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Research article

Existence results of mild solutions for nonlocal fractional delay integro-differential evolution equations via Caputo conformable fractional derivative

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Abstract: In this paper, we investigate the existence of mild solutions for nonlocal delay fractional Cauchy problem with Caputo conformable derivative in Banach spaces. We establish a representation of a mild solution using a fractional Laplace transform. The existence of such solutions is proved under certain conditions, using the Mönch fixed point theorem and a general version of Gronwall's inequality under weaker conditions in the sense of Kuratowski measure of non compactness. Applications illustrating our main abstract results and showing the applicability of the presented theory are also given.

Keywords: fractional Laplace transform; nonlocal delay fractional evolution equation; mild solution; Mönch's fixed point theorem

Mathematics Subject Classification: 34G10, 26A33

1. Introduction

Fractional calculus has shown great effectiveness in describing and modeling many phenomena in many fields such as physics, chemistry, biology, electricity, economics, etc [1–4]. Recently, many researchers focus to introduce and develop fractional operators such as Caputo-Fabrizio, Atangana-Baleanu and conformable derivatives [5–11].

A new extended fractional operator was proposed in [12] as a combination between conformable and Caputo derivatives. The novel fractional derivative has attracted attention in limited papers, see [13–16]. For the first time authors in [13] investigated the existence, uniqueness and Ulam-Hyers stability of solutions for conformable derivatives in Caputo setting with four-point integral conditions by applying suitable fixed point theorems. Baleanu et al. [14] discussed Caputo fractional conformable differential inclusion subject to four-point conditions using some analytical techniques on the α-ψ contractive mappings. In [17] authors applied this fractional derivative to describe the behavior of an electrical circuit model.

Delay differential equations and integro-differential equations in classical and fractional order have been used in modeling many situations from science and engineering. For this reason they have attracted great attention in the last two decades and have been investigated theoretically in many papers [18–27,31–35]. In a previous study [36], Kavitha et al. established the existence results of Hilfer fractional neutral evolution equation with infinite delay by utilizing the semigroup theory, fractional calculus and Mönch fixed point theorem. In [20] authors studied the existence of mild solutions for a class of non local fractional integro-differential equation in neutral type with infinite delay, using the theory of resolvent operators. Valliammal et al. in [31] established some sufficient conditions for the existence of solutions for neutral delay fractional integro-differential systems, where the authors in [37] studied controllability of nonlocal neutral impulsive differential equations with measure of noncompactness.

Inspired by the above research, we consider the following nonlocal fractional integro-differential evolution equation with finite delay:

$$\begin{cases} \mathfrak{D}_{0^+}^{\alpha,\varrho} u(t) = Au(t) + f(t, u_t, \int_0^t h(t, s, u_s)ds), & t \in [0, b]\,, \\ u(t) + (gu)(t) = \varphi(t), & t \in [-\delta, 0]\,, \end{cases} \tag{1.1}$$

where $D_{0^+}^{\alpha,\varrho}$ is the Caputo conformable fractional derivative of order $0 < \alpha < 1$ and type $0 < \varrho \leq 1$, and A is infinitesimal generator of a fractional C_0-ϱ-semigroup $\{T_\varrho(t)\}_{t\geq 0}$ of bounded operators on a Banach space X with the norm $\|.\|$. The functions $\varphi \in C([-\delta, 0], X)$, $f : [0, b) \times C([-\delta, 0], X) \times X \to X$, $h : [0, b) \times [0, +\infty) \times C([-\delta, 0], X) \to X$ and $g : C([-\delta, b], X) \to C([-\delta, 0], X)$ are given abstract functions and $u_t : [-\delta, 0] \to X$ is defined by $u_t(\theta) = u(t + \theta)$.

In this paper, we try to refine the conditions imposed in some previous works [20,31,36,38,39] and prove the existence of a mild solution under weaker and more general conditions than those mentioned in the previous works. For example, when we chose $\varrho = 1, \delta = 0$ and $h \equiv 0$, problem (1.1) improves the results of existence of a mild solution for the nonlocal Caputo fractional evolution equation discussed in [39]:

$$\begin{cases} {}^C D_{0^+}^{\alpha} u(t) = Au(t) + f(t, u(t)), & t \in (0, b] \\ u(0) = g(u)\,. \end{cases}$$

We study problem (1.1) by converting it to an equivalent integral equation using a suitable fractional Laplace transform, then we define the mild solution of (1.1) in terms of two new families of operators. Each family of operators is associated with the wright function and a suitable fractional semigroup. In the second part of this paper, we apply the Mönch fixed point theorem to establish our main results on the existence of mild solutions with the help of the general version of Gronwall's inequality under weaker conditions in the sense of Kuratowski measure of non-compactness.

We organize the contents of our paper as follows: In Section 2, we state some basic definitions, concepts and preliminary results which are used throughout this paper. The representation of mild solution of (1.1) using fractional Laplace transform is given in Section 3. In Section 4, under some sufficient conditions, we prove the existence theorem of a mild solution of (1.1), based on Mönch fixed point theorem. An application of our abstract results is given in the last section.

2. Preliminaries

In this section, we introduce fractional integral, fractional derivative, fractional semigroup and then give the definition of the fractional Laplace transform. Finally, we will give some definitions and lemmas which are used throughout this paper.
Let X be a Banach space with the norm $\|.\|$. Let $\delta, b \in \mathbb{R}^+$. Denote By $C = C([-\delta, 0], X)$ and $C([-\delta, b], X)$ the spaces of continuous X-valued functions on $[-\delta, 0]$ and $[-\delta, b]$ with the norms

$$\|u\|_C = \sup_{t\in[-\delta,0]} \|u(t)\|$$

and

$$\|u\|_\infty = \sup_{t\in[-\delta,b]} \|u(t)\|$$

respectively, and $B(X)$ be the space of all bounded linear operators from X into X with the norm $\|T\|_{B(X)} = \sup_{x\in X, \|X\|\leq 1} \|Tx\|$. Define $L_p^\varrho([0, b], X)$ to be the space of X-valued Bochner functions on $[0, b]$ with the norm $\|u\|_p^\varrho = \left(\int_0^b \|u(s)\|^p \frac{ds}{s^{1-\varrho}}\right)^{\frac{1}{p}}$, $1 \leq p < \infty$.

Definition 2.1. *[5,10] The left conformable derivative with lower point a of the function $f : [a, +\infty) \to \mathbb{R}$ of order $0 < \varrho \leq 1$ is defined by*

$$D_a^\varrho f(t) = \lim_{\epsilon\to 0} \frac{f(t + \epsilon(t - a)^{1-\varrho}) - f(t)}{\epsilon}.$$

If $D_a^\varrho f(t)$ exists on (a, b), then $D_a^\varrho f(a) = \lim_{t\to a^+} D_a^\varrho f(t)$. If f is differentiable, then

$$D_a^\varrho f(t) = (t - a)^{1-\varrho} f'(t).$$

Definition 2.2. *[12] Let $\alpha, \varrho > 0$. The left-sided Riemann-Liouville conformablde fractional integral operator of order α and type ϱ with lower limit a for a function $f : [a, +\infty) \to \mathbb{R}$ is defined by*

$$I_a^{\alpha,\varrho} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{(t - a)^\varrho - (s - a)^\varrho}{\varrho}\right)^{\alpha-1} u(s) \frac{ds}{(s - a)^{1-\varrho}}.$$

Definition 2.3. *[12] Let $\alpha > 0$, $0 < \varrho \leq 1$ and $n = [\alpha] + 1$. The left Caputo conformable fractional derivative with lower limit a of order α and type ϱ of a function $f \in C^n_{\varrho,a}([a, b], X)$ is defined by*

$$\mathfrak{D}_a^{\alpha,\varrho} f(t) = I_a^{n-\alpha,\varrho} \left({}^nD_a^\varrho f\right)(t) \tag{2.1}$$

$$= \frac{1}{n - \alpha} \int_a^t \left(\frac{(t - a)^\varrho - (s - a)^\varrho}{\varrho}\right)^{n-\alpha-1} \frac{D_a^\varrho f(s)}{(s - a)^{1-\varrho}} ds, \tag{2.2}$$

where ${}^nD_a^\varrho = \underbrace{D_a^\varrho D_a^\varrho ... D_a^\varrho}_{n-times}$, and D_a^ϱ is the left conformable derivative given in Definition 2.1.

