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Research article

Fixed point theory in complex valued controlled metric spaces with an application

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Abstract: In this article we have introduced a metric named complex valued controlled metric type space, more generalized form of controlled metric type spaces. This concept is a new extension of the concept complex valued b-metric type space and this one is different from complex valued extended b-metric space. Using the idea of this new metric, some fixed point theorems involving Banach, Kannan and Fisher contractions type are proved. Some examples togetheran application are described to sustain our primary results.

Keywords: complex valued controlled; metric type spaces; complex valued metrics types; unique fixed point; Bannach contraction; integral equation
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1. Introduction and preliminaries

Fixed point theory, famous due to its vast applications in various areas of engineering, physics and mathematical sciences, has become an interesting area for many researchers. For non-linear analysis, techniques of fixed point act as a pivotal tool. Banach [9] made huge involvement in this area by introducing the concept of contraction mapping for a complete metric type space to find fixed point of the stated function.

The classical Banach contraction theorem [9] has been studied by many mathematicians and researchers in different ways, see [1, 2, 4, 11, 12, 15, 17, 26]. During these generalizations, different fixed point theorems were studied for different contractive mappings or sometimes studied after metric space extension. Bakhtin [7] and Czerwick ([13, 14]) introduced a generalized metric with respect to the structure named b-metric space.

Definition 1.1. *[2, 14] Consider $\mathbb{D} \neq 0$ with a real number $t > 1$. The functional $h_b : \mathbb{D} \times \mathbb{D} \to [0, \infty)$ satisfying the following conditions:*

(1) $h_b(d, e) = 0 \iff d = e$,

(2) $h_b(d, e) = h_b(e, d)$,

(3) $h_b(d, f) \leq t[h_b(d, e) + h_b(e, f)]$,

for all $d, e, f \in \mathbb{D}$, called b-metric. Also we call the pair $(\mathbb{D}, h_b)$ a b-metric space.

A b-metric is an usual metric in case of $t = 1$. So, the category of b -metric spaces is appreciably greater than that of classic metric spaces. For instance, see [13, 14, 25, 27, 30]. Many other extensions of this space were used in related literature as platforms for fixed point results, see [3, 8, 16, 18, 29].

Azam et al. [2] was the first one who introduced and presented the notion complex valued metric space, with more generalized form than a metric space. Ullah et al. gave in [31] the concept of complex valued extended b-metric space, which is extension of notion b-metric space which was introduced by Kamran et al. [19]. For examples and applications see [6, 31].

Consider set of complex numbers $\mathbb{C}$ and $d_1, d_2 \in \mathbb{C}$. Since we cannot compare in usual way two complex numbers let us add to the complex set $\mathbb{C}$ the following partial order $\precsim$, known in related literature as lexicographic order

$$d_1 \precsim d_2 \iff Re(d_1) \leq Re(d_2) \text{ or } (Re(d_1) = Re(d_2) \text{ and } Im(d_1) \leq Im(d_2).$$

Considering the previous definition, we may say that $d_1 \precsim d_2$ if one of the next conditions are satisfied:

(1) $Re(d_1) < Re(d_2)$ and $Im(d_1) < Im(d_2)$;

(2) $Re(d_1) < Re(d_2)$ and $Im(d_1) = Im(d_2)$;

(3) $Re(d_1) < Re(d_2)$ and $Im(d_1) > Im(d_2)$;

(4) $Re(d_1) = Re(d_2)$ and $Im(d_1) < Im(d_2)$;

in continuation of generalizations of metric spaces, another form of b-metric space was introduced and presented by Rao et al. [28] in 2013 and named as complex valued b-metric space. This idea was also studied and generalized by many mathematicians and researchers.

Definition 1.2. *[28] Let $\mathbb{D} \neq 0$ and also consider $t > 1$, where t is a real number. A functional $h_b : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$, holding the following properties:*

(CBM_1) $0 \precsim h_b(d,e)$ *and* $h_b(d,e) = 0 \iff d = e$,
(CBM_2) $h_b(e,d) = h_b(d,e)$,
(CBM_3) $h_b(d,f) \precsim t[h_b(d,e) + h_b(e,f)]$.
for all $d, e, f \in \mathbb{D}$, *is denoted by complex valued b-metric on* $\mathbb{D}$ *and the pair* $(\mathbb{D}, h_b)$ *is said to be complex valued b-metric space.*

Further recall the definition of complex valued extended b-metric given by Ullah et al. in [31].

Definition 1.3. *[31] Let* $\mathbb{D} \neq 0$ *and* $\vartheta : \mathbb{D} \times \mathbb{D} \to [1, \infty)$. *The function* $h_e : \mathbb{D} \times \mathbb{D} \to C$ *satisfying the following conditions:*
(CEB_1) $0 \precsim h_e(d,e)$ *and* $h(d,e) = 0 \iff d = e$,
(CEB_2) $h_e(d,e) = h_e(e,d)$,
(CEB_3) $h_e(d,f) \precsim \vartheta(d,f)[h_e(d,e) + h_e(e,f)]$,
for all $d, e, f \in \mathbb{D}$, *is known as c.v. extended b-metric. And the pair* $(\mathbb{D}, h_e)$ *is called a c.v. extended b-metric space.*

Example 1.1. *[31] Let* $\mathbb{D} \neq 0$ *and* $\vartheta : \mathbb{D} \times \mathbb{D} \to [1, \infty)$ *be defined as:*

$$\vartheta(d,e) = \frac{1+d+e}{d+e}.$$

Further, let

(1) $h_e(d,e) = \frac{i}{de}$, *for all* $d, e \in (0,1]$;

(2) $h_e(d,e) = 0 \iff d = e$, *for all* $d, e \in (0,1]$;

(3) $h_e(0,e) = h_e(e,0) = \frac{i}{e}$, *for all* $e \in (0,1]$.

So, $(\mathbb{D}, h_e)$ *is here a complex valued (c.v.) extended b-metric space.*

In [23] Maliki et al. introduced the idea of controlled metric and gave few important fixed point theorems with respect to this new type of metric. Several researchers proved fixed point theorems using this idea (see [5, 20, 21, 24]).

Definition 1.4. *([23]) Let* $\mathbb{D} \neq 0$ *and* $\vartheta : \mathbb{D} \times \mathbb{D} \to [1, \infty)$. *The functional* $h_{cm} : \mathbb{D} \times \mathbb{D} \to [0, \infty)$ *is called controlled metric type if, the given conditions holds:*
(CMT_1) $h_{cm}(d,e) = 0 \iff d = e$,
(CMT_2) $h_{cm}(d,e) = h_{cm}(e,d)$,
(CMT_3) $h_{cm}(d,f) \leq \vartheta(d,e)h_{cm}(d,e) + \vartheta(e,f)h_{cm}(e,f)$,
for all $d, e, f \in \mathbb{D}$. *Also the pair* $(\mathbb{D}, h_{cm})$ *is said controlled metric type space.*

Now moving towards the main definition/concept, in which we have generalized the idea of controlled metric spaces in complex valued spaces as follows.

Definition 1.5. *Let* $\mathbb{D} \neq 0$ *and consider* $\vartheta : \mathbb{D} \times \mathbb{D} \to [1, \infty)$. *The functional* $h_{cvc} : \mathbb{D} \times \mathbb{D} \to [0, \infty)$ *is called complex valued controlled metric type space if, the following conditions holds:*
(CMT_1) $0 \precsim h_{cvc}(d,e)$ *also* $h_{cvc}(d,e) = 0 \iff d = e$,
(CMT_2) $h_{cvc}(d,e) = h_{cvc}(e,d)$,
(CMT_3) $h_{cvc}(d,f) \precsim \vartheta(d,e)h_{cvc}(d,e) + \vartheta(e,f)h_{cvc}(e,f)$,
for all $p, q, r \in \mathbb{D}$, *Then, the pair* $(\mathbb{D}, h_{cvc})$ *is known as a complex valued controlled metric type space.*

Example 1.2. *Let $\mathbb{D} = [0, \infty)$ and $\vartheta : \mathbb{D} \times \mathbb{D} \to [1, \infty)$ be defined as*

$$\vartheta(d, e) = \begin{cases} 1, & \text{if } d, e \in [0, 1], \\ 1 + d + e, & \text{otherwise}. \end{cases}$$

and $h_{cvc} : \mathbb{D} \times \mathbb{D} \to [0, \infty)$ defined as follows

$$h_{cvc}(d, e) := \begin{cases} 0, & d = e \\ i, & d \neq e \end{cases}$$

Then $(\mathbb{D}, h_{cvc})$ is here a complex valued (c.v.) controlled metric type space.

Remark 1.1. *If we take $\vartheta(d, e) = t \geq 1$, for all $d, e \in \mathbb{D}$, then $(\mathbb{D}, h_{cvc})$ is c.v.(complex valued) b-metric space, which means that every complex valued b-metric space is a complex valued controlled metric type space.*

Example 1.3. *Let $\mathbb{D} = V \cup W$ with $V = \{(\frac{1}{n}) | n \in N\}$, W is set of positive integer and $\vartheta : \mathbb{D} \times \mathbb{D} \to [1, \infty)$ be defined for all $d, e \in \mathbb{D}$ as*

$$\vartheta(d, e) = 5k$$

where $k > 0$ and $h_{cvc} : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ defined as follows

$$h_{cvc}(d, e) := \begin{cases} 0, & \iff d = e \\ 2ki, & if d, e \in V \\ \frac{ki}{2}, & \text{otherwise} \end{cases}$$

where $k > 0$.

Now, the conditions $(CCMT_1)$ and $(CCMT_2)$ hold. Also $(CCMT_3)$ hold under the given following cases.

Case 1. *If $d = e$ and $e = f$.*

Case 2. *If $d = e \neq f$ or if $d \neq e = f$ or if $d = f \neq e$ or if $d \neq e \neq f$.*

SubCase 1. *If $d \in V$ and $e, f \in W$.*

SubCase 2. *If $e \in V$ and $d, f \in W$.*

SubCase 3. *If $f \in V$ and $d, e \in W$.*

SubCase 4. *If $d, e \in V$ and $f \in W$.*

SubCase 5. *If $d, f \in V$ and $e \in W$.*

SubCase 6. *If $e, f \in V$ and $d \in W$.*

SubCase 7. *If $d, e, f \in V$.*

SubCase 8. *If $d, e, f \in W$. Then $(\mathbb{D}, h_{cvc})$ is here a complex valued controlled metric type space.*

Remark 1.2. *If $\vartheta(d, e) = \vartheta(e, f)$,(as in above example) for all $d, e, f \in \mathbb{D}$, then $(\mathbb{D}, h_{cvc})$ is complex valued extended b-metric space. From which we can conclude that every complex valued extended b-metric space is a complex valued controlled metric space. But converse may not true in general.*

We will put in evidence the previous remark by the following example.

