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Research article

Error bounds for linear complementarity problems of SDD_1 matrices and SDD_1-B matrices

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Abstract: An upper bound of the infinity norm for the inverse of SDD_1 matrix is presented. We apply the new bound to linear complementarity problems (LCPs) and obtain an alternative error bound for LCPs of SDD_1 matrices and SDD_1-B matrices. In addition, a new lower bound for the smallest singular value is also given. Numerical examples show the validity of the results.

Keywords: error bounds; linear complementarity problems; infinity norm; SDD_1 matrices; SDD_1-B matrices; singular values
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1. Introduction

The linear complementarity problem, denoted by LCP(A, q), is to find a vector $x \in R^n$ such that

$$Ax + q \geq 0, \qquad (Ax + q)^T x = 0, \qquad x \geq 0, \tag{1.1}$$

where $A = (a_{ij}) \in R^{n\times n}$ and $q \in R^n$. The $LCP(A, q)$ often arises form the various scientific areas of computing, economics and enginerring, such as quadratic programs, optimal stopping, Nash equilibrium point of a bimatrix games, the contact problems and the free boundary problem for journal bearing, etc. for details, see [1–3].

The $LCP(A, q)$ has a unique solution for any $q \in R^n$ if and only if A is a P-matrix, see [2]. Some basic definitions for the special matrix are given below: a matrix $A = (a_{ij}) \in R^{n\times n}$ is called a Z-matrix, if $a_{ij} \leq 0$ for any $i \neq j$; a P-matrix, if all its principal minors are positive; an M-matrix, if $A^{-1} \geq 0$ and A is a Z-matrix; an H-matrix, if its comparison matrix $\tilde{A} = (\tilde{a}_{ij})$ is an M-matrix, where

$$\tilde{a}_{ij} = \begin{cases} |a_{ij}|, & if \quad i = j, \\ -|a_{ij}|, & if \quad i \neq j. \end{cases}$$

In [4], Chen et al. gave the following error bound of the $LCP(A, q)$ when A is a P-matrix:

$$\|x - x^*\|_\infty \le \max_{d\in[0,1]^n} \|(I - D + DA)^{-1}\|_\infty \|r(x)\|_\infty, \quad \forall x \in R^n, \tag{1.2}$$

where x^* is the solution of the $LCP(A, q)$, $r(x) = \min\{x, Ax + q\}$, $D = diag(d_i)$ with $0 \le d_i \le 1$, and the min operator $r(x)$ denotes the componentwise minimum of two vectors. When real H-matrix A with positive diagonal entries is a subclass of P-matrices, computation of error bounds for the $LCP(A, q)$ becomes simpler (see formula (2.4) in [4]). In recent years, numerous scholars are keen to study special H-matrices, such as S-QN matrices [5], S-Nekrasov matrices [6], S-$SDDS$ matrices [7] and DZT matrices [8]. The corresponding error bound of $LCP(A, q)$ for S-QN matrices is presented by Li et al. in [5]. And new error bounds of the $LCP(A, q)$ for S-Nekrasov matrices are achieved by Gao et al. in [9] and Dai et al. in [10], which depends only on the entries of the matrix A.

When the class of involved matrices is subclass of P-matrices that are not H-matrices, error bounds of $LCP(A, q)$ also need to be studied, such as, B_π^R-matrices [11], B-Nekrasov matrices [12] and [13], weakly chained diagonally dominant B-matrices [14], CKV-type B-matrices [15]. Specially, some error bounds for linear complementarity problems with B-type matrices have been derived by the papers [16–19].

In this paper, we provide an upper bound for infinity norm of the inverse of SDD_1 matrix and apply the new bound to estimate the error for linear complementarity problems of SDD_1 matrices and SDD_1-B matrices. Meanwhile, a lower bound for the smallest singular value is also provided. Numerical examples are given to show the effectiveness of the obtained results.

2. Preliminaries

In this section, some definitions and lemmas are given. For any positive integer n, N denotes the set $\{1, 2, ..., n\}$. Let $A = (a_{ij})$ be an $n \times n$ matrix.

Definition 1. [20] *Given a matrix $A = (a_{ij}) \in R^{n\times n}$, let the ith deleted absolute row sum be*

$$r_i(A) = \sum_{j=1, j\ne i}^{n} |a_{ij}|, \quad \forall i \in N.$$

Definition 2. [21] *$A = (a_{ij}) \in R^{n\times n}$ is called a strictly diagonally dominant (SDD) matrix if, for all $i \in N$,*

$$|a_{ii}| > r_i(A).$$

Definition 3. [22] *Matrix $A = (a_{ij}) \in R^{n\times n}$ is called an SDD_1 matrix if, for each $i \in N_1(A)$,*

$$|a_{ii}| > P_i(A),$$

where

$$P_i(A) = \sum_{j\in N_1(A)\backslash\{i\}} |a_{ij}| + \sum_{j\notin N_1(A)\cup\{i\}} \frac{r_j(A)}{|a_{jj}|} |a_{ij}|,$$

$$N_1(A) = \{i \in N \,|\, |a_{ii}| \le r_i(A)\}.$$

Definition 4. [22] *A real square matrix A is called an SDD_1-B matrix (or B_1-matrix) if it can be written as*

$$A = B^+ + C, \qquad C = \begin{pmatrix} r_1^+ & \cdots & r_1^+ \\ \vdots & \ddots & \vdots \\ r_n^+ & \cdots & r_n^+ \end{pmatrix}, \tag{2.1}$$

with $r_i^+ = \max\left\{0, a_{ij} | j \neq i\right\}$ and B^+ is an SDD_1 matrix with positive diagonal entries.

Definition 5. [11] *Matrix $A = (a_{ij}) \in R^{n\times n}$ with positive row sums is a B-matrix if all of its off-diagonal elements are bounded above by the corresponding row means, i.e., for all $i \in N$,*

$$\sum_{k=1}^{n} a_{ik} > 0 \quad \text{and} \quad \frac{1}{n}\left(\sum_{k=1}^{n} a_{ik}\right) > a_{ij}, \qquad \forall j \neq i.$$

Now, we will introduce some useful lemmas.

