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*Research article*

# Study of multi term delay fractional order impulsive differential equation using fixed point approach

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**Abstract:** This manuscript is devoted to investigate a class of multi terms delay fractional order impulsive differential equations. Our investigation includes existence theory along with Ulam type stability. By using classical fixed point theorems, we establish sufficient conditions for existence and uniqueness of solution to the proposed problem. We develop some appropriate conditions for different kinds of Ulam-Hyers stability results by using tools of nonlinear functional analysis. We demonstrate our results by an example.

**Keywords:** impulsive delay differential equations; existence theory; Banach theorem; stability analysis
**Mathematics Subject Classification:** 26A33, 34A08

## 1. Introduction

Fractional calculus has gotten considerable attention recently. Derivatives and integrals of non-integer order are increasingly using for different analysis of various problems. Fractional differential operators are global in nature and preserve greater degree of freedom. Therefore, researchers now give preference to use fractional order differential equations (FODEs) in mathematical modelings of various real world process and phenomenons over classical order differential equation. FODEs have multi-dimensional applications in the variety of fields of modern sciences, such as to control the phase difference in oscillators, to accomplish the high frequency oscillation and in electric engineering DC converter models are used to obtain good assessment of the power conversion efficiency. Recently various biological models have been investigated by using fractional order differential equations. In the mentioned study, researchers have established more good results than those already derived for ordinary differential equations. For some more applications of fractions derivatives, for theory and applications of FODEs see [1, 2], for integro-FODEs, see [3]. For basic theory and applications we refer [4]. Some generalized type FODEs have been analyzed in [5, 6]. Some engineering applications have been investigated in [7]. For real world applications by using FODEs, see [8]. Those differential equations which observe impulsive conditions at points of discontinuity of solution are known as impulsive differential equations. Impulsive differential equations are the tools used for modeling of those evolutionary and physical phenomena that containing sudden changes and discontinuous jumps. Therefore, the proposed types of impulsive differential equations play a significant role to models such phenomena, in this regards (see [9]). Some stability results about the said area has been studied in [10]. Also some conformable impulsive FODEs have been studied in [11]). In some circumstance physical problems depends on preceding states of problem and cannot be describes by current time. In order to avoid such circumstance, researches introduced an important class of differential equations (DEs) known as Delay Differential Equations (DDEs). There are verities of DDEs including proportional (pantograph), continuous and discrete type DDEs. The concern types of DEs are widely using to formulate various real world phenomena in different fields, such as dynamics, quantum mechanics [12], biology [13], and electrodynamics [14].

Important aspects of mathematical analysis are existence theory and stability analysis. Researchers have used various tools of nonlinear analysis for investigating the existence and uniqueness of solution to various problems of FODEs. Various fixed point results and degrees theories have been developed to investigate the said area for existence of solutions. In same line stability analysis of FODEs has also been given proper attention recently. The mentioned analysis is important for developing various numerical methods. Stability results have been investigated by using various methods including exponential method, Mittag-Leffler method and Hyers-Ulam method (see some detail in [15]). Here, we remark that the Ulam's type stability analysis has given more attention recently. The aforementioned stability results have been derived for various problems of FODEs in last few years (for instance see [16]). Hyers and Ulam had been introduced the mentioned stability for the first time for functional equations in 1940 (see some detail [17]). Rassias [18] extended the mentioned stability analysis for linear equations. Also Jung [19] extended the Rassias stability results for functional equations in nonlinear analysis.

Motivated from the mentioned work, researchers have been given much attention to investigate the aforesaid stability analysis for various dynamical problems (we refer few as [20, 21]). For boundary value problems of FODEs, the mentioned stability has been studied very well (we refer few results as [22, 23]). Furthermore, results related to existence theory of solutions to various problems of fractional order mathematical models of epidemiology have been investigated very well (for instance see [24]). The said results have been investigated for TB models in [27]. The Green functions theory using FODEs has been established in [25, 26]. Also the mentioned analysis has been studied for those FODEs involving non-singular derivatives (for instance see [28]).

The existence theory has been developed very well for FODEs in last few years. As an example the reader can look at the second order FODE with non local boundary condition on the independent variable [29]. Researchers have been used fixed point theory together with topological degree theory to develop necessary condition for existence of solution for various problems of fractional order differential equations. Furthermore, they have also derived various results related to Ulam type stability for said problems. Here we recall a suitable example which has been studied in [33] as

$$\begin{aligned}&{}^{c}D^{\eta}\mathscr{U}(t) = \mathfrak{F}(t, \mathscr{U}(t)), \quad 1 < \eta \leq 2, \quad t \in [0, 1],\\ &\mathfrak{z}_1\mathscr{U}(0) + \mathfrak{z}_2\mathscr{U}(1) = \mathfrak{P}_1(\mathscr{U}), \quad \mathfrak{z}_3\mathscr{U}'(0) + \mathfrak{z}_4\mathscr{U}'(1) = \mathfrak{P}_2(\mathscr{U}),\end{aligned} \tag{1.1}$$

where $\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3$ and $\mathfrak{z}_4$ are members of the set of real numbers, which satisfy the condition given as: $\mathfrak{z}_1 + \mathfrak{z}_2 \neq 0$ and $\mathfrak{z}_3 + \mathfrak{z}_4 \neq 0$. The function $\mathfrak{F}, \mathfrak{P}_1$ and $\mathfrak{P}_1$ are continuous. The authors initially utilized the tools of fractional calculus as well as nonlinear analysis to transform the aforementioned FDE to corresponding integral equation and then used fixed theory to achieve their aims.

Similarly, in [30] uniqueness and existence of solution have been studied by utilizing the tools of fixed point theory. Authors have investigated the following system of FODEs with anti periodic coupled with non local subsidiary conditions as

$$\begin{aligned}&\alpha_1^c D^{\eta_1}\mathscr{U}(t) + \alpha_2^c D^{\eta_2}\mathscr{U}(t) = \mathfrak{H}_g(t, \mathscr{U}(t), \mathscr{Y}(t)),\\ &\alpha_3^c D^{\eta_3}\mathscr{Y}(t) + \alpha_4^c D^{\eta_4}\mathscr{Y}(t) = \mathfrak{H}_y(t, \mathscr{U}(t), \mathscr{Y}(t)),\\ &\mathscr{U}(0) + \mathscr{U}(1) = \sum_{j=1}^{n} \kappa_j \mathscr{Y}(\beta_j), \quad \mathscr{U}'(0) + \mathscr{U}'(1) = \sum_{j=1}^{n} \lambda_j \mathscr{Y}'(\beta_j),\\ &\mathscr{Y}(0) + \mathscr{Y}(1) = \sum_{j=1}^{n} \kappa_j^* \mathscr{U}(\beta_j), \quad \mathscr{Y}'(0) + \mathscr{Y}'(1) = \sum_{j=1}^{n} \lambda_j^* \mathscr{U}'(\beta_j).\end{aligned}$$

The author have used standard Caputo derivative in the consider problem, where parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \kappa_j, \kappa_j^*, \lambda_j$ and $\lambda_j^*$ are real numbers for $j = 1, 2, \ldots, m$ and $0 < \beta_j < 1$.

Inspired from by the above discussion, in this research work, we take the following system by extending the problem (1.1) utilizing the concept of [30, 31] as

$$\begin{cases}\displaystyle\sum_{i=1}^{p} \sigma_i {}^{c}D^{\alpha_i}\mathscr{U}(t) = \mathscr{H}(t, \mathscr{U}(t), \mathscr{U}(\lambda t)),\\ \qquad \alpha_1 \in (1, 2]\ \alpha_i \in (0, 1], \text{ for } i = 2, 3, \ldots, p, \quad t \in [0, \tau],\\ \Delta(\mathscr{U}(t_k)) = \mathcal{I}_k\mathscr{U}(t_k), \quad \Delta(\mathscr{U}'(t_k)) = \mathcal{I}_k^*\mathscr{U}(t_k), \quad k = 0, 1, \ldots, m,\\ a_1\mathscr{U}(0) + b_1\mathscr{U}(\tau) = g_1(\mathscr{U}), \quad a_2\mathscr{U}'(0) + b_2\mathscr{U}'(\tau) = g_2(\mathscr{U}), \quad a_l, b_l \in R \quad \text{for} \quad l = 1, 2.\end{cases} \tag{1.2}$$

In the consider problem (1.2), $\sigma_i \in R$ for $i = 1, 2, \ldots, p$ with $\sigma_1 \neq 0$, and functions $g_1, g_2 : PC([0, \tau]), R) \mapsto R$ and non linear function $\mathscr{H} : [0, \tau] \times R \times R \mapsto R$ are continuous and $\tau > 0$ is real constant. Furthermore, impulsive operators $\mathcal{I}_k$ and $\mathcal{I}^*{}_k$ are also continuous. In this article, we use tools of fixed point theory and functional analysis to obtain the desired results. Results devoted to the existence and uniqueness of solution are derived by using Banach and Krasnoselskii's fixed point theorems. Also, the results devoted to stability analysis of Ulam type are established by using tools of nonlinear functional analysis. For verification of the obtain results, we give appropriate example.

The rest of the paper is organized as follows: In Section 2, we recall some basic concepts of fractional calculus, while the main results, relying on Krasnoselski's fixed point theorem and Banach contraction principal are presented in Section 3. Section 4 is devoted to stability analysis of the proposed problem (1.2). Section 5 contains illustrative examples for the obtained results. In Section 6, we present conclusion of our findings.

## 2. Auxiliary definitions and results

This section of research, is devoted to basic results, theorems and lemmas of FPT and non-linear analysis, which we need for investigation of the main work.