Example 1.4. *Let $\mathbb{D} = \{1, 2, 3\}$ and $h_{cvc} : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ defined as follows*

$$h_{cvc}(1, 1) = h_{cvc}(2, 2) = h_{cvc}(3, 3) = 0,$$

$$h_{cvc}(1,2) = h_{cvc}(2,1) = 4 + 4i,$$

$$h_{cvc}(2,3) = h_{cvc}(3,2) = 1 + 2i,$$

$$h_{cvc}(1,3) = h_{cvc}(3,1) = 1 - i,$$

and $\vartheta : \mathbb{D} \times \mathbb{D} \to [1, \infty)$ be defined as

$$\vartheta(1,1) = \vartheta(2,2) = \vartheta(3,3) = 3,$$

$$\vartheta(1,2) = \vartheta(2,1) = 2,$$

$$\vartheta(2,3) = \vartheta(3,2) = 4,$$

$$\vartheta(1,3) = \vartheta(3,1) = 1.$$

From above defined functions, the conditions $(CCMT_1)$ and $(CCMT_2)$ hold. Also $(CCMT_3)$ hold under the following cases.
Case 1. *If $d = f$ the condition $(CCMT_3)$ hold immediately.*
Case 2. *If $d = 1$ and $f = 3$ (same as $f = 1$ and $d = 3$) and $e = 2$*

$$\begin{aligned} h_{cvc}(d,f) = |h_{cvc}(1,3)| = |1 - i| \precsim |12 + 16i| &= |2(4 + 4i) + 4(1 + 2i)| \\ &\precsim 2|(4 + 4i)| + 4|(1 + 2i)| = \vartheta(1,2)h_{cvc}(1,2) + \vartheta(2,3)h_{cvc}(2,3) \\ &= \vartheta(d,e)h_{cvc}(d,e) + \vartheta(e,f)h_{cvc}(e,f). \end{aligned}$$

Case 3. *If $d = 1$ and $f = 2$ (same as $f = 1$ and $d = 2$) and $e = 3$*

$$\begin{aligned} h_{cvc}(d,f) = |h_{cvc}(1,2)| = |4 + 4i| \precsim |5 + 7i| &= |1(1 - i) + 4(1 + 2i)| \\ &\precsim 1|(1 - i)| + 4|(1 + 2i)| = \vartheta(1,3)h_{cvc}(1,3) + \vartheta(3,2)h_{cvc}(3,2) \\ &= \vartheta(d,e)h_{cvc}(d,e) + \vartheta(e,f)h_{cvc}(e,f). \end{aligned}$$

Case 4. *If $d = 2$ and $f = 3$ (same as $f = 3$ and $d = 2$) and $e = 1$*

$$\begin{aligned} h_{cvc}(d,f) = |h_{cvc}(2,3)| = |1 + 2i| \precsim |9 + 7i| &= |2(4 + 4i) + 1(1 - i)| \\ &\precsim 2|(4 + 4i)| + 1|(1 - i)| = \vartheta(2,1)h_{cvc}(2,1) + \vartheta(1,3)h_{cvc}(1,3) \\ &= \vartheta(d,e)h_{cvc}(d,e) + \vartheta(e,f)h_{cvc}(e,f). \end{aligned}$$

For contradiction to complex valued (c.v.) extended b-metric, we have the following inequality:

$$\begin{aligned} |h_{cvc}(1,2)| = |4 + 4i| \succ |4 + 2i| & \\ &= 2|(1 - i) + (1 + 2i)| \\ &= \vartheta(1,2)[h_{cvc}(1,3) + h_{cvc}(3,2)]. \end{aligned}$$

In conclusion, $(\mathbb{D}, h_{cvc})$ is a complex valued controlled metric space. It can also be seen that it is not complex valued extended b-metric space.

For complex valued controlled (c.v.c.) metric type spaces, we now define Cauchy sequence and also convergent sequence as below.

Definition 1.6. *Consider* $(\mathbb{D}, h_{cvc})$ *is a complex valued controlled (c.v.c)metric space with* $\{d_n\}_{n\geq 0}$ *a sequence in* $\mathbb{D}$ *and* $d \in \mathbb{D}$. *Then*

(i) A sequence $\{d_n\}$ *in* $\mathbb{D}$ *is convergent and converges to* $d \in \mathbb{D}$ *if for every* $0 \prec c \in \mathbb{C}$ $\exists$ *a natural number N so that* $h_{cvc}(d_n, d) \prec c$ *for every* $n \geq N$. *Then we say* $\lim_{n\to\infty} d_n = d$ *or* $d_n \to d$ *as* $n \to \infty$.

(ii) If, for every $0 \prec c$ *where* $c \in \mathbb{C}$ $\exists$ *a natural number N so that* $h_{cvc}(d_n, d_{n+m}) \prec c$ *for every* $m \in \mathbb{N}$ *and* $n > N$. *Then* $\{d_n\}$ *is said to be Cauchy sequence in* $(\mathbb{D}, h_{cvc})$.

(iii) Complex valued controlled (c.v.c) metric type space $(\mathbb{D}, h_{cvc})$ *is said to be Complete, if every Cauchy sequence in* $\mathbb{D}$ *is convergent in* $\mathbb{D}$.

Definition 1.7. *Consider* $(\mathbb{D}, h_{cvc})$ *be a complex valued controlled (c.v.c.) metric space. Let* $d \in \mathbb{D}$ *and* $0 \prec c$

(i) Define the open ball as $B(d, c)$ *is* $B(d, c) = \{e \in \mathbb{D}, h_{cvc}(d, e) \prec c\}$. *Moreover, the family of open balls forms a basis of some topology* $\tau_{h_{cvc}}$ *on* $\mathbb{D}$.

(ii) The functional/mapping $\Upsilon : \mathbb{D} \to \mathbb{D}$ *is called continuous at* $p \in \mathbb{D}$ *if* $\forall\ 0 \prec c$, $\exists\ 0 \prec \delta^*$ *such that* $\Upsilon(B(d, \delta^*)) \subseteq B(\Upsilon d, c)$.

Obviously, if Υ is taken continuous at d in the complex valued controlled (c.v.c.) metric spaces $(\mathbb{D}, h_{cvc})$ then $d_n \to d$ implies $\Upsilon d_n \to \Upsilon d$ as $n \to \infty$.

One can prove the following lemmas for the specific case of complex valued controlled (c.v.c) metric space, in a similar way as in [28].

Lemma 1.1. *Let* $(\mathbb{D}, h_{cvc})$ *be a complex valued controlled (c.v.c) metric space and assume a sequence* $\{d_n\} \in \mathbb{D}$. *Then* $\{d_n\}$ *is Cauchy sequence* $\iff |h_{cvc}(d_m, d_n)| \to 0$ *as* $m, n \to \infty$, *where* $m, n \in N$.

Proof. Let $\{d_n\}$ be a sequence which is Cauchy in $\mathbb{D}$, then for every $\varepsilon \succ 0$, $\exists\ n_0 \in \mathbb{N}$ such that

$$h_{cvc}(d_m, d_n) \prec \epsilon, \text{ for all } m, n > n_0. \tag{1}$$

Since $\epsilon \in \mathbb{C}$, $\exists\ a, b \in \mathbb{R}$ such that $\epsilon = a + ib$. We consider $\delta^* = |\epsilon| = \sqrt{a^2 + b^2}$, and taking the modulus on both sides of (1), we have $|h_{cvc}(d_m, d_n)| \prec \delta^*$.

Further, using the concept of limit, we have $|h_{cvc}(d_m, d_n)| \to 0$ as $m, n \to \infty$, where $m, n \in N$.

Conversely, suppose that $|h_{cvc}(d_m, d_n)| \to 0$ as $m, n \to \infty$, where $m, n \in N$. Then, result that $|h_{cvc}(d_m, d_n)| \prec \delta^*$, for every $\delta^* > 0$.

Since $h_{cvc}(d_m, d_n) \in \mathbb{C}$, $\exists\ \epsilon \in \mathbb{C}$ such that $\delta^* = |\epsilon|$. Then, we have $h_{cvc}(d_m, d_n) \prec \epsilon$, for every $\epsilon \succ 0$. Thus, $\exists\ n_0 \in \mathbb{N}$ such that $m, n > n_0$, $\{d_n\}$ is a complex valued Cauchy sequence. □

Lemma 1.2. *Suppose* $(\mathbb{D}, h_{cvc})$ *be a complex valued controlled (c.v.c) metric space and* $\{d_n\}$ *be sequence in* $\mathbb{D}$. *Then* $\{d_n\}$ *converges to* $d \iff |h_{cvc}(d_n, d)| \to 0$ *as* $n \to \infty$.

Proof. Letting $m \to \infty$ in proof of the above lemma, we can get the conclusion. □

Remark 1.3. *While taking* $\vartheta(d, e) = \vartheta(e, f)$ *for all* $d, e, f \in \mathbb{D}$, *then a complex valued controlled metric space is reduced to a complex valued extended b-metric space.*

Lemma 1.3. *Let* $(\mathbb{D}, h_{cvc})$ *be a complex valued controlled(c.v.c) metric space. Then a sequence* $\{d_n\}$ *in* $\mathbb{D}$ *is Cauchy sequence, such that* $d_m \neq d_n$, *whenever* $m \neq n$. *Then* $\{d_n\}$ *converges to at most one point.*

Proof. Consider the sequence $\{d_n\}$ with two limit points d^* and $e^* \in \mathbb{D}$ and $\lim_{n\to\infty} h_{cvc}(d_n, d^*) = 0 = \lim_{n\to\infty} h_{cvc}(d_n, e^*)$. Since $\{d_n\}$ is Cauchy, from $(CCMT_3)$, for $d_m \neq d_n$, whenever $m \neq n$, we can write

$$|h_{cvc}(d^*, e^*)| \precsim [\vartheta(d^*, d_n)|h_{cvc}(d^*, d_n)| + \vartheta(d_n, e^*)|h_{cvc}(d_n, e^*)|] \to 0 \text{ as } n \to \infty.$$

We get $|h_{cvc}(d^*, e^*)| = 0$, i.e., $d^* = e^*$. Thus, $\{d_n\}$ converges to at most one point. □

Remark that, in general an usual b-metric is not considered a continuous functional. Concerning this aspect we will discuss next, the continuity of the complex valued controlled metric with respect to the partial order $\precsim$.

Lemma 1.4. *For a given complex valued controlled space $(\mathbb{D}, h_{cvc})$, the complex valued controlled (c.v.c) metric function $h_{cvc} : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ is continuous, with respect to the partial order " $\precsim$ ".*

Proof. Choosing any two arbitrary complex numbers y and x, such that $y \succ x$, then we should show that the set $h^{-1}_{cvc}(x, y)$ given by

$$h^{-1}_{cvc}(x, y) := \{(d, e) \in \mathbb{D} \times \mathbb{D} | y \succ h_{cvc}(d, e) \succ x\},$$

is open in the product topology on $\mathbb{D} \times \mathbb{D}$. A basis for this product topology is the collection of all cartesian products of open balls in $(\mathbb{D}, h_{cvc})$.