Lemma 1. [22] *If matrix $A = (a_{ij}) \in R^{n\times n}$ is an SDD_1 matrix by rows, then is also an H-matrix, if, in addition, A has positive diagonal entries, then det $A > 0$.*

Lemma 2. [22] *If $A = (a_{ij}) \in R^{n\times n}$ is an SDD_1-B matrix, then A is also a P-matrix.*

Remark 1. *From Definitions 2–5, Lemmas 1 and 2, we have the following relationships:*

$$SDD \text{ matrices} \subseteq SDD_1 \text{ matrices} \subseteq H\text{-matrices},$$

$$B\text{-matrices} \subseteq SDD_1\text{-}B \text{ matrices} \subseteq P\text{-matrices}.$$

Remark 2. *In Example 1 we shall see that an SDD_1 matrix is not necessarily an SDD matrix. In Example 7 we shall also see that an SDD_1-B matrix is not necessarily a B-matrix.*

Lemma 3. [23] *Let $\gamma > 0$ and $\eta \geq 0$, then for any $x \in [0, 1]$,*

$$\frac{1}{1 - x + x\gamma} \leq \frac{1}{\min\{\gamma, 1\}}, \quad \frac{\eta x}{1 - x + x\gamma} \leq \frac{\eta}{\gamma}.$$

Lemma 4. *Matrix $A = (a_{ij}) \in R^{n\times n}$ is an SDD_1 matrix with poistive diagonal entries, then $A_D = (\bar{a}_{ij}) = I - D + DA$ is also an SDD_1 matrix, where $D = diag(d_i)$ with $0 \leq d_i \leq 1$.*

Proof. Since $A_D = (\bar{a}_{ij}) = I - D + DA$, then

$$\bar{a}_{ij} = \begin{cases} 1 - d_i + d_i a_{ii}, & if \quad i = j, \\ d_i a_{ij}, & if \quad i \neq j. \end{cases}$$

By Lemma 3, for any $i \in N_1(A)$, we have

$$\begin{aligned} P_i(A_D) &= \sum_{j\in N_1(A)/\{i\}} \left|d_i a_{ij}\right| + \sum_{j\notin N_1(A)\cup\{i\}} \frac{d_j r_j(A)}{1 - d_j + a_{jj} d_j} \left|d_i a_{ij}\right| \\ &= d_i \left(\sum_{j\in N_1(A)/\{i\}} \left|a_{ij}\right| + \sum_{j\notin N_1(A)\cup\{i\}} \frac{d_j r_j(A)}{1 - d_j + a_{jj} d_j} \left|a_{ij}\right| \right) \end{aligned}$$

$$\begin{aligned} &\leq d_i\left(\sum_{j\in N_1(A)/\{i\}}\left|a_{ij}\right| + \sum_{j\notin N_1(A)\cup\{i\}}\frac{r_j(A)}{\left|a_{jj}\right|}\left|a_{ij}\right|\right)\\ &= d_iP_i(A), \end{aligned}$$

and

$$d_iP_i(A) < 1 - d_i + d_i|a_{ii}| = |\bar{a}_{ii}|.$$

That is, for each $i \in N_1(A_D) \subseteq N_1(A)$, $|\bar{a}_{ii}| > P_i(A_D)$, so $A_D = I - D + DA$ is an SDD_1 matrix. □

Lemma 5. [7] *Let $A = (a_{ij}) \in R^{n\times n}$, $n \geq 2$, be a nonsingular matrix. Then there is a vector $x \in C^n$ such that $\|x\|_\infty = 1$ and*

$$\left\|A^{-1}\right\|_\infty^{-1} = \|Ax\|_\infty = \max_{1\leq i\leq n}|(Ax)_i|.$$

3. An upper bound for $\left\|A^{-1}\right\|_\infty$ of SDD_1 matrices

In this section, a new error bound of $\left\|A^{-1}\right\|_\infty$ is presented when A is an SDD_1 matrix.

Theorem 1. *Let $A = (a_{ij}) \in R^{n\times n}$, $n \geq 2$, be an SDD_1 matrix with poistive diagonal entries. Then*

$$\left\|A^{-1}\right\|_\infty \leq \max\left\{\frac{1}{\min\limits_{i\in N_1(A)}\{|a_{ii}| - P_i(A)\}}, \frac{1}{\min\limits_{i\in N_2(A)}\left\{|a_{ii}| - \sum\limits_{j\neq i}\left|a_{ij}\right|\right\}}\right\},$$

where $N_1(A) = \{i \in N|\, |a_{ii}| \leq r_i(A)\}$, $N_2(A) = \{i \in N|\, |a_{ii}| > r_i(A)\}$.

Proof. In accordance with Lemma 5, choose a vector $x = (x_i) \in C^n$ such that

$$\|x\|_\infty = 1 \quad and \quad |(Ax)_i| \leq \left\|A^{-1}\right\|_\infty^{-1} \quad for\ all\ i = 1, \ldots, n. \tag{3.1}$$

Assume that

$$|x_k| = 1 = \|x\|_\infty \quad and \quad |x_j| = \frac{r_j(A)}{|a_{jj}|} \quad for\ all\ j \in N_2(A). \tag{3.2}$$

When $k \in N_1(A)$, we have

$$(Ax)_k = \sum_{j\in N} a_{kj}x_j = a_{kk}x_k + \sum_{j\in N_1(A)/\{k\}} a_{kj}x_j + \sum_{j\notin N_1(A)\cup\{k\}} a_{kj}x_j.$$

implying, in view of (3.1) and (3.2), that

$$\begin{aligned} |a_{kk}| &= |a_{kk}||x_k| = \left|(Ax)_k - \sum_{j\in N_1(A)/\{k\}} a_{kj}x_j - \sum_{j\notin N_1(A)\cup\{k\}} a_{kj}x_j\right|\\ &\leq |(Ax)_k| + \sum_{j\in N_1(A)/\{k\}} |a_{kj}||x_j| + \sum_{j\notin N_1(A)\cup\{k\}} |a_{kj}||x_j|\\ &\leq |(Ax)_k| + \sum_{j\in N_1(A)/\{k\}} \left|a_{kj}\right| + \sum_{j\notin N_1(A)\cup\{k\}} \frac{r_j(A)}{\left|a_{jj}\right|}\left|a_{kj}\right| = |(Ax)_k| + P_k(A). \end{aligned}$$