Lemma 2.1. *[12] Let $\alpha, \beta, \gamma > 0$. Then*

$$I_a^{\alpha,\varrho} \left(I_a^{\beta,\varrho} f\right)(t) = I_a^{\alpha+\beta,\varrho} f(t),$$

and

$$\left(I_a^{\alpha,\varrho} (s - a)^{\alpha(\gamma-1)}\right)(t) = \frac{1}{\alpha^\varrho} \frac{\Gamma(\gamma)}{\Gamma(\varrho + \gamma)} (t - a)^{\alpha(\varrho+\gamma-1)}.$$

Theorem 2.2. *[12] Let* $\alpha > 0$, $0 < \varrho \leq 1$ *and* $n = [\alpha] + 1$. *Then*

(1) If $\alpha \notin N$ *and* $f \in C([a,b], X)$*, then*

$$\mathfrak{D}_a^{\alpha,\varrho}\left(I_a^{\alpha,\varrho} f\right)(t) = f(t).$$

(2) If $\alpha \in N$ *and* $f \in C([a,b], X)$*, then*

$$\mathfrak{D}_a^{\alpha,\varrho}\left(I_a^{\alpha,\varrho} f\right)(t) = f(t) - \frac{I_a^{\alpha-n+1,\varrho} f(a)}{\varrho^{n-\alpha}\Gamma(n-\alpha)}(t-a)^{\varrho(n-\alpha)}.$$

(3) If $f \in C^n_{\varrho,a}([a,b], X)$*, then*

$$I_a^{\alpha,\varrho}\left(\mathfrak{D}_a^{\alpha,\varrho} f\right)(t) = f(t) - \sum_{k=0}^{n-1} \frac{{}^k D_a^{\varrho} f(a)(t-a)^{\varrho k}}{\varrho^k k!}.$$

Definition 2.4. *([5]) Let* $a \in \mathbb{R}$, $0 < \varrho \leq 1$ *and* $f : [a, \infty) \to X$ *be an X-valued Bochner function. The fractional Laplace transform of order* ϱ *started from a is given by*

$$\mathcal{L}_a^{\varrho} f(s) = \int_a^{+\infty} e^{-s\frac{(t-a)^{\varrho}}{\varrho}} f(t) \frac{dt}{(t-a)^{1-\varrho}}. \tag{2.3}$$

Definition 2.5. *Let* $0 < \varrho \leq 1$, *and* u, v *be two X-valued Bochner functions. We define the fractional convolution of u and v of order* ϱ *by*

$$\left(u *_{\varrho} v\right)(t) = \int_0^t u\left((t^{\varrho} - \tau^{\varrho})^{\frac{1}{\varrho}}\right) v(\tau) \frac{d\tau}{\tau^{1-\varrho}}. \tag{2.4}$$

Proposition 2.1. *Let* $0 < \varrho \leq 1$ *and* u, v *be two X-valued functions which are piecewise continuous at each interval* $[a, b]$ *and of* ϱ*-exponential order* ($u(t) \leq M e^{ct^{\varrho}}$). *Then*

(1) For any $c_1, c_2 \in \mathbb{R}$, $\mathcal{L}_a^{\varrho}\{c_1 u + c_2 v\}(s) = c_1 \mathcal{L}_a^{\varrho}\{u\}(s) + c_2 \mathcal{L}_a^{\varrho}\{v\}(s)$.

(2) $\mathcal{L}_a^{\varrho}\{\left(\frac{(t-a)^{\varrho}}{\varrho}\right)^{\alpha}\}(s) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$.

(3) $\mathcal{L}_0^{\varrho}\left(u *_{\varrho} v\right)(s) = \mathcal{L}_0^{\varrho}\{u\}(s) \mathcal{L}_0^{\varrho}\{v\}(s)$.

(4) For $\alpha > 0$, $\mathcal{L}_0^{\varrho}\{I_a^{\alpha,\varrho} u\}(s) = \frac{\mathcal{L}_0^{\varrho} u(s)}{s^{\alpha}}$.

Proof. The proof follows from the argument of [40] by letting $\psi(t) = \frac{t^{\varrho}}{\varrho}$. □

Lemma 2.3. *[41] Let* $\alpha \geq 0$, $0 < \varrho \leq 1$ *Assume that* u, v *are two nonnegative locally integrable real valued functions on* $[0, b]$ *and h is a nonnegative and nondecreasing real valued function on* $[0, b]$. *If*

$$u(t) \leq v(t) + h(t) \int_0^t \left(\frac{t^{\varrho} - s^{\varrho}}{\varrho}\right)^{\alpha-1} u(s) \frac{ds}{s^{1-\varrho}},$$

then

$$u(t) \leq v(t) + \int_0^t \sum_{k=1}^{\infty} \frac{(h(t)\Gamma(\alpha))^k}{\Gamma(k\alpha)} \left(\frac{t^{\varrho} - s^{\varrho}}{\varrho}\right)^{k\alpha-1} v(s) \frac{ds}{s^{1-\varrho}}.$$

Now, we define and give some results about the so called fractional ϱ-semigroup of bounded linear operators, these results can be found in [5, 42–44]

Definition 2.6. *[5] Let $0 < \varrho \leq 1$. A family $\{T_\varrho(t)\}_{t\geq 0} \subseteq B(X)$ is called a fractional ϱ-semigroup of bounded linear operators on Banach space X if*

(i) $T_\varrho(0) = I$,

(ii) $T_\varrho(t+s)^{\frac{1}{\varrho}} = T_\varrho(t^{\frac{1}{\varrho}})T_\varrho(s^{\frac{1}{\varrho}})$ *for all* $t, s \in [0, \infty)$.

An ϱ−semigroup $T_\varrho(t)$ is called a C_0-ϱ-semigroup, if for each $x \in X$, $T_\varrho(t)x \to x$ as $t \to 0^+$.

A linear operator A defined by

$$D(A) = \{x \in X | \lim_{t\to 0^+} D^{\varrho}_{0_+}(T_\varrho)(t)x \ \ exists\},$$

and

$$Ax = \lim_{t\to 0^+} D^{\varrho}_{0_+}(T_\varrho)(t)x$$

is called the ϱ−infinitesimal generator of the fractional ϱ−semigroup $T_\varrho(t)$.

Theorem 2.4. *[42, 43] Let A be the ϱ-infinitesimal generator of a fractional C_0-ϱ-semigroup $\{T_\varrho(t)\}_{t\geq 0}$ where $0 < \varrho \leq 1$. Then*

(1) There exist $M \geq 1$ and $\omega \geq 0$, such that $\|T_\varrho(t)\| \leq Me^{\omega t^{\varrho}}$, $t \geq 0$.

(2) A is closed densely defined.

(3) The resolvent set $\rho(A)$ of A contains the interval $(\omega, +\infty)$ and for any $\lambda > \omega$, we have $\|R(\lambda, A)\|_{B(X)} \leq \frac{M}{\lambda-\omega}$, where the resolvent operator $R(\lambda, A)$ is defined by

$$R(\lambda, A)x = (\lambda I - A)^{-1}\, x = \int_0^{+\infty} e^{-\lambda\frac{\tau^{\varrho}}{\varrho}} T_\varrho(\tau)\frac{d\tau}{\tau^{1-\varrho}}, \ \ \forall x \in X.$$

Throughout this paper, assume that A is the infinitesimal generator of a uniformly bounded C_0-ϱ-semigroup $\{T_\varrho(t)\}_{t\geq 0}$ on X. i.e, there exists $M \geq 1$ such that $M = \sup_{t\in[0,+\infty)} \|T_\varrho(t)\|$.

Next, we define the Kurtawoski measure of noncompact $\mu(.)$ on each bounded subset Λ of Banach space X by

$$\mu(\Lambda) = \inf\{\varepsilon \geq 0 : \Lambda \subseteq \bigcup_{i=1}^{m} B_i, \ \text{ where } \ diam\,(B_i) \leq \varepsilon\},$$

where $diam\,(B)$ is the diameter of B. The Kurtawoski measure of noncompact (KMN) μ satisfies the following basic properties (see [45, 46]):

(1) $\mu\,(\Lambda_1) \leq \mu\,(\Lambda_2)$, for any bounded subsets $\Lambda_1, \Lambda_2 \in X$ such that $\Lambda_1 \subseteq \Lambda_2$.

(2) $\mu\,(\Lambda) = 0$ if and only if Λ is relatively compact in X.

(3) $\mu\,(\{x\} \cup \Lambda) = \mu\,(\Lambda)$ for all $x \in X$ and $\Lambda \subset X$.

(4) $\mu\,(\Lambda_1 \cup \Lambda_2) \leq \max\{\mu\,(\Lambda_1)\,, \mu\,(\Lambda_2)\}$.

(5) $\mu(\Lambda_1 + \Lambda_2) \le \mu(\Lambda_1) + \mu(\Lambda_2)$.

(6) $\mu(\eta\Lambda) \le |\eta|\mu(\Lambda)$ for $\eta \in \mathbb{R}$.

(7) If $\Theta : \Omega \subset X \to X$ is a Lipschitz map with constant K, then $\mu(\Theta(\Lambda)) \le K\mu(\Lambda)$ for any bounded subset $\Lambda \subset \Omega$.

Lemma 2.5. *[47] If $\Lambda \subseteq C([0,b],X)$ is bounded, then $\mu(\Lambda(t)) \le \mu(\Lambda)$ for every $t \in [0,b]$, where $\Lambda(t) = \{u(t), u \in \Lambda\}$. Moreover if Λ is equicontinuous on $[0,b]$, then $t \to \mu(\Lambda(t))$ is continuous real valued function on $[0,b]$, $\mu(\Lambda) = \sup\{\mu(\Lambda(t)), t \in [0,b]\}$. Furthermore $\mu\left(\int_0^t \Lambda(s)ds\right) \le \int_0^t \mu(\Lambda(s))\,ds$.*

Lemma 2.6. *[48] If $\{u_n\}_{n=1}^{\infty}$ is a sequence of Bochner integrable X-valued functions on $[0,b]$ satisfies $\|u_n(t)\| \le \phi(t)$ for almost all $t \in [0,b]$ and every $n \ge 1$, where $\phi \in L^1([0,b],\mathbb{R})$, then the function $\psi(t) = \mu(\{u_n(t) : n \ge 1\}) \in L^1([0,b])$ and satisfies $\mu\left(\{\int_0^t u_n(t)ds : n \ge 1\}\right) \le 2\int_0^t \psi(s)ds$.*

Lemma 2.7. *Let Ω be a closed convex subset of a Banach space X, and $G : \Omega \to \Omega$ be continuous satisfying Mönch's condition, i.e.,*

$$\Lambda \subseteq \Omega \text{ is countable, } \overline{\Lambda} \subseteq \overline{conv}(0 \cup G(\Lambda)) \quad \Rightarrow \quad \Lambda \text{ is compact,}$$

where $conv(\Lambda)$ denotes the convex hall of Λ. Then G has a fixed point.