Then, let $(d, e) \in h^{-1}_{cvc}(x, y)$. We choose $\epsilon = \frac{1}{100} \min(h_{cvc}(d, e) - x, y - h_{cvc}(d, e))$. Then, for any point $(\zeta, \sigma) \in B(d, \epsilon) \times B(e, \epsilon)$ we have

$$h_{cvc}(\zeta, \sigma) \leq h_{cvc}(\zeta, d) + h_{cvc}(d, e) + h_{cvc}(e, \sigma) \prec 2\epsilon + h_{cvc}(d, e) \prec y$$

and

$$x \leq h_{cvc}(d, e) - 2\epsilon \prec h_{cvc}(\zeta, \sigma) + h_{cvc}(d, \zeta) - \epsilon + h_{cvc}(e, \sigma) - \epsilon \prec h_{cvc}(\zeta, \sigma).$$

Then it results in $(d, e) \in B(p, \epsilon) \times B(e, \epsilon) \subseteq h^{-1}_{cvc}(x, y)$. Then, any point $(d, e) \in h^{-1}_{cvc}(x, y) \subseteq \mathbb{D} \times \mathbb{D}$ is surrounded by some open set included in $h^{-1}_{cvc}(x, y)$.

Then, the complex valued metric $h_{cvc}(d, e)$ is continuous in the complex valued controlled (c.v.c) metric space $(\mathbb{D}, h_{cvc})$, with respect to the partial order " $\precsim$ ". □

According with this result, let us give the following lemma.

Lemma 1.5. *Consider $(\mathbb{D}, h_{cvc})$ be a complex valued controlled(c.v.c) metric type space. Limit of every convergent sequence in $\mathbb{D}$ is unique, if the functional $h_{cvc} : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ is continuous.*

The next graph shows the relations between the recently generalizations of complex valued metric space.

$$\begin{array}{ccccccc}
\text{metric space} & \to & b\text{- metric space} & \to & \text{extended } b\text{-metric space} & \to & \text{controlled metric space} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{c.v.metric space} & \to & \text{c.v.}b\text{- metric space} & \to & \text{c.v.extend } b\text{-metric space} & \to & \text{c.v.controlled metric space}
\end{array}$$

In this paper, CVC-metric space will be the notation of complex valued controlled (c.v.c) metric space. Also, $\mathbb{N}^* := \mathbb{N} - \{0\}$ is notation of the set for all natural nonzero numbers. Then, for operator Υ, $Fix\Upsilon := \{d^* \in \mathbb{D} \mid d^* = \Upsilon(d^*)\}$ will be the set of fixed points.

In this paper, we give the notion of complex valued controlled metric space. We will also prove the Banach contraction principle, Kannan and Fisher rational type fixed point theorems in the settings of complex valued controlled metric. Few examples and an application are presented to sustain our main results.

2. Banach type fixed point result on CVC-metric space

For this section let us give a Banach type contraction principle related to the complex valued controlled metric space.

Theorem 2.1. *Let* $(\mathbb{D}, h_{cvc})$ *be a complete CVC-metric space and* $\Upsilon : \mathbb{D} \to \mathbb{D}$ *be a continuous mapping such that*

$$h_{cvc}(\Upsilon d, \Upsilon e) \precsim \lambda h_{cvc}(d, e), \tag{2}$$

for all $d, e \in \mathbb{D}$, *where* $0 < \lambda < 1$. *For* $d_0 \in \mathbb{D}$ *we denote* $d_n = \Upsilon^n d_0$. *Suppose that*

$$\max_{m \geq 1} \lim_{x \to \infty} \frac{\vartheta(d_{x+1}, d_{x+2})}{\vartheta(d_x, d_{x+1})} \vartheta(d_{x+1}, d_m) < \frac{1}{\lambda}, \tag{3}$$

In addition, for every $d \in \mathbb{D}$ *the limits*

$$\lim_{n \to \infty} \vartheta(d_n, d) \text{ and } \lim_{n \to \infty} \vartheta(d, d_n) \text{ exists and are finite.} \tag{4}$$

Then Υ *has a unique fixed point.*

Proof. Suppose the sequence $\{d_n = \Upsilon^n d_0\}$. From (2) we get

$$h_{cvc}(d_n, d_{n+1}) \precsim \lambda h_{cvc}(d_{n-1}, d_n) \precsim \ldots \precsim \lambda^n h_{cvc}(d_0, d_1) \forall n \geq 0.$$

$\forall$ $n < m$, where n and m are natural numbers, we have

$$\begin{aligned}
h_{cvc}(d_n, d_m) &\precsim \vartheta(d_n, d_{n+1}) h_{cvc}(d_n, d_{n+1}) + \vartheta(d_{n+1}, d_m) h_{cvc}(d_{n+1}, d_m) \\
&\precsim \vartheta(d_n, d_{n+1}) h_{cvc}(d_n, d_{n+1}) + \vartheta(d_{n+1}, d_m) \vartheta(d_{n+1}, d_{n+2}) h_{cvc}(d_{n+1}, d_{n+2}) \\
&+ \vartheta(d_{n+1}, d_m) \vartheta(d_{n+2}, d_m) h_{cvc}(d_{n+2}, d_m) \\
&\precsim \vartheta(d_n, d_{n+1}) h_{cvc}(d_n, d_{n+1}) + \vartheta(d_{n+1}, d_m) \vartheta(d_{n+1}, d_{n+2}) h_{cvc}(d_{n+1}, d_{n+2}) \\
&+ \vartheta(d_{n+1}, d_m) \vartheta(d_{n+2}, d_m \vartheta(d_{n+2}, d_{n+3)} h_{cvc}(d_{n+2}, d_{n+3}) \\
&+ \vartheta(d_{n+1}, d_m) \vartheta(d_{n+2}, d_m \vartheta(d_{n+3}, d_m) h_{cvc}(d_{n+3}, d_m) \\
&\precsim \ldots \precsim \vartheta(d_n, d_{n+1}) h_{cvc}(d_n, d_{n+1}) + \sum_{l=n+1}^{m-2} \prod_{j=n+1}^{l} \vartheta(d_j, d_m) \vartheta(d_l, d_{l+1}) h_{cvc}(d_l, d_{l+1}) \\
&+ \prod_{k=n+1}^{m-1} \vartheta(d_k, d_m) h_{cvc}(d_{m-1}, d_m) \\
&\precsim \vartheta(d_n, d_{n+1}) \lambda^n h_{cvc}(d_0, d_1) + \sum_{l=n+1}^{m-2} \prod_{j=n+1}^{l} \vartheta(d_j, d_m) \vartheta(d_l, d_{l+1}) \lambda^l h_{cvc}(d_0, d_1)
\end{aligned}$$

$$+\prod_{k=n+1}^{m-1}\vartheta(d_k,d_m)\lambda^{m-1}h_{cvc}(d_0,d_1)$$

$$\precsim \vartheta(d_n,d_{n+1})\lambda^n h_{cvc}(d_0,d_1)+\sum_{l=n+1}^{m-2}\prod_{j=n+1}^{l}\vartheta(d_j,d_m)\vartheta(d_l,d_{l+1})\lambda^l h_{cvc}(d_0,d_1)$$

$$+\prod_{k=n+1}^{m-1}\vartheta(d_k,d_m)\lambda^{m-1}\vartheta(d_{m-1},d_m)h_{cvc}(d_0,d_1)$$

$$=\vartheta(d_n,d_{n+1}\lambda^n h_{cvc}(d_0,d_1)+\sum_{l=n+1}^{m-1}\prod_{j=n+1}^{l}\vartheta(d_j,d_m)\vartheta(d_l,d_{l+1})\lambda^l h_{cvc}(d_0,d_1)$$

$$\precsim \vartheta(d_n,d_{n+1})\lambda^n h_{cvc}(d_0,d_1)+\sum_{l=n+1}^{m-1}\prod_{j=0}^{l}\alpha(d_j,d_m)\vartheta(d_l,d_{l+1})\lambda^l h_{cvc}(d_0,d_1).$$

Also, using $\vartheta(d,e)\geq 1$. Let

$$S_u=\sum_{l=0}^{u}\prod_{j=0}^{l}\vartheta(d_j,d_m)\vartheta(d_l,d_{l+1})\lambda^l.$$

Hence we have

$$h_{cvc}(d_n,d_m)\precsim h_{cvc}(d_0,d_1)[\lambda^n\vartheta(d_n,d_{n+1})+(S_{m-1},S_n)].\tag{5}$$

By (3) and applying the ratio test, we get that $\underset{m,n\to\infty}{lim}S_n$ exists and so the real sequence $\{S_n\}$ is Cauchy. At the end, taking the limit in the inequality (5) when $m,n\to\infty$ we found that

$$\underset{m,n\to\infty}{lim}h_{cvc}(d_n,d_m)=0.\tag{6}$$

Results $\{d_n\}$ is Cauchy sequence in the complete CVC-metric space$(\mathbb{D},h_cvc)$; then $\{d_n\}$ converges to a $d^*\in\mathbb{D}$. We now show that d^* is fixed point of Υ.

By the continuity of Υ we obtain

$$d^*=\lim_{n\to\infty}d_{n+1}=\lim_{n\to\infty}\Upsilon d_n=\Upsilon(\lim_{n\to\infty}d_n)=\Upsilon d^*.$$

For uniqueness of fixed point, we suppse that Υ has two fixed points, $d^*,e^*\in Fix\Upsilon$. Thus we have

$$h_{cvc}(d^*,e^*)=h_cvc(\Upsilon d^*,\Upsilon e^*)\precsim\vartheta h_{cvc}(d^*,e^*).$$

which holds unless $h_{cvc}(d^*,e^*)=0$; so $d^*=e^*$. Hence Υ has a unique fixed point. □

If we avoid the continuity condition of the mapping Υ in the previous theorem we get a new general result as follows.

Theorem 2.2. *Let $(\mathbb{D},h_{cvc})$ be a complete CVC-metric space and $\Upsilon:\mathbb{D}\to\mathbb{D}$ be a mapping such that*

$$h_{cvc}(\Upsilon d,\Upsilon e)\precsim\vartheta h_{cvc}(d,e),\tag{7}$$

for all $d, e \in \mathbb{D}$, where $0 < \vartheta < 1$. For $d_0 \in \mathbb{D}$ we denote $d_n = \Upsilon^n d_0$. Suppose that

$$\max_{m\geq 1} \lim_{l\to\infty} \frac{\vartheta(d_{l+1}, d_{l+2})}{\vartheta(d_l, d_{l+1})} \vartheta(d_{l+1}, d_m) < \frac{1}{\lambda}. \tag{8}$$

In addition, for every $d \in \mathbb{D}$ the limits

$$\lim_{n\to\infty} \vartheta(d_n, d) \text{ and } \lim_{n\to\infty} \vartheta(d, d_n) \text{ exist and are finite.} \tag{9}$$

Then Υ has a unique fixed point.

