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Research article

Entire solutions for several Fermat type differential difference equations

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Abstract: This paper is devoted to investigate the existence and the forms of entire solutions of several Fermat type quadratic trinomial differential difference equations. Our results improve some results due to Liu and Yang [An. Stiint. Univ. Al. I. Cuza Iasi. Mat., 2016], Han and Lü [J. Contemp. Math. Anal., 2019], Luo, Xu and Hu [Open Math., 2021].

Keywords: entire solution; differential difference equation; quadratic trinomial
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1. Introduction and main results

In this paper, we assume that the reader is familiar with the basic notions of Nevanlinna's value distribution theory, see [6, 7, 18, 19]. In the following, a meromorphic function means meromorphic in the whole complex plane. By $S(r, f)$, we denote any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, possible outside of an exceptional set E with finite logarithmic measure $\int_E dr/r < \infty$. And we define the difference operators of $f(z)$ as $\Delta_c f(z) = f(z + c) - f(z)$, where c is a nonzero constant. For convenience, let

$$A_1 = \frac{1}{2\sqrt{1+\alpha}} - \frac{i}{2\sqrt{1-\alpha}}, A_2 = \frac{1}{2\sqrt{1+\alpha}} + \frac{i}{2\sqrt{1-\alpha}}, \tag{1.1}$$

where α is a constant satisfying $\alpha^2 \neq 0, 1$.

As is known to all, in 17th-century, French mathematician Fermat proposed the famous Fermat conjecture: Let $n \geq 3$, the equation $x^n + y^n = z^n$ has no positive integer solutions. Subsequently, it attracted the interest of many scholars in the mathematics field. After more than three hundred years, in 1995, British mathematician Andrew Wiles proved it with the knowledge of elliptical curves in geometry. Then the conjecture was further extended and developed. Nowadays, people call general equations $x^n + y^n = z^n$ as Fermat type equations. Many scholars have studied this type of equation and has achieved many results, see [1, 2, 8–10, 16, 20].

The classical results on the solutions of the Fermat type function equations

$$f^n(z) + g^n(z) = 1 \tag{1.2}$$

can be stated as follows: The Eq (1.2) has no transcendental meromorphic solutions when $n \geq 4$ [3], and it has no transcendental entire solutions when $n \geq 3$ [15]. If $n = 2$, then the Eq (1.2) has the entire solutions $f(z) = \sin(h(z))$ and $g(z) = \cos(h(z))$, where $h(z)$ is any entire function, no other solutions exists [4].

For the case that $g(z)$ has a special relationship with $f(z)$ in (1.2), Yang et al. [17] considered the solutions of the following equation

$$f(z)^2 + f'(z)^2 = 1, \tag{1.3}$$

and obtained that the transcendental meromorphic solutions of (1.3) must satisfy $f(z) = \frac{1}{2}(Pe^{\lambda z} + \frac{1}{P}e^{-\lambda z})$, where P, λ are nonzero constants.

In 2009, Liu [11] considered the entire solutions of the following equation

$$f(z)^2 + f(z+c)^2 = 1, \tag{1.4}$$

and obtained that the transcendental entire solutions with finite order of (1.4) must satisfy $f(z) = \frac{1}{2}(h_1(z) + h_2(z))$, where $\frac{h_1(z+c)}{h_1(z)} = i$, $\frac{h_2(z+c)}{h_2(z)} = -i$ and $h_1(z)h_2(z) = 1$.

In 2012, Liu et al. [12] obtained that the nonconstant finite order entire solutions of (1.4) must have order one.

In 2016, Liu et al. [13] studied the existence and the form of solutions of some quadratic trinomial functional equations and obtained the following results.

Theorem A. *Equation*

$$f(z)^2 + 2\alpha f(z)f'(z) + f'(z)^2 = 1 \tag{1.5}$$

has no transcendental meromorphic solutions.

Theorem B. *The finite order transcendental entire solutions of equation*

$$f(z)^2 + 2\alpha f(z)f(z+c) + f(z+c)^2 = 1 \tag{1.6}$$

must be of order equal to one.

In 2019, Han et al. [5] gave the description of meromorhic solutions for the functional Eq (1.2) when $g(z) = f'(z)$ and 1 is replaced by $e^{\alpha z+\beta}$, where $\alpha, \beta \in \mathbb{C}$, and obtained the following results.

Theorem C. *Let $f(z)$ be a meromorphic solution with finite order of the following differential equation*

$$f(z)^n + f'(z)^n = e^{\alpha z+\beta}. \tag{1.7}$$

Then $f(z)$ must be an entire function and satisfy one of the following cases:
(1) For $n = 1$, the general solutions of (1.7) are $f(z) = \frac{e^{\alpha z+\beta}}{\alpha+1} + ae^{-z}$ for $\alpha \neq -1$ and $f(z) = ze^{-z+\beta} + ae^{-z}$;
(2) For $n = 2$, either $\alpha = 0$ and the general solutions of (1.7) are $f(z) = e^{\frac{\beta}{2}} sin(z+b)$ or $f(z) = de^{\frac{\alpha z+\beta}{2}}$;
(3) For $n \geq 3$, the general solutions of (1.7) are $f(z) = de^{\frac{\alpha z+\beta}{2}}$.
Here, $a, b, d \in \mathbb{C}$ with $d^n(1 + \frac{\alpha}{n}) = 1$ for $n \geq 1$.

They also proved that all the trivial meromorphic solutions of $f(z)^n + f(z+c)^n = e^{\alpha z+\beta}$ are the functions $f(z) = de^{\frac{\alpha z+\beta}{2}}$ with $d^n(1+e^{\alpha c}) = 1$ for $n \geq 1$ [5].

Motivated by above question, in 2021, Luo et al. [14] considered the case that the right side of Eqs (1.5) and (1.6) were replaced by $e^{g(z)}$ in Theorems A and B, where $g(z)$ is a nonconstant polynomial. They proved:

Theorem D. *Let $g(z)$ be a nonconstant polynomial, and let $f(z)$ be a transcendental entire solution with finite order of the following difference equation*

$$f(z+c)^2 + 2\alpha f(z)f(z+c) + f(z)^2 = e^{g(z)}. \tag{1.8}$$

Then $g(z)$ must be of the form $g(z) = az+b$, where a, b are constants, and $f(z)$ must satisfy one of the following cases:
(1)$f(z) = \frac{1}{\sqrt{2}}(A_1\eta + A_2\eta^{-1})e^{\frac{1}{2}(az+b)}$, where $\eta(\neq 0)$ is a constant and $e^{\frac{1}{2}ac} = \frac{A_2\eta+A_1\eta^{-1}}{A_1\eta+A_2\eta^{-1}}$;
(2)$f(z) = \frac{1}{\sqrt{2}}(A_1e^{a_1z+b_1} + A_2e^{a_2z+b_2})$, where $a_i(\neq 0), b_i(i = 1,2)$ are constants satisfying $a_1 \neq a_2, g(z) = (a_1+a_2)z + b_1 + b_2 = az+b$, and $e^{a_1c} = \frac{A_2}{A_1}, e^{a_2c} = \frac{A_1}{A_2}, e^{ac} = 1$.

