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Research article

Adjacency relations induced by some Alexandroff topologies on $\mathbb{Z}^n$

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Abstract: Let (X, T) be an Alexandroff space. We define the adjacency relation AR_T on X induced by T as the irreflexive relation defined for $x \neq y$ in X by:

$$(x, y) \in AR_T \text{ if and only if } x \in SN_T(y) \text{ or } y \in SN_T(x),$$

where $SN_T(z)$ is the smallest open set containing z in (X, T) and $z \in \{x, y\}$. Two families of Alexandroff topologies $(T_k, k \in \mathbb{Z})$ and $(T'_k, k \in \mathbb{Z})$ have been recently introduced on $\mathbb{Z}$. The aim of this paper is to show that for each nonzero integers k, the topologies T_k, T'_k, T_{-k}, and T'_{-k} are homeomorphic. The adjacency relations induced by the product topologies $(T_k)^n$ and $(T'_k)^n$ are studied and compared with classical ones. We also show that the adjacency relations induced by T_k, T'_k, T_{-k}, and T'_{-k} are isomorphic. Then, note that the adjacency relations on $\mathbb{Z}$ induced by these topologies, $k \neq 0$, are different from each other.

Keywords: adjacency relation; digital space; Khalimsky topology; Alexandroff topology; digital connectivity; $T_{\frac{1}{2}}$-space; digital topology
Mathematics Subject Classification: 54A05, 54J05, 68U05

1. Introduction

In this paper, since we will often use the term "Khalimsky", hereinafter, we will use the notation "K-" for brevity if there is no danger of ambiguity. Besides, we also take the notations $\mathbb{N}$, $\mathbb{Z}_0$, $\mathbb{Z}_1$, and $\mathbb{Z}^n$ to indicate the sets of natural numbers (i.e., positive integers), even integers, odd integers, and the n-fold of Cartesian product of the set of integers $\mathbb{Z}$, respectively. In addition, we use the notation " :=" to introduce a new term. Also, we use the notation $X^\sharp$ and $\aleph_0$ to denote the cardinal number of the given set X and the cardinality of an (infinite) denumerable set, respectively.

Motivated by the n-dimensional K-topological structure on $\mathbb{Z}^n$, denoted by $(\mathbb{Z}^n, \kappa^n)$ (in detail, see later in Section 2), we have the following query: Are there certain topologies on $\mathbb{Z}^n$ that are not equal but homeomorphic to $(\mathbb{Z}^n, \kappa^n)$? Hence a recent paper [12] developed infinitely many types of topological structures on $\mathbb{Z}^n$, say $(\mathbb{Z}^n, (T_k)^n)$, $k \in \mathbb{Z}, n \in \mathbb{N}$, which need not be homeomorphic to the n-dimensional K-topological space, where the topology T_k (or T_{S_k}) is generated by the set

$$S_k := \{S_{k,n} \,|\, S_{k,n} := \{2n, 2n+1, 2n+2k+1\},\ n \in \mathbb{Z}\} \tag{1.1}$$

as a subbase [12], and the topology $(T_k)^n$ (resp. $(T'_k)^n$) on $\mathbb{Z}^n$ is the n-fold product topology of $(\mathbb{Z}, T_k)$ (resp. $(\mathbb{Z}, T'_k)$).

Thus, the papers [12, 13] investigated some properties of $(\mathbb{Z}, T_{-k})$ and $(\mathbb{Z}^n, (T_k)^n)$. Besides, they addressed the following issues.
(1) Characterization of the closures of singletons of $(\mathbb{Z}, T_k)$, $k \in \mathbb{Z}$.
(2) Calculation of the numbers of components of $(\mathbb{Z}, T_{-k})$ and $(\mathbb{Z}, T_k)$, $k \in \mathbb{N} \cup \{0\}$.
(3) Establishment of a homeomorphism between $(\mathbb{Z}, T_{-k})$ and $(\mathbb{Z}, T_k)$, $k \in \mathbb{N}$.
(4) Investigation of a necessary and sufficient condition supporting the connectedness of $(\mathbb{Z}, T_k)$.

The paper [12] also studied some properties of the topology T'_k on $\mathbb{Z}$, where T'_k (or T'_{S_k}) is generated by the set

$$S'_k := \{S'_{k,n} \mid S'_{k,n} := \{2n, 2n+1, 2n+2k\}, n \in \mathbb{Z}\}, \tag{1.2}$$

as a subbase.

In view of (1.1) and (1.2), for $k, n \in \mathbb{Z}$, we observe the following:

$$\left\{\begin{array}{l} (a)\, S_{k,n} \neq S'_{k,n} \text{ if } k \neq 0 \text{ and } S^{\sharp}_{k,n} = (S'_{k,n})^{\sharp}, \\ (b)\, S_k \neq S'_k \text{ if } k \neq 0 \text{ and } S^{\sharp}_k = \aleph_0 = (S'_k)^{\sharp}, \text{ and} \\ (c)\, \mathcal{B}_k \neq \mathcal{B}'_k, \end{array}\right\} \tag{1.3}$$

where $\mathcal{B}_k$(resp. $\mathcal{B}'_k$) is the base generated by the subbase S_k(resp. S'_k).

Recall that a topological space (X, T) is called an Alexandroff space [1, 2] if for each $x \in X$, the intersection of all open sets of X containing x (denoted by $SN_T(x)$) is T-open in X.

According to [12, 13], the following properties hold:
(1) For $k \in \mathbb{Z}$, both $(\mathbb{Z}, T_k)$ and $(\mathbb{Z}, T'_k)$ are Alexandroff spaces.

Note that the 1-dimensional K-topological space $(\mathbb{Z}, \kappa)$ coincides with $(\mathbb{Z}, T_{-1})$ and is homeomorphic to $(\mathbb{Z}, T_1)$.
(2) For $k \in \mathbb{Z} \setminus \{0\}$, in $(\mathbb{Z}, T_k)$ and $(\mathbb{Z}, T'_k)$, each singleton consisting of an odd (resp. even) numbers is an open (resp. closed) set; and consequently, the spaces are $T_{\frac{1}{2}}$-spaces [3, 4].

Now, let us recall that a topology T is called clopen [22] (quasi-discrete [21], pseudo-discrete [22], or indiscrete-generated [16]) if every open set in T is closed. This is precisely a space which is a sum of indiscrete spaces.
(3) The spaces $(\mathbb{Z}, T_0)$ and $(\mathbb{Z}, T'_0)$ are clopen (and not discrete) topological spaces and they are not connected.
(4) For $i \neq j$ in $\mathbb{N} \cup \{0\}$, T_i (resp. T'_i) is not homeomorphic to T_j (resp. T'_j) [12, 13].

Let us recall the notion of a digital space [15], as follows: A digital space is a kind of a relation set (X, π), where X is a nonempty set and π is a binary symmetric relation on X such that X is π-connected, where we say that X is *π-connected* if for any two elements x and y of X, there is a finite sequence $(x_i)_{i \in [0,l]_{\mathbb{Z}}}$ of elements in X such that $x = x_0$, $y = x_l$ and $(x_j, x_{j+1}) \in \pi$ for $j \in [0, l-1]_{\mathbb{Z}}$.