Then, by (3.1), we have

$$\left\|A^{-1}\right\|_{\infty} \leq \frac{1}{|a_{kk}| - P_k(A)} \leq \frac{1}{\min\limits_{i\in N_1(A)} \{|a_{ii}| - P_i\,(A)\}}. \tag{3.3}$$

When $k \in N_2\,(A)$, by (3.1), we get

$$0 < \min_{i\in N_2(A)} \left(|a_{ii}| - \sum_{j\neq i} \left|a_{ij}\right|\right) \leq |a_{kk}| - \sum_{j\neq k} \left|a_{kj}\right|,$$

and

$$\begin{aligned} 0 &< \min_{i\in N_2(A)} \left(|a_{ii}| - \sum_{j\neq i} \left|a_{ij}\right|\right) |x_k| \leq |a_{kk}x_k| - \sum_{j\neq k} \left|a_{kj}\right| \left|x_j\right| \\ &\leq |a_{kk}x_k| - \left|\sum_{j\neq k} a_{kj}x_j\right| \leq \left|\sum_{k\in N_2(A), j\in N} a_{kj}x_j\right| \\ &\leq \max_{i\in N_2(A)} \left|\sum_{j\in N} a_{ij}x_j\right| \leq \left\|A^{-1}\right\|_{\infty}^{-1}. \end{aligned}$$

Then

$$\left\|A^{-1}\right\|_{\infty} \leq \frac{1}{\min\limits_{i\in N_2(A)} \left\{|a_{ii}| - \sum\limits_{j\neq i} \left|a_{ij}\right|\right\}}. \tag{3.4}$$

From (3.3) and (3.4), the conclusion follows. □

We illustrate our results by the following tow examples.

Example 1. *Consider the bound of* $\left\|A_1^{-1}\right\|_{\infty}$ *for an* SDD_1 *matrix* A_1*, where*

$$A_1 = \begin{pmatrix} 9.5 & 5 & 5 & -6 \\ -1 & 20 & 1 & 2 \\ -2 & 1 & 5 & 0 \\ -2 & 1 & 7 & 12 \end{pmatrix}.$$

Then $N_1(A_1) = \{1\}$*,* $N_2(A_1) = \{2, 3, 4\}$*. By Theorem* 1*, we have*

$$|a_{11}| - P_1\,(A_1) = |a_{11}| - \sum_{j\in N_2, j\neq 1} \frac{r_j\,(A_1)}{|a_{jj}|} |a_{1j}| = 0.5,$$

and

$$|a_{22}| - r_2\,(A_1) = 16, \quad |a_{33}| - r_3\,(A_1) = 2, \quad |a_{44}| - r_4\,(A_1) = 3.$$

Thus $\left\|A_1^{-1}\right\|_{\infty} \leq \max\left\{\frac{1}{0.5}, \frac{1}{2}\right\} = 2.$

Example 2. *Consider the bound of* $\left\|A_2^{-1}\right\|_\infty$ *for an* SDD_1 *matrix* A_2*, where*

$$A_2=\begin{pmatrix} 12 & 1 & 2.5 & 1.5 & 0.5 & 2 & 1 & -1.2 & 1.3 & 1.6\\ -2 & 20 & 1 & 3 & 4 & 5 & 2.1 & 1 & 1.2 & 1\\ 1 & 2 & 14 & 0.5 & 1 & 1 & 1.2 & 2 & 0 & 1\\ 3 & -2.1 & 2 & 25 & 6 & 1 & 3 & 2 & -5 & 0.5\\ 2 & 3 & 1 & 4 & 20 & 1 & 1.5 & 0 & 0 & 1\\ 1.5 & 0 & 1 & 4 & 3 & 23 & 1 & 5 & 7 & 1\\ 5 & 1 & 6 & 2 & 3 & 2 & 25 & 1 & 5 & 1\\ 1 & 1.2 & 3 & 2 & 5 & 6 & 4 & 26 & 1 & 2\\ 3 & 5 & 7 & 6 & 4 & 3 & 1 & 2 & 35 & 1\\ 2 & 1 & 1 & 2 & 3 & 1 & 1 & 2 & 1 & 15 \end{pmatrix}.$$

Then $N_1(A_2)=\{1,2,6,7\}$*,* $N_2(A_2)=\{3,4,5,8,9,10\}$*. By Theorem* 1*, we get*

$$|a_{11}|-P_1(A_2)=|a_{11}|-\left(\sum_{j\in N_1,j\neq 1}|a_{1j}|+\sum_{j\in N_2,j\neq 1}\frac{r_j(A_2)}{|a_{jj}|}|a_{1j}|\right)=0.6094,$$

$$|a_{22}|-P_2(A_2)=|a_{22}|-\left(\sum_{j\in N_1,j\neq 2}|a_{2j}|+\sum_{j\in N_2,j\neq 2}\frac{r_j(A_2)}{|a_{jj}|}|a_{2j}|\right)=1.5556,$$

$$|a_{66}|-P_6(A_2)=|a_{66}|-\left(\sum_{j\in N_1,j\neq 6}|a_{6j}|+\sum_{j\in N_2,j\neq 6}\frac{r_j(A_2)}{|a_{jj}|}|a_{6j}|\right)=1.6667,$$

$$|a_{77}|-P_7(A_2)=|a_{77}|-\left(\sum_{j\in N_1,j\neq 7}|a_{7j}|+\sum_{j\in N_2,j\neq 7}\frac{r_j(A_2)}{|a_{jj}|}|a_{7j}|\right)=2.376,$$

and

$$|a_{33}|-r_3(A_2)=4.3,\qquad |a_{44}|-r_4(A_2)=0.4,\qquad |a_{55}|-r_5(A_2)=6.5,$$

$$|a_{88}|-r_8(A_2)=1.2,\qquad |a_{99}|-r_9(A_2)=3,\qquad |a_{1010}|-r_{10}(A_2)=1.$$