In the present work, we use the following space and norm

$$PC(J,R) = X = \{\mathscr{U} : J \mapsto R : \mathscr{U} \in C(J_k),\ k = 0,1,\ldots,m, \text{ and } \mathscr{U}(t^+),\mathscr{U}(t^-) \text{ exist}, k = 1,2,\ldots,m\}$$

with norm define as

$$\|\mathscr{U}\| = \max_{t\in J}\{|\mathscr{U}(t)|\ \mathscr{U} \in PC(J,R) : t \in J\},$$

where

$$J_0 = [0,t_1], J_1 = (t_1,t,2], J_2 = (t_2,t_3],\ldots,J_m = (t_m,\tau] \text{ and } J = [0,\tau].$$

**Definition 2.1.** *The integral of fractional order $\alpha$ of a function $y(t) \in L[0,d]$ is denoted by $I^\alpha y(t)$, and defined as*

$$\mathbb{I}^\alpha y(t) = \int_0^t \frac{y(\chi)}{\Gamma(\alpha)(t-\chi)^{1-\alpha}} d\chi.$$

**Definition 2.2.** *[32] Fractional order Caputo derivative for a function $y(t) \in L^1([0,d],R_+)$ on the interval $[0,d]$ is defined as*

$$^c\mathfrak{D}^\alpha y(t) = \int_0^t \frac{y^n(\chi)}{\Gamma(n-\alpha)(t-\chi)^{\alpha+1-n}} d\chi,$$

*where $n = \lceil\alpha\rceil$ and $\lceil\alpha\rceil$ is defined to be the smallest integer equal or greater than $\alpha$.*

**Lemma 2.1.** *[34] The relation between fractional order integral and derivative is given as*

$$I^\alpha[^c\mathfrak{D}^\alpha y(t)] = A_1 + A_2 t + A_3 t^2 + A_4 t^3 + \cdots + A_n t^{n-1} + y(t),$$

*where $A_i \in R$ for $i = 1,2,\ldots,n$.*

**Definition 2.3.** *The mapping $T : X \longrightarrow Y$ on norm linear spaces is continuous and complete, if for each bounded $M \in X$, $\overline{T(M)} \in Y$ is compact.*

**Definition 2.4.** *[35] Let $F(X)$ be the collection of function (real valued) on $(X,d)$ metric space, be equi-continuous $x \in X$, if for each $\epsilon > 0$, we can find $\delta > 0$, such that for every function $f \in F(X)$ and $x_0 \in X$, we have $|f(x_0) - f(x)| < \epsilon$, whenever $d(x_0 - x) < \delta$.*

**Definition 2.5.** *[35] An operator $\mathcal{T}$ on $(X,d)$ metric space into itself is Lipschitz, if $\exists\ c \geq 0$, and $d(\mathcal{T}(x_1),\mathcal{T}(x_2)) \leq cd(x_2,x_1)$, for each $x_2,x_1 \in X$, where $c$ is called Lipschitz constant and contraction, if $0 < c < 1$.*

**Definition 2.6.** *[35] An operator $\mathcal{T}$ from a metric space $(X,d)$ into itself is contraction, if $\exists\ 0 < c < 1$, such that $d(\mathcal{T}(x_1),\mathcal{T}(x_2)) \leq cd(x_1,x_2), \forall x_1,x_2 \in X$.*

**Theorem 2.7.** *[35] every mapping(self contraction) $\mathcal{T}$ in complete $(X,d)$ metric space has unique fixed point.*

**Theorem 2.8.** *[36] Assume that $\mathcal{H}$ is a non empty, convex, bounded and closed convex bounded subset of a Banach space $\mathcal{X}$. let $\mathscr{J}_1$ and $\mathscr{J}_1$ be two operator provide that $\mathscr{J}_1\mathscr{U}_1 + \mathscr{J}_2\mathscr{U}_2 \in \mathcal{H}$ whenever $\mathscr{U},\mathscr{U}_2 \in \mathcal{H}$, $\mathscr{J}_1$ is continuous and compact and $\mathscr{J}_2$ is contraction map. Then there is $\mathscr{U} \in \mathcal{H}$ provide that, $\mathscr{U} = \mathscr{J}_1\mathscr{U} + \mathscr{J}_2\mathscr{U}$.*

## 3. Qualitative analysis

This section of research work is committed, to integral representation and existence results for the consider class of multi-term fractional delay differential equations. The authors established the expression for integral representation of proposed problem. In order to obtain results for existence and stability analysis the authors used the tools of analysis and fixed point theory.

### *3.1. Integral representations for the proposed problem*

This subsection of the research work, is devoted the integral representation of the consider model (1.2).

**Theorem 3.1.** *Assumed that $\mathscr{Y}(t) \in C(J, R)$, then the solution of multi-term impulsive fractional delay differential equation,*

$$\begin{cases} \sum_{i=1}^{p} \sigma_i {}^{c}D^{\alpha_i}\mathscr{U}(t) = \mathscr{Y}(t), \quad \alpha_1 \in (1,2] \quad \alpha_i \in (0,1], \text{ for } i = 2,3,\ldots,p, \quad t \in [0,\tau], \\ \Delta(\mathscr{U}(t_k)) = \mathcal{I}_k\mathscr{U}(t_k), \quad \Delta(\mathscr{U}'(t_k)) = \mathcal{I}^*_k\mathscr{U}(t_k), \quad k = 0,1,\ldots,m, \\ a_1\mathscr{U}(0) + b_1\mathscr{U}(\tau) = g_1(\mathscr{U}), \quad a_2\mathscr{U}'(0) + b_2\mathscr{U}'(\tau) = g_2(\mathscr{U}), \quad a_l, b_l \in R \quad \text{for} \quad l = 1,2, \end{cases} \tag{3.1}$$

*is equivalent to the integral equation*

$$\mathscr{U}(t) = \begin{cases} \frac{1}{\sigma_1}\left[\frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\mathscr{X})^{\alpha_1-1}\mathscr{Y}(\mathscr{X})d\mathscr{X} - \sum_{i=2}^{p}\sigma_i I^{\alpha_1}{}^{c}D^{\alpha_i}\mathscr{U}(t)\right] + \frac{1}{\sigma_1}\mathscr{D}, \quad for \quad t \in [0,t_1] \\ \frac{1}{\sigma_1}\Bigg[\sigma_1\sum_{j=1}^{k}\mathcal{I}_j\mathscr{U}(t_j) + \sigma_1\sum_{j=1}^{k-1}(t_k-t_j)\mathcal{I}^*_j\mathscr{U}(t_j) + \sigma_1\sum_{j=1}^{k}(t-t_k)\mathcal{I}^*_j\mathscr{U}(t_j) \\ + \sum_{j=1}^{k}\frac{\sigma_1}{\Gamma(\alpha_1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-1}\mathscr{Y}(\mathscr{X})d\mathscr{X} \\ + \sum_{j=1}^{k}\frac{t-t_k}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-2}\mathscr{Y}(\mathscr{X})d\mathscr{X} \\ + \sum_{j=1}^{k-1}\frac{t_k-t_j}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_1-\mathscr{X})^{\alpha_1-2}\mathscr{Y}(\mathscr{X})d\mathscr{X} \\ - \sum_{j=1}^{k}\sum_{i=2}^{p}\frac{\sigma_i}{\Gamma(\alpha_1-\alpha_i)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-1}\mathscr{U}(\mathscr{X})d\mathscr{X} \\ - \sum_{j=1}^{k}\sum_{i=2}^{p}\frac{(t-t_k)\sigma_i}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\mathscr{U}(\mathscr{X})d\mathscr{X} \\ - \sum_{j=1}^{k-1}\sum_{i=2}^{p}\sigma_i\frac{t_k-t_j}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\mathscr{U}(\mathscr{X})d\mathscr{X} \\ + \frac{1}{\Gamma(\alpha_1)}\int_{t_k}^{t}(t-\mathscr{X})^{\alpha_1-1}\mathscr{Y}(\mathscr{X})d\mathscr{X} \\ - \sum_{i=2}^{p}\sigma_i\frac{1}{\Gamma(\alpha_1-\alpha_i)}\int_{t_k}^{t}(t-\mathscr{X})^{\alpha_1-\alpha_i-1}\mathscr{U}(\mathscr{X})d\mathscr{X}\Bigg] + \frac{1}{\sigma_1}\mathscr{D}, \\ \quad t \in (t_k, t_{k+1}] \quad for\ k = 1,2,3,\ldots,m. \end{cases} \tag{3.2}$$

*Where* $g_i(i=1,2): C(J,R)\mapsto R$ *are continuous function,* $a_l+b_l\neq 0$ *for* $l=1,2$ *and* $\alpha_1-\alpha_i-1>0$ *for* $i=2,3,\dots,p$ *and*

