p_2$ are distinct prime numbers and $l_1, l_2 \in \mathbb{N}$. Further, we investigate the D^Q-eigenvalues of zero divisor graphs of local rings and the rings whose associated zero divisor graphs are Hamiltonian. Also, we obtain the trace norm and the Wiener index of $\Gamma(\mathbb{Z}_n)$ for some special values of n.

Keywords: distance signless Laplacian matrix; zero divisor graphs; commutative rings; trace norm
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1. Introduction

Throughout this study, all graphs are simple, finite, and connected. A graph is symbolized by $G = (V(G), E(G))$, where $V(G) = \{w_1, w_2, \ldots, w_n\}$ represents its vertex set, whereas $E(G)$ represents its edge set. Further, the number of elements in $V(G)$ is the *order* n while the *size* m of G is the number of elements in $E(G)$. We write $u \sim v$ if a vertex u is adjacent to a vertex v. The *degree* (*valency*) symbolized by $d_G(v)$ of a vertex v is the number of vertices incident on v. If every vertex of G has the same degree, it is referred to as a *regular* graph. The $n \times n$ matrix $A = (\alpha_{\imath\jmath})$, where $\alpha_{\imath\jmath} = 1$ when $\imath$ is edge connected to $\jmath$, and 0 otherwise, is the adjacency matrix of G. Assume that $Deg(G) = \mathrm{diag}(\delta_1, \delta_2, \dots, \delta_n)$ is the diagonal matrix, where $\delta_i = d_G(v_i)$, $i = 1, 2, \dots, n$ is the vertex degrees of G. The real symmetric positive-definite matrix $Q(G) = Deg(G) + A(G)$ is known as the signless Laplacian matrix, while its eigenvalue set including multiplicities is called the signless Laplacian spectrum of G. We denote the complete graph by K_n, for more notations and terminology, see [8].

The *distance* $d(v, w)$ between two unique vertices $w \neq v$ is specified in G as the length of the smallest path connecting v and w. The *diameter* of G is defined as the greatest distance among any two of its vertices. The matrix $D(G) = (d(v, w))$ is said to be the *distance matrix* of G, while $Tr_G(u_1)$ is the *transmission degree* of u_1 and it is equal to the total of the distances between u_1 and all other vertices in G, i.e., $Tr_G(u_1) = \sum_{w \in V(G)} d(w, u_1)$. If $Tr_G(v_i)$ (or simply Tr_i) is the transmission degree of $v_i \in V(G)$, the sequence $\{Tr_i\}$, $i = 1, 2, \dots, n$ is known as the *transmission degree sequence* of G.

Suppose $Tr(G) = \mathrm{diag}(Tr_1, Tr_2, \dots, Tr_n)$ is the diagonal matrix of vertex transmissions degree sequence of G. The authors of [6] presented the signless Laplacian for the distance matrix of G. The matrix $D^Q(G) = D(G) + Tr(G)$ is known as the *distance signless Laplacian* matrix of G. Also, $D^Q(G)$ is real symmetric positive-definite for $n > 2$, so its eigenvalues are real and may be arranged say $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$, where γ_1 is said to be the D^Q-spectral radius. More about D^Q-matrix can be seen in [7, 15, 16, 19] and references therein.

For a commutative ring R with multiplicative identity $1 (\neq 0)$, the zero divisor graphs of R, represented by $\Gamma(R)$, is a simple, connected and undirected graph whose vertex set is the set of non-zero zero divisors of R, in which two vertices x_1 and x_2 are edge connected whenever $x_1 x_2 = 0$. The zero divisor graphs including their adjacency and (distance) Laplacian eigenvalues have been studied in [5,9,10,16,18,20,21,25]. For eigenvalue analysis of other graphs defined on groups, see [2,3,22–24].

The rest of the manuscript is structured as follows. In Section 2, we start with some essential results and use them in proving our main problems. In Section 3, we deliberate the trace norms of $\Gamma(\mathbb{Z}_n)$ of the $D^Q(G)$-matrix for some special values of n.

2. D^Q-eigenvalues of zero divisor graphs of $\Gamma(\mathbb{Z}_{p_1^{l_1} p_2^{l_2}})$

Assume an $n \times n$ matrix

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,m} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,m} \end{pmatrix},$$

such that, its columns and rows are partitioned according to a partition $\Pi = \{\pi_1, \pi_2, \dots, \pi_m\}$ of $I = \{1, 2, \dots, n\}$. The quotient matrix Q (see [8]) of A is the matrix having m order, where (k, ℓ)-th entry is the average column sums (row sums) of $A_{k,\ell}$. The partition Π is referred to as the *equitable* if every block $A_{k,\ell}$ has some constant column (row) sum, in such case, Q is known as the *equitable quotient matrix*. For equitable partitions, every eigenvalue of Q is also the eigenvalue of A.

Next, we have the definition of the joined union of graphs and state a result about D^Q-spectrum of the joined union.

Definition 2.1. *(Joined union) Assume G is an order n graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and $G_i = G_i(V_i, E_i)$ are disjoint graphs having n_i order, $1 \leq i \leq n$. The joined union $G[G_1, G_2, \ldots, G_n]$ of graphs, is obtained by considering graphs G_i, $i = 1, 2, \ldots, n$ and connect every vertex of G_k to each vertex of G_ℓ, when k and ℓ are connected in G.*

Theorem 2.2. *[16] For a graph G with $V(G) = \{u_1, \ldots, u_n\}$, and G_i is the r_i-regular graphs having n_i order whose adjacency eigenvalues are $r_i = \lambda_{i1} \geq \lambda_{i2} \geq \ldots \geq \lambda_{in_i}$, whenever $i = 1, 2, \ldots, n$. The D^Q-spectrum of $G[G_1, \ldots, G_n]$ contains the eigenvalues $2n_i + n_i' - r_i - \lambda_{ik} - 4$, for $i = 1, \ldots, n$ and $k = 2, 3, \ldots, n_i$, when $n_i' = \sum_{k=1, k\neq i}^{n} n_k d_G(u_i, u_k)$. The other n D^Q-eigenvalues of $G[G_1, \ldots, G_n]$ are the eigenvalues of following equitable quotient matrix:*

$$Q = \begin{pmatrix} 4n_1 + n_1' - 2r_1 - 4 & n_2 d_G(u_1, u_2) & \ldots & n_n d_G(u_1, u_n) \\ n_1 d_G(u_2, u_1) & 4n_2 + n_2' - 2r_2 - 4 & \ldots & n_n d_G(u_2, u_n) \\ \vdots & \vdots & \ddots & \vdots \\ n_1 d_G(u_n, u_1) & n_2 d_G(u_n, u_2) & \ldots & 4n_n + n_n' - 2r_n - 4 \end{pmatrix}. \tag{2.1}$$

In general, it is very non trivial to determine the eigenvalues of any matrix. Here in algebraic theory of graphs, the eigenvalues of matrices corresponding to some special graphs like the complete bipartite graphs, the complete graphs are easily found. So, effort lies in transforming a graph by some operations into some nicely structures, so that the maximum eigenvalues of graph can be obtained. In [10], the authors showed that $\Gamma(\mathbb{Z}_n)$ may be written as the joined union of graphs, where the components are either null graphs or cliques. The authors in [21] have found the structure of $\Gamma(\mathbb{Z}_n)$ with $n = p_1^{l_1} p_2^{l_2}$.