Hence $\left\|A_2^{-1}\right\|_\infty\leq\max\left\{\frac{1}{0.6094},\frac{1}{0.4}\right\}=2.5.$

Example 3. *Consider the bound for* $\left\|A_3^{-1}\right\|_\infty$ *of an* SDD_1 *matrix* A_3*, where*

$$A_3=\begin{pmatrix} 2 & -1 & & & & & & & \\ -1.2 & 2 & -1 & & & & & & \\ & -1.2 & 2 & 0 & & & & & \\ & & -1.2 & 2 & 0 & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & -1.2 & 2 & 0 & & \\ & & & & & -1.2 & 2 & 0 & \\ & & & & & & -1 & 2 & -1.2\\ & & & & & & & -1 & 2 \end{pmatrix}.$$

Then $N_1(A_3) = \{2, n-1\}$, $N_2(A_3) = \{1, 3, \ldots, n-2, n\}$. *By Theorem 1, for* $i \in N_1(A_3)$, *we have*

$$|a_{22}| - P_2(A_3) = |a_{22}| - \left(\sum_{j \in N_1(A_3), j \neq 2} |a_{2j}| + \sum_{j \in N_2(A_3), j \neq 2} \frac{r_j(A_3)}{|a_{jj}|} |a_{2j}| \right) = 0.4,$$

$$|a_{n-1,n-1}| - P_{n-1}(A_3) = |a_{n-1,n-1}| - \left(\sum_{j \in N_1(A_3), j \neq n-1} |a_{n-1,j}| + \sum_{j \in N_2(A_3), j \neq n-1} \frac{r_j(A_3)}{|a_{jj}|} |a_{n-1,j}| \right) = 0.4.$$

For $i \in N_2(A_3)$, *we get*

$$|a_{ii}| - \sum_{j \neq i} |a_{ij}| = 1, \quad i = 1, n,$$

$$|a_{ii}| - \sum_{j \neq i} |a_{ij}| = 0.8, \quad i = 3, 4, \ldots, n-2.$$

Thus $\left\| A_3^{-1} \right\|_\infty \leq \max \left\{ \frac{1}{0.4}, \frac{1}{0.8} \right\} = 2.5000.$

4. Error bound for the linear complementarity problems involving SDD_1 matrices

In this section, new error bound of $LCP(A, q)$ is presented when A is an SDD_1 matrix.

Corollary 1. *Let* $A = (a_{ij}) \in R^{n \times n}$, $n \geq 2$, *be an* SDD_1 *matrix with poistive diagonal entries, and* $A_D = I - D + DA$, *where* $D = diag(d_i)$ *with* $0 \leq d_i \leq 1$. *Then*

$$\max_{d \in [0,1]^n} \left\| A_D^{-1} \right\|_\infty \leq \max \left\{ \frac{1}{\min\limits_{i \in N_1(A)} \{|a_{ii}| - P_i(A), 1\}}, \frac{1}{\min\limits_{i \in N_2(A)} \left\{ |a_{ii}| - \sum\limits_{j \neq i} |a_{ij}|, 1 \right\}} \right\},$$

where $N_1(A) = \{i \in N | \, |a_{ii}| \leq r_i(A)\}$, $N_2(A) = \{i \in N | \, |a_{ii}| > r_i(A)\}$.

Proof. Since A is an SDD_1 matrix, by Lemma 4, $A_D = I - D + DA$ is also an SDD_1 matrix, and

$$A_D = \begin{cases} 1 - d_i + d_i a_{ii}, & i = j, \\ d_i a_{ij}, & i \neq j. \end{cases}$$

By Theorem 1, we obtain

$$\left\| A_D^{-1} \right\|_\infty \leq \max \left\{ \frac{1}{\min\limits_{i \in N_1(A_D)} (|\bar{a}_{ii}| - P_i(A_D))}, \frac{1}{\min\limits_{i \in N_2(A_D)} \left(|\bar{a}_{ii}| - \sum\limits_{i \neq j} |\bar{a}_{ij}| \right)} \right\}.$$

By Lemma 3, for $i \in N_1(A)$, it holds that

$$\frac{1}{|\bar{a}_{ii}| - P_i(A_D)} = \frac{1}{1 - d_i + d_i a_{ii} - \left(d_i \sum\limits_{j \in N_1(A)/\{i\}} |a_{ij}| + \sum\limits_{j \notin N_1(A) \cup \{i\}} \frac{d_j r_j(A)}{1 - d_j + d_j a_{jj}} |d_i a_{ij}| \right)}$$

$$\leq \frac{1}{1-d_i+d_ia_{ii}-d_i\left(\sum\limits_{j\in N_1(A)/\{i\}}\left|a_{ij}\right|+\sum\limits_{j\notin N_1(A)\cup\{i\}}\frac{r_j(A)}{a_{jj}}\left|a_{ij}\right|\right)}$$
$$= \frac{1}{1-d_i+d_i\left(|a_{ii}|-P_i\left(A\right)\right)} \leq \frac{1}{\min\left\{|a_{ii}|-P_i\left(A\right),1\right\}}.$$

For $i \in N_2(A)$, by Lemma 3, we get

$$\frac{1}{|\bar{a}_{ii}|-\sum\limits_{i\neq j}\left|\bar{a}_{ij}\right|} = \frac{1}{1-d_i+d_ia_{ii}-\sum\limits_{j\neq i}\left|d_ia_{ij}\right|} \leq \frac{1}{\min\left\{|a_{ii}|-\sum\limits_{j\neq i}\left|a_{ij}\right|,1\right\}}.$$

The proof is complete. □

Example 4. *Consider the bound for* $\max\limits_{d\in[0,1]^n}\left\|(I-D+DA_4)^{-1}\right\|_\infty$ *of an* SDD_1 *matrix* A_4, *where*

$$A_4 = \begin{pmatrix} 16 & -8 & 4.1 & 8 \\ 0 & 8 & 3.1 & 1 \\ -8 & 8 & 20 & 8 \\ 1 & 1.2 & 5 & 8 \end{pmatrix}.$$