Proof. Using similar steps as we did in the proof of Theorem 2.1, and taking into account Lemma 1.4, we find a Cauchy sequence $\{d_n\}$ in the complete CVC-metric space $(\mathbb{D}, h_{cvc})$. Then the sequence $\{d_n\}$ converges to a $d^* \in \mathbb{D}$. We shall prove that d^* is a fixed point of Υ. The triangular inequality follows that

$$h_{cvc}(d^*, d_{n+1}) \precsim \vartheta(d^*, d_n) h_{cvc}(d^*, d_n) + \vartheta(d_n, d_{n+1}) h_{cvc}(d_n, d_{n+1}).$$

Using (8), (9) and (31), we get

$$\lim_{n\to\infty} h_{cvc}(d^*, d_{n+1}) = 0. \tag{10}$$

Using again the triangular inequality and (7),

$$\begin{aligned} h_{cvc}(d^*, \Upsilon d^*) &\precsim \vartheta(d^*, d_{n+1}) h_{cvc}(d^*, d_{n+1}) + \vartheta(d_{n+1}, \Upsilon d^*) h_{cvc}(d_{n+1}, \Upsilon d^*) \\ &\precsim \vartheta(d^*, d_{n+1}) h_{cvc}(d^*, d_{n+1}) + \lambda \vartheta(d_{n+1}, \Upsilon d^*) h_{cvc}(d_n, \Upsilon d^*). \end{aligned}$$

Taking the limit $n \to \infty$ and by (9) and (32) we found that $h_{cvc}(d^*, \Upsilon d^*) = 0$. Remark that, in view of Lemma 1.3, the sequence $\{d_n\}$ converges uniquely at the point $d^* \in \mathbb{D}$. □

We illustrate this theorem with the help of following example.

Example 2.1. *Consider $\mathbb{D} = \{0, 1, 2\}$ and let $h_{cvc} : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ be a symmetric metric given by*

$$h_{cvc}(d, d) = 0, \text{ for each } d \in \mathbb{D}$$

and

$$h_{cvc}(0, 1) = 1 + i, h_{cvc}(0, 2) = 4 + 4i, h_{cvc}(1, 2) = 1 + i.$$

Also, let $\vartheta : \mathbb{D} \times \mathbb{D} \to [1, \infty)$ be a symmetric function and

$$\vartheta(0, 0) = 2, \vartheta(0, 1) = \frac{3}{2}, \vartheta(0, 2) = \frac{4}{3},$$

$$\vartheta(1, 1) = \frac{4}{3}, \vartheta(1, 2) = \frac{5}{4}, \vartheta(2, 2) = \frac{6}{5}.$$

It's easy to remark that we have a CVC- metric space.

Let us consider the self map Υ on $\mathbb{D}$ as follows $\Upsilon(0) = 0$,$\Upsilon(1) = 0$, $\Upsilon(2) = 0$.

Choosing $\lambda = \frac{2}{5}$, for both cases from the definition of $\vartheta(d, e)$ it is clearly that (7) holds. Also, for any $d_0 \in \mathbb{D}$ the condition(8) is satisfied.

Case 1. *If $d = e = 0, d = e = 1, d = e = 2$, the results hold immediately.*

Case 2. *If* $d = 0, e = 1$, *we have*
$h_{cvc}(\Upsilon d, \Upsilon e) = h_{cvc}(\Upsilon 0, \Upsilon 1) = h_c vc(2, 2) = 0 \precsim \frac{2}{5}(1 + i) = \lambda(h_{cvc}(0, 1)) = \lambda(h_{cvc}(d, e))$.
Case 3. *If* $d = 0, e = 2$, *we have*
$h_{cvc}(\Upsilon d, \Upsilon e) = h_{cvc}(\Upsilon 0, \Upsilon 2) = h_{cvc}(2, 2) = 0 \precsim \frac{2}{5}(4 + 4i) = \lambda(h_{cvc}(0, 2)) = \lambda(h_{cvc}(d, e))$.
Case 4. *If* $d = 1, e = 2$, *we have*
$h_{cvc}(\Upsilon d, \Upsilon e) = h_{cvc}(\Upsilon 1, \Upsilon 2) = h_{cvc}(2, 2) = 0 \precsim \frac{2}{5}(1 + i) = \lambda(h_{cvc}(1, 2)) = \lambda(h_{cvc}(d, e))$.
Then all hypothesis of Theorem (2.2) hold; then Υ *has a unique fixed point, which is* $d^* = 0$.

Further, let us give another example to put in evidence that, the complex valued controlled space is larger than the one of controlled metric space.

Example 2.2. *Let* $\mathbb{D} = \mathbb{C}$ *and* $h_{cvc} : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ *be a metric defined as follows*

$$h_{cvc}(d, e) = \begin{cases} 0, & \text{if } d = e, \\ |d - e| + |d - e|i, & \text{otherwise}. \end{cases}$$

with $d, e \in \mathbb{C}$ *defined as* $d = b^* + c^* i$ *and* $e = d^* + e^* i$, *for every* $b^*, c^*, d^*, e^* \in \mathbb{R}_+ - 0$, *with* $|d| = \sqrt{b^{*2} + c^{*2}}$ *and* $\mathbb{R}_+ - \{0\}$ *represents the set of real positive nonzero numbers.*

Let $\vartheta : \mathbb{D} \times \mathbb{D} \to [1, +\infty)$ *be a function defined as* $\vartheta = 1 + |d + e|^2$ *and let* $\Upsilon : \mathbb{D} \to \mathbb{D}$ *be a mapping defined by* $\Upsilon d = di$. *It is quite easy to check that* $(\mathbb{D}, h_{cvc})$ *is a complete complex valued controlled (c.v.c.) metric space. Next, we will verify the hypothesis of Theorem 2.1. First, we calculate the values of* $h_{cvc}(\Upsilon d, \Upsilon e)$ *and* $h_{cvc}(d, e)$.

$$\begin{aligned} h_{cvc}(\Upsilon d, \Upsilon e) &= |di - ei| + |di - ei|i \\ &= |(b^* i - c^*) - (d^* i - e^*)| + |(b^* i - c^*) - (d^* i - e^*)|i \\ &= |(b^* - d^*)i + (e^* - c^*)| + |(b^* - d^*)i + (e^* - c^*)|i \\ &= \sqrt{(b - d)^2 + (e - c)^2} + i\sqrt{(b - d)^2 + (e - c)^2}. \end{aligned}$$

$$\begin{aligned} h_{cvc}(d, e) &= |d - e| + |d - e|i \\ &= |(b^* - c^* i) - (d^* - e^* i)| + |(b^* - c^* i) - (d^* - e^* i)|i \\ &= |(b^* - d^*) + (c^* - e^*)i| + |(b^* - d^*) + (c^* - e^*)i|i \\ &= \sqrt{(b^* - d^*)^2 + (c^* - e^*)^2} + i\sqrt{(b^* - d^*)^2 + (c^* - e^*)^2}. \end{aligned}$$

Moreover,

$$\begin{aligned} \vartheta(d, e) &= 1 + |d + e|^2 = 1 + |(b^* + c^* i) + (d^* + e^* i)|^2 \\ &= 1 + |(b^* + d^*) + (c^* + e^*)i|^2 \\ &= 1 + \sqrt{(b^* + d^*)^2 + (c^* + e^*)^2}^2 > 1. \end{aligned}$$

Since $b^*, c^*, d^*, e^* \in \mathbb{R}_+ - \{0\}$ *it is obviously that* $(e^* - c^*)^2 = (c^* - e^*)^2$. *Then we get*

$$\begin{aligned} h_{cvc}(\Upsilon d, \Upsilon e) &= \sqrt{(b^* - d^*)^2 + (e^* - c^*)^2} + i\sqrt{(b^* - d^*)^2 + (q^* - c^*)^2} \\ &= \sqrt{(b^* - d^*)^2 + (c^* - e^*)^2} + i\sqrt{(b^* - d^*)^2 + (c^* - e^*)^2} \end{aligned}$$

$$\precsim [1 + \sqrt{(b^* + d^*)^2 + (c^* + e^*)^{2^2}}] \sqrt{(b^* - d^*)^2 + (c^* - e^*)^2} + i \sqrt{(b^* - d^*)^2 + (c^* - e^*)^2}$$
$$= \vartheta(d, e) h_{cvc}(d, e).$$

Then, all hypothesis of Theorem (2.1) hold; then, Υ has a unique fixed point $d = 0 + 0i$.

Corollary 2.1. *Let $(\mathbb{D}, h_{cvc})$ be a complete CVC-metric space and $\Upsilon : \mathbb{D} \to \mathbb{D}$ be a continuous mapping such that*

$$h_{cvc}(\Upsilon^n d, \Upsilon^n e) \precsim \lambda h_{cvc}(d, e), \tag{11}$$

for all $d, e \in \mathbb{D}$, where $0 < \lambda < 1$. For $d_0 \in \mathbb{D}$ we denote $d_n = \Upsilon^n d_0$. Suppose that

$$\max_{m \geq 1} \lim_{l \to \infty} \frac{\vartheta(d_{l+1}, d_{l+2})}{\vartheta(d_l, d_{l+1})} \vartheta(d_{l+1}, d_m) < \frac{1}{\lambda}, \tag{12}$$

In addition, for every $d \in \mathbb{D}$ the limits

$$\lim_{n \to \infty} \vartheta(d_n, d) \text{ and } \lim_{n \to \infty} \vartheta(d, d_n) \text{ exist and are finite.} \tag{13}$$

Then ϑ has a unique fixed point.

Proof. From Theorem 2.1 we have that Υ^n has a unique fixed point f. Since $\Upsilon^n(\Upsilon f) = \Upsilon(\Upsilon^n f) = \Upsilon f$ results Υf is a fixed point of Υ^n. Therefore, $\Upsilon f = f$ by the uniqueness of a fixed point of Υ^n. Therefore, f is also a fixed point of Υ. Since the fixed point of Υ is also a fixed point of Υ^n, then the fixed point of Υ is unique. □

3. Kannan type fixed point results on CVC-metric space

The main result of this section is a Kannan type fixed point theorem for the case of complex valued controlled metric space.

Theorem 3.1. *Let $(\mathbb{D}, h_{cvc})$ be a complete CVC-metric space and $\Upsilon : \mathbb{D} \to \mathbb{D}$ be a continuous mapping such that*

$$h_{cvc}(\Upsilon d, \Upsilon e) \precsim \gamma(h_{cvc}(d, \Upsilon d) + h_{cvc}(e, \Upsilon e)), \tag{14}$$

for all $d, e \in \mathbb{D}$, where $0 \leq \gamma < \frac{1}{2}$. For $d_0 \in \mathbb{D}$ we denote $d_n = \Upsilon^n d_0$. Suppose that

$$\max_{m \geq 1} \lim_{l \to \infty} \frac{\vartheta(d_{l+1}, d_{l+2})}{\vartheta(d_l, d_{l+1})} \vartheta(d_{l+1}, d_m) < \frac{1}{\lambda}, \text{ where } \lambda = \frac{\gamma}{1 - \gamma}. \tag{15}$$

In addition, assume for every $d \in \mathbb{D}$ that the limits

$$\lim_{n \to \infty} \vartheta(d_n, d) \text{ and } \lim_{n \to \infty} \vartheta(d, d_n) \text{ exist and are finite.} \tag{16}$$

Then Υ has a unique fixed point.