3. Representation of mild solution using fractional Laplace transform

According to Theorem 2.2, we can rewrite the nonlocal problem (1.1) in the following equivalent integral equation:

$$\begin{cases} u(t) = \varphi(0) - (gu)(0) + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1}[Au(s) + f(s,u_s,Bu(s))]\frac{ds}{s^{1-\varrho}}, & t \in [0,b] \\ u(t) + (gu)(t) = \varphi(t), & t \in [-\delta, 0] \end{cases} \tag{3.1}$$

where $Bu(t) = \int_0^t h(t,s,u_s)ds$, provided that the integral in (3.1) exists.

To introduce the mild solution of (1.1) we need to define the two families of operators $\{\mathrm{S}_{\alpha,\varrho}(t)\}_{t\ge0}$ and $\{\mathrm{P}_{\alpha,\varrho}(t)\}_{t\ge0}$ by

$$\begin{cases} \mathrm{S}_{\alpha,\varrho}(t)x = \varrho\int_0^\infty \theta^{\varrho-1}\psi_\alpha(\theta)T_\varrho\left(\varrho^{\frac{1-\alpha}{\varrho}}t^\varrho\theta\right)xd\theta, & x \in X, \\ \mathrm{P}_{\alpha,\varrho}(t) = \alpha\varrho\int_0^\infty \theta^{2\varrho-1}\psi_\alpha(\theta)T_\varrho\left(\varrho^{\frac{1}{\varrho}}t^{\frac{\alpha}{\varrho}}\theta\right)xd\theta, & x \in X \end{cases} \tag{3.2}$$

where $0 \le \alpha, \varrho \le 1$ and for $\theta \ge 0$

$$\psi_\alpha(\theta) = \sum_{k=0}^{\infty}\frac{(-\theta)^k}{k!\Gamma(-\alpha(k+1)+1)} = \sum_{k=0}^{\infty}\frac{(-\theta)^k\Gamma(\alpha(k+1))}{k!}\sin(\pi(k+1)\alpha)$$

is the wright type function defined on $(0,\infty)$ which is positive and satisfies

$$\int_0^\infty \psi_\alpha(\theta)d\theta = 1, \tag{3.3}$$

$$\int_0^\infty \theta^\gamma\psi_\alpha(\theta)d\theta = \frac{\Gamma(1+\gamma)}{\Gamma(1+\alpha\gamma)}, \tag{3.4}$$

$$\int_0^\infty e^{-\lambda\theta}\phi_\alpha(\theta)d\theta = e^{-\lambda^\alpha}, \tag{3.5}$$

where $\phi_\alpha(\theta) = \alpha t^{-1-\alpha}\psi_\alpha(t^{-\alpha})$.

Lemma 3.1. *If* (3.1) *holds, then we have*

$$\begin{cases} u(t) = \mathrm{S}_{\alpha,\varrho}(t)\left[\varphi(0) - (gu)(0)\right] + \int_0^t \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \mathrm{P}_{\alpha,\varrho}\left(\frac{t^\varrho - s^\varrho}{\varrho}\right) f(s, u_s, Bu(s)) \frac{ds}{s^{1-\varrho}}, & t \in [0, b], \\ u(t) + (gu)(t) = \varphi(t), & t \in [-\delta, 0] \end{cases} \tag{3.6}$$

where the operators $\mathrm{S}_{\alpha,\varrho}(t)$ *and* $\mathrm{P}_{\alpha,\varrho}(t)$ *are defined in* (3.2).

Proof. Let $\lambda > 0$. By applying the fractional Laplace transform of order $0 < \varrho \leq 1$ to (3.1) for $t \geq 0$, we get:

$$\begin{aligned} U(\lambda) &= \frac{1}{\lambda}\left[\varphi(0) - (gu)(0)\right] + \mathcal{L}_0^\varrho\{I_0^{\alpha,\varrho}\left(Au(t) + f(t, u_t, Bu(t))\right)\}(\lambda) \\ &= \frac{1}{\lambda}\left[\varphi(0) - (gu)(0)\right] + \frac{1}{\lambda^\alpha}\left[AU(\lambda) + F(\lambda)\right] \\ U(\lambda) &= \lambda^{\alpha-1}\left(\lambda^\alpha - A\right)^{-1}\left[\varphi(0) - (gu)(0)\right] + \left(\lambda^\alpha - A\right)^{-1} F(\lambda) \\ &= J_1 + J_2 \end{aligned}$$

where $U(\lambda) = \mathcal{L}_0^\varrho(u)(\lambda)$ and $F(\lambda) = \mathcal{L}_0^\varrho\left(f(t, u_t, Bu(t))\right)(\lambda)$.
Now, differentiating (3.5) with respect to λ, we obtain

$$\int_0^\infty \theta\phi_\alpha(\theta)e^{-\lambda\theta}d\theta = \alpha\lambda^{\alpha-1}e^{-\lambda^\alpha}. \tag{3.7}$$

Using (3.7) and from Theorem 2.4 , we get

$$\begin{aligned} J_1 &= \lambda^{\alpha-1}\left(\lambda^\alpha - A\right)^{-1}\left[\varphi(0) - (gu)(0)\right] \\ &= \lambda^{\alpha-1}\int_0^{+\infty} e^{-\lambda^\alpha \frac{s^\varrho}{\varrho}} T_\varrho(s)\left[\varphi(0) - (gu)(0)\right]\frac{ds}{s^{1-\varrho}} \\ &= \int_0^{+\infty}\left(\frac{s^\varrho}{\varrho}\right)^{\frac{1}{\alpha}-1}\left(\lambda\left(\frac{s^\varrho}{\varrho}\right)^{\frac{1}{\alpha}}\right)^{\alpha-1} e^{-\left(\lambda\left(\frac{s^\varrho}{\varrho}\right)^{\frac{1}{\alpha}}\right)^\alpha} T_\varrho(s)\left[\varphi(0) - (gu)(0)\right]\frac{ds}{s^{1-\varrho}} \\ &= \int_0^{+\infty}\int_0^{+\infty} e^{-\lambda\left(\frac{s^\varrho}{\varrho}\right)^{\frac{1}{\alpha}}\theta}\phi_\alpha(\theta)T_\varrho(s)\left[\varphi(0) - (gu)(0)\right]\frac{1}{\alpha}\left(\frac{s^\varrho}{\varrho}\right)^{\frac{1}{\alpha}-1}\theta d\theta\frac{ds}{s^{1-\varrho}}. \end{aligned} \tag{3.8}$$

By using the substitution $\frac{\tau^\varrho}{\varrho} = \left(\frac{s^\varrho}{\varrho}\right)^{\frac{1}{\alpha}}\theta$ and $\vartheta = \theta^{-\frac{\alpha}{\varrho}}$ in the last equation, we obtain

$$\begin{aligned} J_1 &= \int_0^{+\infty} e^{-\lambda\frac{\tau^\varrho}{\varrho}}\int_0^{+\infty}\phi_\alpha(\theta)T_\varrho(\varrho^{\frac{1-\alpha}{\varrho}}\tau^\alpha\frac{1}{\theta^{\frac{\alpha}{\varrho}}})\left[\varphi(0) - (gu)(0)\right]d\theta\frac{d\tau}{\tau^{1-\varrho}} \\ &= \int_0^{+\infty} e^{-\lambda\frac{\tau^\varrho}{\varrho}}\left[\int_0^{+\infty}\varrho\vartheta^{\varrho-1}\psi_\alpha(\vartheta^\varrho)T_\varrho(\varrho^{\frac{1-\alpha}{\varrho}}\tau^\alpha\vartheta)\left[\varphi(0) - (gu)(0)\right]d\vartheta\right]\frac{d\tau}{\tau^{1-\varrho}}. \end{aligned} \tag{3.9}$$

Similarly, from (3.5) and by using the substitution $\frac{\tau^\varrho}{\varrho} = \left(\frac{s^\varrho}{\varrho}\right)^{\frac{1}{\alpha}}\theta$, we get

$$J_2 = \left(\lambda^\alpha - A\right)^{-1} F(\lambda) = \int_0^{+\infty} e^{-\lambda\frac{s^\varrho}{\varrho}} T_\varrho(s)F(\lambda)\frac{ds}{s^{1-\varrho}}$$

$$
\begin{aligned}
&= \int_0^{+\infty}\int_0^{+\infty} \phi_\alpha(\theta)e^{-\lambda\left(\frac{s^\varrho}{\varrho}\right)^{\frac{1}{\alpha}}\theta}T_\varrho(s)F(\lambda)d(\theta)\frac{ds}{s^{1-\varrho}}\\
&= \int_0^{+\infty} e^{-\lambda\frac{\tau^\varrho}{\varrho}}\int_0^{+\infty}\left(\frac{\tau^\varrho}{\varrho}\right)^{\alpha-1}\frac{\alpha}{\theta^\alpha}\phi_\alpha(\theta)T_\varrho(\varrho^{\frac{1-\alpha}{\varrho}}\tau^\alpha\frac{1}{\theta^{\frac{\alpha}{\varrho}}})F(\lambda)d(\theta)\frac{d\tau}{\tau^{1-\varrho}}\\
&= \int_0^{+\infty} e^{-\lambda\frac{\tau^\varrho}{\varrho}}\int_0^{+\infty}\alpha\varrho\theta^{2\varrho-1}\psi_\alpha(\theta^\varrho)\left(\frac{\tau^\varrho}{\varrho}\right)^{\alpha-1}T_\varrho(\varrho^{\frac{1-\alpha}{\varrho}}\tau^\alpha\theta)F(\lambda)d(\theta)\frac{d\tau}{\tau^{1-\varrho}}\,.
\end{aligned}
\tag{3.10}
$$