Theorem E. *Let $g(z)$ be a polynomial, and let $f(z)$ be a transcendental entire solution with finite order of the following differential equation*

$$f(z)^2 + 2\alpha f(z)f'(z) + f'(z)^2 = e^{g(z)}, \tag{1.9}$$

then $g(z)$ must be the form $g(z) = az+b$, where a, b are constants.

For the differential difference counterpart of Theorem E, they proved

Theorem F. *Let $g(z)$ be a nonconstant polynomial, and let $f(z)$ be a transcendental entire solution with finite order of the following differential difference equation*

$$f(z+c)^2 + 2\alpha f(z+c)f'(z) + f'(z)^2 = e^{g(z)}. \tag{1.10}$$

Then $g(z)$ must be of the form $g(z) = az+b$, where $a(\neq 0), b$ are constants, and $f(z)$ must satisfy one of the following cases:
(1) $f(z) = \frac{\sqrt{2}}{a}\left(A_1\eta^{-1} + A_2\eta\right)e^{\frac{1}{2}(az+b)}$, where $\eta(\neq 0)$ is a constant and $e^{\frac{1}{2}ac} = \frac{a(A_1\eta+A_2\eta^{-1})}{2(A_2\eta+A_1\eta^{-1})}$;
(2) $f(z) = \frac{1}{\sqrt{2}}\left(\frac{A_2}{a_1}e^{a_1z+b_1} + \frac{A_1}{a_2}e^{a_2z+b_2}\right)$, where $a_i(\neq 0), b_i(i = 1,2)$ are constants satisfying $a_1 \neq a_2, g(z) = (a_1+a_2)z + b_1 + b_2 = az+b$, and $e^{a_1c} = \frac{A_2}{A_1}a_1, e^{a_2c} = \frac{A_1}{A_2}a_2, e^{ac} = a_1a_2$.

From Theorems D–F, it is naturally to pose the following question.

Question: *If considering the relationship between $f(z), f'(z), \Delta_c f(z)$, does there similar conclusion exist?*

In this paper, we give a positive answer to this question, and prove the following results.

Theorem 1. *Let $g(z)$ be a nonconstant polynomial, and let $f(z)$ be a transcendental entire solution with finite order of the following difference equation*

$$f(z)^2 + 2\alpha f(z)\Delta_c f(z) + \Delta_c f(z)^2 = e^{g(z)}. \tag{1.11}$$

Then $g(z)$ *must be of the form* $g(z) = az + b$, *and* $f(z) = Ae^{\frac{1}{2}az}$, *where* $a(\neq 0), b, A(\neq 0)$ *are constants satisfying* $A^2[e^{ac} + 2(\alpha - 1)e^{\frac{1}{2}ac} - 2(\alpha - 1)] = e^b$.

Example 1.1. Let $\alpha = -\frac{1}{2}, A = 1, a = 2, b = ln7, c = ln4$, then $f(z) = e^z$. Thus, $f(z)$ is a solution of (1.11) with $g(z) = 2z + ln7$.

This example shows the existence of transcendental entire solutions with finite order of (1.11).

Theorem 2. *Let* $g(z)$ *be a nonconstant polynomial, and let* $f(z)$ *be a transcendental entire solution with finite order of the following difference equation*

$$f(z + c)^2 + 2\alpha f(z + c)\Delta_c f(z) + \Delta_c f(z)^2 = e^{g(z)}. \tag{1.12}$$

Then $g(z)$ *must be of the form* $g(z) = az + b$, *and* $f(z) = Ae^{\frac{1}{2}az}$, *where* $a(\neq 0), b, A(\neq 0)$ *are constants satisfying* $A^2[2(1 + \alpha)e^{ac} - 2(1 + \alpha)e^{\frac{1}{2}ac} + 1] = e^b$.

Example 1.2. $\alpha = -\frac{1}{2}, A = 1, a = 2, b = ln3, c = ln2$, then $f(z) = e^z$. Thus, $f(z)$ is a solution of (1.12) with $g(z) = 2z + ln3$.

This example shows the existence of transcendental entire solutions with finite order of (1.12).

Obviously, Theorem 2 cannot be directly obtained by Theorem 1.

Theorem 3. *Let* $g(z)$ *be a nonconstant polynomial, and let* $f(z)$ *be a transcendental entire solution with finite order of the following differential difference equation*

$$f'(z)^2 + 2\alpha f'(z)\Delta_c f(z) + \Delta_c f(z)^2 = e^{g(z)}. \tag{1.13}$$

Then $g(z)$ *must be of the form* $g(z) = az + b$, *where* $a(\neq 0), b$ *are constants, and* $f(z)$ *must satisfy one of the following cases:*

(1) $f(z) = Ae^{\frac{1}{2}az} + c_1$, *where* $A(\neq 0), c_1$ *are constants satisfying* $A^2[e^{ac} + (\alpha a - 2)e^{\frac{1}{2}ac} + \frac{1}{4}a^2 - \alpha a + 1] = e^b$;

(2) $f(z) = B_1 z + B_2 e^{az} + c_1$, *where* $B_i(\neq 0, i = 1, 2), c_1$ *are constants satisfying*

$$\begin{cases} a^2 + 2\alpha a(e^{ac} - 1) + (e^{ac} - 1)^2 = 0 \\ 1 + 2\alpha c + c^2 = 0 \\ a + \alpha(e^{ac} - 1) + \alpha ac + c(e^{ac} - 1) = \dfrac{e^b}{2B_1 B_2}; \end{cases}$$

(3) $f(z) = B_1 e^{a_1 z} + B_2 e^{(a - a_1)z} + c_1$, *where* $a_1(\neq 0), B_i(\neq 0, i = 1, 2), c_1$ *are constants satisfying*

$$\begin{cases} a_1^2 + 2\alpha a_1(e^{a_1 c} - 1) + (e^{a_1 c} - 1)^2 = 0 \\ (a - a_1)^2 + 2\alpha(a - a_1)(e^{(a - a_1)c} - 1) + (e^{(a - a_1)c} - 1)^2 = 0 \\ a_1(a - a_1) + \alpha a_1(e^{(a - a_1)c} - 1) + \alpha(a - a_1)(e^{a_1 c} - 1) + (e^{a_1 c} - 1)(e^{(a - a_1)c} - 1) = \dfrac{e^b}{2B_1 B_2}. \end{cases}$$

Example 1.3. Let $\alpha = \frac{1}{2}, A = 1, a = 2, b = ln3, c = ln2, c_1 = 0$ then $f(z) = e^z$. Thus, $f(z)$ is a solution of (1.13) with $g(z) = 2z + ln3$.