$$
\begin{aligned}
\mathscr{D} &= \frac{\sigma_1}{a_1+b_1}g_1(\mathscr{U})+\frac{b_2\sigma_1}{a_2+b_2}\Big(t-\frac{\tau b_1}{a_1+b_1}\Big)g_2(\mathscr{U})\\
&-\frac{b_1}{a_1+b_1}\Big[\sigma_1\sum_{j=1}^{m}\mathcal{I}_j\mathscr{U}(t_j)+\sigma_1\sum_{j=1}^{m-1}(t_m-t_j)\mathcal{I}_j^*\mathscr{U}(t_j)+\sigma_1\sum_{j=1}^{m}(t-t_m)\mathcal{I}_j^*\mathscr{U}(t_j)\Big]\\
&-\frac{b_2}{a_2+b_2}\Big(t-\frac{\tau b_1}{a_1+b_1}\Big)\sum_{j=1}^{m}\frac{1}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-2}\mathscr{Y}(\mathscr{X})d\mathscr{X}\\
&+\frac{b_2}{a_2+b_2}\Big(t-\frac{\tau b_1}{a_1+b_1}\Big)\sum_{j=1}^{m}\sum_{i=2}^{p}\sigma_i\frac{1}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\mathscr{U}(\mathscr{X})d\mathscr{X}\\
&-\frac{b_2}{a_2+b_2}\Big(t-\frac{\tau b_1}{a_1+b_1}\Big)\frac{1}{\Gamma(\alpha_1-1)}\int_{t_m}^{\tau}(\tau-\mathscr{X})^{\alpha_1-2}\mathscr{Y}(\mathscr{X})d\mathscr{X}\\
&+\frac{b_2}{a_2+b_2}\Big(t-\frac{\tau b_1}{a_1+b_1}\Big)\sum_{i=2}^{p}\sigma_i\frac{1}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_m}^{\tau}(\tau-\mathscr{X})^{\alpha_1-\alpha_i-2}\mathscr{U}(\mathscr{X})d\mathscr{X}\\
&+\frac{b_1}{a_1+b_1}\Big[-\sum_{j=1}^{m}\frac{1}{\Gamma(\alpha_1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-1}\mathscr{Y}(\mathscr{X})d\mathscr{X}\\
&-\sum_{j=1}^{m}\frac{\tau-t_m}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-2}\mathscr{Y}(\mathscr{X})d\mathscr{X}-\sum_{j=1}^{m-1}\frac{t_m-t_j}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_1-\mathscr{X})^{\alpha_1-2}\mathscr{Y}(\mathscr{X})d\mathscr{X}\\
&+\sum_{j=1}^{m}\sum_{i=2}^{p}\sigma_i\frac{1}{\Gamma(\alpha_1-\alpha_i)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-1}\mathscr{U}(\mathscr{X})d\mathscr{X}\\
&+\sum_{j=1}^{m}\sum_{i=2}^{p}\sigma_i\frac{\tau-t_m}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\mathscr{U}(\mathscr{X})d\mathscr{X}\\
&-\frac{1}{\Gamma(\alpha_1)}\int_{t_m}^{\tau}(\tau-\mathscr{X})^{\alpha_1-1}\mathscr{Y}(\mathscr{X})d\mathscr{X}\\
&+\sum_{j=1}^{m-1}\sum_{i=2}^{p}\sigma_i\frac{t_m-t_j}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\mathscr{U}(\mathscr{X})d\mathscr{X}\\
&-\sum_{i=2}^{p}\sigma_i\int_{t_m}^{\tau}\frac{(\tau-\mathscr{X})^{\alpha_1-\alpha_i-1}}{\Gamma(\alpha_1-\alpha_i)}\mathscr{U}(\mathscr{X})d\mathscr{X}\Big].
\end{aligned}
$$

*Proof.* Applying fractional order integral $I^{\alpha_1}$ on (3.1) and in-view of Lemma 2.1 for $t\in[0,t_1]$, we get

$$
\begin{aligned}
\sigma_1\mathscr{U}(t) &= C_0+C_1t+\frac{1}{\Gamma(\alpha_1)}\int_0^t(t-\mathscr{X})^{\alpha_1-1}\mathscr{Y}(\mathscr{X})d\mathscr{X}\\
&-\sum_{i=2}^{p}\frac{\sigma_i}{\Gamma(\alpha_1-\alpha_i)}\int_0^t(t-\mathscr{X})^{\alpha_1-\alpha_i-1}\mathscr{U}(\mathscr{X})d\mathscr{X}.
\end{aligned}\tag{3.3}
$$

By differentiating (3.3), we get

$$\begin{aligned}\sigma_1\mathscr{U}'(t) = C_1 + \frac{1}{\Gamma(\alpha_1-1)}\int_0^t (t-\mathscr{X})^{\alpha_1-2}\mathscr{Y}(\mathscr{X})d\mathscr{X} \\ -\sum_{i=2}^{p}\frac{\sigma_i}{\Gamma(\alpha_1-\alpha_i-1)}\int_0^t (t-\mathscr{X})^{\alpha_1-\alpha_i-2}\mathscr{U}(\mathscr{X})d\mathscr{X}.\end{aligned} \tag{3.4}$$

Similarly, for $t \in (t_1, t_2]$, the system (3.1) become

$$\begin{aligned}\sigma_1\mathscr{U}(t) = C_{01} + C_{11}(t-t_1) + \frac{1}{\Gamma(\alpha_1)}\int_{t_1}^t (t-\mathscr{X})^{\alpha_1-1}\mathscr{Y}(\mathscr{X})d\mathscr{X} \\ -\sum_{i=2}^{p}\int_{t_1}^t \frac{\sigma_i(t-\mathscr{X})^{\alpha_1-\alpha_i-1}}{\Gamma(\alpha_1-\alpha_i)}\mathscr{U}(\mathscr{X})d\mathscr{X}.\end{aligned} \tag{3.5}$$

By differentiating (3.5), we get

$$\begin{aligned}\sigma_1\mathscr{U}'(t) = C_{11} + \frac{1}{\Gamma(\alpha_1)}\int_{t_1}^t (t-\mathscr{X})^{\alpha_1-1}\mathscr{Y}(\mathscr{X})d\mathscr{X} \\ -\sum_{i=2}^{p}\frac{\sigma_i}{\Gamma(\alpha_1-\alpha_i)}\int_{t_1}^t (t-\mathscr{X})^{\alpha_1-\alpha_i-1}\mathscr{U}(\mathscr{X})d\mathscr{X}.\end{aligned} \tag{3.6}$$

Now to compute $\sigma_1\mathscr{U}(t_1^-)$, $\sigma_1\mathscr{U}'(t_1^-)$, $\sigma_1\mathscr{U}(t_1^+)$ and $\sigma_1\mathscr{U}'(t_1^+)$ using (3.3)–(3.6), we obtain

$$\begin{aligned}\sigma_1\mathscr{U}(t_1^-) &= C_0 + C_1 t_1 + \frac{1}{\Gamma(\alpha_1)}\int_0^{t_1} (t_1-\mathscr{X})^{\alpha_1-1}\mathscr{Y}(\mathscr{X})d\mathscr{X} \\ &\quad -\sum_{i=2}^{p}\int_0^{t_1} \frac{\sigma_i t_1-\mathscr{X})^{\alpha_1-\alpha_i-1}}{\Gamma(\alpha_1-\alpha_i)}(\mathscr{U}(\mathscr{X})d\mathscr{X}, \\ \sigma_1\mathscr{U}'(t_1^-) &= C_1 + \frac{1}{\Gamma(\alpha_1-1)}\int_0^{t_1} (t_1-\mathscr{X})^{\alpha_1-2}\mathscr{Y}(\mathscr{X})d\mathscr{X} \\ &\quad -\sum_{i=2}^{p}\int_0^{t_1} \frac{\sigma_i(t_1-\mathscr{X})^{\alpha_1-\alpha_i-2}}{\Gamma(\alpha_1-\alpha_i-1)}\mathscr{U}(\mathscr{X})d\mathscr{X}, \\ \sigma_1\mathscr{U}(t_1^+) &= C_{01}, \qquad \sigma_1\mathscr{U}'(t_1^+) = C_{11}.\end{aligned} \tag{3.7}$$

Using the impulsive condition

$$\Delta(\mathscr{U}(t_1)) = \mathscr{U}(t_1^+) - \mathscr{U}(t_1^-) = \mathcal{I}_1\mathscr{U}(t_1),$$

and

$$\Delta(\mathscr{U}'(t_1)) = \mathscr{U}'(t_1^+) - \mathscr{U}'(t_1^-) = \mathcal{I}_1^*\mathscr{U}(t_1),$$

we get

$$\begin{aligned} C_{01} &= \sigma_1 \mathcal{I}_1 \mathcal{U}(t_1) + C_0 + C_1 t_1 + \frac{1}{\Gamma(\alpha_1)} \int_0^{t_1} (t_1 - \mathcal{X})^{\alpha_1 - 1} \mathcal{Y}(\mathcal{X}) d\mathcal{X} \\ &- \sum_{i=2}^{p} \sigma_i \frac{1}{\Gamma(\alpha_1 - \alpha_i)} \int_0^{t_1} (t_1 - \mathcal{X})^{\alpha_1 - \alpha_i - 1} \mathcal{U}(\mathcal{X}) d\mathcal{X}, \\ C_{11} &= \sigma_1 \mathcal{I}_1^* \mathcal{U}(t_1) + C_1 + \frac{1}{\Gamma(\alpha_1 - 1)} \int_0^{t_1} (t_1 - \mathcal{X})^{\alpha_1 - 2} \mathcal{Y}(\mathcal{X}) d\mathcal{X} \\ &- \sum_{i=2}^{p} \sigma_i \frac{1}{\Gamma(\alpha_1 - \alpha_i - 1)} \int_0^{t_1} (t_1 - \mathcal{X})^{\alpha_1 - \alpha_i - 2} \mathcal{U}(\mathcal{X}) d\mathcal{X}. \end{aligned} \tag{3.8}$$