Applying the property 3 of Proposition 2.1 yields

$$
\begin{aligned}
J_2 &= \int_0^{+\infty} e^{-\lambda\frac{\tau^\varrho}{\varrho}}\int_0^{+\infty}\alpha\varrho\theta^{2\varrho-1}\psi_\alpha(\theta^\varrho)\left(\frac{\tau^\varrho}{\varrho}\right)^{\alpha-1}T_\varrho(\varrho^{\frac{1-\alpha}{\varrho}}\tau^\alpha\theta)F(\lambda)d(\theta)\frac{d\tau}{\tau^{1-\varrho}}\\
&= \int_0^{+\infty}\int_0^{+\infty} e^{-\lambda\left(\frac{\tau^\varrho}{\varrho}+\frac{s^\varrho}{\varrho}\right)}\int_0^{+\infty}\alpha\varrho\theta^{2\varrho-1}\psi_\alpha(\theta^\varrho)\left(\frac{\tau^\varrho}{\varrho}\right)^{\alpha-1}\\
&\quad\times\ T_\varrho(\varrho^{\frac{1-\alpha}{\varrho}}\tau^\alpha\theta)f(s,u_s,Bu(s))d(\theta)\frac{d\tau}{\tau^{1-\varrho}}\frac{ds}{s^{1-\varrho}}\\
&= \int_0^{+\infty} e^{-\lambda\frac{t^\varrho}{\varrho}}\left[\alpha\int_0^t\int_0^{+\infty}\varrho\theta^{2\varrho-1}\psi_\alpha(\theta^\varrho)\left(\frac{t^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}\right.\\
&\quad\left.\times\, T_\varrho(\varrho^{\frac{1-\alpha}{\varrho}}(\tau^\alpha-s^\varrho)^{\frac{\alpha}{\varrho}}\theta)f(s,u_s,Bu(s))d(\theta)\frac{ds}{s^{1-\varrho}}\right]\frac{dt}{t^{1-\varrho}}.
\end{aligned}
\tag{3.11}
$$

According to (3.9) and (3.11), we have

$$
\begin{aligned}
U(\lambda) &= \int_0^{+\infty} e^{-\lambda\frac{t^\varrho}{\varrho}}\left(\int_0^{+\infty}\varrho\theta^{\varrho-1}\psi_\alpha(\theta^\varrho)T_\varrho(\varrho^{\frac{1-\alpha}{\varrho}}t^\alpha\theta)\left[\varphi(0)-(gu)(0)\right]d\theta\right)\frac{dt}{t^{1-\varrho}}\\
&\quad+\int_0^{+\infty} e^{-\lambda\frac{t^\varrho}{\varrho}}\left(\alpha\int_0^t\int_0^{+\infty}\varrho\theta^{2\varrho-1}\psi_\alpha(\theta^\varrho)\left(\frac{t^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}\right.\\
&\quad\left.\times\, T_\varrho(\varrho^{\frac{1-\alpha}{\varrho}}(\tau^\alpha-s^\varrho)^{\frac{\alpha}{\varrho}}\theta)f(s,u_s,Bu(s))d(\theta)\frac{ds}{s^{1-\varrho}}\right)\frac{dt}{t^{1-\varrho}}
\end{aligned}
\tag{3.12}
$$

Now, by inverting the inverse fractional Laplace transform, we obtain

$$
\begin{aligned}
u(t) &= \int_0^{+\infty}\varrho\theta^{\varrho-1}\psi_\alpha(\theta^\varrho)T_\varrho(\varrho^{\frac{1-\alpha}{\varrho}}t^\alpha\theta)\left[\varphi(0)-(gu)(0)\right]d\theta\\
&\quad+\alpha\int_0^t\int_0^{+\infty}\varrho\theta^{2\varrho-1}\psi_\alpha(\theta^\varrho)\left(\frac{t^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}T_\varrho(\varrho^{\frac{1-\alpha}{\varrho}}(\tau^\alpha-s^\varrho)^{\frac{\alpha}{\varrho}}\theta)f(s,u_s,Bu(s))d(\theta)\frac{ds}{s^{1-\varrho}}\\
&= \mathrm{S}_{\alpha,\varrho}(t)\left[\varphi(0)-(gu)(0)\right]+\int_0^t\left(\frac{t^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}\mathrm{P}_{\alpha,\varrho}\left(\frac{t^\varrho-s^\varrho}{\varrho}\right)f(s,u_s,Bu(s))\frac{ds}{s^{1-\varrho}}
\end{aligned}
$$

□

Definition 3.1. *An X-valued function $u\in C([-\delta,b],X)$ is called a mild solution of the nonlocal Cauchy problem* (1.1)*, if it satisfies:*

$$
\begin{cases}
u(t) = \mathrm{S}_{\alpha,\varrho}(t)\left[\varphi(0)-(gu)(0)\right]+\int_0^t\left(\frac{t^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}\mathrm{P}_{\alpha,\varrho}\left(\frac{t^\varrho-s^\varrho}{\varrho}\right)f(s,u_s,Bu(s))\frac{ds}{s^{1-\varrho}}, & t\in[0,b],\\
u(t)+(gu)(t)=\varphi(t), & t\in[-\delta,0].
\end{cases}
\tag{3.13}
$$

Lemma 3.2. *The family of operators* $\{\mathrm{S}_{\alpha,\varrho}(t)\}_{t\geq 0}$ *and* $\{\mathrm{P}_{\alpha,\varrho}(t)\}_{t\geq 0}$ *satisfy:*

(i) For any fixed $t \geq 0$, $\mathrm{S}_{\alpha,\varrho}(t)$ *and* $\mathrm{P}_{\alpha,\varrho}(t)$ *are linear and bounded.*
(ii) For any $x \in X$, *the* X*-valued functions* $t \to \mathrm{S}_{\alpha,\varrho}(t)x$ *and* $t \to \mathrm{P}_{\alpha,\varrho}(t)x$ *are continuous on* $[0, +\infty)$.

Proof. The linearity is obvious. Since $\|T_\varrho(t)\| \leq M$ for any $t \geq 0$ and from (3.3), we get

$$\begin{aligned}\|\mathrm{S}_{\alpha,\varrho}(t)x\| &\leq \varrho \int_0^{+\infty} \psi_\alpha(\theta^\varrho)\|T_\varrho(\varrho^{\frac{1-\alpha}{\varrho}} t^\alpha \theta)\|\|x\| \frac{d\theta}{\theta^{1-\varrho}} \\ &\leq M\|x\| \int_0^{+\infty} \psi_\alpha(\theta^\varrho) d(\theta^\varrho) \\ &\leq M\|x\| \int_0^{+\infty} \psi_\alpha(\theta) d\theta = M\|x\|.\end{aligned}$$

Similarly:

$$\begin{aligned}\|\mathrm{P}_{\alpha,\varrho}(t)x\| &\leq M\|x\|\alpha\varrho \int_0^{+\infty} \theta^{2\varrho-1}\psi_\alpha(\theta^\varrho) d\theta \\ &\leq \alpha M\|x\| \int_0^{+\infty} \theta\psi_\alpha(\theta) d\theta = \frac{\alpha M}{\Gamma(1+\alpha)}\|x\| = \frac{M}{\Gamma(\alpha)}\|x\|.\end{aligned}$$

For the part (ii), let $t_1, t_2 \geq 0$. Then

$$\|\mathrm{S}_{\alpha,\varrho}(t_1)x - \mathrm{S}_{\alpha,\varrho}(t_2)x\| \leq \varrho \int_0^{+\infty} \psi_\alpha(\theta^\varrho)\|T_\varrho(\varrho^{\frac{1-\alpha}{\varrho}} t_1^\alpha \theta)x - T_\varrho(\varrho^{\frac{1-\alpha}{\varrho}} t_2^\alpha \theta)x\| \frac{d\theta}{\theta^{1-\varrho}}.$$

From the strong continuity of $T_\varrho(t)$ and by using Lesbegue dominated convergence we obtain $\lim_{t_2\to t_1} \|\mathrm{S}_{\alpha,\varrho}(t_1)x - \mathrm{S}_{\alpha,\varrho}(t_2)x\| = 0$, which implies that $\{\mathrm{S}_{\alpha,\varrho}(t)\}_{t\geq 0}$ is strongly continuous. A similar argument enables us to prove the strong continuity of $\{\mathrm{P}_{\alpha,\varrho}(t)\}_{t\geq 0}$.

□

4. Existence results of a mild solution

In this section we will establish the existence results by using the Hausdorff measure of noncompactness. To state and prove our main results for the existence of mild solutions of problem (1.1), we need the following hypotheses:

(H_1) The uniformly bounded C_0-ϱ-semigroup $\{T_\varrho(t)\}_{t\geq 0}$ generated by A is continuous in the uniform operator topology for $t > 0$.

(H_2) The function $f : [0, b] \times C \times X \to X$ satisfies the following

- (i) For each $(v, x) \in C \times X$, $f(., v, x)$ is strongly measurable, $f(t, ., .)$ is continuous a.e. for $t \in [0, b]$.
- (ii) There exists $m \in L_1([0, b], \mathbb{R}^+)$ such that $I_0^{\alpha,\varrho} m \in C((0, b], \mathbb{R}^+)$ and $\lim_{t\to 0^+} I_0^{\alpha,\varrho} m(t) = 0$, satisfying: $\|f(t, x, v)\| \leq m(t)$ for all $(x, v) \in X \times C$ and almost all $t \in [0, b]$.
- (iii) There exists a constant $L \geq 0$ such that for any bounded sets $\Lambda_1 \subset C$, $\Lambda_2 \subset X$

$$\mu(f(t, \Lambda_1, \Lambda_2)) \leq L\left(\sup_{\theta\in[-\delta,0]} \mu(\Lambda_1(t)) + \mu(\Lambda_2)\right), \quad a.e.\ t \in [0, b].$$

(H_3) The function $h : [0,b] \times [0,b] \times C \to X$ satisfies the following

(i) For each $v \in C$, $h(.,.,v)$ is strongly measurable, $h(t,s,.)$ is continuous a.e. for $(t,s) \in [0,b] \times [0,b]$.