This example shows the existence of the conclusion (1) of Theorem 3.

Example 1.4. Let $\alpha = -\frac{1 + ln2}{2(ln2)^{\frac{1}{2}}}, a = c = (ln2)^{\frac{1}{2}}, b = ln\left(-\frac{(ln2 - 1)^2}{(ln2)^{\frac{1}{2}}}\right), c_1 = 0, B_1 = B_2 = 1$, then $f(z) = z + e^{(ln2)^{\frac{1}{2}}z}$. Thus, $f(z)$ is a solution of (1.13) with $g(z) = (ln2)^{\frac{1}{2}}z + ln\left(-\frac{(ln2 - 1)^2}{(ln2)^{\frac{1}{2}}}\right)$.

This example shows the existence of the conclusion (2) of Theorem 3.

Example 1.5. Let $\alpha = -\frac{5}{4}, a_1 = 2, c = \frac{ln2}{2}$, then we have $e^{(a-a_1)c} - 1 = e^{\frac{ln2}{2}(a-2)} - 1$. So equation $(a-a_1)^2 + 2\alpha(a-a_1)(e^{(a-a_1)c} - 1) + (e^{(a-a_1)c} - 1)^2 = 0$ can be written as $h(z)^2 + (3 - \frac{5}{2}z)h(z) + z^2 - \frac{3}{2}z = 0$, where $h(z) = e^{\frac{ln2}{2}(z-2)}$.

By the Nevanlinna's second fundamental, we have

$$\begin{aligned}
&2T(r, h(z)) \\
\leq &T\left(r, h(z)^2 + \left(3 - \frac{5}{2}z\right)h(z) + z^2 - \frac{3}{2}z\right) + S(r, h) \\
\leq &\overline{N}\left(r, \frac{1}{h(z)^2 + \left(3 - \frac{5}{2}z\right)h(z)}\right) + \overline{N}\left(r, \frac{1}{h(z)^2 + (3 - \frac{5}{2}z)h(z) + z^2 - \frac{3}{2}z}\right) + S(r, h) \\
\leq &T(r, h(z)) + \overline{N}\left(r, \frac{1}{h(z)^2 + (3 - \frac{5}{2}z)h(z) + z^2 - \frac{3}{2}z}\right) + S(r, h).
\end{aligned}$$

So we have $T(r, h(z)) \leq \overline{N}\left(r, \frac{1}{h(z)^2 + (3-\frac{5}{2}z)h(z) + z^2 - \frac{3}{2}z}\right) + S(r, h)$, which means that the equation $(a - a_1)^2 + 2\alpha(a - a_1)(e^{(a-a_1)c} - 1) + (e^{(a-a_1)c} - 1)^2 = 0$ must have infinitely many solutions. Then we can chose one a satisfying this equation.

In addition, from $a_1(a - a_1) + \alpha a_1(e^{(a-a_1)c} - 1) + \alpha(a - a_1)(e^{a_1 c} - 1) + (e^{a_1 c} - 1)(e^{(a-a_1)c} - 1) = \frac{e^b}{2B_1B_2}$, we know that there must exist $b, B_i(\neq 0, i = 1, 2)$ satisfying this equation.

This example shows the existence of the conclusion (3) of Theorem 3.

2. Preliminary lemmas

For the proof of our results, we need the following lemmas.

Lemma 1. [18] *Let $f_j(z)(j = 1, 2, 3)$ be meromorphic functions, and let $f_1(z)$ be a nonconstant function. If $\sum_{j=1}^{3} f_j \equiv 1$ and*

$$\sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right) + 2\sum_{j=1}^{3} \overline{N}(r, f_j) < (\lambda + o(1))T(r),$$

where $\lambda < 1$ and $T(r) = \max_{1\leq j\leq 3} T(r, f_j)$, then $f_2(z) \equiv 1$ or $f_3(z) \equiv 1$.

Lemma 2. [6, 18, 19] *Let $f(z)$ be a meromorphic function in the complex plane. If $f \neq 0, \infty$, then there exists an entire function $\alpha(z)$ such that $f(z) = e^{\alpha(z)}$.*

Lemma 3. [6, 18, 19] *Let $f(z)$ be a nonconstant meromorphic function, and let $a(z), b(z)$ be two distinct small functions of $f(z)$. Then*

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f - a}\right) + \overline{N}\left(r, \frac{1}{f - b}\right) + S(r, f).$$

3. Proof of Theorem 1

Suppose that $f(z)$ is a transcendental entire solution with finite order of Eq (1.11).

Let

$$u(z) = \frac{1}{\sqrt{2}}(f(z) + \Delta_c f(z)), v(z) = \frac{1}{\sqrt{2}}(f(z) - \Delta_c f(z)).$$

Then we have

$$f(z) = \frac{1}{\sqrt{2}}(u + v), \Delta_c f(z) = \frac{1}{\sqrt{2}}(u - v).$$

Thus, we know that Eq (1.11) can be written as

$$(1+\alpha)u^2 + (1-\alpha)v^2 = e^{g(z)}. \tag{3.1}$$

It follows from (3.1) that

$$\left(\frac{\sqrt{1+\alpha}u}{e^{\frac{g(z)}{2}}}\right)^2 + \left(\frac{\sqrt{1-\alpha}v}{e^{\frac{g(z)}{2}}}\right)^2 = 1.$$

The above equation leads to

$$\left(\frac{\sqrt{1+\alpha}u}{e^{\frac{g(z)}{2}}} + i\frac{\sqrt{1-\alpha}v}{e^{\frac{g(z)}{2}}}\right)\left(\frac{\sqrt{1+\alpha}u}{e^{\frac{g(z)}{2}}} - i\frac{\sqrt{1-\alpha}v}{e^{\frac{g(z)}{2}}}\right) = 1.$$