Using (3.8) in (3.5), we obtain

$$\begin{aligned} \sigma_1 \mathcal{U}(t) &= tC_1 + C_0 + \sigma_1 \mathcal{I}_1 \mathcal{U}(t_1) + \sigma_1 (t - t_1) \mathcal{I}_1^* \mathcal{U}(t_1) + \frac{1}{\Gamma(\alpha_1)} \int_0^{t_1} (t_1 - \mathcal{X})^{\alpha_1 - 1} \mathcal{Y}(\mathcal{X}) d\mathcal{X} \\ &- \sum_{i=2}^{p} \sigma_i \frac{1}{\Gamma(\alpha_1 - \alpha_i)} \int_0^{t_1} (t_1 - \mathcal{X})^{\alpha_1 - \alpha_i - 1} \mathcal{U}(\mathcal{X}) d\mathcal{X} + \frac{t - t_1}{\Gamma(\alpha_1 - 1)} \int_0^{t_1} (t_1 - \mathcal{X})^{\alpha_1 - 2} \mathcal{Y}(\mathcal{X}) d\mathcal{X} \\ &- \sum_{i=2}^{p} \sigma_i \frac{t - t_1}{\Gamma(\alpha_1 - \alpha_i - 1)} \int_0^{t_1} (t_1 - \mathcal{X})^{\alpha_1 - \alpha_i - 2} \mathcal{U}(\mathcal{X}) d\mathcal{X} + \frac{1}{\Gamma(\alpha_1)} \int_{t_1}^{t} (t - \mathcal{X})^{\alpha_1 - 1} \mathcal{Y}(\mathcal{X}) d\mathcal{X} \\ &- \sum_{i=2}^{p} \sigma_i \frac{1}{\Gamma(\alpha_1 - \alpha_i)} \int_{t_1}^{t} (t - \mathcal{X})^{\alpha_1 - \alpha_i - 1} \mathcal{U}(\mathcal{X}) d\mathcal{X}. \end{aligned}$$

On the same fashion, for $t \in (t_k, t_{k+1}]$, (3.1) becomes

$$\begin{aligned} \sigma_1 \mathcal{U}(t) &= C_0 + tC_1 + \sigma_1 \sum_{j=1}^{k} \mathcal{I}_j \mathcal{U}(t_j) + \sigma_1 \sum_{j=1}^{k-1} (t_k - t_j) \mathcal{I}_j^* \mathcal{U}(t_j) + \sigma_1 \sum_{j=1}^{k} (t - t_k) \mathcal{I}_j^* \mathcal{U}(t_j) \\ &+ \sigma_1 \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha_1)} \int_{t_{j-1}}^{t_j} (t_j - \mathcal{X})^{\alpha_1 - 1} \mathcal{Y}(\mathcal{X}) d\mathcal{X} \\ &+ \sum_{j=1}^{k} \frac{t - t_k}{\Gamma(\alpha_1 - 1)} \int_{t_{j-1}}^{t_j} (t_j - \mathcal{X})^{\alpha_1 - 2} \mathcal{Y}(\mathcal{X}) d\mathcal{X} \\ &+ \sum_{j=1}^{k-1} \frac{t_k - t_j}{\Gamma(\alpha_1 - 1)} \int_{t_{j-1}}^{t_j} (t_1 - \mathcal{X})^{\alpha_1 - 2} \mathcal{Y}(\mathcal{X}) d\mathcal{X} \\ &- \sum_{j=1}^{k} \sum_{i=2}^{p} \sigma_i \frac{1}{\Gamma(\alpha_1 - \alpha_i)} \int_{t_{j-1}}^{t_j} (t_j - \mathcal{X})^{\alpha_1 - \alpha_i - 1} \mathcal{U}(\mathcal{X}) d\mathcal{X} \\ &- \sum_{j=1}^{k} \sum_{i=2}^{p} \frac{\sigma_i (t - t_k)}{\Gamma(\alpha_1 - \alpha_i - 1)} \int_{t_{j-1}}^{t_j} (t_j - \mathcal{X})^{\alpha_1 - \alpha_i - 2} \mathcal{U}(\mathcal{X}) d\mathcal{X} \\ &+ \frac{1}{\Gamma(\alpha_1)} \int_{t_k}^{t} (t - \mathcal{X})^{\alpha_1 - 1} \mathcal{Y}(\mathcal{X}) d\mathcal{X} \end{aligned} \tag{3.9}$$

$$- \sum_{j=1}^{k-1} \sum_{i=2}^{p} \frac{\sigma_i(t_k - t_j)}{\Gamma(\alpha_1 - \alpha_i - 1)} \int_{t_{j-1}}^{t_j} (t_j - \mathscr{X})^{\alpha_1-\alpha_i-2} \mathscr{U}(\mathscr{X}) d\mathscr{X}$$

$$- \sum_{i=2}^{p} \frac{\sigma_i}{\Gamma(\alpha_1 - \alpha_i)} \int_{t_k}^{t} (t - \mathscr{X})^{\alpha_1-\alpha_i-1} \mathscr{U}(\mathscr{X}) d\mathscr{X}.$$

By using the boundary conditions involve in (3.1), we obtain

$$\begin{aligned} C_1 = & -\frac{b_2}{a_2+b_2}\Bigg[ \sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} \frac{(t_j - \mathscr{X})^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \mathscr{Y}(\mathscr{X}) d\mathscr{X} \\ & - \sum_{j=1}^{m} \sum_{i=2}^{p} \int_{t_{j-1}}^{t_j} \frac{\sigma_i(t_j - \mathscr{X})^{\alpha_1-\alpha_i-2}}{\Gamma(\alpha_1-\alpha_i-1)} \mathscr{U}(\mathscr{X}) d\mathscr{X} \\ & - \sum_{i=2}^{p} \int_{t_m}^{\tau} \frac{\sigma_i(\tau - \mathscr{X})^{\alpha_1-\alpha_i-2}}{\Gamma(\alpha_1-\alpha_i-1)} \mathscr{U}(\mathscr{X}) d\mathscr{X} \\ & + \frac{1}{\Gamma(\alpha_1-1)} \int_{t_m}^{\tau} (\tau - \mathscr{X})^{\alpha_1-2} \mathscr{Y}(\mathscr{X}) d\mathscr{X} \Bigg] + \frac{\sigma_1}{a_2+b_2} g_2(\mathscr{U}) \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} C_0 = & \frac{\sigma_1 \tau}{a_1+b_1} \frac{b_2}{a_2+b_2}\Bigg[ \sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} \frac{(t_j - \mathscr{X})^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \mathscr{Y}(\mathscr{X}) d\mathscr{X} \\ & - \sum_{j=1}^{m} \sum_{i=2}^{p} \int_{t_{j-1}}^{t_j} \frac{\sigma_i(t_j - \mathscr{X})^{\alpha_1-\alpha_i-2}}{\Gamma(\alpha_1-\alpha_i-1)} \mathscr{U}(\mathscr{X}) d\mathscr{X} + \frac{1}{\Gamma(\alpha_1-1)} \int_{t_m}^{\tau} (\tau - \mathscr{X})^{\alpha_1-2} \mathscr{Y}(\mathscr{X}) d\mathscr{X} \\ & - \sum_{i=2}^{p} \sigma_i \frac{1}{\Gamma(\alpha_1-\alpha_i-1)} \int_{t_m}^{\tau} (\tau - \mathscr{X})^{\alpha_1-\alpha_i-2} \mathscr{U}(\mathscr{X}) d\mathscr{X} \Bigg] - \frac{\tau b_1}{a_1+b_1} \frac{\sigma_1}{a_2+b_2} g_2(\mathscr{U}) \\ & - \frac{b_1}{a_1+b_1}\Bigg[ \sigma_1 \sum_{j=1}^{m} \mathcal{I}_j \mathscr{U}(t_j) + \sigma_1 \sum_{j=1}^{m-1} (t_m - t_j) \mathcal{I}_j^* \mathscr{U}(t_j) + \sigma_1 \sum_{j=1}^{m} (t - t_m) \mathcal{I}_j^* \mathscr{U}(t_j) \\ & + \sigma_1 \sum_{j=1}^{m} \frac{1}{\Gamma(\alpha_1)} \int_{t_{j-1}}^{t_j} (t_j - \mathscr{X})^{\alpha_1-1} \mathscr{Y}(\mathscr{X}) d\mathscr{X} + \sum_{j=1}^{m} \frac{\tau - t_m}{\Gamma(\alpha_1-1)} \int_{t_{j-1}}^{t_j} (t_j - \mathscr{X})^{\alpha_1-2} \mathscr{Y}(\mathscr{X}) d\mathscr{X} \\ & + \sum_{j=1}^{m-1} \frac{t_m - t_j}{\Gamma(\alpha_1-1)} \int_{t_{j-1}}^{t_j} (t_1 - \mathscr{X})^{\alpha_1-2} \mathscr{Y}(\mathscr{X}) d\mathscr{X} \\ & - \sum_{j=1}^{m} \sum_{i=2}^{p} \sigma_i \frac{1}{\Gamma(\alpha_1-\alpha_i)} \int_{t_{j-1}}^{t_j} (t_j - \mathscr{X})^{\alpha_1-\alpha_i-1} \mathscr{U}(\mathscr{X}) d\mathscr{X} \\ & - \sum_{j=1}^{m} \sum_{i=2}^{p} \sigma_i \frac{\tau - t_m}{\Gamma(\alpha_1-\alpha_i-1)} \int_{t_{j-1}}^{t_j} (t_j - \mathscr{X})^{\alpha_1-\alpha_i-2} \mathscr{U}(\mathscr{X}) d\mathscr{X} \\ & + \frac{1}{\Gamma(\alpha_1)} \int_{t_m}^{\tau} (\tau - \mathscr{X})^{\alpha_1-1} \mathscr{Y}(\mathscr{X}) d\mathscr{X} \end{aligned} \tag{3.11}$$

$$
-\sum_{j=1}^{m-1}\sum_{i=2}^{p}\left(\frac{\sigma_i(t_m-t_j)}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\right.
$$
$$
\left.-\int_{t_m}^{\tau}\frac{\sigma_i(\tau-\mathscr{X})^{\alpha_1-\alpha_i-1}}{\Gamma(\alpha_1-\alpha_i)}\right)\mathscr{U}(\mathscr{X})d\mathscr{X}\Bigg]+\frac{\sigma_1 g_1(\mathscr{U})}{a_1+b_1}.
$$

One can obtain the desired integral form of solution (3.2), by using (3.10) and (3.11) in (3.9) and (3.3). □