The aim of the paper is to show the following results.
(1) For each $k \in \mathbb{Z} \setminus \{0\}$, the spaces $(\mathbb{Z}, T_k)$, $(\mathbb{Z}, T_{-k})$, $(\mathbb{Z}, T'_k)$, and $(\mathbb{Z}, T'_{-k})$ are homeomorphic.
(2) For each $k \in \mathbb{Z} \setminus \{0\}$, the space $(\mathbb{Z}, T'_k)$ has $|k|$ connected components.
(3) Adjacency relations induced by T_k and T'_k on $\mathbb{Z}$ are isomorphic.
(4) Adjacency relations induced by $(T_k)^n$ and $(T'_k)^n$ on $\mathbb{Z}^n$ are isomorphic.

The paper is organized, as follows: Section 2 refers to some notions relating to various structures of Alexandroff spaces and K-topological spaces, and some notions regarding the digital k-connectivity of $\mathbb{Z}^n$. Section 3 proves non-connectedness of the topological spaces $(\mathbb{Z}, T'_k), k \in \mathbb{Z} \setminus \{-1, 1\}$. More precisely, we further study various structures of $(\mathbb{Z}, T'_k)$ and $(\mathbb{Z}^2, T'_k \times T'_k)$ and investigate some topological properties of each singletons in $(\mathbb{Z}, T'_k), k \in \mathbb{Z}$. Besides, it proves that for $k \in \mathbb{Z} \setminus \{0\}$, the topological space $(\mathbb{Z}, T'_{-k})$ has $|k|$ components, which confirms that $(\mathbb{Z}, T'_{-i})$ is not homeomorphic to $(\mathbb{Z}, T'_{-j})$ if $i \neq j, i, j \in \mathbb{N} \cup \{0\}$. Furthermore, we prove that $(\mathbb{Z}, T'_k)$, $(\mathbb{Z}, T_k)$, $(\mathbb{Z}, T'_{-k})$, and $(\mathbb{Z}, T_{-k})$ are mutually homeomorphic for any $k \in \mathbb{N}$. Section 4 develops new types of several adjacencies on $\mathbb{Z}$ come from the infinitely many topological spaces $(\mathbb{Z}, T'_{-k})$, $(\mathbb{Z}, T'_k)$, $(\mathbb{Z}, T_{-k})$, and $(\mathbb{Z}, T_k), k \in \mathbb{N}$. Section 5 establishes new types of adjacencies on $\mathbb{Z}^n$ derived from the infinitely many topological spaces $(\mathbb{Z}^n, (T'_{-k})^n)$, $(\mathbb{Z}^n, (T'_k)^n)$, $(\mathbb{Z}^n, (T_{-k})^n)$, and $(\mathbb{Z}^n, (T_k)^n), k \in \mathbb{Z}$. Besides it refers to some utilities of the topologies T'_k, T_k, T'_{-k}, and T_{-k} on $\mathbb{Z}$. Section 6 concludes the paper and refers to a further work.

2. Preliminaries

The study of digital images [6, 11, 14, 15, 18–20, 25] needs several methods using Alexandroff (A-, for brevity, if there is no danger of ambiguity) topological structures on $\mathbb{Z}^n$ such as Khalimsky, Marcus-Wyse, Han topological structure [5–7, 9, 10, 14, 18, 25] and further, a graph theoretical approach on $\mathbb{Z}^n$ initiated by Rosenfeld [23, 24].

In this section, we refer to several concepts which are used in this paper. Indeed, the Alexandroff topological structure plays an important role in digital topology. Since we will often use the term "Alexandroff topological", hereinafter, we will use the notation "A-" for brevity if there is no danger of ambiguity. Given two A-spaces $X := (X, A_1)$ and $Y := (Y, A_2)$, a map $h : X \to Y$ is called a homeomorphism if h is a continuous bijection and further, $h^{-1} : Y \to X$ is continuous.

As an example of an A-space we can consider the n-dimensional K-topological space [17–20]. Let us now recall basic notions related to the K-topological structure on $\mathbb{Z}^n$. The K-line topology on $\mathbb{Z}$, denoted by $(\mathbb{Z}, \kappa)$, is induced by the set $\{[2n-1, 2n+1]_{\mathbb{Z}} \,|\, n \in \mathbb{Z}\}$ as a subbase [18], where for $a, b \in \mathbb{Z}$, $[a, b]_{\mathbb{Z}} := \{x \in \mathbb{Z} \,|\, a \leq x \leq b\}$. The product topology on $\mathbb{Z}^n$ induced by $(\mathbb{Z}, \kappa)$ is called the K-product topology on $\mathbb{Z}^n$ (or the n-dimensional K-topological space or the Khalimsky nD space), denoted by $(\mathbb{Z}^n, \kappa^n)$ and further, various properties of $(\mathbb{Z}^n, \kappa^n)$ have been investigated [3, 17–20].

In relation to the study of digital images, Rosenfeld [23, 24] initially took a graph theoretical approach to study digital images. Naively, a digital image (X, k) can be considered to be a set $X \subset \mathbb{Z}^n$ with one of the k-adjacency of $\mathbb{Z}^n$ from (2.1) below (or a digital k-graph on $\mathbb{Z}^n$ [9]).

Motivated by the digital k-connectivity for low dimensional digital images $(X, k), X \subset \mathbb{Z}^3$ [23, 24], as a generalization of this approach, the papers [5, 8] initially developed some k-adjacency relations for high dimensional digital images $(X, k), X \subset \mathbb{Z}^n$ (see also (2.1) below). More precisely, the digital k-adjacency relations (or digital k-connectivity) for $X \subset \mathbb{Z}^n, n \in \mathbb{N}$, were initially developed in [8] (see also [5]), as follows:

For a natural number t, $1 \leq t \leq n$, the distinct points $p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n) \in \mathbb{Z}^n$ are $k(t,n)$-adjacent if at most t of their coordinates differ by ± 1 and the others coincide.

According to this statement, the $k(t,n)$-adjacency relations of $\mathbb{Z}^n, n \in \mathbb{N}$, are formulated [8] (see also [5]) as follows:

$$k := k(t,n) = \sum_{i=1}^{t} 2^i C_i^n, \text{where } C_k^n := \frac{n!}{(n-i)!\ i!}. \tag{2.1}$$

For instance, the following are obtained [8]:

$$(n,t,k) \in \begin{Bmatrix} (4,1,8), (4,2,32), (4,3,64), (4,4,80); \text{and} \\ (5,1,10), (5,2,50), (5,3,130), (5,4,210), (5,5,242). \end{Bmatrix}$$

3. Properties of the space $(\mathbb{Z}, T'_k)$

This section investigates some properties of the topological space $(\mathbb{Z}, T'_k)$ (see (1.2)). Owing to the subbase structures of (1.1) and (1.2), we obtain the following:

Remark 3.1. *(1) If $k = 0$, then $T_0 = T'_0$ is a clopen topology, and*

$$SN_{T_0}(2n) = SN_{T_0}(2n+1) = Cl_{T_0}(2n) = Cl_{T_0}(2n+1) = \{2n, 2n+1\},$$

for each integer n. So each of the connected components of $(\mathbb{Z}, T'_0)$ is the set $\{2n, 2n+1\}, n \in \mathbb{Z}$.
(2) For each $k \in \mathbb{Z} \setminus \{0\}$,

$$\mathcal{B} := \{\{2n+1\} \mid n \in \mathbb{Z}\} \cup \{S_{k,n} \mid n \in \mathbb{Z}\}$$

is a basis of $(\mathbb{Z}, T_k)$.
(3) For each $k \in \mathbb{Z} \setminus \{0\}$,

$$\mathcal{B}' := \{\{2n\} \mid n \in \mathbb{Z}\} \cup \{S'_{k,n} \mid n \in \mathbb{Z}\}$$

is a basis of $(\mathbb{Z}, T'_k)$.