Proof. For $d_0 \in \mathbb{D}$, consider a sequence $\{d_n = \Upsilon^n d_0\}$. If there exists $d_0 \in N$ for which $d_{n_0+1} = d_{n_0}$, then $\Upsilon d_{n_0} = d_{n_0}$. So everything will be trivially satisfied. Now we assume that $d_{n+1} \neq d_n$ for all $n \in N$. By using (2) we get

$$h_{cvc}(d_n, d_{n+1}) = h_{cvc}(\Upsilon d_{n-1}, \Upsilon d_n)$$

$$\begin{aligned}&\precsim \gamma(h_{cvc}(d_{n-1},\Upsilon d_{n-1})+h_{cvc}(d_n,\Upsilon d_n))\\&=\gamma(h_{cvc}(d_{n-1},d_n)+h_{cvc}(d_n,d_{n+1})),\end{aligned}$$

which implies $h_{cvc}(d_n,d_{n+1})\precsim(\frac{\gamma}{1-\gamma})h_{cvc}(d_{n-1},d_n)=\lambda h_{cvc}(d_{n-1},d_n)$. In the same way

$$\begin{aligned}h_{cvc}(d_{n-1},d_n)&=h_{cvc}(\Upsilon d_{n-2},\Upsilon d_{n-1})\\&\precsim \gamma(h_{cvc}(d_{n-2},\Upsilon d_{n-2})+h_{cvc}(d_{n-1},\Upsilon d_{n-1}))\\&=\gamma(h_{cvc}(d_{n-2},d_{n-1})+h_{cvc}(d_{n-1},d_n)),\end{aligned}$$

which implies $h_{cvc}(d_{n-1},d_n)\precsim(\frac{\gamma}{1-\gamma})h_{cvc}(d_{n-2},d_{n-1})=\lambda h_{cvc}(d_{n-2},d_{n-1})$.
Continuing in the same way, we have

$$h_{cvc}(d_n,d_{n+1})\precsim\lambda h_{cvc}(d_{n-1},d_n)\precsim\lambda^2h_{cvc}(d_{n-2},d_{n-1})\precsim...\precsim\lambda^nh_{cvc}(d_0,d_1).$$

Thus, $h_{cvc}(d_n,d_{n+1})\precsim\lambda^nh_{cvc}(d_0,d_1)$ for all $n\geq 0$. For all $n<m$, where n and m are natural numbers, we have

$$\begin{aligned}h_{cvc}(d_n,d_m)&\precsim\vartheta(d_n,d_{n+1})h_{cvc}(d_n,d_{n+1})+\vartheta(d_{n+1},d_m)h_{cvc}(d_{n+1},d_m)\\
&\precsim\vartheta(d_n,d_{n+1})h_{cvc}(d_n,d_{n+1})+\vartheta(d_{n+1},d_m)\vartheta(d_{n+1},d_{n+2})h_{cvc}(d_{n+1},d_{n+2})\\
&+\vartheta(d_{n+1},d_m)\vartheta(d_{n+2},d_m)h_{cvc}(d_{n+2},d_m)\\
&\precsim\vartheta(d_n,d_{n+1})h_{cvc}(d_n,d_{n+1})+\vartheta(d_{n+1},d_m)\vartheta(d_{n+1},d_{n+2})h_{cvc}(d_{n+1},d_{n+2})\\
&+\vartheta(d_{n+1},d_m)\vartheta(d_{n+2},d_m\vartheta(d_{n+2},d_{n+3)}h_{cvc}(d_{n+2},d_{n+3})\\
&+\vartheta(d_{n+1},d_m)\vartheta(d_{n+2},d_m\vartheta(d_{n+3},d_m)h_{cvc}(d_{n+3},d_m)\\
&\precsim...\precsim\vartheta(d_n,d_{n+1})h_{cvc}(d_n,d_{n+1})+\sum_{l=n+1}^{m-2}\prod_{j=n+1}^{l}\vartheta(d_j,d_m)\vartheta(d_l,d_{l+1})h_{cvc}(d_l,d_{l+1})\\
&+\prod_{k=n+1}^{m-1}\vartheta(d_k,d_m)h_{cvc}(d_{m-1},d_m)\\
&\precsim\vartheta(d_n,d_{n+1})\lambda^nh_{cvc}(d_0,d_1)+\sum_{l=n+1}^{m-2}\prod_{j=n+1}^{l}\vartheta(d_j,d_m)\vartheta(d_l,d_{l+1})\lambda^lh_{cvc}(d_0,d_1)\\
&+\prod_{k=n+1}^{m-1}\vartheta(d_k,d_m)\lambda^{m-1}h_{cvc}(d_0,d_1)\\
&\precsim\vartheta(d_n,d_{n+1})\lambda^nh_{cvc}(d_0,d_1)+\sum_{l=n+1}^{m-2}\prod_{j=n+1}^{l}\vartheta(d_j,d_m)\vartheta(d_l,d_{l+1})\lambda^lh_{cvc}(d_0,d_1)\\
&+\prod_{k=n+1}^{m-1}\vartheta(d_k,d_m)\lambda^{m-1}\vartheta(d_{m-1},d_m)h_{cvc}(d_0,d_1)\\
&=\vartheta(d_n,d_{n+1}\lambda^nh_{cvc}(d_0,d_1)+\sum_{l=n+1}^{m-1}\prod_{j=n+1}^{l}\vartheta(d_j,d_m)\vartheta(d_l,d_{l+1})\lambda^lh_{cvc}(d_0,d_1)\\
&\precsim\vartheta(d_n,d_{n+1})\lambda^nh_{cvc}(d_0,d_1)+\sum_{l=n+1}^{m-1}\prod_{j=0}^{l}\alpha(d_j,d_m)\vartheta(d_l,d_{l+1})\lambda^lh_{cvc}(d_0,d_1).\end{aligned}$$

Further, using $\vartheta(d,e) \geq 1$. Let

$$S_u = \sum_{l=0}^{u} \prod_{j=0}^{l} \vartheta(d_j, d_m)\vartheta(d_l, d_{l+1})\lambda^l.$$

Hence we have

$$h_{cvc}(d_n, d_m) \precsim h_{cvc}(d_0, d_1)[\lambda^n \vartheta(d_n, d_{n+1}) + (S_{m-1}, S_n)], \tag{17}$$

Condition (3), by aplying the ratio test, ensures that $\underset{m,n\to\infty}{lim} S_n$ exists and so the sequence $\{S_n\}$ is a Cauchy sequence. If we apply the limit of the inequality (17) for $m, n \to \infty$, we can deduce that

$$\underset{m,n\to\infty}{lim} h_{cvc}(d_n, d_m) = 0.$$

Then $\{d_n\}$ is a Cauchy sequence in the complete CVC-metric space $(\mathbb{D}, h_{cvc})$. This means the sequence$\{d_n\}$ converges to some $d^* \in \mathbb{D}$. We now show that d^* is a fix point of Υ.

By the continuity of Υ we obtain

$$d^* = \lim_{n\to\infty} d_{n+1} = \lim_{n\to\infty} \Upsilon d_n = \Upsilon(\lim_{n\to\infty} d_n) = \Upsilon d^*.$$

For the uniqueness assume that Υ has two fix points $d^*, e^* \in Fix\Upsilon$. Thus,

$$h_{cvc}(d^*, e^*) = h_{cvc}(\Upsilon d^*, \Upsilon e^*) \precsim \gamma[h_{cvc}(d^*, \Upsilon d^*) + h_{cvc}(e^*, \Upsilon e^*)] \precsim \gamma[h_{cvc}(d^*, d^*) + h_{cvc}(e^*, e^*)] = 0.$$

Since $h_{cvc}(d^*, e^*) = 0$ then $d^* = e^*$. Hence Υ has a unique fixed point. □

If we consider Υ a mapping not necessary continuous, we get the following general fixed point result.

Theorem 3.2. *Let $(\mathbb{D}, h_{cvc})$ be a complete CVC-metric space and $\Upsilon : \mathbb{D} \to \mathbb{D}$ be a mapping such that*

$$h_{cvc}(\Upsilon d, \Upsilon e) \precsim \gamma(h_{cvc}(d, \Upsilon d) + h_{cvc}(e, \Upsilon e)) \tag{18}$$

for all $d, e \in \mathbb{D}$ where $0 \leq \gamma < \frac{1}{2}$. For $d_0 \in \mathbb{D}$ we denote $d_n = \Upsilon^n d_0$. Suppose that

$$\underset{m\geq 1}{max} \underset{l\to\infty}{lim} \frac{\vartheta(d_{l+1}, d_{l+2})}{\vartheta(d_l, d_{l+1})} \vartheta(d_{l+1}, d_m) < \frac{1}{\lambda}, \text{ where } \lambda = \frac{\gamma}{1-\gamma}. \tag{19}$$

In addition, assume for every $d \in \mathbb{D}$ that the limits

$$\underset{n\to\infty}{lim} \vartheta(d_n, d) \text{ and } \underset{n\to\infty}{lim} \vartheta(d, d_n) \text{ exist and are finite.} \tag{20}$$

Then Υ has a unique fixed point.

Proof. Using similar steps as followed in the proof of Theorem 3.1 and taking into account Lemma 1.4, we find a Cauchy sequence $\{d_n\}$ in the complete CVC-metric space $(\mathbb{D}, h_{cvc})$. Then the sequence $\{d_n\}$ converges to a $d^* \in \mathbb{D}$. We must prove that d^* is a fixed point of Υ.

Using the triangular inequality we get

$$h_{cvc}(d^*, d_{n+1}) \precsim \vartheta(d^*, d_n)h_{cvc}(d^*, d_n) + \vartheta(d_n, d_{n+1})h_{cvc}(d_n, d_{n+1})$$

Using (3), (4) and (31) we deduce

$$\lim_{n\to\infty} d_{cvc}(d^*, d_{n+1}) = 0.$$

Using again the triangular inequality and (2) we obtain

$$\begin{aligned} h_{cvc}(d^*, \Upsilon d^*) &\precsim \vartheta(d^*, d_{n+1}) h_{cvc}(d^*, d_{n+1}) + \vartheta(d_{n+1}, \Upsilon d^*) h_{cvc}(d_{n+1}, \Upsilon d^*) \\ &\precsim \vartheta(d^*, d_{n+1}) h_{cvc}(d^*, d_{n+1}) + \vartheta(d_{n+1}, \Upsilon d^*)[\gamma(h_{cvc}(d_n, d_{n+1}) + h_{cvc}(d^*, \Upsilon d^*))]. \end{aligned}$$

Taking the limit as $n \to \infty$ and by (4) and (32) we deduce that $h_{cvc}(d^*, \Upsilon d^*) = 0$. Remark that, in view of Lemma 1.3, the sequence $\{d_n\}$ converges uniquely at the point $d^* \in \mathbb{D}$. □

Further we will present some special cases concerning this new type of results. We will show that there exists a close connections between CVC-metric space and other different types of spaces.

Corollary 3.1. *Let $(\mathbb{D}, h_e)$ be a complete complex valued extended b-metric space and $\Upsilon : \mathbb{D} \to \mathbb{D}$ be a continuous mapping such that*

$$h_e(\Upsilon d, \Upsilon e) \precsim \gamma(h_e(d, \Upsilon d) + h_e(e, \Upsilon e)) \tag{21}$$

for all $d, e \in \mathbb{D}$, where $0 \le \gamma < \frac{1}{2}$. For $d_0 \in \mathbb{D}$ we denote $d_n = \Upsilon^n d_0$. Suppose that

$$\max_{m\geq 1} \lim_{l\to\infty} \frac{\vartheta(d_{l+1}, d_{l+2})}{\vartheta(d_l, d_{l+1})} \vartheta(d_{l+1}, d_m) < \frac{1}{\lambda}, \text{ where } \lambda = \frac{\gamma}{1-\gamma}. \tag{22}$$

In addition to this, suppose that for every $d \in \mathbb{D}$, we have

$$\lim_{n\to\infty} \vartheta(d_n, d) \text{ and } \lim_{n\to\infty} \vartheta(d, d_n) \text{ exists and are finite.} \tag{23}$$

Then Υ has a unique fixed point.