**Corollary 3.1.** *In view of Theorem 3.1 the solution of the given multi-terms fractional delay differential equation* (1.2) *is given by*

$$
\mathscr{U}(t)=\begin{cases}
\frac{1}{\sigma_1}\left[\frac{1}{\Gamma(\alpha_1)}\int_0^t(t-\mathscr{X})^{\alpha_1-1}\mathscr{H}(\mathscr{X},\mathscr{U}(\mathscr{X}),\mathscr{U}(\lambda\mathscr{X}))d\mathscr{X}\right.\\
\left.-\sum_{i=2}^{p}\sigma_i I^{\alpha_1 c}D^{\alpha_i}\mathscr{U}(t)\right]+\frac{1}{\sigma_1}\mathscr{D},\ for\, t\in[0,t_1]\\
\frac{1}{\sigma_1}\left[\sigma_1\sum_{j=1}^{k}\mathcal{I}_j\mathscr{U}(t_j)+\sigma_1\sum_{j=1}^{k-1}(t_k-t_j)\mathcal{I}_j^*\mathscr{U}(t_j)+\sigma_1\sum_{j=1}^{k}(t-t_k)\mathcal{I}_j^*\mathscr{U}(t_j)\right.\\
+\sum_{j=1}^{k}\left(\frac{\sigma_1}{\Gamma(\alpha_1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-1}+\frac{t-t_k}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-2}\right)\mathscr{H}(\mathscr{X},\mathscr{U}(\mathscr{X}),\mathscr{U}(\lambda\mathscr{X}))d\mathscr{X}\\
+\left(\sum_{j=1}^{k-1}\frac{t_k-t_j}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_1-\mathscr{X})^{\alpha_1-2}+\frac{1}{\Gamma(\alpha_1)}\int_{t_k}^{t}(t-\mathscr{X})^{\alpha_1-1}\right)\mathscr{H}(\mathscr{X},\mathscr{U}(\mathscr{X}),\mathscr{U}(\lambda\mathscr{X}))d\mathscr{X}\\
-\sum_{j=1}^{k}\sum_{i=2}^{p}\frac{(t-t_k)\sigma_i}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\mathscr{U}(\mathscr{X})d\mathscr{X}\\
-\sum_{j=1}^{k}\sum_{i=2}^{p}\int_{t_{j-1}}^{t_j}\frac{\sigma_i(t_j-\mathscr{X})^{\alpha_1-\alpha_i-1}}{\Gamma(\alpha_1-\alpha_i)}\mathscr{U}(\mathscr{X})d\mathscr{X}\\
-\sum_{j=1}^{k-1}\sum_{i=2}^{p}\frac{\sigma_i(t_k-t_j)}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\mathscr{U}(\mathscr{X})d\mathscr{X}\\
\left.-\sum_{i=2}^{p}\sigma_i\frac{1}{\Gamma(\alpha_1-\alpha_i)}\int_{t_k}^{t}(t-\mathscr{X})^{\alpha_1-\alpha_i-1}\mathscr{U}(\mathscr{X})d\mathscr{X}\right]+\frac{1}{\sigma_1}\mathscr{D},\\
\quad t\in(t_k,t_{k+1}]\qquad for\ k=1,2,3,\ldots,m.
\end{cases}
\tag{3.12}
$$

*Where* $(g_i(i=1,2):C(J,R)\mapsto R)$ *are continuous functions,* $a_l+b_l\neq 0$ *for* $l=1,2$ *and* $\alpha_1-\alpha_i-1>0$ *for* $i=2,3,\ldots,p$ *and*

$$
\mathscr{D}=\frac{\sigma_1}{a_1+b_1}g_1(\mathscr{U})+\frac{b_2\sigma_1}{a_2+b_2}\left(t-\frac{\tau b_1}{a_1+b_1}\right)g_2(\mathscr{U})
$$
$$
-\frac{b_1}{a_1+b_1}\left[\sigma_1\sum_{j=1}^{m}\mathcal{I}_j\mathscr{U}(t_j)+\sigma_1\sum_{j=1}^{m-1}(t_m-t_j)\mathcal{I}_j^*\mathscr{U}(t_j)+\sigma_1\sum_{j=1}^{m}(t-t_m)\mathcal{I}_j^*\mathscr{U}(t_j)\right]
$$

$$
\begin{aligned}
&-\frac{b_2}{a_2+b_2}\Big(t-\frac{\tau b_1}{a_1+b_1}\Big)\sum_{j=1}^{m}\frac{1}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-2}\mathscr{H}(\mathscr{X},\mathscr{U}(\mathscr{X}),\mathscr{U}(\lambda\mathscr{X}))d\mathscr{X}\\
&+\frac{b_2}{a_2+b_2}\Big(t-\frac{\tau b_1}{a_1+b_1}\Big)\sum_{j=1}^{m}\sum_{i=2}^{p}\sigma_i\frac{1}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\mathscr{U}(\mathscr{X})d\mathscr{X}\\
&-\frac{b_2}{a_2+b_2}\Big(t-\frac{\tau b_1}{a_1+b_1}\Big)\frac{1}{\Gamma(\alpha_1-1)}\int_{t_m}^{\tau}(\tau-\mathscr{X})^{\alpha_1-2}\mathscr{H}(\mathscr{X},\mathscr{U}(\mathscr{X}),\mathscr{U}(\lambda\mathscr{X}))d\mathscr{X}\\
&+\frac{b_2}{a_2+b_2}\Big(t-\frac{\tau b_1}{a_1+b_1}\Big)\sum_{i=2}^{p}\sigma_i\frac{1}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_m}^{\tau}(\tau-\mathscr{X})^{\alpha_1-\alpha_i-2}\mathscr{U}(\mathscr{X})d\mathscr{X}\\
&+\frac{b_1}{a_1+b_1}\Bigg[\Big(-\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\frac{(t_j-\mathscr{X})^{\alpha_1-1}}{\Gamma(\alpha_1)}\\
&-\sum_{j=1}^{m}\frac{\tau-t_m}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-2}\Big)\mathscr{H}(\mathscr{X},\mathscr{U}(\mathscr{X}),\mathscr{U}(\lambda\mathscr{X}))d\mathscr{X}\\
&-\Big(\sum_{j=1}^{m-1}\frac{t_m-t_j}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_1-\mathscr{X})^{\alpha_1-2}+\frac{1}{\Gamma(\alpha_1)}\int_{t_m}^{\tau}(\tau-\mathscr{X})^{\alpha_1-1}\Big)\mathscr{H}(\mathscr{X},\mathscr{U}(\mathscr{X}),\mathscr{U}(\lambda\mathscr{X}))d\mathscr{X}\\
&+\sum_{j=1}^{m}\sum_{i=2}^{p}\sigma_i\Big(\frac{1}{\Gamma(\alpha_1-\alpha_i)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-1}+\frac{\tau-t_m}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\Big)\mathscr{U}(\mathscr{X})d\mathscr{X}\\
&+\sum_{j=1}^{m-1}\sum_{i=2}^{p}\frac{\sigma_i(t_m-t_j)}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\mathscr{U}(\mathscr{X})d\mathscr{X}\\
&-\sum_{i=2}^{p}\frac{\sigma_i}{\Gamma(\alpha_1-\alpha_i)}\int_{t_m}^{\tau}(\tau-\mathscr{X})^{\alpha_1-\alpha_i-1}\mathscr{U}(\mathscr{X})d\mathscr{X}\Bigg].
\end{aligned}
$$

### *3.2. Data dependence results for proposed problem*

In this subsection of the research work, we represent the desired solution for MIFDDE (1.2), in the form of operator equation and provides some assumptions for investigation of existence results for proposed problem.