Remark 3.1 enables us to state the following result.

Theorem 3.2. *Let $k \in \mathbb{Z} \setminus \{0\}$. Then the following properties hold.*
(1) T_k is an Alexandroff topology, and for each $n \in \mathbb{Z}$ we have

$$\begin{Bmatrix} SN_{T_k}(2n+1) = \{2n+1\}, SN_{T_k}(2n) = S_{k,n}, \text{ and} \\ Cl_{T_k}(2n) = \{2n\}, Cl_{T_k}(2n+1) = \{2n, 2n+1, 2n-2k\} = S'_{-k,n}. \end{Bmatrix}$$

(2) T'_k is an Alexandroff topology, and for each $n \in \mathbb{Z}$ we have

$$\begin{Bmatrix} SN_{T'_k}(2n) = \{2n\}, SN_{T'_k}(2n+1) = S'_{k,n}, \text{ and} \\ Cl_{T'_k}(2n+1) = \{2n+1\}, Cl_{T'_k}(2n) = S_{-k,n}. \end{Bmatrix}$$

Corollary 3.3. *Let $k \in \mathbb{Z} \setminus \{0\}$. Then each of the spaces $(\mathbb{Z}, T_k)$ and $(\mathbb{Z}, T'_k)$ is a $T_{\frac{1}{2}}$-space.*

Since $(\mathbb{Z}, T_0)$ and $(\mathbb{Z}, T'_0)$ are obviously homeomorphic, the main theorem of the paper is the following.

Theorem 3.4. *Let $k \in \mathbb{Z} \setminus \{0\}$. Then the following properties hold.*
(1) The spaces $(\mathbb{Z}, T_k)$ and $(\mathbb{Z}, T_{-k})$ are homeomorphic.
(2) The spaces $(\mathbb{Z}, T_k)$ and $(\mathbb{Z}, T'_k)$ are homeomorphic.

Proof. (1) Consider the map $\phi : (\mathbb{Z}, T_k) \to (\mathbb{Z}, T_{-k})$, defined by

$$\begin{Bmatrix} \phi(x) = x \text{ if } x \text{ is even, and} \\ \phi(x) = x - 2k \text{ if } x \text{ is odd.} \end{Bmatrix}$$

Then ϕ is a bijection. One may check that

$$\begin{Bmatrix} \phi^{-1}(SN_{T_{-k}}(2n+1)) = SN_{T_k}(2(n+k)+1), \text{ and} \\ \phi^{-1}(SN_{T_{-k}}(2n)) = SN_{T_k}(2n). \end{Bmatrix}$$

This shows, in particular, that ϕ is a continuous open map. Therefore ϕ is a homeomorphism.
(2) Let $\psi : (\mathbb{Z}, T_k) \to (\mathbb{Z}, T'_k)$, defined by

$$\begin{Bmatrix} \psi(x) = x - 1 \text{ if } x \text{ is even, and} \\ \psi(x) = x + 1 \text{ if } x \text{ is odd.} \end{Bmatrix}$$

So it is clear that ψ is a bijection. One may check easily that (see Figure 1 for the case of $k = 2$)

$$\begin{Bmatrix} \psi^{-1}(SN_{T'_k}(2n)) = SN_{T_k}(2n+1), \text{ and} \\ \psi^{-1}(SN_{T'_k}(2n+1)) = SN_{T_k}(2n). \end{Bmatrix}$$

Therefore ψ is a continuous open map and consequently, ψ is a homeomorphism. □

Remark 3.5. *Owing to Theorem 3.4, we obtain that $(\mathbb{Z}, T'_{-k})$ and $(\mathbb{Z}, T'_k)$ are homeomorphic.*

Proof. Let us consider the map $h : (\mathbb{Z}, T'_{-k}) \to (\mathbb{Z}, T'_k)$ defined by

$$h(x) = \begin{Bmatrix} x + 2k, x \in \mathbb{Z}_0, \\ x, x \in \mathbb{Z}_1. \end{Bmatrix}$$

Then h is a continuous open map and consequently, h is a homeomorphism (see Figure 2). □

Example 3.1. *(1) $(\mathbb{Z}, T'_{-1})$ is homeomorphic to $(\mathbb{Z}, T'_1)$ (see Figure 2).*
(2) $(\mathbb{Z}, T'_{-2})$ is homeomorphic to $(\mathbb{Z}, T'_2)$ (see Figure 3).

Now combining Theorem 3.4 and the Theorem 4.3 of [13], we get the following (see Figures 4 and 5).

Theorem 3.6. *Let $k \in \mathbb{Z} \setminus \{0\}$. Then the following properties hold.*
(1) Each of the spaces $(\mathbb{Z}, T_k)$, $(\mathbb{Z}, T'_k)$, and $(\mathbb{Z}, T_{-k})$ has $|k|$ connected components.
(2) If $i, j \in \mathbb{Z}$, and $|i| \neq |j|$, then T_i, T_j, T'_i, T'_j are mutually non-homeomorphic.