Proof. If we choose $\vartheta(d, e) = \vartheta(e, f)$ in Theorem 3.1 we get the conclusion. □

Corollary 3.2. *Let $(\mathbb{D}, h_b)$ be a complete complex valued b-metric space and $\Upsilon : \mathbb{D} \to \mathbb{D}$ be a continuous mapping such that*

$$h_b(\Upsilon d, \Upsilon e) \precsim \gamma(h_b(d, \Upsilon d) + h_b(e, \Upsilon e)) \tag{24}$$

for all $d, e \in \mathbb{D}$, where $0 \le \gamma < \frac{1}{2}$. For $d_0 \in \mathbb{D}$ we denote $d_n = \Upsilon^n d_0$. Suppose that

$$\max_{m\geq 1} \lim_{l\to\infty} \frac{\vartheta(d_{l+1}, d_{l+2})}{\vartheta(d_l, d_{l+1})} \vartheta(d_{l+1}, d_m) < \frac{1}{\lambda}, \text{ where } \lambda = \frac{\gamma}{1-\gamma}. \tag{25}$$

In addition to this, suppose that for every $p \in \mathbb{D}$, we have

$$\lim_{n\to\infty} \vartheta(d_n, d) \text{ and } \lim_{n\to\infty} \vartheta(d, d_n) \text{ exist and are finite.} \tag{26}$$

Then Υ has a unique fixed point.

Proof. Taking $\vartheta(d, e) = \vartheta(e, f) = t > 1$ in Theorem 3.1 we get the conclusion. □

Corollary 3.3. *Let* $(\mathbb{D}, h_{cm})$ *be a complete complex valued metric space and* $\Upsilon : \mathbb{D} \to \mathbb{D}$ *be a continuous mapping such that*

$$h_{cm}(\Upsilon d, \Upsilon e) \precsim \gamma(h_{cm}(d, \Upsilon d) + h_{cm}(e, \Upsilon e)) \tag{27}$$

for all $d, e \in \mathbb{D}$, *where* $0 \leq \gamma < \frac{1}{2}$. *For* $d_0 \in \mathbb{D}$ *we denote* $d_n = \Upsilon^n d_0$. *Suppose that*

$$\max_{m\geq 1}\lim_{l\to\infty} \frac{\vartheta(d_{l+1}, d_{l+2})}{\vartheta(d_l, d_{l+1})}\vartheta(d_{l+1}, d_m) < \frac{1}{\lambda}, \text{ where } \lambda = \frac{\gamma}{1-\gamma}. \tag{28}$$

In addition, assume that for every $d \in \mathbb{D}$, *we have*

$$\lim_{n\to\infty} \vartheta(d_n, d) \text{ and } \lim_{n\to\infty} \vartheta(d, d_n) \text{ exist and are finite.} \tag{29}$$

Then Υ *has a unique fixed point.*

Proof. Replacing $\vartheta(d, e) = \vartheta(e, f) = 1$ in Theorem 3.1 we get the conclusion. □

Let us give the illustrative example as follows.

Example 3.1. *Let us consider* $\mathbb{D} = \{0, 1, 2\}$ *and let* $h_{cvc} : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ *be a symmetric metric given as follows*

$$h_{cvc}(d, d) = 0 \text{ for each } d \in \mathbb{D}$$

and

$$h_{cvc}(0, 1) = 1 + i, h_{cvc}(0, 2) = 4 + 4i, h_{cvc}(1, 2) = 1 + i.$$

Also, let $\vartheta : \mathbb{D} \times \mathbb{D} \to [1, \infty)$ *be a symmetric function and*

$$\vartheta(0, 0) = 5, \vartheta(0, 1) = 3, \vartheta(0, 2) = \frac{7}{3},$$

$$\vartheta(1, 1) = \frac{7}{3}, \vartheta(1, 2) = 2, \vartheta(2, 2) = \frac{9}{5}.$$

Let us consider the self-map Υ *on* $\mathbb{D}$ *as* $\Upsilon(0) = \Upsilon(1) = \Upsilon(2) = 2$.

Choosing $\gamma = \frac{2}{5}$, *for both cases from the definition of* ϑ *it is clearly that (18) holds. Also, for any* $p_0 \in \mathbb{D}$ *the condition(19) is satisfied.*

Case 1. *If* $d = e = 0, d = e = 1, d = e = 2$, *then the results hold immediately.*

Case 2. *If* $d = 0, e = 1$, *we have*
$h_{cvc}(\Upsilon d, \Upsilon e) = h_{cvc}(\Upsilon 0, \Upsilon 1) = h_{cvc}(2, 2) = 0 \precsim \frac{10}{5}(1+i) = \frac{2}{5}(4+4i+(1+i)) = \gamma(h_{cvc}(0, 2)+h_{cvc}(1, 2)) = \gamma(h_{cvc}(d, \Upsilon d) + h_{cvc}(e, \Upsilon e))$.

Case 3. *If* $d = 0, e = 2$, *we have*
$h_{cvc}(\Upsilon d, \Upsilon e) = h_{cvc}(\Upsilon 0, \Upsilon 2) = h_{cvc}(2, 2) = 0 \precsim \frac{8}{5}(1 + i) = \frac{2}{5}(4 + 4i + 0) = \gamma(h_{cvc}(0, 2) + h_{cvc}(2, 2)) = \gamma(h_{cvc}(d, \Upsilon d) + h_{cvc}(e, \Upsilon e))$.

Case 4. *If* $d = 1, e = 2$, *we have*
$h_{cvc}(\Upsilon d, \Upsilon e) = h_{cvc}(\Upsilon 1, \Upsilon 2) = h_{cvc}(2, 2) = 0 \precsim \frac{2}{5}(1 + i) = \frac{2}{5}((1 + i) + 0) = \gamma(h_{cvc}(1, 2) + h_{cvc}(2, 2)) = \gamma(h_{cvc}(d, \Upsilon d) + h_{cvc}(e, \Upsilon e))$.

Then all hypothesis of Theorem (3.2) hold; then T *has a unique fixed point, which is* $d^* = 2$.

4. Fisher type fixed point result on CVC-metric space

In [20] D. Lateef gave some fixed point result for rational functions, Fisher type, in controlled metric space. In this section let us present a generalisation of D. Lateef result in the settings of the CVC-metric space.

Theorem 4.1. *Let $(\mathbb{D}, h_{cvc})$ be a complete CVC-metric space and $\Upsilon : \mathbb{D} \to \mathbb{D}$ be continuous mapping such that*

$$h_{cvc}(\Upsilon d, \Upsilon e) \precsim \xi h_{cvc}(d, e) + \varsigma \frac{h_{cvc}(d, \Upsilon d) h_{cvc}(e, \Upsilon e)}{1 + h_{cvc}(d, e)}, \tag{30}$$

for all $d, e \in \mathbb{D}$, where $\xi, \varsigma \in [0, 1)$ such that $\eta = \frac{\xi}{1-\varsigma} < 1$. For $d_0 \in \mathbb{D}$ we denote $d_n = \Upsilon^n d_0$. Suppose that

$$\max_{m \geq 1} \lim_{i \to \infty} \frac{\vartheta(d_{i+1}, d_{i+2})}{\vartheta(d_i, d_{i+1})} \vartheta(d_{i+1}, d_m) < \frac{1}{\eta}, \tag{31}$$

In addition, suppose that for every $p \in \mathbb{D}$ we have

$$\lim_{n \to \infty} \vartheta(d_n, d) \text{ and } \lim_{n \to \infty} \vartheta(d, d_n) \text{ exist and are finite.} \tag{32}$$

Then Υ has a unique fixed point.

Proof. For $d_0 \in \mathbb{D}$ consider the sequence $\{d_n = \Upsilon^n d_0\}$. If there exists $d_0 \in N$ for which $d_{n_0+1} = d_{n_0}$, then $\Upsilon d_{n_0} = d_{n_0}$. So everything will be trivially satisfied. Now we assume that $d_{n+1} \neq d_n$ for all $n \in N$. By using (2), we get

$$\begin{aligned}
h_{cvc}(d_n, d_{n+1}) &= h_{cvc}(\Upsilon d_{n-1}, \Upsilon d_n) \\
&\precsim \xi h_{cvc}(d_{n-1}, d_n) + \varsigma \frac{h_{cvc}(d_{n-1}, \Upsilon d_{n-1}) h_{cvc}(d_n, \Upsilon d_n)}{1 + h_{cvc}(d_{n-1}, d_n)} \\
&= \xi h_{cvc}(d_{n-1}, d_n) + \varsigma \frac{h_{cvc}(d_{n-1}, d_n) h_{cvc}(p_n, d_{n+1})}{1 + h_{cvc}(d_{n-1}, d_n)} \\
&\precsim \xi h_{cvc}(p_{n-1}, d_n) + \varsigma h_{cvc}(d_n, d_{n+1})
\end{aligned}$$

which implies

$$h_{cvc}(d_n, d_{n+1}) \precsim (\frac{\xi}{1 - \varsigma}) h_{cvc}(d_{n-1}, d_n) = \eta h_{cvc}(d_{n-1}, d_n)$$

In the same way

$$\begin{aligned}
h_{cvc}(d_{n-1}, d_n) &= h_{cvc}(\Upsilon d_{n-2}, \Upsilon d_{n-1}) \\
&\precsim \xi h_{cvc}(d_{n-2}, d_{n-1}) + \varsigma \frac{h_{cvc}(d_{n-2}, \Upsilon d_{n-2}) h_{cvc}(d_{n-1}, \Upsilon d_{n-1})}{1 + h_{cvc}(d_{n-2}, d_{n-1})} \\
&= \xi h_{cvc}(d_{n-2}, d_{n-1}) + \varsigma \frac{h_{cvc}(d_{n-2}, d_{n-1}) h_{cvc}(d_{n-1}, d_n)}{1 + h_{cvc}(d_{n-2}, d_{n-1})} \\
&\precsim \xi h_{cvc}(d_{n-2}, d_{n-1}) + \varsigma h_{cvc}(d_{n-1}, d_n)
\end{aligned}$$

which implies

$$h_{cvc}(d_{n-1}, d_n) \precsim (\frac{\xi}{1-\varsigma})h_{cvc}(d_{n-2}, d_{n-1}) = \eta(d_{n-2}, d_{n-1})$$

Continuing the same way, we have

$$h_{cvc}(d_n, d_{n+1}) \precsim \eta h_{cvc}(d_{n-1}, d_n) \precsim \eta^2 h_{cvc}(d_{n-2}, d_{n-1}) \precsim ... \precsim \eta^n h_{cvc}(d_0, d_1). \tag{33}$$