Lets define $\mathcal{T}: X \mapsto X$, such that

$$
\begin{aligned}
\mathcal{T}(\mathscr{U})=\frac{1}{\sigma_1}\Bigg[&\sigma_1\sum_{j=1}^{k}\mathcal{I}_j\mathscr{U}(t_j)+\sigma_1\sum_{j=1}^{k-1}(t_k-t_j)\mathcal{I}_j^*\mathscr{U}(t_j)+\sigma_1\sum_{j=1}^{k}(t-t_k)\mathcal{I}_j^*\mathscr{U}(t_j)\\
&+\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\Big(\frac{\sigma_1(t_j-\mathscr{X})^{\alpha_1-1}}{\Gamma(\alpha_1)}+\frac{t-t_k}{\Gamma(\alpha_1-1)}(t_j-\mathscr{X})^{\alpha_1-2}\Big)\mathscr{H}(\mathscr{X},\mathscr{U}(\mathscr{X}),\mathscr{U}(\lambda\mathscr{X}))d\mathscr{X}\\
&+\Big(\sum_{j=1}^{k-1}\frac{t_k-t_j}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_1-\mathscr{X})^{\alpha_1-2}+\frac{1}{\Gamma(\alpha_1)}\int_{t_k}^{t}(t-\mathscr{X})^{\alpha_1-1}\Big)\mathscr{H}(\mathscr{X},\mathscr{U}(\mathscr{X}),\mathscr{U}(\lambda\mathscr{X}))d\mathscr{X}\\
&-\sum_{j=1}^{k}\sum_{i=2}^{p}\int_{t_{j-1}}^{t_j}\frac{\sigma_i(t_j-\mathscr{X})^{\alpha_1-\alpha_i-1}}{\Gamma(\alpha_1-\alpha_i)}\mathscr{U}(\mathscr{X})d\mathscr{X}
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{j=1}^{k}\sum_{i=2}^{p}\frac{\sigma_i(t-t_k)}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\mathscr{U}(\mathscr{X})d\mathscr{X}\\
&-\sum_{j=1}^{k-1}\sum_{i=2}^{p}\frac{\sigma_i(t_k-t_j)}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\mathscr{U}(\mathscr{X})d\mathscr{X}\\
&-\sum_{i=2}^{p}\int_{t_k}^{t}\frac{\sigma_i(t-\mathscr{X})^{\alpha_1-\alpha_i-1}}{\Gamma(\alpha_1-\alpha_i)}\mathscr{U}(\mathscr{X})d\mathscr{X}\Bigg]+\frac{g_1(\mathscr{U})}{a_1+b_1}+\frac{b_2g_2(\mathscr{U})}{a_2+b_2}\Big(t-\frac{\tau b_1}{a_1+b_1}\Big)\\
&-\frac{b_1}{a_1+b_1}\Bigg[\sum_{j=1}^{m}\mathcal{I}_j\mathscr{U}(t_j)+\sum_{j=1}^{m-1}(t_m-t_j)\mathcal{I}_j^*\mathscr{U}(t_j)+\sum_{j=1}^{m}(t-t_m)\mathcal{I}_j^*\mathscr{U}(t_j)\Bigg]\\
&+\frac{b_2}{\sigma_1(a_2+b_2)}\Big(t-\frac{\tau b_1}{a_1+b_1}\Big)\Bigg[-\sum_{j=1}^{m}\frac{1}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-2}\mathscr{H}(\mathscr{X},\mathscr{U}(\mathscr{X}),\mathscr{U}(\lambda\mathscr{X}))d\mathscr{X}\\
&+\sum_{j=1}^{m}\sum_{i=2}^{p}\int_{t_{j-1}}^{t_j}\frac{\sigma_i(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}}{\Gamma(\alpha_1-\alpha_i-1)}\mathscr{U}(\mathscr{X})d\mathscr{X}\\
&-\frac{1}{\Gamma(\alpha_1-1)}\int_{t_m}^{\tau}(\tau-\mathscr{X})^{\alpha_1-2}\mathscr{H}(\mathscr{X},\mathscr{U}(\mathscr{X}),\mathscr{U}(\lambda\mathscr{X}))d\mathscr{X}\\
&+\sum_{i=2}^{p}\int_{t_m}^{\tau}\frac{\sigma_i(\tau-\mathscr{X})^{\alpha_1-\alpha_i-2}}{\Gamma(\alpha_1-\alpha_i-1)}\mathscr{U}(\mathscr{X})d\mathscr{X}\Bigg]\\
&+\frac{b_1}{\sigma_1(a_1+b_1)}\Bigg[\sum_{j=1}^{m}\sum_{i=2}^{p}\frac{\sigma_i}{\Gamma(\alpha_1-\alpha_i)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-1}\mathscr{U}(\mathscr{X})d\mathscr{X}\\
&-\sum_{j=1}^{m}\Big(\frac{1}{\Gamma(\alpha_1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-1}+\frac{\tau-t_m}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-2}\Big)\mathscr{H}(\mathscr{X},\mathscr{U}(\mathscr{X}),\mathscr{U}(\lambda\mathscr{X}))d\mathscr{X}\\
&-\sum_{j=1}^{m-1}\frac{t_m-t_j}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_1-\mathscr{X})^{\alpha_1-2}\mathscr{H}(\mathscr{X},\mathscr{U}(\mathscr{X}),\mathscr{U}(\lambda\mathscr{X}))d\mathscr{X}\\
&-\sum_{i=2}^{p}\int_{t_m}^{\tau}\frac{\sigma_i(\tau-\mathscr{X})^{\alpha_1-\alpha_i-1}}{\Gamma(\alpha_1-\alpha_i)}\mathscr{U}(\mathscr{X})d\mathscr{X}\\
&-\int_{t_m}^{\tau}\frac{(\tau-\mathscr{X})^{\alpha_1-1}}{\Gamma(\alpha_1)}\mathscr{H}(\mathscr{X},\mathscr{U}(\mathscr{X}),\mathscr{U}(\lambda\mathscr{X}))d\mathscr{X}\\
&+\sum_{j=1}^{m-1}\sum_{i=2}^{p}\frac{\sigma_i(t_m-t_j)}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\mathscr{U}(\mathscr{X})d\mathscr{X}\\
&+\sum_{j=1}^{m}\sum_{i=2}^{p}\frac{\sigma_i(\tau-t_m)}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\mathscr{U}(\mathscr{X})d\mathscr{X}\Bigg].
\end{aligned}
\tag{3.13}
$$

We consider the following assumptions, which we needs for further correspondence in this work.
$(H_1)$ For $\mathscr{U}_1,\mathscr{U}_2\in X$, there exist $\mathscr{L}_{\mathfrak{F}_g},\mathscr{L}_{\mathfrak{F}_y}\geq 0$, i.e

$$
|\mathscr{H}(t,\mathscr{U}_1(t),\mathscr{U}_2(\lambda t))-\mathscr{H}(t,\mathscr{U}_2(\lambda t),\mathscr{U}_2(t))|\leq\mathscr{L}_{\mathfrak{F}_1}\|\mathscr{U}_1-\mathscr{U}_2\|+\mathscr{L}_{\mathfrak{F}_2}\|\mathscr{U}_1-\mathscr{U}_2\|. \tag{3.14}
$$

$(H_2)$ For $\mathscr{U}_1, \mathscr{U}_2 \in X$, there exist $\mathscr{L}_{g_i} > 0$, for i=1,2, such that

$$|g_i(\mathscr{U}_1) - g_i(\mathscr{U}_2)| \leq \mathscr{L}_{g_i}\|\mathscr{U}_1 - \mathscr{U}_2\|. \tag{3.15}$$

$(H_3)$ For $\mathscr{U}_1, \mathscr{U}_2 \in X$, there exist $\mathscr{L}_I > 0$, such that

$$\left|\mathcal{I}_k\mathscr{U}_1(t) - \mathcal{I}_k\mathscr{U}_2(t)\right| \leq \mathscr{L}_I\|\mathscr{U}_1 - \mathscr{U}_2\|. \tag{3.16}$$

$(H_4)$ For $\mathscr{U}_1, \mathscr{U}_2 \in X$, there exist $\mathscr{L}_{I^*} > 0$, such that

$$\left|\mathcal{I}^*_k\mathscr{U}_1(t) - \mathcal{I}^*_k\mathscr{U}_2(t)\right| \leq \mathscr{L}_{I^*}\|\mathscr{U}_1 - \mathscr{U}_2\|. \tag{3.17}$$

$(H_5)$ For any $\mathscr{U} \in X$, there exist $\mathscr{B}_{\mathscr{F}} : C(J, R^+)$ such that

$$|\mathscr{H}(t, \mathscr{U}(t), \mathscr{U}(\lambda t))| \leq \mathscr{B}_{\mathscr{F}}(t). \tag{3.18}$$

$(H_6)$ For any $\mathscr{U} \in X$, there exist $\mathscr{B}_{g_i} : C(J, R^+)$ for $i = 1, 2$, such that

$$\left|g_i(\mathscr{U})\right| \leq \mathscr{B}_{g_i}(t). \tag{3.19}$$

$(H_7)$ For any $\mathscr{U} \in X$, there exist $\mathscr{B}_I : C(J, R^+)$, such that

$$\left|\mathcal{I}_k\mathscr{U}(t)\right| \leq \mathscr{B}_I(t). \tag{3.20}$$

$(H_8)$ For any $\mathscr{U} \in X$, there exist $\mathscr{B}_{I^*} : C(J, R^+)$, such that

$$\left|\mathcal{I}^*_k\mathscr{U}(t)\right| \leq \mathscr{B}_{I^*}(t). \tag{3.21}$$

For computational convenience, we introduce the following notation:

$$\mathscr{B}_I = \left(\frac{|b_1|}{|a_1 + b_1|} + 1\right)m. \tag{3.22}$$

$$\mathscr{B}_{I^*} = \left(\frac{|b_1|}{|a_1 + b_1|} + 1\right)\tau(2m - 1). \tag{3.23}$$

$$\mathscr{B}_{g_2} = \frac{|b_2|}{|a_2 + b_2|} \sup_{t\in J}\left\{\left|t - \frac{\tau b_1}{a_1 + b_1}\right|\right\}. \tag{3.24}$$

$$\begin{aligned}\mathscr{B}_{\mathscr{L}_{\mathscr{F}}} = & \frac{\tau^{\alpha_1}}{|\sigma_1|\Gamma(\alpha_1)}\left(\frac{|\sigma_1|m + 1}{\alpha_1} + 2m - 1\right) + \frac{|b_2|}{|\sigma_1||a_2 + b_2|} \sup_{t\in J}\left\{\left|t - \frac{\tau b_1}{a_1 + b_1}\right|\right\}\frac{(m + 1)\tau^{\alpha_1 - 1}}{\Gamma(\alpha_1)} \\ & + \frac{|b_1|}{|\sigma_1||a_1 + b_1|}\frac{\tau^{\alpha_1}}{\Gamma(\alpha_1)}\left(\frac{m + 1}{\alpha_1} + (2m - 1)\right).\end{aligned} \tag{3.25}$$

$$\begin{aligned}\mathscr{B}_{\mathscr{C}} = & \sum_{i=2}^{p} \frac{|\sigma_i|\tau^{\alpha_1 - \alpha_i}}{|\sigma_1|\Gamma(\alpha_1 - \alpha_i)}\left(\frac{m + 1}{\alpha_1 - \alpha_i} + 2m - 1\right) \\ & + \frac{|b_2|}{|\sigma_1||a_2 + b_2|} \sup_{t\in J}\left\{\left|t - \frac{\tau b_1}{a_1 + b_1}\right|\right\} \sum_{i=2}^{p} \frac{(m + 1)|\sigma_i|}{\Gamma(\alpha_1 - \alpha_i)}\tau^{\alpha_1 - \alpha_i - 1} \\ & + \frac{|b_1|}{|\sigma_1||a_1 + b_1|} \sum_{i=2}^{p} |\sigma_i|\frac{1}{\Gamma(\alpha_1 - \alpha_i)}\tau^{\alpha_1 - \alpha_i}\left(\frac{m + 1}{\alpha_1 - \alpha_i} + 2m - 1)\right).\end{aligned} \tag{3.26}$$

$$\mathscr{L} = \mathscr{B}_I\mathscr{L}_I + \mathscr{B}_{I^*}\mathscr{L}_{I^*} + \frac{\mathscr{L}_{g_1}}{|a_1+b_1|} + \mathscr{B}_{g_2}\mathscr{L}_{g_2} + \mathscr{B}_{\mathscr{L}_\mathscr{F}}\mathscr{L}_{\mathfrak{F}} + \mathscr{B}_{\mathscr{C}}. \tag{3.27}$$

$$\mathscr{L}_d = \mathscr{B}_{\mathscr{L}_\mathscr{F}}\mathscr{L}_{\mathfrak{F}} + \mathscr{B}_{\mathscr{C}}. \tag{3.28}$$

$$\mathscr{B}_{\mathscr{L}_\mathscr{R}} = \mathscr{B}_{\mathscr{L}_\mathscr{F}} - \frac{\tau^{\alpha_1}}{|\sigma_1|\Gamma(\alpha_1+1)}. \tag{3.29}$$