Theorem 2.3. *[21] For the zero divisor graph $\Gamma(\mathbb{Z}_n)$ with $n = p_1^{l_1} p_2^{l_2}$, where $p_1 < p_2$ are distinct primes and both $l_1 = 2s_1$, and $l_2 = 2s_2$ are positive even integers, where $s_1, s_2 \geq 1$ are positive integers. The structure of $\Gamma(\mathbb{Z}_n)$ is given as:*

$$\begin{aligned}\Gamma(\mathbb{Z}_n) = \Upsilon_n\Big[&\overline{K}_{\phi(p_1^{l_1-1} p_2^{l_2})}, \ldots, \overline{K}_{\phi(p_1^{s_1} p_2^{l_2})}, \ldots, \overline{K}_{\phi(p_2^{l_2})}, \overline{K}_{\phi(p_1^{l_1} p_2^{l_2-1})}, \ldots, \overline{K}_{\phi(p_1^{l_1} p_2^{s_2})}, \ldots, \\ &\overline{K}_{\phi(p_1^{l_1})}, \overline{K}_{\phi(p_1^{l_1-1} p_2^{l_2-1})}, \ldots, \overline{K}_{\phi(p_1^{l_1-1} p_2^{s_2})}, \ldots, \overline{K}_{\phi(p_1^{l_1-1})}, \ldots, \overline{K}_{\phi(p_1^{s_1} p_2^{l_2-1})}, \ldots, \\ &\overline{K}_{\phi(p_1^{s_1} p_2^{s_2-1})}, K_{\phi(p_1^{s_1} p_2^{s_2})}, \ldots, K_{\phi(p_1^{s_1})}, \ldots, K_{\phi(p_2^{l_2-1})}, \ldots, K_{\phi(p_2^{s_2-1})}, K_{\phi(p_2^{s_2})}, \ldots, K_{\phi(p_2)}\Big],\end{aligned}$$

where Υ_n is referred to as the divisor graph since its vertices are defined as proper divisors of n and two vertices are connected if their product is a multiple of n.

From the Theorems 2.2 and 2.3, we see that out of $n - 1 - \phi(n)$ number of D^Q-eigenvalues of $\Gamma(\mathbb{Z}_n)$, $n - 1 - t - \phi(n)$ are positive integer, where t is the order of Υ_n, the other t D^Q-eigenvalues of the graph $\Gamma(\mathbb{Z}_n)$ are the eigenvalues of the equitable quotient matrix.

Next, we will illustrate the D^Q-eigenvalues of the graph $\Gamma(\mathbb{Z}_n)$ with $n = p_1^{l_1} p_2^{l_2}$, where $p_1 < p_2$ are distinct primes and $l_1 \leq l_2$ are positive even integers. This generalize the results of [16] in a natural setting.

Theorem 2.4. *For the graph* $\Gamma(\mathbb{Z}_{p_1^{l_1}p_2^{l_2}})$*, the* D^Q*-spectrum of* $\Gamma(\mathbb{Z}_n)$ *comprises of the eigenvalues*

$$
\begin{aligned}
\gamma_i =& 2N + \phi(p_1^{l_1})p_2^{l_2-1} - p_1^{j} - 3, \textit{ for } i = j = 1,2,\ldots,s_1,\ldots,l_1-1,\\
\gamma_{l_1} =& 2N + \phi(p_1^{l_1})(p_2^{l_2-1}-1) - p_1^{l_1} - 3,\\
\gamma_i =& 2N + p_1^{l_1-1}\phi(p_2^{l_2}) - p_2^{j} - 3, \textit{ for } j = 1,2,\ldots,l_2-1, \textit{ and } i = l_1+1,\ldots,l_1+l_2-1,\\
\gamma_{l_1+l_2} =& 2N + (p_1^{l_1-1}-1)\phi(p_2^{l_2}) - p_2^{l_2} - 3,\\
\gamma_i =& 2N - p_1p_2^{j} - 3, \textit{ for } j = 1,2,\ldots,l_2, \textit{ and } i = l_1+l_2+1,\ldots,l_1+2l_2,\\
&\vdots\\
\gamma_i =& 2N - p_1^{s_1}p_2^{j} - 3, \textit{ for } j = 1,2,\ldots,s_2-1, \textit{ and }\\
& \qquad i = l_1+s_2l_2+1,\ldots,l_1+s_2l_2+s_2-1,\\
\gamma_i =& 2N - p_1^{s_1}p_2^{j} - 1, \textit{ for } j = s_2,\ldots,l_2, \textit{ and } i = l_1+s_2l_2+s_2,\ldots,l_1+(s_2+1)l_2\\
&\vdots\\
\gamma_i =& 2N - p_1^{l_1}p_2^{j} - 1, \textit{ for } j = 1,2,\ldots,s_2-1, \textit{ and } i = l_1+l_1l_2+1,\ldots,l_1+l_1l_2\\
\gamma_i =& 2N + \phi(p_2^{l_2-j}) - p_1^{l_1}p_2^{j} - 1, \textit{ for } j = 1,2,\ldots,s_2-1, \textit{ and }\\
& \qquad i = l_1+l_1l_2+1,\ldots,+s_2-1\\
\gamma_i =& 2N + \phi(p_2^{l_2-j}) - p_1^{l_1}p_2^{j} - 1, \textit{ for } j = s_2,\ldots,l_2, \textit{ and }\\
& \qquad i = l_1+l_1l_2+s_2,\ldots,l_1+l_1l_2+l_2-1
\end{aligned}
$$

with multiplicities $\phi(p_1^{l_1-i}p_2^{l_2})-1, \phi(p_1^{l_1}p_2^{l_2-j})-1, \phi(p_1^{l_1-i}p_2^{l_2-j})-1, \ldots, \phi(p_1^{s_1}p_2^{l_2-j})-1, \ldots, \phi(p_2^{l_2-j})-1$*, respectively, where* $i = 1,\ldots,l_1$ *and* $j = 1,\ldots,l_2$*. The persisting* D^Q*-eigenvalues of* $\Gamma(\mathbb{Z}_n)$ *are the eigenvalues of matrix* (2.1).