Thus, $h_{cvc}(d_n, d_{n+1}) \precsim \eta^n h_{cvc}(d_0, d_1)$ for all $n \geq 0$. For all $n < m$, where n and m are natural numbers, we have

$$\begin{aligned}
h_{cvc}(d_n, d_m) &\precsim \vartheta(d_n, d_{n+1})h_{cvc}(d_n, d_{n+1}) + \vartheta(d_{n+1}, d_m)h_{cvc}(d_{n+1}, d_m) \\
&\precsim \vartheta(d_n, d_{n+1})h_{cvc}(d_n, d_{n+1}) + \vartheta(d_{n+1}, d_m)\vartheta(d_{n+1}, d_{n+2})h_{cvc}(d_{n+1}, d_{n+2}) \\
&+ \vartheta(d_{n+1}, d_m)\vartheta(d_{n+2}, d_m)h_{cvc}(d_{n+2}, d_m) \\
&\precsim \vartheta(d_n, d_{n+1})h_{cvc}(d_n, d_{n+1}) + \vartheta(d_{n+1}, d_m)\vartheta(d_{n+1}, d_{n+2})h_{cvc}(d_{n+1}, d_{n+2}) \\
&+ \vartheta(d_{n+1}, d_m)\vartheta(d_{n+2}, d_m\vartheta(d_{n+2}, d_{n+3)}h_{cvc}(d_{n+2}, d_{n+3}) \\
&+ \vartheta(d_{n+1}, d_m)\vartheta(d_{n+2}, d_m\vartheta(d_{n+3}, d_m)h_{cvc}(d_{n+3}, d_m) \\
&\precsim ... \precsim \vartheta(d_n, d_{n+1})h_{cvc}(d_n, d_{n+1}) + \sum_{l=n+1}^{m-2}(\prod_{j=n+1}^{l}\vartheta(d_j, d_m))\vartheta(d_l, d_{l+1})h_{cvc}(d_l, d_{l+1}) \\
&+ \prod_{k=n+1}^{m-1}\vartheta(d_k, d_m)h_{cvc}(d_{m-1}, d_m) \\
&\precsim \vartheta(d_n, d_{n+1})\lambda^n h_{cvc}(d_0, d_1) + \sum_{l=n+1}^{m-2}(\prod_{j=n+1}^{l}\vartheta(d_j, d_m))\vartheta(d_l, d_{l+1})\lambda^l h_{cvc}(d_0, d_1) \\
&+ \prod_{k=n+1}^{m-1}\vartheta(d_k, d_m)\lambda^{m-1}h_{cvc}(d_0, d_1) \\
&\precsim \vartheta(d_n, d_{n+1})\lambda^n h_{cvc}(d_0, d_1) + \sum_{l=n+1}^{m-2}(\prod_{j=n+1}^{l}\vartheta(d_j, d_m))\vartheta(d_l, d_{l+1})\lambda^l h_{cvc}(d_0, d_1) \\
&+ (\prod_{k=n+1}^{m-1}\vartheta(d_k, d_m))\lambda^{m-1}\vartheta(d_{m-1}, d_m)h_{cvc}(d_0, d_1) \\
&= \vartheta(d_n, d_{n+1}\lambda^n h_{cvc}(d_0, d_1) + \sum_{l=n+1}^{m-1}(\prod_{j=n+1}^{l}\vartheta(d_j, d_m))\vartheta(d_l, d_{l+1})\lambda^l h_{cvc}(d_0, d_1) \\
&\precsim \vartheta(d_n, d_{n+1})\lambda^n h_{cvc}(d_0, d_1) + \sum_{l=n+1}^{m-1}(\prod_{j=0}^{l}\alpha(d_j, d_m))\vartheta(d_l, d_{l+1})\lambda^l h_{cvc}(d_0, d_1).
\end{aligned}$$

Further, using $\vartheta(d, e) \geq 1$. Let

$$S_u = \sum_{l=0}^{u}(\prod_{j=0}^{l}\vartheta(d_j, d_m))\vartheta(d_l, d_{l+1})\lambda^l.$$

Hence we have

$$h_{cvc}(d_n, d_m) \precsim h_{cvc}(d_0, d_1)[\eta^n \vartheta(d_n, d_{n+1}) + (S_{m-1}, S_n)]. \tag{34}$$

Condition (3), by using the ratio test, ensure that $\underset{m,n\to\infty}{lim} S_n$ exists and hence, the real sequence $\{S_n\}$ is a Cauchy sequence. Finally, if we apply the limit in the inequality (34) as $m, n \to \infty$, we deduce that

$$\underset{m,n\to\infty}{lim} h_{cvc}(d_n, d_m) = 0,$$

that is, $\{d_n\}$ is a Cauchy sequence in the complete CVC-metric space $(\mathbb{D}, h_{cvc})$. Then, the sequence $\{d_n\}$ converges to an $d^* \in \mathbb{D}$. We shall show that d^* is a fixed point of Υ.

By the continuity of Υ we obtain

$$d^* = \lim_{n\to\infty} d_{n+1} = \lim_{n\to\infty} \Upsilon d_n = \Upsilon(\lim_{n\to\infty} d_n) = \Upsilon d^*.$$

For uniqueness let us assume that $d^*, e^* \in Fix\Upsilon$ are two fixed points of Υ. Then we get

$$\begin{aligned} h_{cvc}(\Upsilon d^*, \Upsilon e^*) &\precsim \xi h_{cvc}(d^*, e^*) + \varsigma \frac{h_{cvc}(d^*, \Upsilon e^*) h_{cvc}(d^*, \Upsilon e^*)}{1 + h_{cvc}(d^*, e^*)} \\ &\precsim \xi h_{cvc}(d^*, e^*) + \varsigma \frac{h_{cvc}(d^*, d^*) h_{cvc}(e^*, e^*)}{1 + h_{cvc}(d^*, e^*)} \\ &\precsim \xi h_{cvc}(d^*, e^*). \end{aligned}$$

which holds $h_{cvc}(d^*, e^*) = 0$; then $d^* = e^*$. Hence Υ has a unique fixed point. □

If we consider a mapping Υ not continuous we get a more general result for rational type mapping as follows.

Theorem 4.2. *Let $(\mathbb{D}, h_{cvc})$ be a complete CVC-metric space and $\Upsilon : \mathbb{D} \to \mathbb{D}$ be a mapping such that*

$$h_{cvc}(\Upsilon d, \Upsilon e) \precsim \xi h_{cvc}(d, e) + \varsigma \frac{h_{cvc}(d, \Upsilon d) h_{cvc}(e, \Upsilon e)}{1 + h_{cvc}(d, e)}, \tag{35}$$

for all $d, e \in \mathbb{D}$, where $\xi, \varsigma \in [0, 1)$ such that $\eta = \frac{\xi}{1-\varsigma} < 1$. For $d_0 \in \mathbb{D}$ we denote $d_n = \Upsilon^n d_0$. Suppose that

$$\underset{m\geq 1}{max} \underset{i\to\infty}{lim} \frac{\vartheta(d_{i+1}, d_{i+2})}{\vartheta(d_i, d_{i+1})} \vartheta(d_{i+1}, d_m) < \frac{1}{\eta}, \tag{36}$$

In addition, assume that for every $d \in \mathbb{D}$ we have

$$\underset{n\to\infty}{lim} \vartheta(d_n, d) \text{ and } \underset{n\to\infty}{lim} \vartheta(d, d_n) \text{ exist and are finite.} \tag{37}$$

Then Υ has a unique fixed point.

Proof. Using similar steps as in the proof of Theorem 4.1 and taking into account Lemma 1.4, we get a Cauchy sequence $\{d_n\}$ which converges to an $d^* \in \mathbb{D}$. We must show that d^* is a fixed point of Υ. Using triangle inequality, we get

$$h_{cvc}(d^*, d_{n+1}) \precsim \vartheta(d^*, d_n) h_{cvc}(d^*, d_n) + \vartheta(d_n, d_{n+1}) h_{cvc}(d_n, d_{n+1}).$$

Using (3), (4) and (36), we deduce that

$$\lim_{n\to\infty} h_{cvc}(d^*, d_{n+1}) = 0.$$

Using again the triangular inequality and (2),

$$\begin{aligned} h_{cvc}(d^*, \Upsilon d^*) &\precsim \vartheta(d^*, d_{n+1})h_{cvc}(d^*, d_{n+1}) + \vartheta(d_{n+1}, \Upsilon^*)h_{cvc}(d_{n+1}, \Upsilon d^*) \\ &\precsim \vartheta(d^*, d_{n+1})h_{cvc}(d^*, d_{n+1}) + \vartheta(d_{n+1}, \Upsilon d^*)[\xi h_{cvc}(d_n, d^*) + \varsigma \frac{h_{cvc}(d_n, \Upsilon d_n)h_{cvc}(d^*, \Upsilon d^*)}{1 + h_{cvc}(d_n, d^*)}] \\ &= \vartheta(d^*, d_{n+1})h_{cvc}(d^*, d_{n+1}) + \vartheta(d_{n+1}, \Upsilon d^*)[\xi h_{cvc}(d_n, d^*) + \varsigma \frac{h_{cvc}(d_n, d_{n+1})h_{cvc}(d^*, \Upsilon d^*)}{1 + h_{cvc}(d_n, d^*)}]. \end{aligned}$$

For $n \to \infty$ and using (4) and (37), we deduce that $h_{cvc}(d^*, \Upsilon d^*) = 0$. Remark that, in view of Lemma 1.3, the sequence $\{d_n\}$ converges uniquely at the point $d^* \in \mathbb{D}$. □

We illustrate this theorem giving the following example.

Example 4.1. *Let us consider* $\mathbb{D} = \{0, 1, 2\}$ *and let* $h_{cvc} : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ *be a symmetric metric given as*

$$h_{cvc}(d, d) = 0 \textit{ for each } d \in \mathbb{D}$$

and

$$h_{cvc}(0, 1) = 1 + i, h_{cvc}(0, 2) = 4 + 4i, h_{cvc}(1, 2) = 1 + i.$$

Also, let $\vartheta : \mathbb{D} \times \mathbb{D} \to [1, \infty)$ *be symmetric and be defined as*

$$\vartheta(0, 0) = 5, \vartheta(0, 1) = 3, \vartheta(0, 2) = \frac{7}{3},$$

$$\vartheta(1, 1) = \frac{7}{3}, \vartheta(1, 2) = 2, \vartheta(2, 2) = \frac{9}{5}.$$

As $(\mathbb{D}, h_{cvc})$ *is complex valued controlled metric space, let us consider the self-map* Υ *on* $\mathbb{D}$ *as* $\Upsilon(0) = \Upsilon(1) = \Upsilon(2) = 1$.

If we choose $\xi = \varsigma = \frac{1}{5}$ *we have*

Case 1. *If* $d = e = 0, d = e = 1, d = e = 2$ *we have* $h_{cvc}(\Upsilon d, \Upsilon e) = 0$.

Case 2. *If* $d = 0, e = 1$, *we have* $h_{cvc}(\Upsilon d, \Upsilon e) = 0 \precsim \xi h_{cvc}(d, e) + \varsigma \frac{h_{cvc}(d,\Upsilon d)h_{cvc}(e,\Upsilon e)}{1+h_{cvc}(d,e)}$.

Case 3. *If* $d = 0, e = 2$,*we have* $h_{cvc}(\Upsilon d, \Upsilon e) = 0 \precsim \xi h_{cvc}(d, e) + \varsigma \frac{h_{cvc}(d,\Upsilon d)h_{cvc}(e,\Upsilon e)}{1+h_{cvc}(d,e)}$.