**Theorem 3.2.** *Consider* $(H_1)$–$(H_4)$ *holds and* $\mathscr{L} < 1$*, then problem* (1.2) *has at most one fixed point, where* $\mathscr{L}$ *is defined by (3.27).*

*Proof.* Consider that $\mathscr{U}, \mathscr{U}^* \in X$ and $t \in J_k$ where k=1,2..., m.

$$\begin{aligned}
&|\mathcal{T}(\mathscr{U}(t)) - \mathcal{T}(\mathscr{U}^*(t))| \\
&\leq \frac{1}{|\sigma_1|}\Bigg[|\sigma_1|\sum_{j=1}^{k}\mathscr{L}_I\|\mathscr{U}_1-\mathscr{U}_2\| + |\sigma_1|\sum_{j=1}^{k-1}(t_k-t_j)\mathscr{L}_{I^*}\|\mathscr{U}_1-\mathscr{U}_2\| \\
&+ |\sigma_1|\sum_{j=1}^{k}\sup_{t\in J}\Big\{|t-t_k|\Big\}\mathscr{L}_{I^*}\|\mathscr{U}_1-\mathscr{U}_2\| + |\sigma_1|\sum_{j=1}^{k}\frac{1}{\Gamma(\alpha_1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-1}\Big(\mathscr{L}_{\mathfrak{F}_1}\|\mathscr{U}_1-\mathscr{U}_2\| \\
&+ \mathscr{L}_{\mathfrak{F}_2}\|\mathscr{U}_1-\mathscr{U}_2\|\Big)d\mathscr{X} + \sum_{j=1}^{k}\frac{|t-t_k|}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-2}\Big(\mathscr{L}_{\mathfrak{F}_1}\|\mathscr{U}_1-\mathscr{U}_2\| + \mathscr{L}_{\mathfrak{F}_2}\|\mathscr{U}_1-\mathscr{U}_2\|\Big)d\mathscr{X} \\
&+ \sum_{j=1}^{k-1}\frac{t_k-t_j}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_1-\mathscr{X})^{\alpha_1-2}\Big(\mathscr{L}_{\mathfrak{F}_1}\|\mathscr{U}_1-\mathscr{U}_2\| + \mathscr{L}_{\mathfrak{F}_2}\|\mathscr{U}_1-\mathscr{U}_2\|\Big)d\mathscr{X} \\
&+ \Bigg(\sum_{j=1}^{k}\sum_{i=2}^{p}\int_{t_{j-1}}^{t_j}\frac{|\sigma_i|(t_j-\mathscr{X})^{\alpha_1-\alpha_i-1}}{\Gamma(\alpha_1-\alpha_i)} + \frac{|\sigma_i||t-t_k|}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\Bigg)\|\mathscr{U}-\mathscr{U}^*\|d\mathscr{X} \\
&+ \sum_{j=1}^{k-1}\sum_{i=2}^{p}\frac{|\sigma_i|(t_k-t_j)}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\|\mathscr{U}-\mathscr{U}^*\|d\mathscr{X} \\
&+ \sup_{t\in J}\Bigg\{\sum_{i=2}^{p}\int_{t_k}^{t}\frac{|\sigma_i|(t-\mathscr{X})^{\alpha_1-\alpha_i-1}}{\Gamma(\alpha_1-\alpha_i)}\|\mathscr{U}-\mathscr{U}^*\|d\mathscr{X}\Bigg\} \\
&+ \sup_{t\in J}\Bigg\{\frac{1}{\Gamma(\alpha_1)}\int_{t_k}^{t}(t-\mathscr{X})^{\alpha_1-1}\Big(\mathscr{L}_{\mathfrak{F}_1}\|\mathscr{U}_1-\mathscr{U}_2\| + \mathscr{L}_{\mathfrak{F}_2}\|\mathscr{U}_1-\mathscr{U}_2\|\Big)d\mathscr{X}\Bigg\}\Bigg] \\
&+ \frac{1}{|a_1+b_1|}\Big|g_1(\mathscr{U})-g_1(\mathscr{U}^*)\Big| + \frac{|b_2|}{|a_2+b_2|}\Big|t-\frac{\tau b_1}{a_1+b_1}\Big|\Big|g_2(\mathscr{U})-g_2(\mathscr{U}^*)\Big| \\
&+ \frac{|b_1|}{|a_1+b_1|}\Bigg[\sum_{j=1}^{m}\mathscr{L}_I\|\mathscr{U}_1-\mathscr{U}_2\| + \sum_{j=1}^{m-1}(t_m-t_j)\mathscr{L}_{I^*}\|\mathscr{U}_1-\mathscr{U}_2\| + \sum_{j=1}^{m}\sup_{t\in J}\Big\{|t-t_m|\Big\}\mathscr{L}_{I^*}\|\mathscr{U}_1-\mathscr{U}_2\|\Bigg] \\
&+ \frac{|b_2|}{|\sigma_1||a_2+b_2|}\sup_{t\in J}\Bigg\{\Big|t-\frac{\tau b_1}{a_1+b_1}\Big|\Bigg\}\Bigg[\sum_{j=1}^{m}\frac{1}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-2}\Big(\mathscr{L}_{\mathfrak{F}_1}\|\mathscr{U}_1-\mathscr{U}_2\| \\
&+ \mathscr{L}_{\mathfrak{F}_2}\|\mathscr{U}_1-\mathscr{U}_2\|\Big)d\mathscr{X} + \sum_{j=1}^{m}\sum_{i=2}^{p}|\sigma_i|\frac{1}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\|\mathscr{U}-\mathscr{U}^*\|d\mathscr{X}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha_1-1)}\int_{t_m}^{\tau}(\tau-\mathscr{X})^{\alpha_1-2}\Big(\mathscr{L}_{\mathfrak{F}1}\|\mathscr{U}_1-\mathscr{U}_2\|+\mathscr{L}_{\mathfrak{F}2}\|\mathscr{U}_1-\mathscr{U}_2\|\Big)d\mathscr{X}\\
&+\sum_{i=2}^{p}|\sigma_i|\frac{1}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_m}^{\tau}(\tau-\mathscr{X})^{\alpha_1-\alpha_i-2}\|\mathscr{U}-\mathscr{U}^*\|d\mathscr{X}\Bigg]\\
&+\frac{|b_1|}{|\sigma_1||a_1+b_1|}\Bigg[\sum_{j=1}^{m}\frac{1}{\Gamma(\alpha_1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-1}\Big(\mathscr{L}_{\mathfrak{F}1}\|\mathscr{U}_1-\mathscr{U}_2\|+\mathscr{L}_{\mathfrak{F}2}\|\mathscr{U}_1-\mathscr{U}_2\|\Big)d\mathscr{X}\\
&+\sum_{j=1}^{m}\frac{\tau-t_m}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-2}\Big(\mathscr{L}_{\mathfrak{F}1}\|\mathscr{U}_1-\mathscr{U}_2\|+\mathscr{L}_{\mathfrak{F}2}\|\mathscr{U}_1-\mathscr{U}_2\|\Big)d\mathscr{X}\\
&+\sum_{j=1}^{m-1}\frac{t_m-t_j}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_1-\mathscr{X})^{\alpha_1-2}\Big(\mathscr{L}_{\mathfrak{F}1}\|\mathscr{U}_1-\mathscr{U}_2\|+\mathscr{L}_{\mathfrak{F}2}\|\mathscr{U}_1-\mathscr{U}_2\|\Big)d\mathscr{X}\\
&+\sum_{j=1}^{m}\sum_{i=2}^{p}|\sigma_i|\frac{1}{\Gamma(\alpha_1-\alpha_i)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-1}\|\mathscr{U}-\mathscr{U}^*\|d\mathscr{X}\\
&+\sum_{j=1}^{m}\sum_{i=2}^{p}|\sigma_i|\frac{\tau-t_m}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\|\mathscr{U}-\mathscr{U}^*\|d\mathscr{X}\\
&+\frac{1}{\Gamma(\alpha_1)}\int_{t_m}^{\tau}(\tau-\mathscr{X})^{\alpha_1-1}\Big(\mathscr{L}_{\mathfrak{F}1}\|\mathscr{U}_1-\mathscr{U}_2\|+\mathscr{L}_{\mathfrak{F}2}\|\mathscr{U}_1-\mathscr{U}_2\|\Big)d\mathscr{X}\\
&+\sum_{j=1}^{m-1}\sum_{i=2}^{p}|\sigma_i|\frac{t_m-t_j}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\|\mathscr{U}-\mathscr{U}^*\|d\mathscr{X}\\
&+\sum_{i=2}^{p}|\sigma_i|\frac{1}{\Gamma(\alpha_1-\alpha_i)}\int_{t_m}^{\tau}(\tau-\mathscr{X})^{\alpha_1-\alpha_i-1}\|\mathscr{U}-\mathscr{U}^*\|d\mathscr{X}\Bigg]. \qquad (3.30)
\end{aligned}
$$