(ii) There exists a function $m_1 : [0,b] \times [0,b] \to \mathbb{R}^+$, such that $\sup_{t\in[0,b]} \int_0^t m_1(t,s)ds = m_1^* < \infty$ and $\|h(t,s,v)\| \leq m_1(t,s)\|v\|_C$, for all $t,s \in [0,b]$ and $v \in C$.

(iii) There exists a function $\gamma : [0,b] \times [0,b] \to \mathbb{R}^+$, such that $\sup_{t\in[0,b]} \int_0^t \gamma(t,s)ds = \gamma^* < \infty$ and

$$\mu\left(h\left(t,s,\Lambda\right)\right) \leq \gamma(t,s) \sup_{\theta\in[-\delta,0]} \mu\left(\Lambda(t)\right)$$

for each bounded subset $\Lambda \in C$ and almost all $t,s \in [0,b]$.

$H_4)$ The operator $g : C\left([-\delta,b],X\right) \to C$ satisfies

(i) For each $t \in [-\delta,0]$, the operator $\Upsilon_t : C\left([-\delta,b],X\right) \to X$ defined by $\Upsilon_t(u) = (gu)(t)$ is continuous. There exists a constant $L_3 \in \left(0,\frac{1}{M}\right)$ such that $\|g(u)\|_C \leq L_3\|u\|_\infty$ for all $u \in C\left([-\delta,b],X\right)$, and the subset $g\left(\Lambda\right) \subset C$ is equicontinuous for each bounded set $\Lambda \subset C\left([-\delta,b],X\right)$

(ii) There exists a constant $L_4 \in [0,1)$ such that $\mu\left(\Upsilon_t\left(\Lambda\right)\right) \leq L_4\mu\left(\Lambda(t)\right)$ for each bounded set $\Lambda \subset C\left([-\delta,b],X\right)$ and all $t \in [-\delta,0]$.

Lemma 4.1. *If (H_1) holds, then the family of operators $\{\mathrm{S}_{\alpha,\varrho}(t)\}_{t\geq0}$ and $\{\mathrm{P}_{\alpha,\varrho}(t)\}_{t\geq0}$ are continuous in the uniform operator topology for $t > 0$.*

Proof. Let $t_1, t_2 \geq 0$. For $\varepsilon > 0$ we have

$$\begin{aligned}
&\|\mathrm{S}_{\alpha,\varrho}(t_1)x - \mathrm{S}_{\alpha,\varrho}(t_2)x\| \\
&\quad\leq \varrho \int_\varepsilon^{+\infty} \psi_\alpha(\theta^\varrho)\|T_\varrho(\varrho^{\frac{1-\alpha}{\varrho}}t_1^\alpha\theta)x - T_\varrho(\varrho^{\frac{1-\alpha}{\varrho}}t_2^\alpha\theta)x\|\frac{d\theta}{\theta^{1-\varrho}} + \varrho M\int_0^\varepsilon \psi_\alpha(\theta^\varrho)\frac{d\theta}{\theta^{1-\varrho}} \\
&\quad\leq \varrho \int_\varepsilon^{+\infty} \psi_\alpha(\theta^\varrho)\|T_\varrho(\varrho^{\frac{1-\alpha}{\varrho}}t_1^\alpha\theta) - T_\varrho(\varrho^{\frac{1-\alpha}{\varrho}}t_2^\alpha\theta)\|\|x\|\frac{d\theta}{\theta^{1-\varrho}} + \varrho M\int_0^{\varepsilon^\varrho} \psi_\alpha(\theta)d\theta. \qquad (4.1)
\end{aligned}$$

Applying the Lebesgue dominated convergence by using the continuity of T_ϱ in the uniform operator topology and the Eq (3.3), we obtain

$$\varrho \int_\varepsilon^{+\infty} \psi_\alpha(\theta^\varrho)\|T_\varrho(\varrho^{\frac{1-\alpha}{\varrho}}t_1^\alpha\theta) - T_\varrho(\varrho^{\frac{1-\alpha}{\varrho}}t_2^\alpha\theta)\|\|x\|\frac{d\theta}{\theta^{1-\varrho}} \to 0 \quad \text{as } t_2 \to t_1.$$

Then for any $x \in X$, $\|x\| \leq 1$

$$\lim_{t_2\to t_1} \|\mathrm{S}_{\alpha,\varrho}(t_1)x - \mathrm{S}_{\alpha,\varrho}(t_2)x\| \leq \varrho M\int_0^{\varepsilon^\varrho} \psi_\alpha(\theta)d\theta.$$

From (3.3), and since ε is arbitrary, then

$$\int_0^{\varepsilon^\varrho} \psi_\alpha(\theta)d\theta \to 0 \quad \text{as } \varepsilon \to 0,$$

and therefore

$$\lim_{t_2\to t_1} \|\mathrm{S}_{\alpha,\varrho}(t_1)x - \mathrm{S}_{\alpha,\varrho}(t_2)x\| = 0$$

which implies that the continuity in the uniform operator topology of $S_{\alpha,\varrho}(t)$ for $t>0$.
Using the similar argument we can prove that $P_{\alpha,\varrho}(t)$ is continuous in the uniform operator topology for $t>0$. $\square$

Let $\mathbf{B}_r=\{u\in C([-\delta,b],X),\|u\|_\infty\le r\}$, where $r\ge 0$. Then $\mathbf{B}_r$ is clearly a bounded closed and convex subset in $C([-\delta,b],X)$. We define the operator Φ by

$$(\Phi_1 u)(t)=\begin{cases} S_{\alpha,\varrho}(t)\left[\varphi(0)-(gu)(0)\right]+\int_0^t\left(\frac{t^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}P_{\alpha,\varrho}\left(\frac{t^\varrho-s^\varrho}{\varrho}\right)f(s,u_s,Bu(s))\frac{ds}{s^{1-\varrho}}, & t\in[0,b],\\ \varphi(t)-(gu)(t), & t\in[-\delta,0].\end{cases}\tag{4.2}$$

Obviously, $u\in\mathbf{B}_r$ is a mild solution of (1.1) if and only if the operator Φ has a fixed point on $\mathbf{B}_r$, i.e., there exists $u\in\mathbf{B}_r$ satisfies $u=\Phi u$.

Lemma 4.2. *If* (H_1)–(H_4) *hold, then* $\{\Phi u,\ u\in\mathbf{B}_r\}$ *is equicontinuous.*

Proof. Let $u\in\mathbf{B}_r$. For $-\delta\le t_1\le t_2\le 0$, we have

$$\|\Phi u(t_2)-\Phi u(t_1)\|\le\|\varphi(t_2)-\varphi(t_1)\|+\|gu(t_2)-gu(t_1)\|.$$

Since $\varphi\in C$ and from (H_4)(i), we obtain

$$\|\Phi u(t_2)-\Phi u(t_1)\|\to 0 \text{ independently for } u\in\mathbf{B}_r \text{ as } t_2\to t_1.$$

For $-\delta\le t_1\le 0<t_2\le b$, then from (H_2)(ii) and Lemma 3.2, we get

$$\begin{aligned}\|\Phi_2 u(t_2)-\Phi_2 u(t_1)\|=&\Big\|\varphi(t_1)-gu(t_1)-S_{\alpha,\varrho}(t_2)\left[\varphi(0)-(gu)(0)\right]\\ &-\int_0^{t_2}\left(\frac{t_2^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}P_{\alpha,\varrho}\left(\frac{t_2^\varrho-s^\varrho}{\varrho}\right)f(s,u_s,Bu(s))\frac{ds}{s^{1-\varrho}}\Big\|\\ \le&\|\varphi(t_1)-\varphi(0)\|+\|S_{\alpha,\varrho}(t_2)\varphi(0)-\varphi(0)\|+\|gu(t_2)-gu(0)\|\\ &+\|S_{\alpha,\varrho}(t_2)gu(0)-gu(0)\|+\frac{M}{\Gamma(\alpha)}\int_0^{t_2}\left(\frac{t_2^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}m(s)\frac{ds}{s^{1-\varrho}}\\ \le&I_1+I_2+I_3+I_4+I_5.\end{aligned}$$

Since $\varphi\in C$ and from (H_2)(ii), (H_4)(i) and Lemma 4.1, we find $I_1,...,I_5\to 0$ as $t_1,t_2\to 0$ and hence

$$\|\Phi u(t_2)-\Phi u(t_1)\|\to 0 \text{ independently for } u\in\mathbf{B}_r \text{ as } t_2\to t_1.$$