Proof. Let $n = p_1^{l_1}p_2^{l_2}$, where $2 < p_1 < p_2$ are primes and $2 \leq l_1 = 2s_1 \leq 2s_2 = l_2$, where s_1 and s_2 are positive even integers. Then by Theorem 2.3, we have

$$
\begin{aligned}
\Gamma(\mathbb{Z}_n) =& \Upsilon_n[\overline{K}_{\phi(p_1^{l_1-1}p_2^{l_2})},\ldots,\overline{K}_{\phi(p_1^{s_1}p_2^{l_2})},\ldots,\overline{K}_{\phi(p_2^{l_2})},\overline{K}_{\phi(p_1^{l_1}p_2^{l_2-1})},\ldots,\overline{K}_{\phi(p_1^{l_1}p_2^{s_2})},\ldots,\\
&\overline{K}_{\phi(p_1^{l_1})},\overline{K}_{\phi(p_1^{l_1-1}p_2^{l_2-1})},\ldots,\overline{K}_{\phi(p_1^{l_1-1}p_2^{s_2})},\ldots,\overline{K}_{\phi(p_1^{l_1-1})},\ldots,\overline{K}_{\phi(p_1^{s_1}p_2^{l_2-1})},\ldots,K_{\phi(p_1^{s_1}p_2^{s_2})},\\
&\ldots,K_{\phi(p_1^{s_1})},\ldots,K_{\phi(p_2^{l_2-1})},\ldots,K_{\phi(p_2^{s_2})},\ldots,K_{\phi(p_2)}].
\end{aligned}
$$

We shall now use Theorem 2.2, for calculating the D^Q-eigenvalues of $\Gamma(\mathbb{Z}_n)$. For that, we first need to know the values of $n_i^{'}$'s. It is well established that zero divisor graphs of rings have a maximum diameter of three, so $p_1^i \sim p_2^i$ if and only if $n = i = j$, otherwise $p_1^i \sim p_1^kp_2^n,\ k+i \geq n$ and $p_2^j \sim p_1^np_2^h,\ h+j \geq n$ and finally $p_1^kp_2^n \sim p_1^np_2^h,\ k \geq 1, h \geq 1$. This means, $d(p_1^i, p_2^j) = 3$, if $1 \leq j, i \leq n-1$ in Υ_n, likewise distance between other vertices is at most 2. Now,

$$
\begin{aligned}
n_1^{'} =& 2(\phi(p_1^{l_1-2}p_2^{l_2}) + \cdots + \phi(p_1^{s_1}p_2^{l_2}) + \cdots + \phi(p_2^{l_2})) + 3(\phi(p_1^{l_1}p_2^{l_2-1}) + \cdots + (p_1^{l_1}p_2^{s_2}) + \ldots\\
&+\phi(p_1^{l_1})) + 2(\phi(p_1^{l_1-1}p_2^{l_2-1}) + \cdots + \phi(p_1^{l_1-1}p_2^{s_2}) + \cdots + \phi(p_1^{l_1-1})) + \cdots + 2(\phi(p_1^{s_1}p_2^{l_2-1})
\end{aligned}
$$

$$+\cdots+\phi(p_1^{s_1}p_2^{s_2})+\cdots+\phi(p_1^{s_1}))+\cdots+2\big(\phi(p_2^{l_2-1})+\cdots+\phi(p_2^{s_2})+\cdots+\phi(p_2)\big)-\phi(p_1),$$

where by definition of $n_1^{'}$, $\phi(p_1^{l_1-1}p_2^{l_2})$ is removed and $p_1 \sim p_2^{l_1-1}p_2^{l_2}$, so we subtract $\phi(p_1)$. As $\sum_{d|l}\phi(d)=l$, so order of $\Gamma(\mathbb{Z}_n)$ is $N=n-\phi(n)-1=\sum_{1,n\neq d|n}\phi(n)$. By applying Theorem 2.2, and using the number theory identities $\sum_{i=1}^{l}\phi(p_1^i)=p_1^l-1$ and $\phi(z_1,z_2)=\phi(z_1)\phi(z_2)$, if and only if $(z_1,z_2)=1$, we simplify the form of $n_1^{'}$ as:

$$\begin{aligned}
n_1^{'} =&2(N-\phi(p_1^{l_1-1}p_2^{l_2}))+\Big(\phi(p_1^{l_1}p_2^{l_2-1})+\cdots+\phi(p_1^{l_1}p_2^{s_2})+\cdots+\phi(p_1^{l_1})\Big)-\phi(p_1)\\
=&2(N-\phi(p_1^{l_1-1}p_2^{l_2}))+\phi(p_1^{l_1})\Big(\phi(p_2^{l_2-1})+\cdots+\phi(p_2^{s_2})+\cdots+\phi(p_2)+1\Big)-\phi(p_2)\\
=&2(N-\phi(p_1^{l_1-1}p_2^{l_2}))+\phi(p_1^{l_1})p_2^{l_2-1}-\phi(p_1).
\end{aligned}$$

Now, by Theorem 2.2, the D^Q-eigenvalues of $\Gamma(\mathbb{Z}_n)$ are given as:

$$\begin{aligned}
2n_1+n_1^{'}-r_1-\lambda_{1k}-4 &= 2\phi(p_1^{l_1-1}p_2^{l_2})+2(N-\phi(p_1^{l_1-1}p_2^{l_2})+\phi(p_1^{l_1})p_2^{l_2-1}-\phi(p_1)-0-0-4\\
&= 2N+\phi(p_1^{l_1})p_2^{l_2-1}-\phi(p_1)-4.
\end{aligned}$$

Thus, $2N+\phi(p_1^{l_1})p_2^{l_2-1}-\phi(p_1)-4$ is the D^Q-eigenvalue with multiplicity $\phi(p_1^{l_1-1}p_2^{l_2})-1$. Continuing in the same manner, other $n_i^{'}$'s are given by:

$$\begin{aligned}
n_i^{'} =&2(N-\phi(p_1^{l_1-j}p_2^{l_2}))+\phi(p_1^{l_1})p_2^{l_2-1}-(p_1^j-1),\ \text{for } i=j=2,\ldots,s_1,\ldots,l_1-1,\\
n_{l_1}^{'} =&2(N-\phi(p_2^{l_2}))+\phi(p_1^{l_1})(p_2^{l_2-1}-1)-(p_1^{l_1}-1),\\
n_i^{'} =&2(N-\phi(p_1^{l_1}p_2^{l_2-j}))+\phi(p_2^{l_2})p_1^{l_1-1}-(p_2^j-1)\ \text{for } i=l_1+1,\ldots,l_1+l_2-1\\
&\text{and } j=1,\ldots,s_2,\ldots,l_2-1,\\
n_{l_1+l_2}^{'} =&2(N-\phi(p_1^{l_1}))+\phi(p_2^{l_2})(p_1^{l_1-1}-1)-(p_2^{l_2}-1),\\
n_i^{'} =&2(N-\phi(p_1^{l_1-1}p_2^{l_2-j}))-(p_1p_2^j-1),\ \text{for } i=l_1+l_2+1,\ldots,l_1+2l_2\\
&\text{and } j=1,\ldots,s_2,\ldots,l_2,\\
&\vdots\\
n_i^{'} =&2(N-\phi(p_1^{s_1}p_2^{l_2-j}))-(p_1^{s_1}p_2^j-1),\ \text{for } i=l_1+s_1l_2+1,\ldots,l_1+s_1l_2+s_2-1\\
&\text{and } j=1,\ldots,s_2-1,\\
n_i^{'} =&2N-\phi(p_1^{s_1}p_2^{l_2-j})-(p_1^{s_1}p_2^j-1),\ \text{for } i=l_1+s_1l_2+s_2,\ldots,l_1+(s_1+1)l_2\\
&\text{and } j=s_2,\ldots,l_2,\\
&\vdots\\
n_i^{'} =&2(N-\phi(p_2^{l_2-j}))-(p_1^{l_1}p_2^j-1),\ \text{for } i=l_1+l_1l_2+1,\ldots,l_1+l_1l_2+s_2-1\\
&\text{and } j=1,\ldots,s_2-1,\\
n_i^{'} =&2N-\phi(p_2^{l_2-j})-(p_1^{l_1}p_2^j-1),\ \text{for } i=l_1+l_1l_2+s_2,\ldots,l_1+l_1l_2+l_2-1\\
&\text{and } j=s_2,\ldots,l_2-1.
\end{aligned}$$

Now, using the values of these $n_i^{'}$'s and the Theorem 2.2, the other D^Q-eigenvalues can be calculated as in statement. The rest D^Q-eigenvalues of the graph $\Gamma(\mathbb{Z}_n)$ are presented in the matrix (2.1). □

In particular if $l_2 = 0$, we have the sequel consequence of Theorem 2.4.