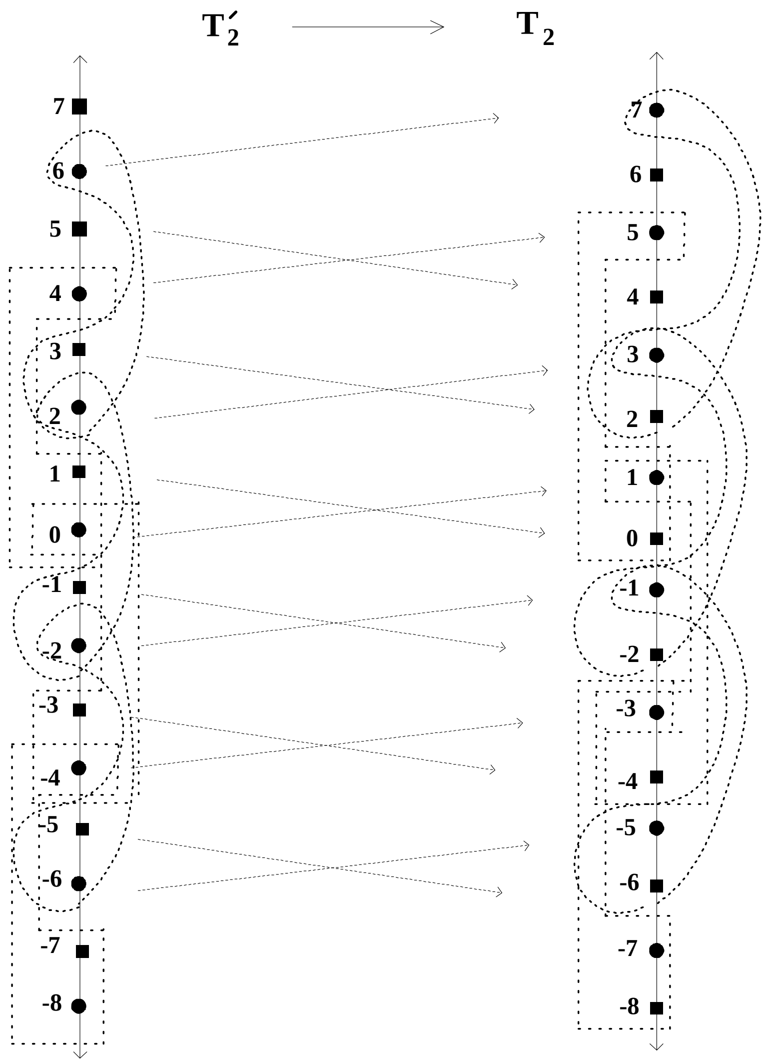

Figure 1. A homeomorphism $\psi^{-1} : (\mathbb{Z}, T_2') \to (\mathbb{Z}, T_2)$ related to Theorem 3.4.

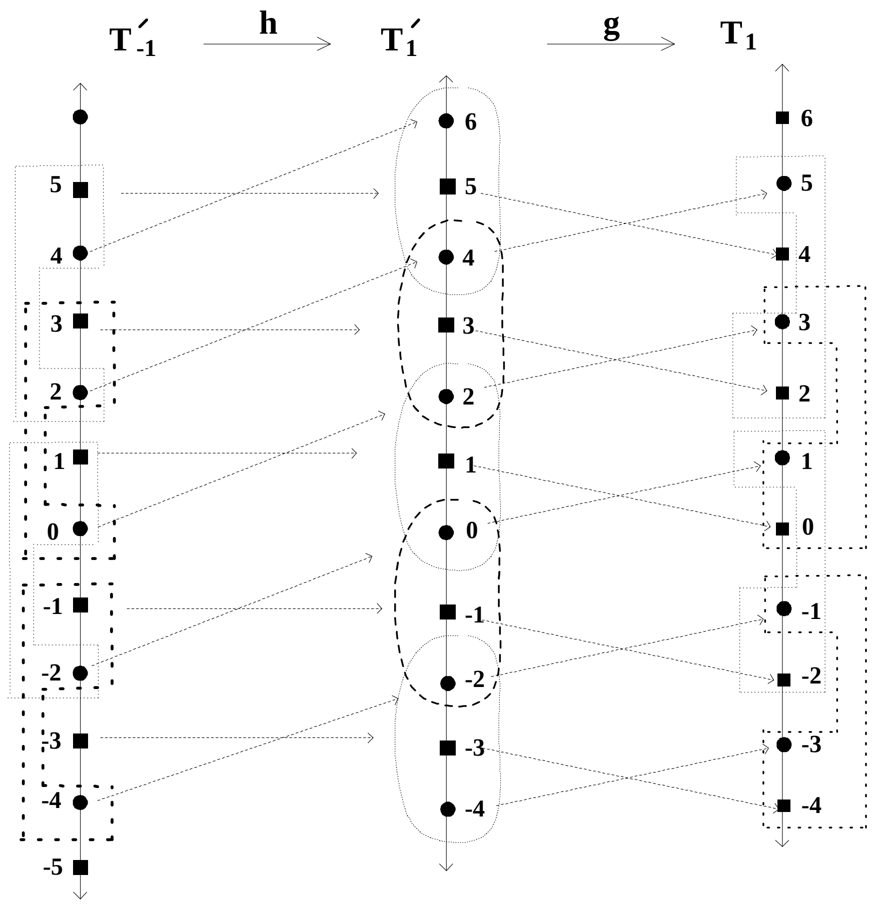

Figure 2. Configuration of a homeomorphism $h : (\mathbb{Z}, T'_{-1}) \to (\mathbb{Z}, T'_1)$ relating to Remark 3.5. Besides, existence of the homeomorphism $g := \psi^{-1} = (\mathbb{Z}, T'_1) \to (\mathbb{Z}, T_1)$ relating to Theorem 3.4(2).