Then $N_1(A_4) = \{1,3\}$, $N_2(A_4) = \{2,4\}$. *By Corollary* 1, *we have*

$$a_{11}-P_1\left(A_4\right) = a_{11}-\left(|a_{13}|+\frac{r_2\left(A_4\right)}{|a_{22}|}|a_{12}|+\frac{r_4\left(A_4\right)}{|a_{44}|}|a_{14}|\right) = 0.6,$$

$$a_{33}-P_3\left(A_4\right) = a_{33}-\left(|a_{31}|+\frac{r_2\left(A_4\right)}{|a_{22}|}|a_{32}|+\frac{r_4\left(A_4\right)}{|a_{44}|}|a_{34}|\right) = 0.7,$$

and

$$|a_{22}|-\sum_{j\neq 2}\left|a_{2j}\right| = 3.9, \qquad |a_{44}|-\sum_{j\neq 4}\left|a_{4j}\right| = 0.8.$$

So $\max\limits_{d\in[0,1]^4}\left\|(I-D+DA_4)^{-1}\right\|_\infty \leq \max\left\{\frac{1}{0.6},\frac{1}{0.8}\right\} = 1.25.$

Example 5. *Consider the bound for* $\max\limits_{d\in[0,1]^n}\left\|(I-D+DA_5)^{-1}\right\|_\infty$ *of an* SDD_1 *matrix* A_5, *where*

$$A_5 = \begin{pmatrix} 15 & 2 & 1 & 0.5 & 1 & 1 & 1.2 & 2 & 0 & 1 \\ 3 & 25 & -2.1 & 2 & 6 & 1 & 3 & 2 & -5 & 0.5 \\ 2.5 & 1 & 12 & 1.5 & 0.5 & 2 & 1 & -1.2 & 1.3 & 1.6 \\ 2 & 3 & 1 & 20 & 4 & 1 & 1.5 & 0 & 0 & 1 \\ -2 & 1 & 3 & 4 & 20 & 1 & 2.1 & 1 & 1.2 & 5 \\ 1.5 & 0 & 1 & 4 & 3 & 23 & 1 & 5 & 7 & 1 \\ 5 & 1 & 6 & 2 & 3 & 2 & 25 & 1 & 5 & 1 \\ 1 & 1.2 & 3 & 2 & 5 & 6 & 4 & 26 & 1 & 2 \\ 3 & 5 & 7 & 6 & 4 & 3 & 1 & 2 & 35 & 1 \\ 2 & 1 & 1 & 2 & 3 & 1 & 1 & 2 & 2 & 17 \end{pmatrix}.$$

Then $N_1(A_5) = \{3,5,6,7\}$*,* $N_2(A_5) = \{1,2,4,8,9,10\}$*. By Corollary* 1*, we have*

$$a_{33} - P_3(A_5) = 1.1233, \qquad a_{55} - P_5(A_5) = 3.5446,$$

$$a_{66} - P_6(A_5) = 2.1814, \qquad a_{77} - P_7(A_5) = 1.9997,$$

and

$$a_{11} - r_1(A_5) = 5.3, \qquad a_{22} - r_2(A_5) = 0.4, \qquad a_{44} - r_4(A_5) = 6.5,$$

$$a_{88} - r_8(A_5) = 0.8, \qquad a_{99} - r_9(A_5) = 3, \qquad a_{10,10} - r_{10,10}(A_5) = 2.$$

So $\max_{d\in[0,1]^{10}} \left\|(I - D + DA_5)^{-1}\right\|_\infty \le 2.5.$

Example 6. *Consider the bound for* $\max_{d\in[0,1]^n} \left\|(I - D + DA_6)^{-1}\right\|_\infty$ *of an* SDD_1 *matrix* A_6*, where*

$$A_6 = \begin{pmatrix} 2 & -1 & & & & & & & \\ -1 & 2 & -1 & & & & & & \\ & -1 & 2.1 & -1 & & & & & \\ & & -1 & 2.1 & -1 & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & -1 & 2.1 & -1 & & \\ & & & & & -1 & 2.1 & -1 & \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -1 & 2 \end{pmatrix}.$$

Then $N_1(A_6) = \{2, n-1\}$*,* $N_2(A_6) = \{1,3,\dots,n-2,n\}$*. By Corollary* 1*, for* $i \in N_1(A_6)$*, we have*

$$|a_{22}| - P_2(A_6) = |a_{22}| - \left(\sum_{j\in N_1(A_6),\neq 2} |a_{2j}| + \sum_{j\in N_2(A_6)} \frac{r_j(A_6)}{|a_{jj}|} |a_{2j}| \right) = 0.5476,$$

$$\left|a_{n-1,n-1}\right| - P_{n-1}(A_6) = \left|a_{n-1,n-1}\right| - \left(\sum_{j\in N_1(A_6),\neq n-1} \left|a_{n-1,j}\right| + \sum_{j\in N_2(A_6)} \frac{r_j(A_6)}{|a_{jj}|} \left|a_{n-1,j}\right| \right) = 0.5476.$$

For $i \in N_2(A_6)$*, we get*

$$|a_{ii}| - \sum_{j\neq i} |a_{ij}| = 1, \quad i = 1, n,$$

$$|a_{ii}| - \sum_{j\neq i} |a_{ij}| = 0.1, \quad i = 3,\dots,n-2.$$

Thus $\max_{d\in[0,1]^n} \left\|(I - D + DA_6)^{-1}\right\|_\infty \le \max\left\{\frac{1}{0.5476}, \frac{1}{0.1}\right\} = 10.$

5. Error bound for linear complementarity problems involving SDD_1-B matrices

In this section, a new error bound of $LCP(A,q)$ is presented when A is an SDD_1-B matrix.