Corollary 2.5. *For $n = p_1^{2m}$, $m \geq 2$ is a positive integer, the D^Q-spectrum of $\Gamma(\mathbb{Z}_n)$ contains of the eigenvalue $2N - p_1^i - 3$ having multiplicity $\phi(p_1^{2m-i}) - 1$, where $i = 1, \ldots, m-1$, and the eigenvalue $N + (p_1^{2m-1} - p_1^i) - 2$, for $i = m, \ldots, 2m-1$. The other D^Q-eigenvalues of $\Gamma(\mathbb{Z}_n)$ are the eigenvalues of the matrix* (2.3).

Proof. The proper divisor set of n is $\{p_1, p_1^2, \ldots, p_1^{2m-1}\}$, we see that the vertex p_1^i is connected to p_1^j in Υ_n for any $j \geq 2m - i$ where $1 \leq i \leq 2m-1$ and $j \neq i$. As n does not divide $(p_1^i)^2$, for $i = 1, \ldots, m-1$, so

$$G_i = \begin{cases} \overline{K}_{\phi(p_1^{2m-i})} & \text{for } i = 1, 2, \ldots, m-1, \\ K_{\phi(p_1^{2m-i})} & \text{for } i = m, \ldots, 2m-1. \end{cases} \tag{2.2}$$

From Eq (2.2), it follows that $n_i = \phi(p_1^{2m-i})$, where $i = 1, \ldots, 2m-1$ and $N = \sum_{i=1}^{2m-1} n_i$. Also, by definition of $n_i^{'}$, we get

$$n_1^{'} = 2n_2 + 2n_3 + \cdots + 2n_{2m-2} + n_{2m-1} = 2\sum_{i=2}^{2m-1} n_i - n_{2m-1}.$$

Similarly, we obtain

$$n_i^{'} = 2\sum_{j=1\ j\neq i}^{2m-1} n_j - \sum_{j=1}^{i} n_{2m-j}, \text{ for } i = 1, \ldots, m-1,$$

and

$$n_i^{'} = \sum_{j=1\ j\neq i}^{2m-1} n_j + \sum_{j=1}^{2m-1-i} n_j, \text{ for } i = m, \ldots, 2m-1.$$

By Eq (2.2), we note that $n_1 = \overline{K}_{\phi(p_1^{2m-1})}$, so by Theorem 2.2, we see that

$$\gamma_1 = 2n_1 + n_1^{'} - r_1 - \lambda_{1k} - 4 = 2\sum_{i=1}^{2m-1} n_i - n_{2m-1} - 0 - 0 - 4 = 2N - \phi(p_1) - 4 = 2N - p_1 - 3$$

is the D^Q-eigenvalue with multiplicity $\phi(p_1^{2m-1}) - 1$. For $i = 2, 3, \ldots, m-1$, proceeding as above with $n_i = \overline{K}_{\phi(p_1^{2m-i})}$, we get

$$\gamma_i = 2N - \sum_{j=1}^{i} \phi(p_1^j) - 4 = 2N - p_1^i - 3,$$

having multiplicities $\phi(p_1^{2m-i}) - 1$, where we use the property $\sum_{i=1}^{r} \phi(p_1^i) = p_1^r - 1$ is used. Similarly, for $i = m, \ldots, 2m-1$, with $G_i = K_{\phi(p_1^{2m-i})}$, $r_i = \phi(p_1^{2m-i}) - 1$ and $\lambda_{ik} = -1$, the other D^Q-eigenvalues are

$$\gamma_i = 2n_i + n_i^{'} - r_i - \lambda_{ik} - 4$$

$$= 2\phi(p_1^{2m-i}) + n_i^{'} - \phi(p_1^{2m-i}) + 1 + 1 - 4$$
$$= \sum_{j=1}^{2m-1} n_j + \sum_{j=1}^{2m-1-i} n_j - 2$$
$$= N + (p_1^{2m-1} - p_1^{i}) - 2,$$

having multiplicities $\phi(p_1^{2m-i}) - 1$. The rest D^Q-eigenvalues are of the subsequent matrix:

$$\begin{pmatrix} d_1 & 2\phi(p_1^{2m-2}) & \cdots & 2\phi(p_1^{m+1}) & 2\phi(p_1^{m}) & 2\phi(p_1^{m-1}) & \cdots & 2\phi(p_1^{2}) & \phi(p_1) \\ 2\phi(p_1^{2m-1}) & d_2 & \cdots & 2\phi(p_1^{m+1}) & 2\phi(p_1^{m}) & 2\phi(p_1^{m-1}) & \cdots & n_{2m-2} & \phi(p_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2\phi(p_1^{2m-1}) & 2\phi(p_1^{2m-2}) & \cdots & d_{m-1} & 2\phi(p_1^{m}) & \phi(p_1^{m-1}) & \cdots & \phi(p_1^{2}) & \phi(p_1) \\ 2\phi(p_1^{2m-1}) & 2\phi(p_1^{2m-2}) & \cdots & \phi(p_1^{m+1}) & d_m & \phi(p_1^{m-1}) & \cdots & \phi(p_1^{2}) & \phi(p_1) \\ 2\phi(p_1^{2m-1}) & 2\phi(p_1^{2m-2}) & \cdots & \phi(p_1^{m+1}) & \phi(p_1^{m}) & d_{m+1} & \cdots & \phi(p_1^{2}) & \phi(p_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2\phi(p_1^{2m-1}) & \phi(p_1^{2m-2}) & \cdots & \phi(p_1^{m+1}) & \phi(p_1^{m}) & \phi(p_1^{m-1}) & \cdots & d_{2m-2} & \phi(p_1) \\ 2\phi(p_1^{2m-1}) & \phi(p_1^{2m-2}) & \cdots & \phi(p_1^{m+1}) & \phi(p_1^{m}) & \phi(p_1^{m-1}) & \cdots & \phi(p_1^{2}) & d_{2m-1} \end{pmatrix} \quad (2.3)$$

where $d_i = \begin{cases} 4n_i + n_i^{'} - 4, & \text{for } i = 1, 2, \dots, m-1, \\ 4n_i + n_i^{'} - 2r_i - 4 = 2n_i + n_i^{'} - 2, & \text{for } i = m, m+1, \dots, 2m-1. \end{cases}$ □

The topological indices are molecular descriptors used in the developments of quantitative structure activity relationships (QSARs), where molecular activities are related to the chemical structures of graphs. There are several well known topological indices, one such is the Wiener index introduced by Harry Wiener and has applications in chemical graph theory and computer networks (see [4, 14]).