Since f is a finite order transcendental entire function and g is a polynomial, then by Lemma 2, there exists a polynomial $p(z)$ such that

$$\begin{cases} \dfrac{\sqrt{1+\alpha}u}{e^{\frac{g(z)}{2}}} + i\dfrac{\sqrt{1-\alpha}v}{e^{\frac{g(z)}{2}}} = e^{p(z)}, \\ \dfrac{\sqrt{1+\alpha}u}{e^{\frac{g(z)}{2}}} - i\dfrac{\sqrt{1-\alpha}v}{e^{\frac{g(z)}{2}}} = e^{-p(z)}. \end{cases} \tag{3.2}$$

Denote

$$r_1(z) = \frac{g(z)}{2} + p(z), r_2(z) = \frac{g(z)}{2} - p(z). \tag{3.3}$$

By combining with (3.2) and (3.3), we have

$$\sqrt{1+\alpha}u = \frac{e^{r_1(z)} + e^{r_2(z)}}{2}, \sqrt{1-\alpha}v = \frac{e^{r_1(z)} - e^{r_2(z)}}{2i}.$$

This leads to

$$\begin{aligned} f(z) = \frac{1}{\sqrt{2}}(u+v) =& \frac{1}{\sqrt{2}}\left(\frac{e^{r_1(z)} + e^{r_2(z)}}{2\sqrt{1+\alpha}} + \frac{e^{r_1(z)} - e^{r_2(z)}}{2\sqrt{1-\alpha}i}\right) \\ =& \frac{1}{\sqrt{2}}(A_1 e^{r_1(z)} + A_2 e^{r_2(z)}), \end{aligned} \tag{3.4}$$

$$\Delta_c f(z) = \frac{1}{\sqrt{2}}(u-v) = \frac{1}{\sqrt{2}}\left(\frac{e^{r_1(z)} + e^{r_2(z)}}{2\sqrt{1+\alpha}} - \frac{e^{r_1(z)} - e^{r_2(z)}}{2\sqrt{1-\alpha}i}\right)$$

$$=\frac{1}{\sqrt{2}}(A_2e^{r_1(z)}+A_1e^{r_2(z)}),\tag{3.5}$$

where A_1, A_2 are defined as (1.1).

It follows from (3.4) that

$$\triangle_c f(z)=\frac{1}{\sqrt{2}}\left[(A_1e^{r_1(z+c)}+A_2e^{r_2(z+c)})-(A_1e^{r_1(z)}+A_2e^{r_2(z)})\right].\tag{3.6}$$

Combing with (3.5) and (3.6), we have

$$\left(\frac{A_2}{A_1}+1\right)e^{r_1(z)-r_1(z+c)}+\left(\frac{A_2}{A_1}+1\right)e^{r_2(z)-r_1(z+c)}-\frac{A_2}{A_1}e^{r_2(z+c)-r_1(z+c)}=1.\tag{3.7}$$

Next we consider the following two cases.

Case 1. $r_2(z)-r_1(z+c)$ is a constant.

In the following, we consider the following two subcases.

Case 1.1. $r_1(z)-r_1(z+c)$ is a constant.

It follows that $r_2(z)-r_1(z+c)$ and $r_1(z)-r_1(z+c)$ are constants.

Combing with (3.3), we have

$$r_1(z)-r_2(z)=2p(z).\tag{3.8}$$

From (3.8), we know that $p(z)$ is a constant.

Let $\eta=e^p$. Substituting this into (3.4) and (3.5), we have

$$f(z)=\frac{1}{\sqrt{2}}(A_1\eta+A_2\eta^{-1})e^{\frac{1}{2}g(z)},\tag{3.9}$$

$$\Delta_c f(z)=\frac{1}{\sqrt{2}}(A_2\eta+A_1\eta^{-1})e^{\frac{1}{2}g(z)}.\tag{3.10}$$

From (3.9), we get

$$\Delta_c f(z)=\frac{1}{\sqrt{2}}(A_1\eta+A_2\eta^{-1})(e^{\frac{1}{2}g(z+c)}-e^{\frac{1}{2}g(z)}).\tag{3.11}$$

Combing with (3.10) and (3.11), we obtain

$$(A_1\eta+A_2\eta^{-1})e^{\frac{1}{2}g(z+c)-\frac{1}{2}g(z)}=(A_1\eta+A_2\eta^{-1})+(A_2\eta+A_1\eta^{-1}).\tag{3.12}$$

In view of $\alpha^2\neq 0,1$ and $e^{\frac{g(z+c)-g(z)}{2}}$ have no zeros and poles, it follows that $A_1\eta+A_2\eta^{-1}=0$ and $(A_1\eta+A_2\eta^{-1})+(A_2\eta+A_1\eta^{-1})=0$ cannot hold at the same time. Hence, we have $A_1\eta+A_2\eta^{-1}\neq 0$ and $(A_1+A_2)\eta+(A_1+A_2)\eta^{-1}\neq 0$.

Since $g(z)$ is a polynomial, then (3.12) implies that $g(z+c)-g(z)$ is a constant. Otherwise, we obtain a contradiction from the left of the above equation is transcendental but the right is not transcendental.

So we get $g(z)=az+b$, where a,b are constants satisfying $e^{\frac{1}{2}ac}=\frac{(A_1+A_2)(\eta+\eta^{-1})}{A_1\eta+A_2\eta^{-1}}$.

Furthermore, we obtain

$$f(z) = \frac{1}{\sqrt{2}}(A_1\eta + A_2\eta^{-1})e^{\frac{1}{2}(az+b)}.$$

Since Eq (1.11) and $f(z)$ are only related to a, b, c, α, then $f(z)$ can be written as $f(z) = Ae^{\frac{1}{2}az}$, where A is a nonzero constant. Substituting it into Eq (1.11), we know that a, b, c, A, α must satisfy $A^2[e^{ac} + 2(\alpha-1)e^{\frac{1}{2}ac} - 2(\alpha-1)] = e^b$.

Case 1.2. $r_1(z) - r_1(z+c)$ is not a constant.

If $e^{r_2(z+c)-r_1(z+c)}$ is a constant, we have $e^{r_1(z)-r_1(z+c)}$ is a constant, a contradiction. So we know that $e^{r_2(z+c)-r_1(z+c)}$ is not a constant.

Let

$$\xi = \frac{A_1 + A_2}{A_1}e^{r_2(z)-r_1(z+c)}. \tag{3.13}$$

From (3.7), we get

$$\left(\frac{A_2}{A_1} + 1\right)e^{r_1(z)-r_1(z+c)} - \frac{A_2}{A_1}e^{r_2(z+c)-r_1(z+c)} = 1 - \xi. \tag{3.14}$$

Next we consider the following two subcases.