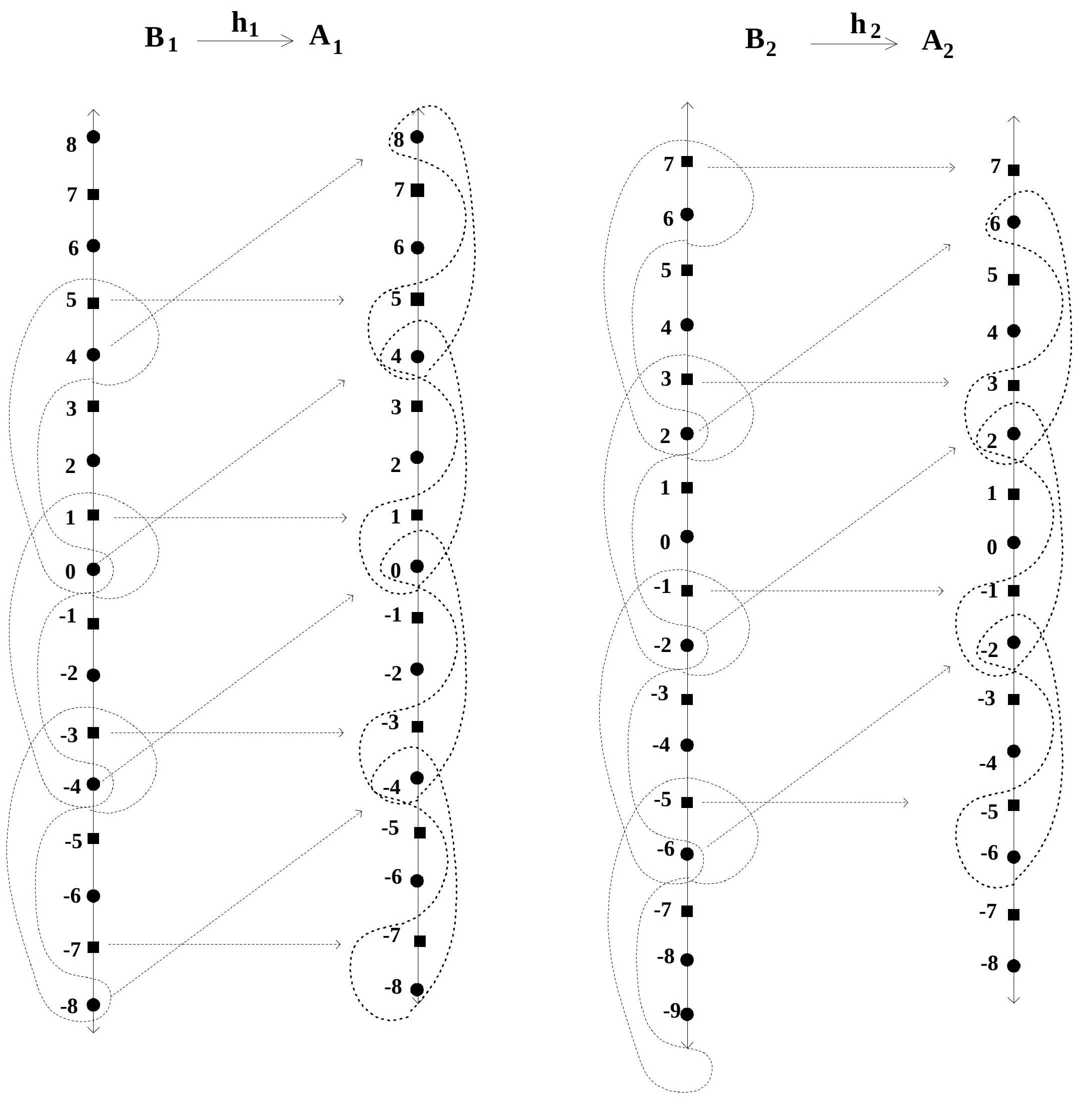

Figure 3. A homeomorphism between $(\mathbb{Z}, T'_{-2})$ and $(\mathbb{Z}, T'_2)$ formulated by the two homeomorphisms h_1 and h_2 associated with Example 3.1. Naively, $h : (\mathbb{Z}, T'_{-2}) \to (\mathbb{Z}, T'_2)$ can be considered as a union of the two homeomorphisms h_1 and h_2, i.e., $h := h_1 \cup h_2$, where $A_1 \cup A_2 = \mathbb{Z} = B_1 \cup B_2$ and $(A_1 \cup A_2, T'_2)$ and $(B_1 \cup B_2, T'_{-2})$.

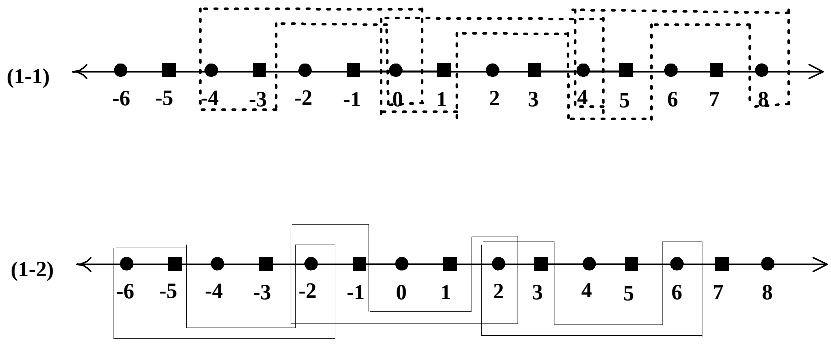

Figure 4. In $(\mathbb{Z}, T_2')$, in relation to Theorem 3.6(1), the only two components are considered as in (1-1) and (1-2).

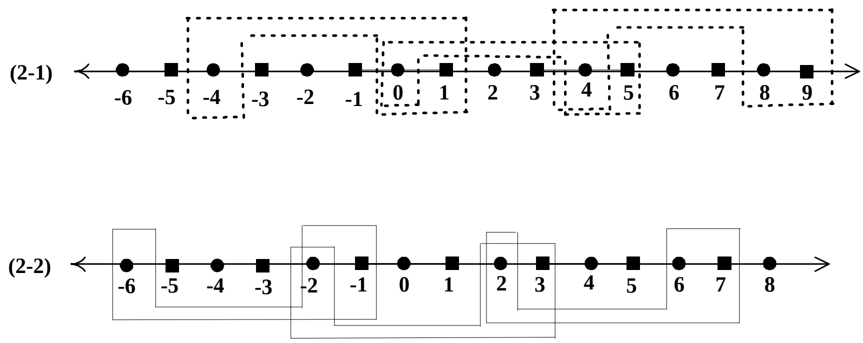

Figure 5. In the topological space $(\mathbb{Z}, T'_{-2})$, as mentioned in Theorem 3.6(1), two components are considered as in (2-1) and (2-2).

4. Adjacency relations induced by some Alexandroff topologies on $\mathbb{Z}^n$

This section first recalls that a connected A-space (X, T) induces a digital space. Given an A-space (X, T), consider the smallest open neighborhood of a point $x \in X$, i.e., $SN_T(x)$ as a subset of X. Then we have a relation set (X, AR_T) using this neighborhood as follows: We say that two distinct points x and y in X have a relation AR_T in X, denoted by $(x, y) \in AR_T$, if and only if

$$x \in SN_T(y) \text{ or } y \in SN_T(x). \tag{4.1}$$

Then, it is clear that the relation AR_T of (4.1) is symmetric. Since each point x of an A-space (X, T) has the smallest open set, using the relation of (4.1), we have the following adjacency neighborhood of x in (X, AR_T).

$$AN_T(x,1) := \{y \in X \,|\, (x,y) \in AR_T\} \cup \{x\}. \tag{4.2}$$

In view of (4.2), it is clear that the 2-adjacency of $\mathbb{Z}$ is exactly the adjacency induced by the Khalimsky topology on $\mathbb{Z}$ or equivalently $(\mathbb{Z}, T_{-1})$ and $(\mathbb{Z}, T'_1)$.

Based on the neighborhood of (4.2), given two A-spaces (X, T_1) and (X, T_2), assume two corresponding adjacency relation sets, e.g., $X := (X, AR_{T_1})$ and $Y := (Y, AR_{T_2})$. Then, let us recall the relation preserving map $f : X \to Y$ at a point $x \in X$ satisfying

$$f(AN_{T_1}(x,1)) \subset AN_{T_2}(f(x),1). \tag{4.3}$$

Then we can further consider an isomorphism between two relation sets, as follows: Assume the two relation sets X and Y. A map $h : X \to Y$ is called an isomorphism if h is a relation preserving bijection. Then, we say that X is isomorphic to Y and use the notation $X \approx_{(AR_{T_1}, AR_{T_2})} Y$.

Based on Theorem 3.4, we need to explore some relationships among digital connectivities on $\mathbb{Z}$ induced by the infinitely many topologies T'_{-k}, T'_k, T_{-k}, and T_k. Furthermore, we need to investigate some properties of adjacencies (or connectivities) on $\mathbb{Z}$ determined by the topologies T'_{-k}, T'_k, T_{-k}, and T_k.

Remark 4.1. *(1) Let $k \in \mathbb{N} \setminus \{1\}$. Then AR_{T_k}, $AR_{T_{-k}}$, $AR_{T'_k}$, $AR_{T'_{-k}}$, and AR_κ are mutually not equivalent (i.e., not equal).*
(2) The adjacency relations AR_{T_1}, $AR_{T'_{-1}}$, and AR_κ are mutually equivalent.
(3) The adjacency relations $AR_{T_{-1}}$, $AR_{T'_1}$, and AR_κ are mutually equivalent (i.e., equal).