As sum of the eigenvalues of $D^Q(G)$ is equal to twice the Wiener index, that is, Trace$(D^Q(G)) = 2W$, thus, we compute the Wiener index of $\Gamma(\mathbb{Z}_n)$ using Theorem 2.4 and Corollary 2.5. First, we shall compute the Wiener index of $\Gamma(\mathbb{Z}_{p_1^{2m}})$.

From Corollary 2.5, the spectrum of $\Gamma(\mathbb{Z}_{p_1^{2m}})$ consists

$$\Big\{(2N - p_1 - 3)^{[\phi(p_1^{2m-1})-1]}, (2N - p_1^2 - 3)^{[\phi(p_1^{2m-2})-1]}, \dots, (2N - p_1^{m-1} - 3)^{[\phi(p_1^{m+1})-1]}$$
$$(2N - 1 - p_1^{m})^{[\phi(p_1^{m})-1]}, (2N - p_1^{m+1} - 1)^{[\phi(p_1^{m-1})-1]}, \dots, (2N - p_1^{2m-2} - 1)^{[\phi(p_1^{2})-1]}, (N-2)^{[\phi(p_1)-1]}\Big\}$$

together with the eigenvalues of the matrix (2.3), where

$$N = \sum_{i=1}^{2m-1} \phi(p_1^{2m-i}) = \sum_{1, n\neq d|n} \phi(d) = n - \phi(n) - 1 = p_1^{2m} - \phi(p_1^{2m}) - 1 = p_1^{2m-1} - 1.$$

Now, the trace of the matrix (2.3) is $d_1 + d_2 + \cdots + d_{m-1} + d_m + d_{m+1} \cdots + d_{2m-2} + d_{2m-1}$, where

$$d_i = \begin{cases} 4n_i + n_i^{'} - 4, & \text{for } i = 1, \dots, m-1, \\ 4n_i + n_i^{'} - 2r_i - 4 = 2n_i + n_i^{'} - 2, & \text{for } i = m, m+1, \dots, 2m-1. \end{cases}$$

Also, $d_1 = 4n_1 + n_1^{'} - 4 = 4\phi(p_1^{2m-1}) + 2\sum_{i=2}^{2m-1}\phi(p_1^{2m-i}) - \phi(p_1) - 4 = 2N + 2\phi(p_1^{2m-1}) - p_1 - 3$. Similarly, other d_i's are

$$d_i = \begin{cases} 2N + 2\phi(p_1^{2m-i}) - p_1^i - 3 & \text{for } i = 2, 3, \ldots, m-1, \\ 2N + \phi(p_1^{2m-i}) - p_1^i - 1 & \text{for } i = m, \ldots, 2m-1. \end{cases}$$

Therefore, the trace of the matrix $D^Q(\Gamma(\mathbb{Z}_{p_1^{2m}}))$ is given by:

$$\begin{aligned} \text{Trace}(D^Q(\Gamma(\mathbb{Z}_{p_1^{2m}}))) &= (2N - p_1^i - 3)(\phi(p_1^{2m-i}) - 1) + (2N - p_1^j - 1)(\phi(p_1^{2m-j}) - 1) \\ &+ 4N + 2\phi(p_1^{2m-i}) + \phi(p_1^{2m-j}) - p_1^i - p_1^j - 4. \end{aligned}$$

Thus the Wiener index of $\Gamma(\mathbb{Z}_{p_1^{2m}})$ is $\frac{1}{2}\text{Trace}(D^Q(\Gamma(\mathbb{Z}_{p_1^{2m}})))$.

Proceeding as above, the Wiener index of $\Gamma(\mathbb{Z}_{p_1^{2s_1}p_2^{2s_2}})$ can be found from its D^Q-spectrum given in Theorem 2.4.

Similar to Theorem 2.4 and Corollary 2.5, the D^Q-spectrum of $\Gamma(\mathbb{Z}_{p_1^{l_1}p_2^{l_2}})$ can be discussed, when both l_1 and l_2 are odd and when one of them is even and other is odd.

The following result demonstrates the D^Q-eigenvalues of zero divisor graphs of some local rings. But before proceeding further, we need the following results.

Theorem 2.6. *[7] The spectrum of $D^Q(K_n)$ is given below:*

$$\left\{(2n-2), (n-2)^{[n-1]}\right\}$$

and that of $D^Q(K_{a,b})$ is given by:

$$\left\{\frac{5n - 8 \pm \sqrt{9(a-b)^2 + 4ab}}{2}, (2n-b-4)^{[a-1]}, (2n-a-4)^{[b-1]}\right\}$$

where $n = a + b$.

A complete split graph, represented by $CS_{\omega,n-\omega}$, is a graph that consists of a clique on ω vertices while an independent set on the rest of $n-\omega$ vertices, so that any vertex of the clique is connected with each vertex of the independent set.

Theorem 2.7. *[16] The D^Q-eigenvalues of $CS_{\omega,n-\omega}$ are given as:*

$$\left\{(n-2)^{[\omega-1]}, (2n-\omega-4)^{[n-\omega-1]}, \frac{1}{2}\left(5n - 2\omega - 6 \pm \sqrt{4\omega n - 6\omega^2 + 8\omega - 3n - 2}\right)\right\}.$$

Theorem 2.8. [Theorems 6 and 7, [1]] *For a finite commutative ring R, if all the possible vertices of $\Gamma(R)$ (or $\overline{\Gamma}(R)$) have the equal degrees, then either $R \cong \mathbb{F} \times \mathbb{F}$ or $Z(R)^2 = \{0\}$, for some finite field $\mathbb{F}$.*

Theorem 2.9. *Suppose R is a finite commutative ring with unity* $1(\neq 0)$. *We have*

(i) If $|R| = p_1^2$, where p_1 is any prime, then the D^Q-spectrum of $\Gamma(R)$ is either $\{(2p_1-4), (p_1-3)^{[p_1-2]}\}$ *or* $\{(7p_1-11), (3p_1-7)^{[2p_1-3]}\}$.

(ii) If R is local having order p_1^3, then the D^Q-spectrum of $\Gamma(R)$ *is either* $\{(2p_1^2-3),(p_1^2-3)^{[p_1^2-2]}\}$ *or*

$$\left\{(2p_1^2-p_1-5)^{[p_1^2-p_1-1]},(p_1^2-3)^{[p_1-2]},\frac{5p_1^2-2p_1-9\pm\sqrt{9p_1^4-20p_1^3+2p_1^2+12p_1+1}}{2}\right\}.$$

Proof. (i) If R is local, then either $R\cong\mathbb{Z}_{p_1^2}$ or $R\cong\dfrac{\mathbb{Z}_{p_1}[x]}{(x^2)}$ and in either case, $\Gamma(R)$ is a complete graph whose order is p_1-1. Thus, by Theorem 2.6, we get

$$D^Q(\Gamma(R),x)=(x-2p_1+4)(x-p_1+3)^{p_1-2}.$$

Therefore, $spec(\Gamma(R))$ is as desired. If R is reduced, then $R\cong\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_1}$, and hence $\Gamma(R)$ is complete bipartite. Thus, by Theorem 2.6, we get

$$D^Q(\Gamma(R),x)=(x-(5p_1-9\pm 2(p_1-1)))\,(x-3p_1+7)^{2p_1-4}.$$

Thus, the D^Q-spectrum is as desired.