Corollary 2. *Let $A=(a_{ij})\in R^{n\times n}$, $n\geq 2$, be an SDD_1-B matrix with poistive diagonal entries, and let $B^+=(b_{ij})$ be the matrix as* (2.1). *Denote $A_D=I-D+DA$, where $D=diag\,(d_i)$ with $0\leq d_i\leq 1$. Then*

$$\max_{d\in[0,1]^n}\left\|(I-D+DA)^{-1}\right\|_\infty\leq(n-1)\max\left\{\frac{1}{\min\limits_{i\in N_1(B^+)}\{|b_{ii}|-P_i\left(B^+\right),1\}},\frac{1}{\min\limits_{i\in N_2(B^+)}\left\{|b_{ii}|-\sum\limits_{i\neq j}\left|b_{ij}\right|,1\right\}}\right\},$$

where

$$\begin{aligned}N_1\left(B^+\right)&=\left\{i\in N|\,|b_{ii}|\leq r_i\left(B^+\right)\right\},\\ N_2\left(B^+\right)&=\left\{i\in N|\,|b_{ii}|>r_i\left(B^+\right)\right\},\\ P_i\left(B^+\right)&=\sum_{j\in N_1(B^+)/\{i\}}\left|b_{ij}\right|+\sum_{j\notin N_1(B^+)\cup\{i\}}\frac{r_j\left(B^+\right)}{\left|b_{jj}\right|}\left|b_{ij}\right|.\end{aligned}$$

Proof. By the assumption, A is an SDD_1-B matrix and $A=B^++C$ is given as (2.1), B^+ is an SDD_1 matrix with positive diagonal entries. Thus for each diagonal matrix $D=diag\,(d_i)$ with $0\leq d_i\leq 1$ $(i\in N)$, we have

$$A_D=I-D+DA=(I-D+DB^+)+DC=B_D^++C_D,$$

where $B_D^+=\left(\bar{b}_{ij}\right)=I-D+DB^+$ and $C_D=DC$. Similar to the proof of Theorem 2.2 in [24], we obtain

$$\left\|A_D^{-1}\right\|_\infty\leq\left\|\left[I+(B_D^+)^{-1}C_D\right]^{-1}\right\|_\infty\cdot\left\|(B_D^+)^{-1}\right\|_\infty\leq(n-1)\left\|(B_D^+)^{-1}\right\|_\infty.$$

We now bound $\left\|\left(B_D^+\right)^{-1}\right\|_\infty$, notice that B^+ is an SDD_1 matrix, by Lemma 4, $B_D^+=I-D+DB^+$ is also an SDD_1 matrix. Hence, by Theorem 1, it holds that

$$\left\|(B_D^+)^{-1}\right\|_\infty\leq\left\{\frac{1}{\min\limits_{i\in N_1\left(B_D^+\right)}\left(\left|\bar{b}_{ii}\right|-P_i\left(B_D^+\right)\right)},\ \frac{1}{\min\limits_{i\in N_2\left(B_D^+\right)}\left(\left|\bar{b}_{ii}\right|-\sum\limits_{j\neq i}\left|\bar{b}_{ij}\right|\right)}\right\}.$$

By Lemma 3, for $i\in N_1(B^+)$, it holds that

$$\begin{aligned}\frac{1}{\left|\bar{b}_{ii}\right|-P_i\left(B_D^+\right)}&=\frac{1}{1-d_i+d_i|b_{ii}|-\left(d_i\sum\limits_{j\in N_1(B^+)/\{i\}}\left|b_{ij}\right|+\sum\limits_{j\notin N_1(B^+)\cup\{i\}}\frac{d_jr_j(B^+)}{1-d_j+d_jb_{jj}}\left|d_ib_{ij}\right|\right)}\\&\leq\frac{1}{1-d_i+d_i|b_{ii}|-d_i\left(\sum\limits_{j\in N_1(B^+)/\{i\}}\left|b_{ij}\right|+\sum\limits_{j\notin N_1(B^+)\cup\{i\}}\frac{r_j(B^+)}{|b_{jj}|}\left|b_{ij}\right|\right)}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{1-d_i+d_i\left(|b_{ii}|-P_i\left(B^+\right)\right)}\\
&\le \frac{1}{\min\left\{|b_{ii}|-P_i\left(B^+\right),1\right\}}.
\end{aligned}
$$

For $i \in N_2(B^+)$, we have

$$
\frac{1}{\left|\bar{b}_{ii}\right|-\sum\limits_{j\neq i}\left|\bar{b}_{ij}\right|}=\frac{1}{1-d_i+d_i|b_{ii}|-\sum\limits_{j\neq i}\left|d_ib_{ij}\right|}\le\frac{1}{\min\left\{|b_{ii}|-\sum\limits_{j\neq i}\left|b_{ij}\right|,1\right\}}.
$$

Therefore, we derive

$$
\left\|\left(B_D^+\right)^{-1}\right\|_\infty \le \left\{\frac{1}{\min\limits_{i\in N_1(B^+)}\left\{|b_{ii}|-P_i\left(B^+\right),1\right\}},\frac{1}{\min\limits_{i\in N_2(B^+)}\left\{|b_{ii}|-\sum\limits_{j\neq i}\left|b_{ij}\right|,1\right\}}\right\}.
$$

The proof is complete. □

Example 7. *Consider the bound for* $\max\limits_{d\in[0,1]^n}\left\|(I-D+DA_7)^{-1}\right\|_\infty$ *of an* SDD_1*-*B *matrix* A_7*, where*

$$
A_7=\begin{pmatrix}16&-8&-4&-8\\5&13&2&4\\-8.5&-8&20&-8\\-1&-1&-5.4&8\end{pmatrix},\quad B^+=\begin{pmatrix}16&-8&-4&-8\\0&8&-3&-1\\-8.5&-8&20&-8\\-1&-1&-5.4&8\end{pmatrix}.
$$

Then $N_1(A_7)=\{1,3\}$*,* $N_2(A_7)=\{2,4\}$*. By Corollary* 2*, we have*

$$
b_{11}-P_1\left(B^+\right)=b_{11}-\left(|b_{13}|+\frac{r_2\left(B^+\right)}{|b_{22}|}|b_{12}|+\frac{r_4\left(B^+\right)}{|b_{44}|}|b_{14}|\right)=1,
$$

$$
b_{33}-P_3\left(B^+\right)=b_{33}-\left(|b_{31}|+\frac{r_2\left(B^+\right)}{|b_{22}|}|b_{32}|+\frac{r_4\left(B^+\right)}{|b_{44}|}|b_{43}|\right)=0.5,
$$

and

$$
|b_{22}|-\sum_{j\neq 2}\left|b_{2j}\right|=4,\qquad |b_{44}|-\sum_{j\neq 4}\left|b_{4j}\right|=0.6.
$$