Case 1.2.1. $1 - \xi = 0$.

Combing with (3.13) and (3.14) we have

$$e^{r_1(z+c)-r_2(z)} = \frac{A_1 + A_2}{A_1}, e^{r_2(z+c)-r_1(z)} = \frac{A_1 + A_2}{A_2},$$

which means that $r_1(z+c) - r_2(z), r_2(z+c) - r_1(z)$ are constants.

It follows from (3.3) that

$$e^{r_1(z+c)+r_2(z+c)-r_1(z)-r_2(z)} = e^{g(z+c)-g(z)} = \frac{(A_1 + A_2)^2}{A_1A_2}.$$

Thus, we obtain $g(z) = az + b$, where $a(\neq 0), b$ are constants satisfying $e^{ac} = \frac{(A_1+A_2)^2}{A_1A_2}$.

Combing with $r_1(z) + r_2(z) = az + b$ and $r_2(z) - r_1(z+c)$ is a constant, we get $r_1(z) = a_1z + b_1, r_2(z) = a_2z + b_2$, where $a_i, b_i (i = 1, 2)$ are constants satisfying $a_1 + a_2 = 0$.

Substituting this into (3.4), we have

$$f(z) = \frac{1}{\sqrt{2}}(A_1e^{a_1z+b_1} + A_2e^{a_2z+b_2}).$$

Similarly, $f(z)$ can be written as $f(z) = B_1e^{a_1z} + B_2e^{(a-a_1)z}$, where $B_i (i = 1, 2)$ are nonzero constants. Substituting it into Eq (1.11), we get a contradiction.

Case 1.2.2. $1 - \xi \neq 0$.

By Lemma 3 and (3.14), we have

$$T\left(r, e^{r_1(z)-r_1(z+c)}\right) \leq \overline{N}\left(r, e^{r_1(z)-r_1(z+c)}\right) + \overline{N}\left(r, \frac{1}{e^{r_1(z)-r_1(z+c)}}\right)$$

$$+\overline{N}\left(r,\frac{1}{e^{r_1(z)-r_1(z+c)}-\frac{A_1}{A_1+A_2}(1-\xi)}\right)+S\left(r,e^{r_1(z)-r_1(z+c)}\right)$$
$$=S\left(r,e^{r_1(z)-r_1(z+c)}\right),$$

a contradiction.

Case 2. $r_2(z)-r_1(z+c)$ is not a constant.

Since $r_1(z), r_2(z)$ are polynomials and $e^{r_2(z)-r_1(z+c)}$ is not a constant, by Lemma 1 and (3.7), we deduce that either $\left(\frac{A_2}{A_1}+1\right)e^{r_1(z)-r_1(z+c)}\equiv 1$ or $-\frac{A_2}{A_1}e^{r_2(z+c)-r_1(z+c)}\equiv 1$.

If $-\frac{A_2}{A_1}e^{r_2(z+c)-r_1(z+c)}\equiv 1$. It follows from (3.7) that

$$\left(\frac{A_2}{A_1}+1\right)e^{r_1(z)-r_1(z+c)}+\left(\frac{A_2}{A_1}+1\right)e^{r_2(z)-r_1(z+c)}\equiv 0,$$

which means that $-e^{r_1(z)-r_2(z)}\equiv 1$.

From (3.3), we have $-e^{2p(z)}\equiv 1$. Combing with $-\frac{A_2}{A_1}e^{r_2(z+c)-r_1(z+c)}\equiv 1$, we have $-e^{2p(z+c)}\equiv 1$ and $-\frac{A_2}{A_1}e^{-2p(z+c)}\equiv 1$. So we get $A_1^2=A_2^2$. This is a contradiction with $\alpha^2\neq 0,1$.

If $\left(\frac{A_2}{A_1}+1\right)e^{r_1(z)-r_1(z+c)}\equiv 1$. It follows that $r_1(z)=a_1z+b_1$, where a_1, b_1 are constants satisfying $e^{a_1c}=\frac{A_1+A_2}{A_1}$.

From (3.7), we have

$$\frac{A_1+A_2}{A_2}e^{r_2(z)-r_2(z+c)}=1.$$

This means that $r_2(z)=a_2z+b_2$, where a_2, b_2 are constants satisfying $e^{a_2c}=\frac{A_1+A_2}{A_2}$. Since $e^{r_2(z)-r_1(z+c)}$ is not a constant, it follows that $a_1\neq a_2$.

So we have $g(z)=(a_1+a_2)z+b_1+b_2=az+b$ and $e^{ac}=\frac{(A_1+A_2)^2}{A_1A_2}$.

Substituting this into (3.4), we have

$$f(z)=\frac{1}{\sqrt{2}}(A_1e^{a_1z+b_1}+A_2e^{a_2z+b_2}).$$

Similarly, we can get a contradiction, which means that the above format of $f(z)$ does not exist.

Therefore, this completes the proof of Theorem 1. □

4. Proof of Theorem 2

Suppose that $f(z)$ is a transcendental entire solution with finite order of Eq (1.12). By using the same argument as the proof of Theorem 1, we have

$$f(z+c)=\frac{1}{\sqrt{2}}(A_1e^{r_1(z)}+A_2e^{r_2(z)}), \tag{4.1}$$
$$\Delta_cf(z)=\frac{1}{\sqrt{2}}(A_2e^{r_1(z)}+A_1e^{r_2(z)}), \tag{4.2}$$

where A_1, A_2 are defined as (1.1).

It follows from (4.1) that

$$\Delta_c f(z) = \frac{1}{\sqrt{2}}(A_1 e^{r_1(z)} + A_2 e^{r_2(z)}) - \frac{1}{\sqrt{2}}(A_1 e^{r_1(z-c)} + A_2 e^{r_2(z-c)}).$$

Then combing with (4.2), we have

$$e^{r_2(z)-r_1(z)} + \frac{A_2}{A_1 - A_2} e^{r_2(z-c)-r_1(z)} + \frac{A_1}{A_1 - A_2} e^{r_1(z-c)-r_1(z)} = 1. \tag{4.3}$$

Next we consider the following two cases.

Case 1. $r_2(z) - r_1(z)$ is a constant.

Combing with (3.3), we know that $p(z)$ is a constant.