(ii) If R is local and $|R|=p_1^3$, then R is isomorphic to any of the subsequent: $\dfrac{\mathbb{F}_{p_1}[x,y]}{(x,y)^2}$, $\dfrac{\mathbb{F}_{p_1}[x]}{(x^3)}$, $\dfrac{\mathbb{Z}_{p_1^2}[x]}{(p_1x,x^2)}$, or $\dfrac{\mathbb{Z}_{p_1^2}[x]}{(p_1x,x^2-\bar{s}p_1)}$, where $\bar{s}\in\mathbb{Z}_{p_1}$ is a non-square element. If $R\cong\dfrac{\mathbb{F}_{p_1}[x,y]}{(x,y)^2}$, then $Z^*(R)=\{uy\}\cup\{ux\}\cup\{xu+yu'\}$, where $u',u\in\mathbb{F}_{p_1}\setminus\{0\}$. Therefore, $\left|\Gamma\left(\dfrac{\mathbb{F}_{p_1}[x,y]}{(x,y)^2}\right)\right|=p_1^2-1$, and for every $u,v\in Z^*(R)$, we have $uv=0$. Thus, $\Gamma(R)=K_{p_1^2-1}$. Also, if $R\cong\dfrac{\mathbb{Z}_{p_1^2}[x]}{(p_1x,x^2)}$, then $Z^*(R)=\{xu\}\cup\{p_1u\}$, where $u\in\mathbb{Z}_{p_1}\setminus\{0\}$, so $\Gamma(R)\cong K_{p_1^2-1}$. Thus in either case, when $R\cong\dfrac{\mathbb{F}_{p_1}[x,y]}{(x,y)^2}$ or $\dfrac{\mathbb{Z}_{p_1^2}[x]}{(p_1x,x^2)}$, then D^Q-spectrum of $\Gamma(R)$ is $\{(2p_1^2-3),(p_1^2-3)^{[p_1^2-2]}\}$. Next, if $R\cong\dfrac{\mathbb{F}_{p_1}[x]}{(x^3)}$, then $Z^*(R)$ can be partitioned into two subsets; $Z_1=\{ux^2|u\in\mathbb{F}_{p_1}\setminus\{0\}\}$ and $Z_2=\{ax+bx^2\mid a\in\mathbb{F}_{p_1}\setminus\{0\},b\in\mathbb{Z}_{p_1}\}$. Then Z_1 induces a clique having p_1-1 vertices while Z_2 is an independent subset. Further, for every $z_1\in Z_1$ and $z_2\in Z_2$, we have $z_1z_2=0$. Finally, if $R\cong\dfrac{\mathbb{Z}_{p_1^2}[x]}{(p_1x,x^2-\bar{s}p_1)}$, where $\bar{s}$ is a non-square element in $\mathbb{Z}_{p_1}$, then the vertex set of $\Gamma\left(\dfrac{\mathbb{Z}_{p_1^2}[x]}{(p_1x,x^2-\bar{s}p_1)}\right)$ can be expressed as disjoint union of the sets S_1 and S_2, where, $S_1=\{up_1\mid u\in\mathbb{Z}_{p_1}\setminus\{0\}\}$ and $S_2=\{ux\}\cup\{up_1+u'x\mid u,u'\in\mathbb{Z}_{p_1}\setminus\{0\}\}$. Then, $\forall\ s_1,s_1'\in S_1$ and $s_2,s_2'\in S_2$, we have $s_1s_1'=0$, $s_1s_2=0$ and $s_2s_2'\neq 0$. Thus, in each of these cases, $\Gamma(R)$ is a complete split graph $CS_{p_1-1,p_1^2-p_1}$. Therefore, by Theorem 2.7, D^Q-spectrum of $\Gamma(R)$ is $\{(2p_1^2-3),(p_1^2-3)^{[p_1^2-2]}\}$ or $\left\{(2p_1^2-p_1-5)^{[p_1^2-p_1-1]},(p_1^2-3)^{[p_1-2]},\dfrac{5p_1^2-2p_1-9\pm\sqrt{9p_1^4-20p_1^3+2p_1^2+12p_1+1}}{2}\right\}$. □

Theorem 2.10. *Suppose R is a finite commutative ring. If either* $\Gamma(R)$ *(or* $\overline{\Gamma}(R)$*) is regular, then the D^Q-spectrum of* $\Gamma(R)$ *is either* $\{(2|Z^*(R)|-4),(|Z^*(R)|-3)^{[|Z^*(R)|-2]}\}$ *or* $\left\{(7|Z^*(R)|-11),(3|Z^*(R)|-7)^{[2|Z^*(R)|-3]}\right\}$.

Proof. If either $\Gamma(R)$ (or $\overline{\Gamma}(R)$) is regular, then using Theorem 2.8, either $Z(R)^2 = 0$ or there is a field $\mathbb{F}$ such that $R \cong \mathbb{F} \times \mathbb{F}$. If $Z(R)^2 = 0$, then $\Gamma(R) \cong K_{|Z^*(R)|-1}$, then by Theorem 2.6, the D^Q-spectrum of $\Gamma(R)$ is obtained by replacing n by $|Z^*(R)| - 1$. Further, if $R \cong \mathbb{F} \times \mathbb{F}$, then $\Gamma(R) \cong K_{|\mathbb{F}-1|,|\mathbb{F}-1|}$, and hence D^Q-spectrum of $\Gamma(R)$ is $\{(7|Z^*(R)| - 11), (3|Z^*(R)| - 7)^{[2|Z^*(R)|-3]}\}$. □

If a graph G contains a cycle that transverses each vertex, then G is said to be Hamiltonian.

Theorem 2.11. *Suppose $R \cong R_1 \times R_2$ is a finite commutative ring whose zero divisor graph is Hamiltonian, then the D^Q-spectrum is given as:*

$$\left\{5|R| - 18 \pm \sqrt{\frac{1}{2}(9|R|^2 - 4|R| - 32|R_1||R_2| + 1)}, (2|R| + |R_i| - 1)^{[|R_j|-1]}\right\},$$

where $i, j \in \{1, 2\}$, and $i \neq j$.