For $0<t_1\le t_2\le b$, we have

$$\begin{aligned}\|\Phi_2 u(t_2)-\Phi_2 u(t_1)\|\le&\left\|\left(S_{\alpha,\varrho}(t_2)-S_{\alpha,\varrho}(t_2)\right)(\varphi(0)-gu(0))\right\|\\ &+\left\|\int_{t_1}^{t_2}\left(\frac{t_2^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}P_{\alpha,\varrho}\left(\frac{t_2^\varrho-s^\varrho}{\varrho}\right)f(s,u_s,Bu(s))\frac{ds}{s^{1-\varrho}}\right\|\\ &+\Big\|\int_0^{t_1}\left(\frac{t_2^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}P_{\alpha,\varrho}\left(\frac{t_2^\varrho-s^\varrho}{\varrho}\right)f(s,u_s,Bu(s))\frac{ds}{s^{1-\varrho}}\\ &-\int_0^{t_1}\left(\frac{t_1^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}P_{\alpha,\varrho}\left(\frac{t_2^\varrho-s^\varrho}{\varrho}\right)f(s,u_s,Bu(s))\frac{ds}{s^{1-\varrho}}\Big\|\end{aligned}$$

$$
\begin{aligned}
&+\left\|\int_0^{t_1}\left(\frac{t_1^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}\mathrm{P}_{\alpha,\varrho}\left(\frac{t_2^\varrho-s^\varrho}{\varrho}\right)f(s,u_s,Bu(s))\frac{ds}{s^{1-\varrho}}\right.\\
&\left.-\int_0^{t_1}\left(\frac{t_1^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}\mathrm{P}_{\alpha,\varrho}\left(\frac{t_1^\varrho-s^\varrho}{\varrho}\right)f(s,u_s,Bu(s))\frac{ds}{s^{1-\varrho}}\right\| \qquad (4.3)\\
\leq&\|\mathrm{S}_{\alpha,\varrho}(t_2)-\mathrm{S}_{\alpha,\varrho}(t_1)\|_{B(X)}\|\varphi(0)-gu(0)\|\\
&+\frac{M}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\frac{t_2^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}m(s)\frac{ds}{s^{1-\varrho}}+\frac{M}{\Gamma(\alpha)}\int_0^{t_1}\left[\left(\frac{t_1^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}\right.\\
&\left.-\left(\frac{t_2^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}\right]m(s)\frac{ds}{s^{1-\varrho}}\\
&+\int_0^{t_1}\left(\frac{t_1^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}\left\|\mathrm{P}_{\alpha,\varrho}\left(\frac{t_2^\varrho-s^\varrho}{\varrho}\right)f(s,u_s,Bu(s))\right.\\
&\left.-\mathrm{P}_{\alpha,\varrho}\left(\frac{t_1^\varrho-s^\varrho}{\varrho}\right)f(s,u_s,Bu(s))\right\|\frac{ds}{s^{1-\varrho}} \qquad (4.4)\\
\leq&\|\mathrm{S}_{\alpha,\varrho}(t_2)-\mathrm{S}_{\alpha,\varrho}(t_2)\|_{B(X)}\|\varphi(0)-gu(0)\|\\
&+\frac{M}{\Gamma(\alpha)}\left|\int_0^{t_2}\left(\frac{t_2^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}m(s)\frac{ds}{s^{1-\varrho}}-\int_0^{t_1}\left(\frac{t_1^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}m(s)\frac{ds}{s^{1-\varrho}}\right|\\
&+\frac{2M}{\Gamma(\alpha)}\int_0^{t_1}\left[\left(\frac{t_1^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}-\left(\frac{t_2^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}\right]m(s)\frac{ds}{s^{1-\varrho}}\\
&+\int_0^{t_1}\left(\frac{t_1^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}\left\|\mathrm{P}_{\alpha,\varrho}\left(\frac{t_2^\varrho-s^\varrho}{\varrho}\right)-\mathrm{P}_{\alpha,\varrho}\left(\frac{t_1^\varrho-s^\varrho}{\varrho}\right)\right\|_{B(X)}m(s)\frac{ds}{s^{1-\varrho}} \qquad (4.5)\\
\leq&J_1+J_2+J_3+J_4. \qquad (4.6)
\end{aligned}
$$

Applying Lemma 4.1, we get $J_1 \to 0$ as $t_2 \to t_1$. According to (H_2)(ii), we find $J_2 \to 0$ as $t_2 \to t_1$. For $t_1 < t_2$ and since

$$J_3 \leq \frac{2M}{\Gamma(\alpha)}\int_0^{t_1}\left(\frac{t_1^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}m(s)\frac{ds}{s^{1-\varrho}},$$

then from Lebesgue dominated convergence, we get $J_3 \to 0$ as $t_2 \to t_1$. For $\varepsilon > 0$ small enough, we have

$$
\begin{aligned}
J_4 \leq&\int_0^{t_1-\varepsilon}\left(\frac{t_1^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}\left\|\mathrm{P}_{\alpha,\varrho}\left(\frac{t_2^\varrho-s^\varrho}{\varrho}\right)-\mathrm{P}_{\alpha,\varrho}\left(\frac{t_1^\varrho-s^\varrho}{\varrho}\right)\right\|_{B(X)}m(s)\frac{ds}{s^{1-\varrho}}\\
&+\int_{t_1-\varepsilon}^{t_1}\left(\frac{t_1^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}\left\|\mathrm{P}_{\alpha,\varrho}\left(\frac{t_2^\varrho-s^\varrho}{\varrho}\right)-\mathrm{P}_{\alpha,\varrho}\left(\frac{t_1^\varrho-s^\varrho}{\varrho}\right)\right\|_{B(X)}m(s)\frac{ds}{s^{1-\varrho}}\\
\leq&\int_0^{t_1-\varepsilon}\left(\frac{t_1^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}\left\|\mathrm{P}_{\alpha,\varrho}\left(\frac{t_2^\varrho-s^\varrho}{\varrho}\right)-\mathrm{P}_{\alpha,\varrho}\left(\frac{t_1^\varrho-s^\varrho}{\varrho}\right)\right\|_{B(X)}m(s)\frac{ds}{s^{1-\varrho}}\\
&+\frac{2M}{\Gamma(\alpha)}\left|\int_0^{t_1}\left(\frac{t_1^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}m(s)\frac{ds}{s^{1-\varrho}}-\int_0^{t_1-\varepsilon}\left(\frac{(t_1-\varepsilon)^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}m(s)\frac{ds}{s^{1-\varrho}}\right|
\end{aligned}
$$

$$+\frac{2M}{\Gamma(\alpha)}\int_0^{t_1-\varepsilon}\left[\left(\frac{(t_1-\varepsilon)^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}-\left(\frac{t_1^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}\right]m(s)\frac{ds}{s^{1-\varrho}}$$
$$\leq J_{41}+J_{42}+J_{43}.$$

Since

$$J_{41}\leq\frac{2M}{\Gamma(\alpha)}\int_0^{t_1-\varepsilon}\left(\frac{t_1^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}m(s)\frac{ds}{s^{1-\varrho}}$$

and from Lemma 4.1, $\mathrm{P}_{\alpha,\varrho}(t)$ is continuous in the uniform operator topology, then by using Lebesgue dominated convergence we find $J_{41}\to 0$ as $t_2\to t_1$. Using the same manner in J_2 and J_3 we get $J_{42},J_{43}\to 0$ as $\varepsilon\to 0$, and consequently J_4 converges to zero independently for $u\in\mathbf{B}_r$ as $t_2\to t_1$. Therefore

$$\|\Phi_2u(t_2)-\Phi_2u(t_1)\|\text{ independently for }u\in\mathbf{B}_r\text{ as }t_2\to t_1,$$

which means that $\{\Phi u,\ u\in\mathbf{B}_r\}$ is equicontinuous. □

Lemma 4.3. *If (H_1)–(H_4) hold, then Φ is continuous in $\mathbf{B}_r$ and maps $\mathbf{B}_r$ into $\mathbf{B}_r$ for any $r\geq 0$ satisfies*

$$\frac{M}{1-L_3M}\left(\|\varphi\|_C+\sup_{t\in[0,b]}\left\{\frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}m(s)\frac{ds}{s^{1-\varrho}}\right\}\right)\leq r. \tag{4.7}$$

Proof. **Claim:** Φ maps $\mathbf{B}_r$ into $\mathbf{B}_r$.
Obviously, from Lemma 4.2, $\Phi u\in C([-\delta,b],X)$. For $t\in[0,b]$ and for any $u\in\mathbf{B}_r$, by using (H_1), (H_2)(ii) and (H_4)(i), we get

$$\begin{aligned}\|\Phi u(t)\|\leq&\left\|\mathrm{S}_{\alpha,\varrho}(t)\left[\varphi(0)-(gu)(0)\right]\right\|+\left\|\int_0^t\left(\frac{t^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}\mathrm{P}_{\alpha,\varrho}\left(\frac{t^\varrho-s^\varrho}{\varrho}\right)f(s,u_s,Bu(s))\frac{ds}{s^{1-\varrho}}\right\|\\ \leq&M\left(\|\varphi(0)\|+L_3\|u\|_\infty\right)+\frac{M}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}\|f(s,u_s,Bu(s))\|\frac{ds}{s^{1-\varrho}}\\ \leq&M\left(\|\varphi\|_C+L_3r+\sup_{t\in[0,b]}\left\{\frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}m(s)\frac{ds}{s^{1-\varrho}}\right\}\right)\leq r.\end{aligned}$$

For $t\in[-\delta,0]$, we get

$$\begin{aligned}\|\Phi u(t)\|\leq&\|\varphi(t)\|+L_3\|u\|_\infty\\ \leq&\|\varphi\|_C+L_3r\\ \leq&M\left(\|\varphi\|_C+L_3r\right)\leq r.\end{aligned}$$