Let $\eta = e^p$. Substituting this into (4.1) and (4.2), we have

$$f(z+c) = \frac{1}{\sqrt{2}}(A_1\eta + A_2\eta^{-1})e^{\frac{1}{2}g(z)}, \tag{4.4}$$

$$\Delta_c f(z) = \frac{1}{\sqrt{2}}(A_2\eta + A_1\eta^{-1})e^{\frac{1}{2}g(z)}. \tag{4.5}$$

From (4.4), we know that

$$\begin{aligned}\Delta_c f(z) =& f(z+c) - f(z)\\ =& \frac{1}{\sqrt{2}}(A_1\eta + A_2\eta^{-1})e^{\frac{1}{2}g(z)} - \frac{1}{\sqrt{2}}(A_1\eta + A_2\eta^{-1})e^{\frac{1}{2}g(z-c)}\\ =& \frac{1}{\sqrt{2}}(A_1\eta + A_2\eta^{-1})(e^{\frac{1}{2}g(z)} - e^{\frac{1}{2}g(z-c)}).\end{aligned}$$

Combing with above equation and (4.5), we have

$$(A_1\eta + A_2\eta^{-1})e^{\frac{1}{2}(g(z-c)-g(z))} = (A_1\eta + A_2\eta^{-1}) - (A_2\eta + A_1\eta^{-1}). \tag{4.6}$$

It follows from $\alpha^2 \neq 1$ that $A_1\eta + A_2\eta^{-1} = 0$ and $(A_1\eta + A_2\eta^{-1}) - (A_2\eta + A_1\eta^{-1}) = 0$ cannot hold at the same time. Hence, we have $A_1\eta + A_2\eta^{-1} \neq 0$ and $(A_1\eta + A_2\eta^{-1}) - (A_2\eta + A_1\eta^{-1}) \neq 0$.

By (4.6), we know that $g(z-c) - g(z)$ is a constant. Since $g(z)$ is a polynomial, it follows that $g(z) = az + b$, where a, b are constants satisfying $e^{\frac{1}{2}ac} = \frac{A_1\eta + A_2\eta^{-1}}{(A_1 - A_2)(\eta - \eta^{-1})}$.

By (4.4), we have

$$f(z+c) = \frac{1}{\sqrt{2}}(A_1\eta + A_2\eta^{-1})e^{\frac{1}{2}(az+b)}.$$

From above equation, we obtain

$$f(z) = \frac{1}{\sqrt{2}}(A_1\eta + A_2\eta^{-1})e^{\frac{1}{2}[a(z-c)+b]}.$$

Since Eq (1.12) and $f(z)$ are only related to a, b, c, α, then $f(z)$ can be written as $f(z) = Ae^{\frac{1}{2}az}$, where a, A are nonzero constants. Substituting it into Eq (1.12), we know that a, b, c, A, α must satisfy $A^2[2(1+\alpha)e^{ac} - 2(1+\alpha)e^{\frac{1}{2}ac} + 1] = e^b$.

Case 2. $r_2(z) - r_1(z)$ is not a constant.

Since $r_1(z), r_2(z)$ are polynomials and $e^{r_2(z)-r_1(z)}$ is not a constant, by Lemma 1 and (4.3), we deduce that either $\frac{A_2}{A_1-A_2}e^{r_2(z-c)-r_1(z)} \equiv 1$ or $\frac{A_1}{A_1-A_2}e^{r_1(z-c)-r_1(z)} \equiv 1$.

Case 2.1. $\frac{A_2}{A_1-A_2}e^{r_2(z-c)-r_1(z)} \equiv 1$.

From (4.3), we get

$$\frac{A_2 - A_1}{A_1}e^{r_2(z)-r_1(z-c)} \equiv 1.$$

Combing with $\frac{A_2}{A_1-A_2}e^{r_2(z-c)-r_1(z)} \equiv 1$ and (3.3), we obtain

$$e^{2p(z-c)+2p(z)} \equiv -\frac{A_2}{A_1},$$

which imply that $p(z)$ is a constant. So we have $r_2(z) - r_1(z)$ is a constant. This is a contradiction with $r_2(z) - r_1(z)$ is not a constant.

Case 2.2. $\frac{A_1}{A_1-A_2}e^{r_1(z-c)-r_1(z)} \equiv 1$.

Then it follows that $r_1(z) = a_1 z + b_1$, where a_1, b_1 are constants satisfying $e^{a_1 c} = \frac{A_1}{A_1-A_2}$.

Moreover, it follows from (4.3) that

$$e^{r_2(z)-r_1(z)} + \frac{A_2}{A_1 - A_2}e^{r_2(z-c)-r_1(z)} = 0.$$

So we have

$$\frac{A_2}{A_2 - A_1}e^{r_2(z-c)-r_2(z)} \equiv 1.$$

This means $r_2(z) = a_2 z + b_2$, where a_2, b_2 are constants satisfying $e^{a_2 c} = \frac{A_2}{A_2-A_1}$.

Since $e^{r_2(z)-r_1(z)}$ is not a constant, it follows that $a_1 \neq a_2$. Thus, we have $g(z) = (a_1 + a_2)z + b_1 + b_2 = az + b$ and $e^{ac} = -\frac{A_1 A_2}{(A_1-A_2)^2}$.

Substituting this into (4.1), we have

$$f(z + c) = \frac{1}{\sqrt{2}}(A_1 e^{a_1 z+b_1} + A_2 e^{a_2 z+b_2}).$$

So we obtain

$$f(z) = \frac{1}{\sqrt{2}}(A_1 e^{a_1(z-c)+b_1} + A_2 e^{a_2(z-c)+b_2}).$$

Similarly, $f(z)$ can be written as $f(z) = B_1 e^{a_1 z} + B_2 e^{(a-a_1)z}$, where $B_i (i = 1, 2)$ are nonzero constants. Substituting it into Eq (1.12), we get a contradiction, which means that the above format of $f(z)$ does not exist.

Therefore, this completes the proof of Theorem 2. □

5. Proof of Theorem 3

Suppose that $f(z)$ is a transcendental entire solution with finite order of Eq (1.13). By using the same argument as the proof of Theorem 1, we have

$$f'(z) = \frac{1}{\sqrt{2}}(A_1 e^{r_1(z)} + A_2 e^{r_2(z)}), \tag{5.1}$$

$$\Delta_c f(z) = \frac{1}{\sqrt{2}}(A_2 e^{r_1(z)} + A_1 e^{r_2(z)}), \tag{5.2}$$

where A_1, A_2 are defined as (1.1).