Proof. We prove that both R_1 and R_2 must be integral domains. If not, let $Z_1 = \{0\} \times Z^*(R_2)$ and $Z_2 = (R_1 - Z(R_1)) \times Z^*(R_2)$. Then, Z_2 is independent while there is $z_1 \in Z_1$ and $z_2 \in Z_2$ such that $z_1 z_2 = 0$. Now, a Hamiltonian cycle in $\Gamma(R)$ containing all vertices of Z_2 and therefore containing a matching among Z_1 and Z_2. Since, Z_2 is an independent set, it means $|Z_2| \leq |Z_1|$. This means that $|R_1 - Z(R_1)| \leq 1$, implying that the only unit in R_1 is the identity element. Thus, $R_1 \cong \Pi\mathbb{Z}_2^k$ for some $k \in \mathbb{N}$. Consider $z' = (1, 1, \cdots, 1, 0) \in R_1$, then $(z', 1) \in V(\Gamma(R_1 \times R_2))$ is the only vertex which is connected to $z'' = (0, 0, \cdots, 0, 1, 0)$, which is the contradiction with the fact that $\Gamma(R)$ is Hamiltonian. As a result, both R_1 and R_2 are integral domains. Now, as R is finite, so $\Gamma(R) \cong K_{|R_1|-1,|R_2|-1}$. Therefore, by Theorem 2.6, the D^Q-spectrum of $\Gamma(R)$ is

$$\left\{5|R| - 18 \pm \sqrt{1/2(9|R|^2 - 4|R| - 32|R_1||R_2| + 1)}, (2|R| + |R_i| - 1)^{[|R_j|-1]}\right\},$$

where $i, j \in \{1, 2\}$, and $i \neq j$. □

3. Minimal distance signless Laplacian energy for zero divisor graphs of $\mathbb{Z}_n$

Suppose $\mathbb{M}_n(\mathbb{C})$ is the set of all $n \times n$ square matrices over the complex field $\mathbb{C}$. For $M \in \mathbb{M}_n(\mathbb{C})$, the square roots of the eigenvalues of M^*M or MM^* are called the *singular values*, where M^* is the complex conjugate of M. As MM^* is positive semi-definite, so the singular values of M are non-negative, denoted by $s_1(M) \geq s_2(M) \geq \cdots \geq s_n(M)$. The *trace norm* of $M \in \mathbb{M}_n(\mathbb{C})$ is specified as the sum of singular values, that is,

$$||M||_n = s_1(M) + s_2(M) + \cdots + s_n(M),$$

and the sum of the first k singular values is the *Ky Fan k-norm*, that is,

$$||M||_k = s_1(M) + s_2(M) + \cdots + s_k(M).$$

$||M||_1$ is the largest singular value of M and is called the *spectral norm*. It is obvious that for a *Hermitian matrix* M, $s_i(M) = |\lambda_i(M)|$, and for a positive semi-definite matrix M, $s_i(M) = \lambda_i(M)$, where the eigenvalues of M are $\lambda_i(M)$, $i = 1, \ldots, n$.

The trace norm of the symmetric matrix $D^Q(G) - \frac{2W(G)}{n} I_n$, is studied under the name distance signless Laplacian energy of G in the algebraic graph theory, where I_n is the identity matrix. For the symmetric matrix $D^Q(G) - \frac{2W(G)}{n}I_n$, we have $s_i(D^Q(G) - \frac{2W(G)}{n}I_n) = |\lambda_i(D^Q(G) - \frac{2W(G)}{n}I_n)|$ and the trace norm (distance signless Laplacian energy or D^Q-energy) [11] of G is given below:

$$DSLE(G) = \sum_{i=1}^{n} \left|\gamma_i^Q - \frac{2W(G)}{n}\right|. \tag{3.1}$$

Suppose σ is the largest positive integer such that $\gamma_\sigma^Q \geq \frac{2W(G)}{n}$ and $S_k(G) = \sum_{i=1}^{b} \gamma_i$ is the Ky Fan k-norm (sum of k largest D^Q-eigenvalues) of the matrix $D^Q(G)$. Then using $\sum_{i=1}^{n} \gamma_i = 2W(G)$, Eq (3.1) can be written in terms of Ky Fan k-norm [11] as shown below

$$DSLE(G) = 2\left(\sum_{i=1}^{\sigma} \gamma_i - \frac{2\sigma W(G)}{n}\right) = 2\max_{1\leq j\leq n}\left(\sum_{i=1}^{j} \gamma_i(G) - \frac{2jW(G)}{n}\right).$$

For some latest works on $DSLE(G)$, see [11, 13].

Assume that G-uv is the connected graph attained from G by removing an edge uv. The next result states that the D^Q-spectrum of G decreases upon edge deletion.

Lemma 3.1. *[6] Suppose G is a simple graph whose order and size are n and m, respectively, where $n \leq m$ and $G' = G - e$ is a connected graph attained from G by removing an edge. If $\gamma_1(G) \geq \gamma_2(G) \geq \cdots \geq \gamma_n(G)$ and $\gamma_1(G') \geq \gamma_2(G') \geq \cdots \geq \gamma_n(G')$ are respectively the D^Q-eigenvalues of G and G'. Then $\gamma_i(G') \geq \gamma_i(G)$ satisfies for every $1 \leq i \leq n$.*

Lemma 3.2. *[16] Consider $\Gamma(\mathbb{Z}_n)$ is the zero divisor graph. Then the following hold.*

(i) The D^Q-spectrum of $\Gamma(\mathbb{Z}_{p_1^2})$ is $\{2p_1 - 4, (p_1 - 3)^{[p_1-2]}\}$.
(ii) The D^Q-spectrum of $\Gamma(\mathbb{Z}_{p_1^3})$ is

$$\left\{(2p_1^2 - p_1 - 5)^{[p_1^2-p_1-1]}, (p_1^2 - 3)^{[p_1-2]}, \frac{1}{2}\left(5p_1^2 - 2p_1 - 9 \pm \sqrt{9p_1^4 - 20p_1^3 + 2p_1^2 + 12p_1 + 1}\right)\right\}.$$

The following result says that $\Gamma(\mathbb{Z}_{p_1^2})$ has minimal D^Q-energy among all the zero divisor graphs of order $p_1 - 1$, where p_1 is prime.

Theorem 3.3. *Suppose $\Gamma(R)$ is a zero divisor graph of ring R. Then*

$$DSLE(\Gamma(R)) \geq 2\left(k(p_1 - 3) + p_1 - 1 - \frac{2kW(R)}{n}\right),$$

the equality holds iff $n = p_1^2$, where p_1 is prime.

Proof. As we know $\Gamma(\mathbb{Z}_n)$ is complete if and only if $n = p_1^2$. Thus by Lemma 3.1, $\gamma_i(R) \geq \gamma_i(\Gamma(\mathbb{Z}_{p_1^2}))$ $\forall\ i = 1, 2, \ldots, n$. So,

$$S_k(\Gamma(R)) \geq S_k(\Gamma(\mathbb{Z}_{p_1^2})) = k(p_1 - 3) + p_1 - 1. \tag{3.2}$$

The equality holds if and only if $n = p_1^2$. Let $\sigma \geq 0$ such that $\gamma_\sigma \geq \frac{2W(G))}{n}$. Then using Eq (3.2) and the definition of D^Q-energy, we get

$$\begin{aligned} DSLE(\Gamma(R)) &= 2\left(\sum_{i=1}^{\sigma} \gamma_i^Q(\Gamma(R)) - \frac{2\sigma W(\Gamma(R))}{n}\right) = 2\max_{1\leq j\leq n}\left(\sum_{i=1}^{j} \gamma_i^Q(\Gamma(R)) - \frac{2jW(R)}{n}\right) \\ &\geq 2\max_{1\leq j\leq n}\left(\sum_{i=1}^{j} \gamma_i^Q(\Gamma(\mathbb{Z}_{p_1^2})) - \frac{2jW(\Gamma(R))}{n}\right) \\ &= 2\left(k(p_1-3) + p_1 - 1 - \frac{2(b-1)W(\Gamma(R))}{n}\right), \end{aligned}$$

with equality as in (3.2). □

Since $\dfrac{2W(\Gamma(\mathbb{Z}_{p_1^2}))}{n} = p_1 - 2$ and by Lemma 3.2, it is easy to see that $\sigma = p_1 - 1$. The next consequence of Theorem 3.3 provides the D^Q-energy of $\Gamma(\mathbb{Z}_{p_1^2})$.