Let us further deal with the following topics.
• What types of digital adjacencies on $\mathbb{Z}$ are obtained from the topological structures T'_{-k}, T'_k, T_{-k}, and T_k ?
• As one of the important problems, we need to examine if two adjacency relation sets induced by two homeomorphic topologies on $\mathbb{Z}$ have some relationships between them.

Based on the digital connectivities mentioned in (2.1), we now investigate some relationships among the digital adjacencies induced by each of $(\mathbb{Z}, T'_{-k})$, $(\mathbb{Z}, T'_k)$, $(\mathbb{Z}, T_{-k})$, and $(\mathbb{Z}, T_k)$ and the typical 2-adjacency of $\mathbb{Z}$.

Remark 4.2. *In case $k \in \mathbb{N} \setminus \{1\}$, the adjacency derived from each of $(\mathbb{Z}, T'_{-k})$, $(\mathbb{Z}, T'_k)$, $(\mathbb{Z}, T_{-k})$, and $(\mathbb{Z}, T_k)$ is different from the other and it is not equivalent to (or different from) the typical 2-adjacency of $\mathbb{Z}$.*

Based on (4.2), in case $k = 1$, let $(\mathbb{Z}, AR_{T'_{-1}})$, $(\mathbb{Z}, AR_{T'_1})$, $(\mathbb{Z}, AR_{T'_{-1}})$, and $(\mathbb{Z}, AR_{T_1})$ be digital adjacency relation sets derived from each of the topological spaces $(\mathbb{Z}, T'_{-1})$, $(\mathbb{Z}, T'_1)$, $(\mathbb{Z}, T_{-1})$, and $(\mathbb{Z}, T_1)$, respectively. Then we need to examine if there are some relationships between the adjacencies on $\mathbb{Z}$ and the typical 2-adjacency of $(\mathbb{Z}, 2)$ (see (2.1)).

Remark 4.3. *(1) Each of $(\mathbb{Z}, T'_{-1})$ and $(\mathbb{Z}, T_1)$ establishes $AR_{T'_{-1}}$- and AR_{T_1}-adjacency on $\mathbb{Z}$ which are different from each other and further, it is not equivalent to the typical 2-adjacency of $\mathbb{Z}$.*
(2) Each of $(\mathbb{Z}, T'_1)$ and $(\mathbb{Z}, T_{-1})$ produces $AR_{T'_1}$- and $AR_{T_{-1}}$-adjacency on $\mathbb{Z}$ which is equivalent to the typical 2-adjacency of $\mathbb{Z}$.

Unlike Remark 4.3, Theorems 3.4 and 3.7, and Remark 3.5 enable us to get the following:

Theorem 4.4. *Assume the relation sets* $(\mathbb{Z}, AR_{T_k})$, $(\mathbb{Z}, AR_{T'_k})$, $(\mathbb{Z}, AR_{T_{-k}})$, *and* $(\mathbb{Z}, AR_{T'_{-k}})$. *Then, for* $k \in \mathbb{Z}$, *each of them is isomorphic to the other.*

5. New digital connectivities on $\mathbb{Z}^n$ coming from infinitely many product topologies on $\mathbb{Z}^n$

In view of (1.3), for $k \in \mathbb{N}$, it is clear that $T'_k \neq T_k \neq T'_{-k} \neq T_{-k}$ on $\mathbb{Z}$. After considering the product topologies $(T'_k)^n, (T_k)^n, (T'_{-k})^n$, and $(T_{-k})^n$ on $\mathbb{Z}^n$, in this section we will deal with digital connectivities on $\mathbb{Z}^n$ coming from these infinitely many product topologies on $\mathbb{Z}^n$. Namely, based on some results proposed in Section 4, this section deals with the following topics.
(1) Examination of some relationships between two adjacency relations in $\mathbb{Z}^n$ induced by two homeomorphic product topologies on $\mathbb{Z}^n$.
(2) What types of digital adjacencies on $\mathbb{Z}^n$ are obtained from the product topological structures generated by T'_{-k}, T'_k, T_{-k}, and T_k ?
(3) Comparison between the digital adjacencies referred to in (2) and the typical $k(t, n)$-adjacencies of $\mathbb{Z}^n$ of (2.1).

Hereinafter, we note that the topologies $(T'_{-k})^n$, $(T'_k)^n$, $(T_{-k})^n$, and $(T_k)^n$ on $\mathbb{Z}^n$ are the n-fold product topologies induced by the topologies T'_{-k}, T'_k, T_{-k}, and T_k on $\mathbb{Z}$, respectively.

By Theorems 3.4 and 3.6, we obtain homeomorphisms among the topological spaces $(\mathbb{Z}^n, (T'_{-k})^n)$, $(\mathbb{Z}^n, (T'_k)^n)$, $(\mathbb{Z}^n, (T_{-k})^n)$, and $(\mathbb{Z}^n, (T_k)^n)$ which play important roles in this paper.

Let us now examine if each of the topological spaces $(\mathbb{Z}^n, (T'_{-k})^n)$, $(\mathbb{Z}^n, (T'_k)^n)$, $(\mathbb{Z}^n, (T_{-k})^n)$, and $(\mathbb{Z}^n, (T_k)^n)$ is a $T_{\frac{1}{2}}$-space, as follows: Unlike Corollary 3.3, we obtain the following:

Lemma 5.1. *Each of the topological spaces* $(\mathbb{Z}^n, (T'_{-k})^n)$, $(\mathbb{Z}^n, (T'_k)^n)$, $(\mathbb{Z}^n, (T_{-k})^n)$, *and* $(\mathbb{Z}^n, (T_k)^n)$ *is not a* $T_{\frac{1}{2}}$*-space,* $k \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N} \setminus \{1\}$.

Proof. Based on the properties of (1.1) and (1.2), we now prove the assertion.
Case 1. In case $k = 0$, by Remark 3.1, each of the given topological spaces is not a $T_{\frac{1}{2}}$-space at all.
Case 2. In case $k \neq 0$, with the topological space $(\mathbb{Z}^n, (T_k)^n)$, $n \in \mathbb{N} \setminus \{1\}$, consider the point $q := (2m_1, 2m_2 + 1, \cdots, 2m_n + 1) \in \mathbb{Z}^n$. Then, by the property of (1.1), the singleton $\{q\}$ is neither open nor closed in $(\mathbb{Z}^n, (T_k)^n)$. For instance, we now show that $(\mathbb{Z}^2, (T_1)^2)$ is not a $T_{\frac{1}{2}}$-space. Consider the point $r := (0, 1) \in \mathbb{Z}^2$ in $(\mathbb{Z}^2, (T_1)^2)$ (see Figure 6(4)). Then it is neither open nor closed in $(\mathbb{Z}^2, (T_1)^2)$.