So

$$
\max_{d\in[0,1]^4}\left\|(I-D+DA_7)^{-1}\right\|_\infty\le(4-1)\max\left\{\frac{1}{0.5},\frac{1}{0.6}\right\}=6.
$$

Example 8. *Consider the bound for* $\max\limits_{d\in[0,1]^n}\left\|(I-D+DA_8)^{-1}\right\|_\infty$ *of an* SDD_1*-*B *matrix* A_8*, where*

$$
A_8=\begin{pmatrix}8&-2&-1&-1\\4&13&4&5\\-8&-8&15&-8\\-4&-4&-2&6\end{pmatrix},\quad B^+=\begin{pmatrix}8&-2&-1&-1\\-1&8&-1&0\\-8&-8&15&-8\\-4&-4&-2&6\end{pmatrix}.
$$

Then $N_1(A_8) = \{3, 4\}$, $N_2(A_8) = \{1, 2\}$. *By Corollary 2, we get*

$$b_{33} - P_3(B^+) = b_{33} - \left(|b_{34}| + \frac{r_1(B^+)}{|b_{11}|}|b_{31}| + \frac{r_2(B^+)}{|b_{22}|}|b_{32}|\right) = 1,$$

$$b_{44} - P_4(B^+) = b_{44} - \left(|b_{43}| + \frac{r_1(B^+)}{|b_{11}|}|b_{41}| + \frac{r_2(B^+)}{|b_{22}|}|b_{42}|\right) = 1,$$

and

$$|b_{11}| - \sum_{j\neq 1}\left|b_{1j}\right| = 4, \qquad |b_{22}| - \sum_{j\neq 2}\left|b_{2j}\right| = 6.$$

So

$$\max_{d\in[0,1]^4}\left\|(I - D + DA_8)^{-1}\right\|_\infty \leq (4-1)\max\left\{\frac{1}{1}, \frac{1}{4}\right\} = 3.$$

Example 9. *Consider the bound for* $\max_{d\in[0,1]^n}\left\|(I - D + DA_9)^{-1}\right\|_\infty$ *of an* SDD_1*-B matrix* A_9*, where*

$$A_9 = \begin{pmatrix} 3.2 & 0 & 1.2 & 1.2 & 1.2 & 1.2 & \cdots & 1.2 & 1.2 \\ 0.1 & 3.2 & 0 & 1.2 & 1.2 & 1.2 & \cdots & 1.2 & 1.2 \\ 1.2 & 0.1 & 4.2 & 0 & 1.2 & 1.2 & \cdots & 1.2 & 1.2 \\ 1.2 & 1.2 & 0.1 & 4.2 & 0 & 1.2 & \cdots & 1.2 & 1.2 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 1.2 & 1.2 & \cdots & 1.2 & 0.1 & 4.2 & 0 & 1.2 & 1.2 \\ 1.2 & 1.2 & \cdots & 1.2 & 1.2 & 0.1 & 4.2 & 0 & 1.2 \\ 1.2 & 1.2 & \cdots & 1.2 & 1.2 & 1.2 & 0.1 & 3.2 & 0 \\ 1.1 & 1.1 & \cdots & 1.1 & 1.1 & 1.1 & 1.1 & 0 & 3.1 \end{pmatrix},$$

and

$$B^+ = \begin{pmatrix} 2 & -1.2 & & & & & & & \\ -1.1 & 2 & -1.2 & & & & & & \\ & -1.1 & 3 & -1.2 & & & & & \\ & & -1.1 & 3 & -1.2 & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & -1.1 & 3 & -1.2 & & \\ & & & & & -1.1 & 3 & -1.2 & \\ & & & & & & -1.1 & 2 & -1.2 \\ & & & & & & & -1.1 & 2 \end{pmatrix}.$$

Then $N_1(B^+) = \{2, n-1\}$, $N_2(B^+) = \{1, 3, \ldots, n-2, n\}$. *By Corollary 2, for* $i \in N_1(B^+)$*, we get*

$$|b_{22}| - P_2(B^+) = |b_{22}| - \left(\sum_{j\in N_1(B^+), j\neq 2}\left|a_{2j}\right| + \sum_{j\in N_2(B^+)}\frac{r_j(B^+)}{\left|b_{jj}\right|}\left|b_{2j}\right|\right) = 0.42,$$

$$\left|b_{n-1,n-1}\right| - P_{n-1}(B^+) = \left|b_{n-1,n-1}\right| - \left(\sum_{j\in N_1(B^+), \neq n-1}\left|b_{n-1,j}\right| + \sum_{j\in N_2(B^+)}\frac{r_j(B^+)}{\left|b_{jj}\right|}\left|b_{n-1,j}\right|\right) = 0.4967.$$

For $i \in N_2(B^+)$, we have

$$|b_{11}| - \sum_{j\neq 1} \left|b_{1j}\right| = 0.8, \quad |b_{nn}| - \sum_{j\neq n} \left|b_{nj}\right| = 0.9,$$

$$|a_{ii}| - \sum_{j\neq i} \left|a_{ij}\right| = 0.7, \quad i = 3, \ldots, n-2.$$

So

$$\max_{d\in[0,1]^n} \left\|(I - D + DA_9)^{-1}\right\|_\infty \leq (n-1) \max\left\{\frac{1}{0.42}, \frac{1}{0.7}\right\} = 2.381\,(n-1)\,.$$

6. A lower bound of the smallest singular value for SDD_1 matrices

The singular values of a matrix A are the eigenvalues of $(AA^*)^{\frac{1}{2}}$, and are denoted as

$$\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A) \geq 0.$$

The smallest singular value plays a special role in expressing properties of matrices, see [25,26] for details. The singular value $\sigma_n(A)$ indicates not only whether A is nonsingular, but also how far (in the spectral norm) from the singular matrices A is. In addition, it is also a key ingredient in the spectral condition number $\sigma_1(A)/\sigma_n(A)$, as $\sigma_1(A) \leq \sqrt{\|A\|_1\|A\|_\infty}$ is a simple upper of $\sigma_1(A)$ and hence if a lower bound of $\sigma_n(A)$ is obtained, then one can give an upper bound for $\sigma_1(A)/\sigma_n(A)$. The spectral condition number $\sigma_1(A)/\sigma_n(A)$ is commonly used in studying numerical calculations involving A [27]. Hence, lower bounds for the smallest singular value of matrices are of interest. Based on Theorem 1, by a similar proof to that of Theorem 7 in [28], a new lower bound for $\sigma_n(A)$ can be obtained.