Hence, $\|\Phi u\|_\infty\leq r$ for all $u\in\mathbf{B}_r$.
Claim: Φ is continuous in $\mathbf{B}_r$.
Let $\{u_n\}_{n=0}^\infty\subset\mathbf{B}_r$ such that $\lim_{n\to\infty}\|u_n-u\|_\infty=0$.
For $t\in[0,b]$, we have

$$\|\Phi u_n(t)-\Phi u(t)\|\leq\left\|\mathrm{S}_{\alpha,\varrho}(t)\left[(gu_n)(0)-(gu)(0)\right]\right\|$$

$$
\begin{aligned}
&+\left\|\int_0^t \left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1} \mathrm{P}_{\alpha,\varrho}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)\left(f(s,(u_n)_s\,,Bu_n(s))-f(s,u_s,Bu(s))\right)\frac{ds}{s^{1-\varrho}}\right\| \\
\leq& M\|\left(gu_n\right)(0)-\left(gu\right)(0)\| \\
&+\frac{M}{\Gamma(\alpha)}\int_0^t \left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\|f(s,(u_n)_s\,,Bu_n(s))-f(s,u_s,Bu(s))\|\frac{ds}{s^{1-\varrho}}
\end{aligned}
$$

From conditions (H_2)(i),(ii) and (H_3)(i),(ii) we get

$$
\lim_{n\to\infty} f(s,(u_n)_s\,,Bu_n(s)) = f(s,u_s,Bu(s))
$$

and

$$
\frac{1}{s_{1-\varrho}}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\|f(s,(u_n)_s\,,Bu_n(s))-f(s,u_s,Bu(s))\| \leq \frac{2}{s_{1-\varrho}}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1} m(s)\,.
$$

Then by using Lesbegue dominated convergence, we obtain

$$
\int_0^t \left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\|f(s,(u_n)_s\,,Bu_n(s))-f(s,u_s,Bu(s))\|\frac{ds}{s^{1-\varrho}} \to 0 \;\text{ as }\; n\to\infty.
$$

From (H_4)(i), we obtain

$$
\|\left(gu_n\right)(0)-\left(gu\right)(0)\| \to 0 \;\text{ as }\; n\to\infty.
$$

Hence

$$
\Phi u_n(t) \to \Phi u(t) \;\text{ as }\; n\to\infty. \tag{4.8}
$$

From (H_4)(i), we find $\Phi u_n \to \Phi u$ pointwise on $[-\delta,0]$ and consequently $\Phi u_n \to \Phi u$ pointwise on $[-\delta,b]$, so the sequence $\{\Phi u_n\}_{n=0}^{\infty}$ is pointwise relatively compact on $[-\delta,b]$. From Lemma 4.2, $\{\Phi u_n\}_{n=0}^{\infty}$ is equicontinuous, then by Ascoli-Arzela theorem, $\{\Phi u_n\}_{n=0}^{\infty}$ is relatively compact, i.e., there exists subsequence of $\{\Phi u_n\}_{n=0}^{\infty}$ converge uniformly, clearly, to Φu as $n\to\infty$, and since $C\left([-\delta,b]\right)$ is compete, $\Phi u_n \to \Phi u$ uniformly on $[-\delta,b]$, as $n\to\infty$, and so Φ is continuous. □

Theorem 4.4. *Assume that* (H_1)–(H_4) *are hold. Then the nonlocal Cauchy problem* (1.1) *has at least a mild solution on* $\mathbf{B}_r$*, where r satisfies* (4.7)*.*

Proof. We know that $\mathbf{B}_r$ is closed and convex. From Lemmas 4.2 and 4.3, we know that Φ is a continuous map from $\mathbf{B}_r$ into $\mathbf{B}_r$ and the set $\{\Phi u,\ u\in\mathbf{B}_r\}$ is equicontinuous. We shall prove that Φ satisfies the Mönch condition $\mathbf{B}_r$. Let $\Lambda = \{u_n\}_{n=0}^{\infty}$ be a countable subset of $\mathbf{B}_r$ such that $\Lambda \subseteq conv\left(0\cup\Phi\left(\Lambda\right)\right)$. Then Λ is bounded and equicontinuous and therefore the function $t\to\varpi(t) = \mu\left(\Lambda(t)\right)$ is continuous on $[-\delta,b]$. From (H_4)(ii), we have, for any $t\in[-\delta,0]$,

$$
\begin{aligned}
\varpi(t) \leq& \mu\left(conv\left(0\cup\Phi\left(\Lambda(t)\right)\right)\right) = \mu\left(0\cup\Phi\left(\Lambda(t)\right)\right) \\
\leq& \mu\left(\Phi\left(\Lambda(t)\right)\right) \\
\leq& \mu\left(\{gu_n(t)\}_{n=1}^{\infty}\right) \\
\leq& L_4\mu\left(\{u_n(t)\}_{n=1}^{\infty}\right) = L_4\varpi(t).
\end{aligned}
$$

Since $L_4 < 1$, then $\varpi(t) = 0$ for all $t \in [-\delta, 0]$. For $t \in [0, b]$, then from (H_2)(iii), (H_3)(iii), (H_4)(ii) and by using Lemma 2.6 and properties of the measure μ, we obtain

$$\begin{aligned}
\varpi(t) \leq& \mu\left(conv\left(0 \cup \Phi\left(\Lambda(t)\right)\right)\right) \leq \mu\left(\Phi\left(\Lambda(t)\right)\right) \\
\leq& \mu\left(\{gu_n(t)\}_{n=1}^{\infty}\right) + \mu\left(\int_0^t \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \mathrm{P}_{\alpha,\varrho}\left(\frac{t^\varrho - s^\varrho}{\varrho}\right)\right. \\
& \left. f\left(s, \{(u_n)_s\}_{n=1}^{\infty}, \{Bu_n(s)\}_{n=1}^{\infty}\right)\frac{ds}{s^{1-\varrho}}\right) \\
\leq& L_4\mu\left(\{u_n(0)\}_{n=1}^{\infty}\right) + \frac{2M}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \mu\left(f\left(s, \{(u_n)_s\}_{n=1}^{\infty}, \{Bu_n(s)\}_{n=1}^{\infty}\right)\right)\frac{ds}{s^{1-\varrho}} \\
\leq& L_4 \sup_{0\leq\theta\leq t}\mu\left(\{u_n(\theta)\}_{n=1}^{\infty}\right) + \frac{2ML}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1}\left[\sup_{-\delta\leq\theta\leq 0}\mu\left(\{u_n(s+\theta)\}_{n=1}^{\infty}\right)\right. \\
& \left. + \mu\left(\{Bu_n(s)\}_{n=1}^{\infty}\right)\right]\frac{ds}{s^{1-\varrho}} \\
\leq& L_4 \sup_{0\leq\theta\leq t}\mu\left(\{u_n(\theta)\}_{n=1}^{\infty}\right) + \frac{2ML(1+2\gamma^*)}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1}\sup_{-\delta\leq\theta\leq 0}\mu\left(\{u_n(s+\theta)\}_{n=1}^{\infty}\right)\frac{ds}{s^{1-\varrho}} \\
\leq& L_4 \sup_{0\leq\theta\leq t}\mu\left(\{u_n(\theta)\}_{n=1}^{\infty}\right) + \frac{2ML(1+2\gamma^*)}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1}\sup_{0\leq\theta\leq s}\mu\left(\{u_n(\theta)\}_{n=1}^{\infty}\right)\frac{ds}{s^{1-\varrho}}.
\end{aligned}$$

From the last equation and by using the properties of supremum, we get

$$\sup_{0\leq\theta\leq t}\varpi(\theta) \leq \frac{2ML(1+2\gamma^*)}{(1-L_4)\Gamma(\alpha)}\int_0^t \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1}\sup_{0\leq\theta\leq s}\varpi(\theta)\frac{ds}{s^{1-\varrho}}.$$

Then from Lemma 2.3, we obtain $\sup_{0\leq\theta\leq t}\varpi(\theta) = 0$ for all $t \in [0, b]$. Hence $\varpi \equiv 0$ on $[-\gamma, b]$. This implies that $\Lambda(t)$ is relatively compact for each $t \in [-\gamma, b]$. From Ascoli-Arzela theorem, Λ is relatively compact on $\mathbf{B}_r$. Hence from Lemma 2.7 , Φ has a fixed point in $\mathbf{B}_r$, i.e., the nonlocal Cauchy problem (1.1) has at least mild solution on $\mathbf{B}_r$. □

5. Applications

Consider the following nonlocal integro-differential equation:

$$\begin{cases}
\partial_t^{\frac{1}{4},\varrho} v(t,x) = \partial_x^2 v(t,x) + e^{-t}\arctan\left(\int_{-\delta}^0 \sin\left(|v_t(\theta,x)|\right)d\theta\right) + \int_0^\pi \left(1 + \left|\int_0^t \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{-\frac{4}{5}}\right.\right. \\
\left.\left.\int_{-\delta}^0 \zeta(\theta)\left(1 - \exp\left(-\left(\frac{s^\varrho}{\varrho}\right)^{-\frac{1}{5}}\right)s^{1-\varrho}\frac{|v_t(\theta,x)|}{1+|v_t(\theta,x)|}\right)d\theta ds\right|\right)^{-1} dx, \ t \in [0,b], \ x \in [0,\pi], \\
v(t,0) = v(t,\pi) = 0, \quad t \in [0,b], \\
v(t,x) + \int_0^b \zeta_1(\theta)\cos\left(\frac{\pi}{2} + |v_t(\theta,x)|\right)d\theta = \psi(t,x), \quad t \in [-\delta,0], \quad t \in [0,\pi],
\end{cases} \tag{5.1}$$

where $0 < \varrho \leq 1$, $\delta > 0$, and $v_t(\theta,x) = v(t+\theta,x)$. The following conditions hold:

(1) The function $\zeta : [-\delta, 0] \to \mathbb{R}$ is integrable, i.e., $\int_{-\delta}^0 |\zeta(\theta)|d\theta < \infty$.