Thus, it follows from (5.1) and (5.2) that

$$\begin{aligned}(\Delta_c f(z))' =& f'(z+c) - f'(z)\\ =& \frac{1}{\sqrt{2}}[(A_1 e^{r_1(z+c)} + A_2 e^{r_2(z+c)}) - (A_1 e^{r_1(z)} + A_2 e^{r_2(z)})]\\ =& \frac{1}{\sqrt{2}}(A_2 r_1'(z) e^{r_1(z)} + A_1 r_2'(z) e^{r_2(z)}).\end{aligned}$$

Then we have

$$\left(\frac{A_2}{A_1} r_1'(z) + 1\right) e^{r_1(z) - r_1(z+c)} + \left(\frac{A_2}{A_1} + r_2'(z)\right) e^{r_2(z) - r_1(z+c)} - \frac{A_2}{A_1} e^{r_2(z+c) - r_1(z+c)} = 1. \tag{5.3}$$

Next, we discuss the following two cases.

Case 1. $r_2(z+c) - r_1(z+c)$ is a constant.

From (3.3), we know that $r_1(z+c) - r_2(z+c) = 2p(z+c)$. So $p(z)$ is a constant.

Let $\eta = e^p$. Substituting this into (5.1) and (5.2), it follows that

$$f'(z) = \frac{1}{\sqrt{2}}(A_1\eta + A_2\eta^{-1}) e^{\frac{1}{2}g(z)}, \tag{5.4}$$

$$\Delta_c f(z) = \frac{1}{\sqrt{2}}(A_2\eta + A_1\eta^{-1}) e^{\frac{1}{2}g(z)}. \tag{5.5}$$

Thus, we can deduce from (5.4) and (5.5) that

$$\begin{aligned}(\Delta_c f(z))' =& f'(z+c) - f'(z)\\ =& \frac{1}{\sqrt{2}}(A_1\eta + A_2\eta^{-1}) e^{\frac{1}{2}g(z+c)} - \frac{1}{\sqrt{2}}(A_1\eta + A_2\eta^{-1}) e^{\frac{1}{2}g(z)}\\ =& \frac{1}{\sqrt{2}}(A_2\eta + A_1\eta^{-1}) e^{\frac{1}{2}g(z)} \cdot \frac{1}{2} g'(z).\end{aligned}$$

So we have

$$(A_1\eta + A_2\eta^{-1}) + \frac{1}{2}(A_2\eta + A_1\eta^{-1}) g'(z) = (A_1\eta + A_2\eta^{-1}) e^{\frac{1}{2}(g(z+c) - g(z))}. \tag{5.6}$$

Next we consider the following two subcases.

Case 1.1. $\deg(g) \geq 2$.

It follows that $g'(z) \not\equiv 0$ and $g(z+c) - g(z)$ is not a constant. Eq (5.6) implies that $A_2\eta + A_1\eta^{-1} = 0$ and $A_1\eta + A_2\eta^{-1} = 0$.

Otherwise, if $A_1\eta + A_2\eta^{-1} \neq 0$, we have

$$e^{\frac{1}{2}(g(z+c)-g(z))} = \frac{1}{2}g'(z)\frac{A_2\eta + A_1\eta^{-1}}{A_1\eta + A_2\eta^{-1}} + 1. \tag{5.7}$$

The left of Eq (5.7) is transcendental, but the right of Eq (5.7) is a polynomial. This is a contradiction.

If $A_1\eta + A_2\eta^{-1} = 0$, by (5.6), we obtain $A_2\eta + A_1\eta^{-1} = 0$. So we can deduce that $A_1^2 = A_2^2$, which is a contradiction with $\alpha^2 \neq 0, 1$.

Case 1.2. $\deg(g) = 1$.

That is $g(z) = az + b$, where $a(\neq 0), b$ are constants. It follows from (5.6) that

$$e^{\frac{1}{2}ac} = \frac{1}{2}\frac{A_2\eta + A_1\eta^{-1}}{A_1\eta + A_2\eta^{-1}}a + 1.$$

Combing with (5.4), we have

$$f'(z) = \frac{1}{\sqrt{2}}(A_1\eta + A_2\eta^{-1})e^{\frac{1}{2}(az+b)}.$$

So we obtain

$$f(z) = \frac{\sqrt{2}}{a}(A_1\eta + A_2\eta^{-1})e^{\frac{1}{2}(az+b)} + c_1,$$

where c_1 is a constant.

Since Eq (1.13) and $f(z)$ is only related to a, b, c, α, then $f(z)$ can be written as $f(z) = Ae^{\frac{1}{2}az} + c_1$ where $A(\neq 0), c_1$ are constants. Substituting it into Eq (1.13), we know that a, b, c, A, α must satisfy $A^2[e^{ac} + (\alpha a - 2)e^{\frac{1}{2}ac} + \frac{1}{4}a^2 - \alpha a + 1] = e^b$.

Thus, we get the conclusion (1) of Theorem 3.

Case 2. $r_2(z+c) - r_1(z+c)$ is not a constant.

It follows from (3.3) that $p(z)$ is not a constant.

Next, we consider the following four subcases.

Case 2.1. $r_1'(z) \equiv 0, r_2'(z) \equiv 0$.

It follows that $r_1(z)$ and $r_2(z)$ are constants. Hence, $r_2(z+c), r_1(z+c)$ are constants. So we get $r_2(z+c) - r_1(z+c)$ is a constant, a contradiction.

Case 2.2. $r_1'(z) \equiv 0, r_2'(z) \not\equiv 0$.

It follows from (5.3) that

$$e^{r_2(z+c)-r_2(z)} = 1 + \frac{A_1}{A_2}r_2'(z). \tag{5.8}$$

If $\deg(r_2) \geq 2$, we have a contradiction from the left of Eq (5.8) is transcendental, but the right of Eq (5.8) is a polynomial. Thus, we have $r_2(z) = a_2z + b_2$ and $r_1(z) = b_1$, where $a_2(\neq 0), b_1, b_2$ are constants.

Combing with (5.1), we obtain

$$f'(z) = \frac{1}{\sqrt{2}}(A_1e^{b_1} + A_2e^{a_2z+b_2}).$$

Thus, we have

$$f(z) = \frac{1}{\sqrt{2}}(A_1e^{b_1}z + \frac{A_2}{a_2}e^{a_2z+b_2}) + c_1,$$

where c_1 is a constant.