Theorem 2. *Let $A = (a_{ij}) \in C^{n\times n}$, $n \geq 2$. If A and its transpose A^T are all SDD_1 matrices with poistive diagonal entries, then*

$$\sigma_n(A) \geq \sqrt{\frac{1}{Bnd_{S.C}(A) \cdot Bnd_{S.C}(A^T)}},$$

where

$$Bnd_{S.C}(A) = \max\left\{\frac{1}{\min\limits_{i\in N_1(A)} (|a_{ii}| - P_i(A))}, \frac{1}{\min\limits_{i\in N_2(A)}\left(|a_{ii}| - \sum\limits_{j\neq i}\left|a_{ij}\right|\right)}\right\},$$

and

$$N_1(A) = \{i \in N|\,|a_{ii}| \leq r_i(A)\},$$

$$N_2(A) = \{i \in N|\,|a_{ii}| > r_i(A)\},$$

$$P_i(A) = \sum_{j\in N_1(A)/\{i\}} \left|a_{ij}\right| + \sum_{j\notin N_1(A)\cup\{i\}} \frac{r_j(A)}{a_{jj}} \left|a_{ij}\right|.$$

Proof. Since A^T is an SDD_1 matrix, it follows that

$$\left\|A^{-1}\right\|_1 = \left\|\left(A^{-1}\right)^T\right\|_\infty = \left\|\left(A^T\right)^{-1}\right\|_\infty \leq Bnd_{S.C}\left(A^T\right).$$

By the well-known inequality (see[15]):

$$\left\|A^{-1}\right\|_2^2 \leq \left\|A^{-1}\right\|_1 \left\|(A)^{-1}\right\|_\infty,$$

it holds that

$$\left\|A^{-1}\right\|_2^{-1} \geq \sqrt{\frac{1}{Bnd_{S.C}(A) \cdot Bnd_{S.C}(A^T)}}.$$

From $\left\|A^{-1}\right\|_2^{-1} = \sigma_n(A)$, the conclusion follows. □

Example 10. *Consider the SDD_1 matrix,*

$$A_{10} = (a_{ij}) = \begin{pmatrix} 8 & 2 & 1 & 1 \\ 1 & 8 & 1 & 0 \\ 8 & 8 & 15 & 4 \\ 4 & 4 & 2 & 6 \end{pmatrix}, \quad A_{10}^T = (\alpha_{ij}) = \begin{pmatrix} 8 & 1 & 8 & 4 \\ 2 & 8 & 8 & 4 \\ 1 & 1 & 15 & 2 \\ 1 & 0 & 4 & 6 \end{pmatrix}.$$

Obviously, $N_1(A_{10}) = \{3,4\}$, $N_2(A_{10}) = \{1,2\}$, $N_1\left(A_{10}^T\right) = \{1,2\}$ and $N_2\left(A_{10}^T\right) = \{3,4\}$. By Theorem 2, we have

$$|a_{11}| - \sum_{j\neq 1} |a_{1j}| = 4, \quad |a_{22}| - \sum_{j\neq 2} |a_{2j}| = 6,$$

$$a_{33} - P_3(A_{10}) = a_{33} - \left(|a_{34}| + \frac{r_1(A_{10})}{|a_{11}|}|a_{31}| + \frac{r_2(A_{10})}{|b_{22}|}|a_{32}|\right) = 5,$$

$$a_{44} - P_4(A_{10}) = a_{44} - \left(|a_{43}| + \frac{r_1(A_{10})}{|a_{11}|}|a_{41}| + \frac{r_2(A_{10})}{|b_{22}|}|a_{42}|\right) = 1,$$

and

$$\alpha_{11} - P_1\left(A_{10}^T\right) = \alpha_{11} - \left(|\alpha_{12}| + \frac{r_3\left(A_{10}^T\right)}{|\alpha_{33}|}|\alpha_{13}| + \frac{r_4\left(A_{10}^T\right)}{|\alpha_{44}|}|\alpha_{14}|\right) = 1.5334,$$

$$\alpha_{22} - P_2\left(A_{10}^T\right) = \alpha_{22} - \left(|\alpha_{21}| + \frac{r_3\left(A_{10}^T\right)}{|\alpha_{33}|}|\alpha_{23}| + \frac{r_4\left(A_{10}^T\right)}{|\alpha_{44}|}|\alpha_{24}|\right) = 0.5334,$$

$$|\alpha_{33}| - \sum_{j\neq 3} |\alpha_{3j}| = 11, \quad |\alpha_{44}| - \sum_{j\neq 4} |\alpha_{4j}| = 1.$$

So

$$Bnd_{S.C}(A_{10}) = 1, \quad Bnd_{S.C}\left(A_{10}^T\right) = 1.8748,$$

$$\sigma_n(A_{10}) \geq \sqrt{\frac{1}{Bnd_{S.C}(A_{10}) \cdot Bnd_{S.C}\left(A_{10}^T\right)}} = 0.7303.$$

7. Conclusions

Based on the fact that $I - D + DA$ is an SDD_1 matrix when A is an SDD_1 matrix, we present an upper bound for the inverse of SDD_1 matrix and give alternative bounds for $\max\limits_{d\in[0,1]^n} \left\|(I - D + DA)^{-1}\right\|_\infty$ when A is an SDD_1 matrix or an SDD_1-B matrix. A lower bound for smallest singular value is also provided. Also we illustrate the results with numerical examples. Finding computable global error bounds of the extended vertical linear complementarity problems for SDD_1 matrices and SDD_1-B matrices is an interesting problem. It is worth studying in the future.

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Conflict of interest

The authors declare that they have no competing interests.

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