(2) The function $\zeta_1 : [0, b] \to \mathbb{R}$ is integrable, and $\int_0^b |\zeta_2(\theta)|d\theta < 1$.

(3) The function $\psi : [-\delta, 0] \times [0, \pi] \to \mathbb{R}$ is measurable and saisfies

$$\lim_{t_2 \to t_1} \int_0^{\pi} |\psi(t_2, x) - \psi(t_2, x)|^2 dx = 0,$$

for all $t_1, t_2 \in [-\delta, 0]$.

Let $X = L^2([0, \pi])$. Consider the operator $A = -\frac{\partial^2}{\partial x^2}$ in X with domain

$$D(A) = H^2([0, \pi]) \cap H_0^1([0, \pi])$$

where $H^2([0, \pi])$ and $H_0^1([0, \pi])$ are the classical Sobolev spaces. Eigenvalues and the corresponding normalized eigenfunctions of A are given by n^2, $\upsilon_n = \sqrt{\frac{2}{\pi}} \sin nx$, $n \in \mathbb{N}$. The family of eigenfunctions $\{\upsilon_n\}_{n=0}^{\infty}$ forms an orthonormal basis in X with inner product $(\omega, \nu) = \int_0^1 \omega(x)\overline{\nu(x)}dx$.

Define the family of linear operators $\left\{T_\varrho(t)\right\}_{t \geq 0}$ by

$$T_\varrho(t)\omega = \Sigma_{n=1}^{\infty} e^{-n^2 \frac{t^\varrho}{\varrho}} (\omega, \upsilon_n)\upsilon_n,$$

for $\omega \in X$ given by $\omega = \Sigma_{n=1}^{\infty}(\omega, \upsilon_n)\upsilon_n$. This family satisfies the following

(1) $T_\varrho(t)$ is a bounded linear operator, with $\|T_\varrho(t)\| \leq 1$ for $t \geq 0$.

(2) For $s, t \geq 0$ and $\omega \in X$ we get the semigroup property $T_\varrho(t^{\frac{1}{\varrho}})T_\varrho(s^{\frac{1}{\varrho}})\omega = T_\varrho(t^\varrho + s^\varrho)^{\frac{1}{\varrho}}\omega$.

(3) For $s, t \geq 0$, $\|T_\varrho(s) - T_\varrho(t)\| \to 0$ when $s \to t$.

(4) For $\omega \in D(A)$, $D_{0^+}^{\varrho} T_\varrho(t)\omega = AT_\varrho(t)\omega$. In particular $\lim_{t \to 0^+} D_{0^+}^{\varrho} T_\varrho(t)\omega = A\omega$.

Clearly, $\left\{T_\varrho(t)\right\}_{t \geq 0}$ is a uniformly bounded C_0-ϱ-semigroup which is continuous in the uniform operator topology for $t \geq 0$, and A its generator. For $x \in [0, \pi]$ and $\phi \in C([-\delta, 0], X)$, we set

$$\begin{aligned}
u(t)(x) &= \nu(t, x) \\
\varphi(t)(x) &= \psi(t, x) \\
f(t, \phi, \omega)(x) &= e^{-t} \arctan\left(\int_{-\delta}^{0} \sin(|\phi(\theta)(x)|)\, d\theta\right) + \int_0^{\pi} (1 + |\omega(x)|)^{-1}\, dx. \\
h(t, s, \phi)(x) &= \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{-\frac{4}{5}} \int_{-\delta}^{0} \zeta(\theta) \left(1 - \exp\left(-\left(\frac{s^\varrho}{\varrho}\right)^{-\frac{1}{5}}\right) s^{1-\varrho} \frac{|\phi(\theta)(x)|}{1 + |\phi(\theta)(x)|}\right) d\theta \\
g(u)(t)(x) &= \int_0^b \zeta_1(\theta) \cos\left(\frac{\pi}{2} + |\phi_t(\theta)(x)|\right) d\theta.
\end{aligned}$$

Then Eq (5.1) can be transformed to the abstract form (1.1).

For $t \in [0, b]$, we can obtain

$$\|f(t, \phi, \omega)\| \leq \pi^{\frac{3}{2}} \left(\frac{e^{-t}}{2} + 1\right) = m(t)$$

where $I_0^{\alpha,\varrho} m \in C((0, b], \mathbb{R}^+)$ and $\lim_{t \to 0^+} I_0^{\alpha,\varrho} m(t) = 0$.

For any $\phi, \tilde{\phi} \in C$ and $\omega, \tilde{\omega} \in X$, by straightforward calculations we get

$$\left\| f(t,\phi,\omega) - f\left(t,\tilde{\phi},\tilde{\omega}\right) \right\| \leq \delta e^{-t} \|\phi - \tilde{\phi}\|_C + \pi \|\omega - \tilde{\omega}\|.$$

Then for any bounded sets $\Lambda_1 \subset C$, $\Lambda \subset X$

$$\mu\left(f\left(t,\Lambda_1,\Lambda_2\right)\right) \leq L\left(\sup_{\theta\in[-\delta,0]} \mu\left(\Lambda_1(t)\right) + \mu\left(\Lambda_2\right) \right),$$

where $L = \delta + \pi$ and $t \in [0,b]$.

For each $t, s \in [0,b]$, $\phi \in C$, we obtain

$$\begin{aligned} \|h(t,s,\phi)\| &\leq \left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{-\frac{4}{5}} \int_{-\delta}^{0} \zeta(\theta) \left\| 1 - \exp\left(-\left(\frac{s^{\varrho}}{\varrho}\right)^{-\frac{1}{5}} s^{1-\varrho} \frac{|\phi(\theta)(x)|}{1+|\phi(\theta)(x)|} \right) \right\| d\theta \\ &\leq \left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{-\frac{4}{5}} \left(\frac{s^{\varrho}}{\varrho}\right)^{-\frac{1}{5}} s^{1-\varrho} \int_{-\delta}^{0} |\zeta(\theta)| d\theta \, \|\phi\|_C \\ &\leq m_1(t,s) \, \|\phi\|_C \, , \end{aligned}$$

where $m_1(t,s) = \left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{-\frac{4}{5}} \left(\frac{s^{\varrho}}{\varrho}\right)^{-\frac{1}{5}} s^{1-\varrho} \int_{-\delta}^{0} |\zeta(\theta)| d\theta$ satisfies

$$\begin{aligned} m_1^* = \sup_{t\in[0,b]} \int_0^t m_1(t,s) ds &= \int_{-\delta}^{0} |\zeta(\theta)| d\theta \sup_{t\in[0,b]} \int_0^t \left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{-\frac{4}{5}} \left(\frac{s^{\varrho}}{\varrho}\right)^{-\frac{1}{5}} \frac{ds}{s^{\varrho-1}} \\ &= \int_{-\delta}^{0} |\zeta(\theta)| d\theta \int_0^t t^{-\frac{4}{5}} (1-t)^{-\frac{1}{5}} \, dt = \beta\left(\frac{1}{5}, \frac{4}{5}\right) \int_{-\delta}^{0} |\zeta(\theta)| d\theta. \end{aligned}$$

For any $t, s \in [0,b]$, $\phi, \tilde{\phi} \in C([-\delta,0],X)$

$$\left\| h(t,s,\phi) - h\left(t,s,\tilde{\phi}\right) \right\| \leq \left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{-\frac{4}{5}} \left(\frac{s^{\varrho}}{\varrho}\right)^{-\frac{1}{5}} s^{1-\varrho} \int_{-\delta}^{0} |\zeta(\theta)| d\theta \|\phi - \tilde{\phi}\|_C.$$

Hence, for any bounded set $\Lambda \subset C$,

$$\mu\left(h(t,s,\Lambda)\right) \leq \gamma(s,t) \sup_{\theta\in[-\delta,0]} \mu\left(\Lambda_1(t)\right)$$

where $\gamma(s,t) = 2\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{-\frac{4}{5}} \left(\frac{s^{\varrho}}{\varrho}\right)^{-\frac{1}{5}} s^{1-\varrho} \int_{-\delta}^{0} |\zeta(\theta)| d\theta$, and $\gamma^* = 2\beta\left(\frac{1}{5}, \frac{4}{5}\right) \int_{-\delta}^{0} |\zeta(\theta)| d\theta$.

For all $t \in [-\delta,0]$, $\phi, \tilde{\phi} \in C([-\delta,b],X)$, we have

$$\|g\phi\|_C \leq L_3 \, \|\phi\|_\infty \, ,$$

and

$$\left\| g\phi(t) - g\tilde{\phi}(t) \right\| \leq \left\| \phi - \tilde{\phi} \right\|_\infty L_4$$

where $L_3 = L_4 = \int_0^b |\zeta_2(\theta)| \, d\theta$. Then $g(.)(t) : C([-\delta,b],X) \to X$ is continuous for any $t \in [-\delta,0]$, and therefore

$$\mu\left(g\left(\Lambda\right)(t)\right) \leq L_4 \mu\left(\Lambda(t)\right).$$

Since all conditions of Theorem 4.4 are satisfied, problem (5.1) has at least a mild solution.

6. Conclusions

In this manuscript, the existence results of mild solutions for non local fractional evolution equations with finite delay in the sense of Caputo conformable fractional derivative have been successfully investigated under some sufficient conditions on Kuratowski measure of non compactness. To the best of our knowledge, this type of problems supplemented with newly defined Caputo conformable fractional operator has not been investigated in any literature. All the obtained results are supported by an application showing the applicability of the presented theory.

Conflict of interest

The authors declare no conflict of interest

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