Using a method similar to this approach, with each of the topologies $(\mathbb{Z}^n, (T'_{-k})^n)$, $(\mathbb{Z}^n, (T'_k)^n)$, and $(\mathbb{Z}^n, (T_{-k})^n)$, it is clear that each of these topological spaces is not a $T_{\frac{1}{2}}$-space because the separation axiom $T_{\frac{1}{2}}$ is a topological property. □

Let us compare each adjacency induced by each of $(\mathbb{Z}^n, (T'_{-k})^n)$, $(\mathbb{Z}^n, (T'_k)^n)$, $(\mathbb{Z}^n, (T_{-k})^n)$, and $(\mathbb{Z}^n, (T_k)^n)$ and the typical $k(t, n)$-adjacency of $\mathbb{Z}^n$ in (2.1), as follows:

Theorem 5.2. *For* $n \in \mathbb{N} \setminus \{1\}$, *consider the* $AR_{(T'_{-k})^n}$-, $AR_{(T'_k)^n}$-, $AR_{(T_{-k})^n}$-, *and* $AR_{(T_k)^n}$*-adjacency on* $\mathbb{Z}^n$. *Then we obtain the following.*
(1) In case $k \in \mathbb{N}$, *each of them is different from the others.*
(2) In case $k \in \mathbb{N} \cup \{0\}$, *each of them is a new one which is not equivalent to (or different from) the typical* $k(t, n)$*-adjacency of* $\mathbb{Z}^n$ *in (2.1).*

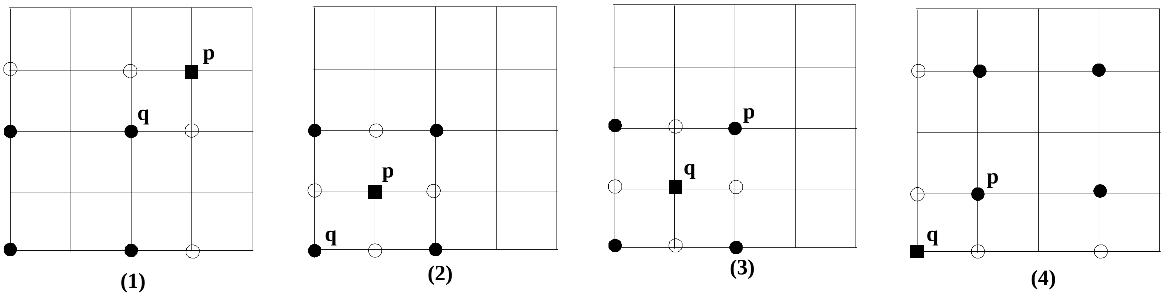

Figure 6. Configuration of the smallest open sets of the given points p and q according to the given topological spaces $(\mathbb{Z}^2, (T'_{-1})^2)$, $(\mathbb{Z}^2, (T'_1)^2)$, $(\mathbb{Z}^2, (T_{-1})^2)$, $(\mathbb{Z}^2, (T_1)^2)$. (1) $SN_{(T'_{-1})^2}(p)$ in $(\mathbb{Z}^2, (T'_{-1})^2)$ establishing an $AN_{(T'_{-1})^2}(p)$ (2) $SN_{(T'_1)^2}(p)$ in $(\mathbb{Z}^2, (T'_1)^2)$ establishing $AN_{(T'_1)^2}(p)$ (3) $SN_{(T_{-1})^2}(q)$ in $(\mathbb{Z}^2, (T_{-1})^2)$ developing $AN_{(T_{-1})^2}(q)$ (4) $SN_{(T_1)^2}(q)$ in $(\mathbb{Z}^2, (T_1)^2)$ constructing $AN_{(T_1)^2}(q)$.

Proof. (1) Case 1. Assume $(\mathbb{Z}^n, (T'_{-k})^n)$, $k \in \mathbb{N}$.

Consider the point $p := (2m_1+1, 2m_2+1, \cdots, 2m_n+1) \in \mathbb{Z}^n$. Then, the smallest open set containing the point p is the following set

$$SN_{(T'_{-k})^n}(p) := \{2m_1 - 2k, 2m_1, 2m_1 + 1\} \times \cdots \times \{2m_n - 2k, 2m_n, 2m_n + 1\}. \tag{5.1}$$

Besides, for the point $q := (2m_1, 2m_2, \cdots, 2m_n) \in \mathbb{Z}^n$, we obtain the smallest open set containing the point q, as follows:

$$SN_{(T'_{-k})^n}(q) := \{q\}.$$

Thus, using these smallest open sets of the points p, q and the property of (4.1), an $AR_{(T'_{-k})^n}$-adjacency derived from $(\mathbb{Z}^n, (T'_{-k})^n)$ is obtained.

Case 2. Assume $(\mathbb{Z}^n, (T'_k)^n)$, $k \in \mathbb{N}$.

Consider the point $p := (2m_1+1, 2m_2+1, \cdots, 2m_n+1) \in \mathbb{Z}^n$. Then, the smallest open set containing the point p is the following set

$$SN_{(T'_k)^n}(p) := \{2m_1, 2m_1 + 1, 2m_1 + 2k\} \times \cdots \times \{2m_n, 2m_n + 1, 2m_n + 2k\}. \tag{5.2}$$

Besides, consider the point $q := (2m_1, 2m_2, \cdots, 2m_n) \in \mathbb{Z}^n$, we obtain the smallest open set containing the point q, as follows:

$$SN_{(T'_k)^n}(q) := \{q\}.$$

Thus, using these smallest open sets and the property of (4.1), an the $AR_{(T'_k)^n}$-adjacency derived from $(\mathbb{Z}^n, (T'_k)^n)$ is obtained.

Case 3. Assume $(\mathbb{Z}^n, (T_{-k})^n)$, $k \in \mathbb{N}$.

Consider the point $q := (2m_1, 2m_2, \cdots, 2m_n) \in \mathbb{Z}^n$. Then, the smallest open set containing the point q is the following set

$$SN_{(T_{-k})^n}(q) := \{2m_1 - 2k + 1, 2m_1, 2m_1 + 1\} \times \cdots \times \{2m_n - 2k + 1, 2m_n, 2m_n + 1\}. \tag{5.3}$$

Besides, for the point $p := (2m_1 + 1, 2m_2 + 1, \cdots, 2m_n + 1) \in \mathbb{Z}^n$, Then, we obtain the smallest open set containing the point p, as follows:

$$SN_{(T_{-k})^n}(p) := \{p\}.$$

Thus, using these smallest open sets and the property of (4.1), an $AR_{(T_{-k})^n}$-adjacency derived from $(\mathbb{Z}^n, (T_{-k})^n)$ is obtained.
Case 4. Assume $(\mathbb{Z}^n, (T_k)^n)$, $k \in \mathbb{N}$.