Corollary 3.4. *The D^Q-energy of $\Gamma(\mathbb{Z}_{p_1^2})$ is*

$$DSLE(\Gamma(\mathbb{Z}_{p_1^2})) = p_1^2 - 3p_1 + 2.$$

The next result states that among the class of zero divisor graphs of order $N = p_1^2 - 1$ with independence and clique number is $p_1^2 - p_1$ and $p_1 - 1$, respectively, the graph $\Gamma(\mathbb{Z}_{p_1^3})$ of $\mathbb{Z}_{p_1^3}$ has the minimal trace norm.

Theorem 3.5. *Assume that $\Gamma(R)$ is a zero divisor graph of R with independence number $p_1^2 - p_1$, with prime p_1. Then*

$$DSLE(\Gamma(R)) \geq 2\left(\sqrt{D} + 2p_1^4 - 3p_1^3 - p_1^2 + 4p_1 - 4 - \frac{2(p_1^2 - p_1)W(\Gamma(R))}{N}\right),$$

where $D = 9p_1^4 - 20p_1^3 + 2p_1^2 + 12p_1 + 1$, and equality occurs iff $\Gamma(R) \cong \Gamma(\mathbb{Z}_{p_1^3})$.

Proof. For $n = p_1^3$, the only proper divisors of n are p_1 and p_1^2, so by the definition of zero divisor graph, $\Gamma(\mathbb{Z}_{p_1^3}) \cong K_{p_1-1} \nabla \overline{K}_{p_1^2-p_1}$, i.e., $\Gamma(\mathbb{Z}_{p_1^3})$ is the complete split graph with independence number $p_1^2 - p_1$. Thus by Lemma 3.1, $\gamma_i^Q(\Gamma(R)) \geq \gamma_i^Q(\Gamma(\mathbb{Z}_{p_1^3}))$ for all $i = 1, 2, \ldots, (p_1^2 - 2), (p_1^2 - 1)$. Also, by Lemma 3.2, the D^Q-spectrum of $\Gamma(\mathbb{Z}_{p_1^3})$ is

$$\left\{(2p_1^2 - p_1 - 5)^{[p_1^2 - p_1 - 1]}, (p_1^2 - 3)^{p_1 - 2}, \frac{1}{2}\left(5p_1^2 - 2p_1 - 9 \pm \sqrt{9p_1^4 - 20p_1^3 + 2p_1^2 + 12p_1 + 1}\right)\right\}.$$

Besides, it is easy to see that $\dfrac{2W}{N} = \dfrac{2p_1^4 - 2p_1^3 - 3p_1^2 + p_1 + 2}{p_1^2 - 1}$. Let $\sigma \in \mathbb{N}$ such that $\gamma_\sigma^Q \geq \frac{2W(\Gamma(R))}{n}$. Since, $\frac{1}{2}\left(5p_1^2 - 2p_1 - 9 + \sqrt{9p_1^4 - 20p_1^3 + 2p_1^2 + 12p_1 + 1}\right)$ is the D^Q-spectral radius of $\Gamma(\mathbb{Z}_{p_1^3})$ and is always greater than $\dfrac{2W}{N}$. Again, $2p_1^2 - p_1 - 5 \geq \dfrac{2p_1^4 - 2p_1^3 - 3p_1^2 + p_1 + 2}{p_1^2 - 1}$ implies that $p_1^3 - 4p_1^2 + 3 \geq 0$, which is true for $p_1 > 3$. Now, if

$$\frac{1}{2}\left(5p_1^2 - 2p_1 - 9 - \sqrt{9p_1^4 - 20p_1^3 + 2p_1^2 + 12p_1 + 1}\right) < \frac{2p_1^4 - 2p_1^3 - 3p_1^2 + p_1 + 2}{p_1^2 - 1},$$

then we obtain $(p_1^2-1)\sqrt{9p_1^4-20p_1^3+2p_1^2+12p_1+1}-p_1^4-2p_1^3+8p_1^2-5\geq 0$, the inequality is true for $p_1>3$. Similarly, the smallest D^Q-eigenvalue p_1^2-3 is always less than average of the eigenvalues. Thus, $\sigma=1+p_1^2-p_1-1=p_1^2-p_1$ and by definition of the D^Q-energy, we have

$$\begin{aligned} DSLE(\Gamma(R)) &= 2\left(\sum_{i=1}^{\sigma}\gamma_i^Q(R)-\frac{2\sigma W(R)}{N}\right)\geq 2\max_{1\leq j\leq n}\left(\sum_{i=1}^{j}\gamma_i^Q(\Gamma(\mathbb{Z}_{p_1^3}))-\frac{2jW(R)}{N}\right)\\ &= 2\left(5p_1^2-2p_1-9+\sqrt{D}+(p_1^2-p_1-1)(2p_1^2-p_1-5)-\frac{2(p_1^2-p_1)W(\Gamma(R))}{N}\right)\\ &= 2\left(\sqrt{D}+2p_1^4-3p_1^3-p_1^2+4p_1-4-\frac{2(p_1^2-p_1)W(\Gamma(R))}{N}\right), \end{aligned}$$

where $D=9p_1^4-20p_1^3+2p_1^2+12p_1+1$ and the equality holds if and only if $\Gamma(R)\cong\Gamma(\mathbb{Z}_{p_1^3})$. □

The trace norm of $\Gamma(\mathbb{Z}_{p_1^3})$ is obtained as a result of the previous theorem.

Corollary 3.6. *The D^Q-energy of $\Gamma(\mathbb{Z}_{p_1^3})$ is given as:*

$$DSLE(\Gamma(\mathbb{Z}_{p_1^2}))=\sqrt{9p_1^4-20p_1^3+2p_1^2+12p_1+1}+\frac{2p_1^4-7p_1^3+p_1^2+79+1}{p_1^2-1}.$$

The parameter σ is very well studied [12] for different types of matrices associated with graphs. It is a very non trivial problem to characterize classes of graphs with particular σ and more interesting is to relate it with the parameters of a graph. There are rare graphs, where σ coincides with the independence number of graph. From Theorem 3.5, we see that $\Gamma(\mathbb{Z}_{p^3})$ is one such family with σ same as the independence number.

4. Conclusions

The present article studies the distance signless Laplacian eigenvalues of the zero divisor graph $\Gamma(\mathbb{Z}_{p_1^{l_1}p^{l_2}})$ and the results are more general than in [16]. However, there are still gaps in the articel as all eigevalues of corresponding equitable quotient matrices cannot be found and in general the technique cannot be used for finding the distance signless Laplcain eigenvalues of $\Gamma(\mathbb{Z}_n)$ as calculations become very hectic and majority of non integral distance signless Lpalacian eigenvalues of the corresponding matrix remains unknown. Some numerial methods may help in approximating the eigenvalue of equitable quotient matrix.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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