Case 4. *If* $d = 1, e = 2$, *we have* $h_{cvc}(\Upsilon d, \Upsilon e) = 0 \precsim \xi h_{cvc}(d, e) + \varsigma \frac{h_{cvc}(d,\Upsilon d)h_{cvc}(e,\Upsilon e)}{1+h_{cvc}(d,e)}$.

Clearly (35) holds. For any $d_0 \in \mathbb{D}$ *(36) is satisfied.*

Then all hypothesis of Theorem (4.2) hold. Results that Υ *has a unique fixed point, which is* $d^* = 1$.

Corollary 4.1. *Let* $(\mathbb{D}, h_e)$ *be a complete complex valued extended b-metric space and* $\Upsilon : \mathbb{D} \to \mathbb{D}$ *be continuous mapping such that*

$$h_e(\mathbb{D}d, \mathbb{D}e) \precsim \xi h_e(d, e) + \varsigma \frac{h_e(d, \mathbb{D}d)h_e(e, \mathbb{D}e)}{1 + h_e(d, e)},$$

for all $d, e \in \mathbb{D}$, *where* $\xi, \varsigma \in [0, 1)$ *such that* $\eta = \frac{\xi}{1-\varsigma} < 1$. *For* $d_0 \in \mathbb{D}$ *we denote* $d_n = \mathbb{D}^n d_0$. *Suppose that*

$$\max_{m\geq 1}\lim_{i\to\infty}\frac{\vartheta(d_{i+1}, d_{i+2})}{\vartheta(d_i, d_{i+1})}\vartheta(d_{i+1}, d_m) < \frac{1}{\lambda}$$

In addition to this, suppose that for every $d \in \mathbb{D}$, *we have*

$$\lim_{n\to\infty}\vartheta(d_n, d) \text{ and } \lim_{n\to\infty}\vartheta(d, d_n) \text{ exists and finite.}$$

Then Υ *has a unique fixed point.*

Proof. If we choose $\vartheta(d, e) = \vartheta(e, f)$ in Theorem 4.1 we get the conclusion. □

Corollary 4.2. *Let* $(\mathbb{D}, h_b)$ *be a complete complex valued b-metric space and* $\Upsilon : \mathbb{D} \to \mathbb{D}$ *be continuous mapping such that*

$$h_b(\mathbb{D}d, \mathbb{D}e) \precsim \xi h_b(d, e) + \varsigma\frac{h_b(d, \mathbb{D}d)h_b(e, \mathbb{D}e)}{1 + h_b(d, e)},$$

for all $d, e \in \mathbb{D}$, *where* $\xi, \varsigma \in [0, 1)$ *such that* $\eta = \frac{\xi}{1-\varsigma} < 1$. *For* $d_0 \in \mathbb{D}$ *we denote* $d_n = \mathbb{D}^n d_0$. *Suppose that*

$$\max_{m\geq 1}\lim_{i\to\infty}\frac{\vartheta(d_{i+1}, d_{i+2})}{\vartheta(d_i, d_{i+1})}\vartheta(d_{i+1}, d_m) < \frac{1}{\lambda}$$

In addition to this, suppose that for every $p \in \mathbb{D}$, *we have*

$$\lim_{n\to\infty}\vartheta(d_n, d) \text{ and } \lim_{n\to\infty}\vartheta(d, d_n) \text{ exists and finite.}$$

Then Υ *has a unique fixed point.*

Proof. If we choose $\vartheta(d, e) = \vartheta(e, f) = t > 1$ in Theorem 4.1 we get the conclusion. □

Corollary 4.3. *Let* $(\mathbb{D}, h_{cm})$ *be a complete complex valued metric space and* $\Upsilon : \mathbb{D} \to \mathbb{D}$ *be continuous mapping such that*

$$h_{cm}(\mathbb{D}d, \mathbb{D}e) \precsim \xi h_{cm}(d, e) + \varsigma\frac{h_{cm}(d, \mathbb{D}d)h_{cm}(e, \mathbb{D}e)}{1 + h_{cm}(d, e)},$$

for all $d, e \in \mathbb{D}$, *where* $\xi, \varsigma \in [0, 1)$ *such that* $\eta = \frac{\xi}{1-\varsigma} < 1$. *For* $d_0 \in \mathbb{D}$ *we denote* $d_n = \mathbb{D}^n d_0$. *Suppose that*

$$\max_{m\geq 1}\lim_{i\to\infty}\frac{\vartheta(d_{i+1}, d_{i+2})}{\vartheta(d_i, d_{i+1})}\vartheta(d_{i+1}, d_m) < \frac{1}{\lambda}$$

In addition to this , suppose that for every $d \in \mathbb{D}$, *we have*

$$\lim_{n\to\infty}\vartheta(d_n, d) \text{ and } \lim_{n\to\infty}\vartheta(d, d_n) \text{ exists and finite.}$$

Then Υ *has a unique fixed point.*

Proof. If we choose $\vartheta(d, e) = \vartheta(e, f) = 1$ in Theorem 4.1 we get the conclusion. □

5. Application to an integral type equation

During this section we suppose the following type of integral equation.

$$p(u) = x(u) + \int_0^u R(u,v)g(v,p(v))dv, \quad u \in [0,1], \; p(u) \in \mathbb{D}, \tag{38}$$

where $g(u,p(u)) : [0,1] \times \mathbb{R} \to \mathbb{R}, x(u) : [0,1] \to \mathbb{R}$ be two bounded and continuous functions and $R : [0,1] \times [0,1] \to [0,\infty)$ be a function such that $R(u,\cdot) \in L^1([0,1])$ for all $u \in [0,1]$.
In order to show the existence of solution for integral equation (38) we use Theorem 4.1. In this situation, let we present the result as follows.

Theorem 5.1. *Let $\mathbb{D} = C([0,1],\mathbb{R})$ is the set of all continuous and real-valued functions which are defined on $[0,1]$. Also let $\Upsilon : \mathbb{D} \to \mathbb{D}$ be an operator of the form:*

$$\mathbb{D}p(u) = x(u) + \int_0^u R(u,v)g(v,p(v))dv, \quad u \in [0,1], \; p(u) \in \mathbb{D}. \tag{39}$$

Suppose the following conditions hold:

(i) *the functions $g(u,p(u)) : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $x(u) : [0,1] \to \mathbb{R}$ are continuous;*

(ii) *$R : [0,1] \times [0,1] \to [0,\infty)$ be a function with $R(u,\cdot) \in L^1([0,1])$ $\forall$ $u \in [0,1]$ we have:*

$$\| \int_0^u R(u,v)dv \| < 1.$$

(iii) *$\mid g(u,p(u)) - g(u,q(u)) \mid \leq \frac{1}{\omega \cdot e^{i\omega u}} \mid p(u) - q(u) \mid$, for all $p,q \in \mathbb{D}$ and $\omega \in (1, \frac{1}{\lambda}]$ with $\lambda \in (0,1)$.*

Then the equation (38) has a unique solution.

Proof. Let $\mathbb{D} = C([0,1],\mathbb{R})$ and $h_{cvc} : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ a complex valued metric such that,

$$h_{cvc}(p,q) = \|p\|_\infty = \sup_{u\in[0,1]} \mid p(u) \mid e^{-i\omega u}, \text{ with } \omega \in (1, \frac{1}{\lambda}], \lambda \in (0,1) \text{ and } i = \sqrt{-1} \in \mathbb{C}.$$

Let $\vartheta_u : \mathbb{D} \times \mathbb{D} \to [1,\infty)$ defined as follows, for every the same $\omega \in (1, \frac{1}{\lambda}]$ and $\lambda \in (0,1)$.

$$\vartheta_u(p,q) = \begin{cases} 1, & \text{if } p,q \in [0,1], \\ \max\{p(u), q(u)\} + \omega, & \text{otherwise.} \end{cases}$$

It is easy to conclude that $(\mathbb{D}, h_{cvc})$ is a complete CVC-metric space. Then the problem (38) can be again resumed to find the element $p^* \in \mathbb{D}$ which one is a fixed point for the operator Υ.
Now we have the following estimation

$$\mid \Upsilon p(u) - \Upsilon q(u) \mid \leq \mid \int_0^u [R(u,v)g(v,p(v)) - R(u,v)g(v,q(v))]dv \mid$$

$$
\begin{aligned}
&\leq \int_0^u | R(u,v)[g(v,p(v)) - g(v,q(v))] | \, dv \\
&\leq \left(\int_0^u R(u,v) dv \right) \int_0^u | [g(v,p(v)) - g(v,q(v))] | \, dv \\
&\leq \frac{1}{\tau \cdot e^{i\omega u}} \int_0^u | p(v) - q(v) | \, dv \left(\int_0^u R(u,v) dv \right) \\
&= \frac{e^{i\omega u}}{\tau \cdot e^{i\omega u} e^{-i\omega u}} \int_0^u | p(v) - q(v) | \, e^{-i\omega t} dv \left(\int_0^u R(u,v) e^{-i\omega u} dv \right).
\end{aligned}
$$

Taking the supremum in the previous inequality we obtain

$$
[\sup_{u \in [0,1]} | \Upsilon p(u) - \Upsilon q(u) | \, e^{-i\omega u}] \leq \frac{1}{\omega} [\sup_{u \in [0,1]} | p(u) - q(u) | \, e^{-i\omega u}] \cdot \left(\int_0^u \sup_{u \in [0,1]} R(u,v) e^{-i\omega u} dv \right).
$$

Using the hypothesis (ii) we have

$$
h_{cvc}(\Upsilon p, \Upsilon q) = \|\Upsilon p - \Upsilon q\|_\infty \leq \frac{1}{\omega} \|p - q\|_\infty = \frac{1}{\omega} h_{cvc}(p,q).
$$

It is easy to check that, for both cases of the expression of $\vartheta(p,q)$, when $p, q \in [0,1]$ and otherwise, the conditions (3) and (4) are true. Then, for $0 < \delta = \frac{1}{\omega} < 1$, all the hypothesis of Theorem 2.1 holds. In this conditions we get that equation (38) has a unique solution. □

6. Conclusions

In this article the concept of complex valued controlled metric type space is introduced. We study the connection between this new type of space and any other similar spaces with complex values, as complex valued b-metric space, or extended complex valued space, proving by some illustrative examples that our new type of space is larger than the others. The fixed point theory is used in our paper by given some fixed point theorems for Banach, Kannan and Fisher contractions type. Moreover, an application in integral equation type with complex values, is given to sustain our results.

Open questions

(1) A first open problem involve the concept of C^*-algebra valued metric spaces. This notion was presented in 2014 by Ma et al. (see [22]), by interchanging the range set $\mathbb{R}$ by unital C^*-algebra, giving more generalized class than that of metric spaces. Many other generalizations of such a structure were given in related literature, and an interesting point would be to consider the case of controlled C^*-algebra valued spaces and to obtain similar fixed point results in this space.

(2) We may also write the interesting results about the concept of cyclic contraction in view to obtain some results for best proximity points in [10]. Similar results can be obtain for the case of CVC-metric space.

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Conflict of interest

The authors declare no conflicts of interest.

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