Consider the point $q := (2m_1, 2m_2, \cdots, 2m_n) \in \mathbb{Z}^n$. Then, the smallest open set containing the point q is the following set

$$SN_{(T_k)^n}(q) := \{2m_1, 2m_1 + 1, 2m_1 + 2k + 1, \} \times \cdots \times \{2m_n, 2m_n + 1, 2m_n + 2k + 1\}. \tag{5.4}$$

Besides, for the point $p := (2m_1 + 1, 2m_2 + 1, \cdots, 2m_n + 1) \in \mathbb{Z}^n$, the smallest open set containing the point p is obtained, as follows:

$$SN_{(T_k)^n}(p) := \{p\}.$$

Thus, using these smallest open sets and the property of (4.1), an $AR_{(T_k)^n}$-adjacency derived from $(\mathbb{Z}^n, (T_k)^n)$ is obtained.

In view of these four cases, we complete the proof.

(2) In view of the obtained smallest open sets in (5.1)–(5.4), each of the adjacencies of $(\mathbb{Z}^n, AR_{(T'_{-k})^n})$, $(\mathbb{Z}^n, AR_{(T'_k)^n})$, $(\mathbb{Z}^n, AR_{(T_{-k})^n})$, and $(\mathbb{Z}^n, AR_{(T_k)^n})$ is not equivalent to (or different from) the typical $k(t, n)$-adjacency of $\mathbb{Z}^n$ in (2.1). □

Remark 5.3. *As shown in Figure 7, we have isomorphisms derived from the homeomorphisms between* $(\mathbb{Z}^2, (T'_{-1})^2)$ *and* $(\mathbb{Z}^2, (T'_1)^2)$*, and further,* $(\mathbb{Z}^2, (T_{-1})^2)$ *and* $(\mathbb{Z}^2, (T_1)^2)$*. (a)* $H_1 : SN_{(T'_{-1})^2}(p) \to SN_{(T'_1)^2}(p)$ *(b)* $H_2 : SN_{(T_{-1})^2}(q) \to SN_{(T_1)^2}(q)$ *(c)* $H_3 : SN_{(T'_1)^2}(p) \to SN_{(T_1)^2}(q)$*.*

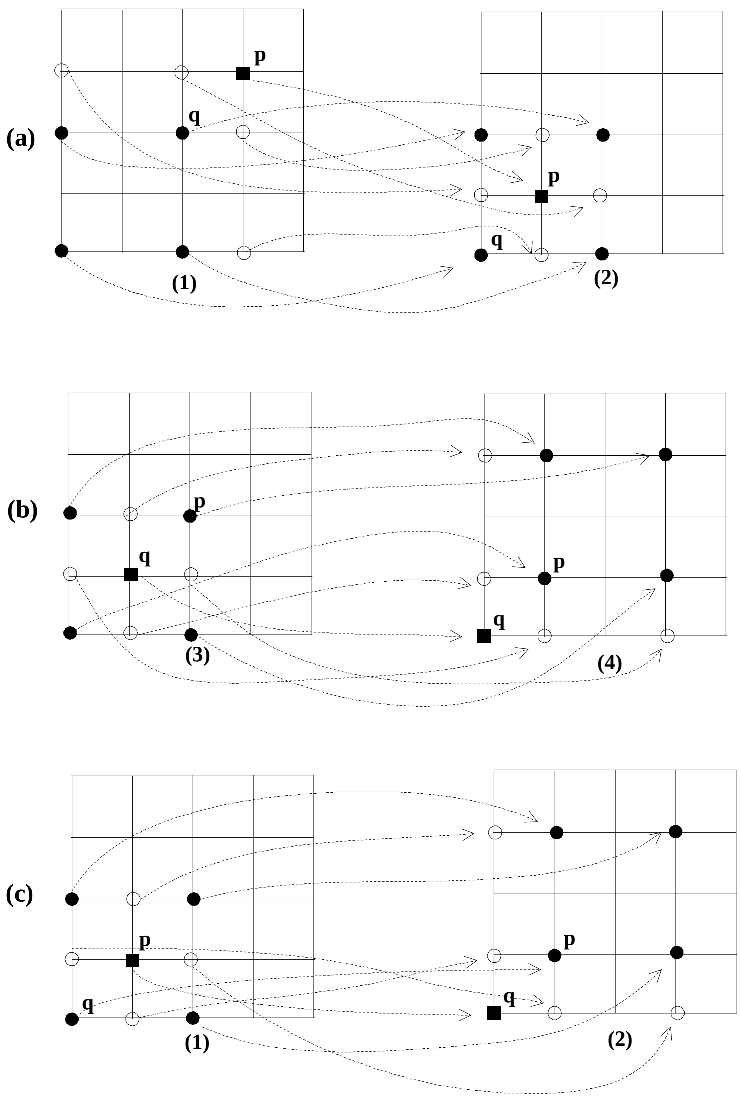

Figure 7. Configuration of the isomorphisms derived from the homeomorphisms between $(\mathbb{Z}^2, (T'_{-1})^2)$ and $(\mathbb{Z}^2, (T'_1)^2)$, and further, $(\mathbb{Z}^2, (T_{-1})^2)$ and $(\mathbb{Z}^2, (T_1)^2)$. (a) $H_1 : SN_{(T'_{-1})^2}(p) \to SN_{(T'_1)^2}(p)$ (b) $H_2 : SN_{(T_{-1})^2}(q) \to SN_{(T_1)^2}(q)$ (c) $H_3 : SN_{(T'_1)^2}(p) \to SN_{(T_1)^2}(q)$, where (1) is equal to (2) of (a) and (2) is equal to (4) of (b).

6. Conclusions and further work

Using the topological structures T'_k, T_k, T'_{-k}, and T_{-k} on $\mathbb{Z}$, $k \in \mathbb{N}$, first we established countably many kinds of adjacencies on $\mathbb{Z}$. In particular, except for $(\mathbb{Z}, T_{-1})$, each of $(\mathbb{Z}, T'_k)$, $(\mathbb{Z}, T_k)$, $(\mathbb{Z}, T'_{-k})$, and $(\mathbb{Z}, T_{-k})$, $k \in \mathbb{N}$, is not equal to $(\mathbb{Z}, \kappa)$. Furthermore, based on the product topological structures $(T'_k)^n$, $(T_k)^n$, $(T'_{-k})^n$, and $(T_{-k})^n$ on $\mathbb{Z}^n$, $k, n \in \mathbb{N}$, we also proposed countably many kinds of adjacencies on $\mathbb{Z}^n$ which are different from each other. Based on this approach, it turns out that we have countably many kinds of adjacencies on $\mathbb{Z}^n$ which are different from K-adjacency induced from the n-dimensional K-topological space. Based on this approach, we can use the above newly-established adjacencies on $\mathbb{Z}^n$ in the field of digital geometry and rough set theory.

As a further work, motivated by the locally finite covering approximation system (LFC-system for brevity in [7, 11]), using the given topological structures $(T'_k)^n$, $(T_k)^n$, $(T'_{-k})^n$, and $(T_{-k})^n$ on $\mathbb{Z}^n$, we can propose countably many kinds of LFC-system from the viewpoint of rough set theory and use them in the fields of applied sciences. In addition, this establishment facilitates various studies in digital topology and computer science related to writing parallel algorithms on subsets of $\mathbb{Z}^n$.

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Conflicts of interest

The author declares no conflict of interest.

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