Similarly, $f(z)$ can be written as $f(z) = B_1z + B_2e^{az} + c_1$, where $B_i(\neq 0, i = 1, 2), c_1$ are constants and $a = a_1$. Substituting it into Eq (1.13), we know that $a, b, c, \alpha, B_i(i = 1, 2)$ must satisfy

$$\begin{cases} a^2 + 2\alpha a(e^{ac} - 1) + (e^{ac} - 1)^2 = 0 \\ 1 + 2\alpha c + c^2 = 0 \\ a + \alpha(e^{ac} - 1) + \alpha ac + c(e^{ac} - 1) = \dfrac{e^b}{2B_1B_2}. \end{cases}$$

Thus, we get the conclusion (2) of Theorem 3.

Case 2.3. $r_1'(z) \not\equiv 0$ and $r_2'(z) \equiv 0$.

It follows from (5.3) that

$$\left(\frac{A_2}{A_1}r_1'(z) + 1\right)e^{r_1(z)-r_1(z+c)} \equiv 1. \tag{5.9}$$

If $\deg(r_1) \geq 2$, we have a contradiction from the left of Eq (5.9) is transcendental, but the right of Eq (5.9) is a polynomial. Thus, $r_1(z) = a_1z + b_1, r_2(z) = b_2$ and $e^{a_1c} = 1 + \frac{A_2}{A_1}a_1$, where $a_1(\neq 0), b_1, b_2$ are constants.

Substituting this into (5.1), we get

$$f'(z) = \frac{1}{\sqrt{2}}(A_1e^{a_1z+b_1} + A_2e^{b_2}).$$

So we have

$$f(z) = \frac{1}{\sqrt{2}}\left(\frac{A_1}{a_1}e^{a_1z+b_1} + A_2e^{b_2}z\right) + c_1,$$

where c_1 is a constant.

Similarly, $f(z)$ can be written as $f(z) = B_1z + B_2e^{az} + c_1$, where $B_i(\neq 0, i = 1, 2), c_1$ are constants and $a = a_1$. Substituting it into Eq (1.13), we can also get the conclusion (2) of Theorem 3.

Case 2.4. $r_1'(z) \not\equiv 0, r_2'(z) \not\equiv 0$.

By Lemma 1, we deduce that either $\left(\frac{A_2}{A_1}r_1'(z) + 1\right)e^{r_1(z)-r_1(z+c)} \equiv 1$ or $(\frac{A_2}{A_1} + r_2'(z))e^{r_2(z)-r_1(z+c)} \equiv 1$.

If $\left(\frac{A_2}{A_1}r_1'(z) + 1\right)e^{r_1(z)-r_1(z+c)} \equiv 1$.

From Case 2.3, we know that $r_1(z) = a_1z + b_1$ and $e^{a_1c} = 1 + \frac{A_2}{A_1}a_1$, where $a_1(\neq 0), b_1$ are constants.

In view of (5.3), it follows that $\frac{A_1}{A_2}\left(\frac{A_2}{A_1} + r_2'(z)\right)e^{r_2(z)-r_2(z+c)} = 1$, which implies that $r_2(z)$ is a linear form of $r_2(z) = a_2z + b_2$ and $e^{a_2c} = 1 + \frac{A_1}{A_2}a_2$, where $a_2(\neq 0), b_2$ are constants.

Since $r_1(z + c) - r_2(z + c)$ is not a constant, it follows that $a_1 \neq a_2$. In view of (5.1) and (5.2), it follows that $g(z) = r_1(z) + r_2(z) = (a_1 + a_2)z + b_1 + b_2 = az + b$ and

$$f'(z) = \frac{1}{\sqrt{2}}(A_1e^{a_1z+b_1} + A_2e^{a_2z+b_2}), \tag{5.10}$$

where A_1, A_2 are defined in (1.1).

So we have $e^{ac} = (1 + \frac{A_2}{A_1}a_1)(1 + \frac{A_1}{A_2}a_2)$.

From (5.10), we have

$$f(z) = \frac{1}{\sqrt{2}}\left(\frac{A_1}{a_1}e^{a_1z+b_1} + \frac{A_2}{a_2}e^{a_2z+b_2}\right) + c_1,$$

where c_1 is a constant.

Similarly, $f(z)$ can be written as $f(z) = B_1e^{a_1z} + B_2e^{(a-a_1)z} + c_1$, where $B_i(\neq 0, i = 1, 2), c_1$ are constants. Substituting it into Eq (1.13), we know that $a, b, c, a_1, \alpha, B_i(i = 1, 2)$ must satisfy

$$\begin{cases} a_1^2 + 2\alpha a_1(e^{a_1c} - 1) + (e^{a_1c} - 1)^2 = 0 \\ (a - a_1)^2 + 2\alpha(a - a_1)(e^{(a-a_1)c} - 1) + (e^{(a-a_1)c} - 1)^2 = 0 \\ a_1(a - a_1) + \alpha a_1(e^{(a-a_1)c} - 1) + \alpha(a - a_1)(e^{a_1c} - 1) + (e^{a_1c} - 1)(e^{(a-a_1)c} - 1) = \dfrac{e^b}{2B_1B_2}. \end{cases}$$

Thus, we get the conclusions (3) of Theorem 3.

If $(\frac{A_2}{A_1} + r_2'(z))e^{r_2(z)-r_1(z+c)} \equiv 1$.

This means that

$$r_2(z) - r_1(z + c) = \varepsilon_1, \tag{5.11}$$

where ε_1 is a constant.

In view of (5.3), it follows that

$$\left(\frac{A_1}{A_2} + r_1'(z)\right)e^{r_1(z)-r_2(z+c)} = 1.$$

So we have

$$r_1(z) - r_2(z + c) = \varepsilon_2 \tag{5.12}$$

where ε_2 is a constant.

In view of (5.11) and (5.12), it yields that

$$r_1(z) - r_2(z) + r_1(z + c) - r_2(z + c) = \varepsilon_2 - \varepsilon_1.$$

By combing with (3.3), we have

$$p(z) + p(z + c) = \frac{1}{2}(\varepsilon_2 - \varepsilon_1).$$

This is a contradiction with the assumption that $r_1(z + c) - r_2(z + c) = 2p(z + c)$ is not a constant.

Therefore, this completes the proof of Theorem 3. □

6. Conclusions

By using the theory of meromorphic functions and Nevanlinna theory, this paper deduce several new theorems including Theorems 1–3, which discuss the specific forms of g(z) and the transcendental entire solutions of three Fermat type equations 1.11–1.13 respectively. It is obvious that Theorems 1–3 do develop the related results by Liu and Yang [13], Han and Lü [5], Luo, Xu and Hu [14] to a certain extent.

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Conflict of interest

The authors declare that none of the authors have any competing interests in the manuscript.

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