Let us assume $\mathscr{L}_{\mathfrak{F}}=\mathscr{L}_{\mathfrak{F}_1}+\mathscr{L}_{\mathfrak{F}_2}$ and evaluate the integral involve in (3.30), we have

$$
\begin{aligned}
&|\mathcal{T}(\mathscr{U}(t))-\mathcal{T}(\mathscr{U}^*(t))|\\
&\leq\frac{\|\mathscr{U}_1-\mathscr{U}_2\|}{|\sigma_1|}\Bigg[|\sigma_1|\sum_{j=1}^{k}\mathscr{L}_I+|\sigma_1|\sum_{j=1}^{k-1}(t_k-t_j)\mathscr{L}_{I^*}+|\sigma_1|\sum_{j=1}^{k}\tau\mathscr{L}_{I^*}+|\sigma_1|\sum_{j=1}^{k}\frac{(t_j-t_{j-1})^{\alpha_1}}{\Gamma(\alpha_1+1)}\mathscr{L}_{\mathfrak{F}}\\
&+\sum_{j=1}^{k}\frac{\tau}{\Gamma(\alpha_1)}(t_j-t_{j-1})^{\alpha_1-1}\mathscr{L}_{\mathfrak{F}}+\sum_{j=1}^{k-1}\frac{\tau}{\Gamma(\alpha_1)}(t_j-t_{j-1})^{\alpha_1-1}\mathscr{L}_{\mathfrak{F}}+\sum_{j=1}^{k}\sum_{i=2}^{p}|\sigma_i|\frac{(t_j-t_{j-1})^{\alpha_1-\alpha_i}}{\Gamma(\alpha_1-\alpha_i+1)}\\
&+\sum_{j=1}^{k}\sum_{i=2}^{p}|\sigma_i|\frac{\tau}{\Gamma(\alpha_1-\alpha_i)}(t_j-t_{j-1})^{\alpha_1-\alpha_i-1}+\sum_{j=1}^{k-1}\sum_{i=2}^{p}|\sigma_i|\frac{t_k-t_j}{\Gamma(\alpha_1-\alpha_i)}(t_j-t_{j-1})^{\alpha_1-\alpha_i-1}\\
&+\sup_{t\in J}\left\{\frac{1}{\Gamma(\alpha_1+1)}(t-t_k)^{\alpha_1}\mathscr{L}_{\mathfrak{F}}\right\}+\sup_{t\in J}\left\{\sum_{i=2}^{p}\frac{|\sigma_i|}{\Gamma(\alpha_1-\alpha_i+1)}(t-t_k)^{\alpha_1-\alpha_i}\right\}\Bigg]\\
&+\frac{1}{|a_1+b_1|}\mathscr{L}_{g_1}\|\mathscr{U}-\mathscr{U}^*\|+\frac{|b_2|}{|a_2+b_2|}\sup_{t\in J}\left\{\left|t-\frac{\tau b_1}{a_1+b_1}\right|\right\}\mathscr{L}_{g_2}\|\mathscr{U}-\mathscr{U}^*\|
\end{aligned}
$$

$$+\frac{|b_1|\|\mathscr{U}-\mathscr{U}^*\|}{|a_1+b_1|}\Bigg[\sum_{j=1}^{m}\mathscr{L}_I\|\mathscr{U}_1-\mathscr{U}_2\|+\sum_{j=1}^{m-1}(t_m-t_j)\mathscr{L}_{I^*}+\sum_{j=1}^{m}\sup_{t\in J}\Big\{|t-t_m|\Big\}\mathscr{L}_{I^*}\Bigg]$$
$$+\frac{|b_2|}{|\sigma_1||a_2+b_2|}\sup_{t\in J}\left\{\left|t-\frac{\tau b_1}{a_1+b_1}\right|\right\}\Bigg[\sum_{j=1}^{m}\frac{1}{\Gamma(\alpha_1)}(t_j-t_{j-1})^{\alpha_1-1}\mathscr{L}_{\mathfrak{F}}+\sum_{j=1}^{m}\sum_{i=2}^{p}|\sigma_i|\frac{(t_j-t_{j-1})^{\alpha_1-\alpha_i-1}}{\Gamma(\alpha_1-\alpha_i)}$$
$$+\frac{1}{\Gamma(\alpha_1)}(\tau-t_m)^{\alpha_1-1}\mathscr{L}_{\mathfrak{F}}+\sum_{i=2}^{p}|\sigma_i|\frac{1}{\Gamma(\alpha_1-\alpha_i)}(\tau-t_m)^{\alpha_1-\alpha_i-1}\Bigg]$$
$$+\frac{|b_1|\|\mathscr{U}-\mathscr{U}^*\|}{|\sigma_1||a_1+b_1|}\Bigg[\sum_{j=1}^{m}\frac{1}{\Gamma(\alpha_1+1)}(t_j-t_{j-1})^{\alpha_1}\mathscr{L}_{\mathfrak{F}}+\sum_{j=1}^{m}\frac{\tau}{\Gamma(\alpha_1)}(t_j-t_{j-1})^{\alpha_1-1}\mathscr{L}_{\mathfrak{F}}$$
$$+\sum_{j=1}^{m-1}\frac{\tau}{\Gamma(\alpha_1)}(t_j-t_{j-1})^{\alpha_1-1}\mathscr{L}_{\mathfrak{F}}+\sum_{j=1}^{m}\sum_{i=2}^{p}|\sigma_i|\frac{1}{\Gamma(\alpha_1-\alpha_i+1)}(t_j-t_{j-1})^{\alpha_1-\alpha_i}$$
$$+\sum_{j=1}^{m}\sum_{i=2}^{p}|\sigma_i|\frac{\tau}{\Gamma(\alpha_1-\alpha_i)}(t_j-t_{j-1})^{\alpha_1-\alpha_i-1}+\frac{1}{\Gamma(\alpha_1+1)}(\tau-t_m)^{\alpha_1}\mathscr{L}_{\mathfrak{F}}$$
$$+\sum_{j=1}^{m-1}\sum_{i=2}^{p}|\sigma_i|\frac{\tau}{\Gamma(\alpha_1-\alpha_i)}(t_j-t_{j-1})^{\alpha_1-\alpha_i-1}+\sum_{i=2}^{p}|\sigma_i|\frac{1}{\Gamma(\alpha_1-\alpha_i+1)}(\tau-t_m)^{\alpha_1-\alpha_i}\Bigg]. \tag{3.31}$$

It is quite clearly that $t_{j-1}-t_j\le\tau$, hence by using the aforementioned inequality equation (3.31) can be express as

$$\begin{aligned}
&|\mathcal{T}(\mathscr{U}(t))-\mathcal{T}(\mathscr{U}^*(t))|\\
&\le\frac{\|\mathscr{U}_1-\mathscr{U}_2\|}{|\sigma_1|}\Bigg[\mathscr{L}_I|\sigma_1|m\\
&+\mathscr{L}_{I^*}|\sigma_1|\tau(2m-1)+\mathscr{L}_{\mathfrak{F}}\frac{\tau^{\alpha_1}}{\Gamma(\alpha_1)}\left(\frac{|\sigma_1|m+1}{\alpha_1}+2m-1\right)\\
&+\sum_{i=2}^{p}\frac{|\sigma_i|\tau^{\alpha_1-\alpha_i}}{\Gamma(\alpha_1-\alpha_i)}\left(\frac{m+1}{\alpha_1-\alpha_i}+2m-1\right)\Bigg]+\frac{\mathscr{L}_{g_1}\|\mathscr{U}-\mathscr{U}^*\|}{|a_1+b_1|}\\
&+\frac{|b_2|\mathscr{L}_{g_2}\|\mathscr{U}-\mathscr{U}^*\|}{|a_2+b_2|}\sup_{t\in J}\left\{\left|t-\frac{\tau b_1}{a_1+b_1}\right|\right\}+\frac{|b_1|\|\mathscr{U}-\mathscr{U}^*\|}{|a_1+b_1|}\Big(m\mathscr{L}_I+(2m-1)\tau\mathscr{L}_{I^*}\Big)\\
&+\frac{|b_2|}{|\sigma_1||a_2+b_2|}\sup_{t\in J}\left\{\left|t-\frac{\tau b_1}{a_1+b_1}\right|\right\}\Bigg[\frac{(m+1)\tau^{\alpha_1-1}}{\Gamma(\alpha_1)}\mathscr{L}_{\mathfrak{F}}+\sum_{i=2}^{p}\frac{(m+1)|\sigma_i|}{\Gamma(\alpha_1-\alpha_i)}\tau^{\alpha_1-\alpha_i-1}\Bigg]\\
&+\frac{|b_1|\|\mathscr{U}-\mathscr{U}^*\|}{|\sigma_1||a_1+b_1|}\Bigg[\frac{\tau^{\alpha_1}}{\Gamma(\alpha_1)}\left(\frac{m+1}{\alpha_1}+(2m-1)\right)\mathscr{L}_{\mathfrak{F}}\\
&+\sum_{i=2}^{p}|\sigma_i|\frac{1}{\Gamma(\alpha_1-\alpha_i)}\tau^{\alpha_1-\alpha_i}\left(\frac{m+1}{\alpha_1-\alpha_i}+2m-1)\right)\Bigg].
\end{aligned}\tag{3.32}$$

By rearranging term containing $\mathscr{L}_I$, $\mathscr{L}_{I^*}$, $\mathscr{L}_{g_1}$, $\mathscr{L}_{g_2}$ and $\mathscr{L}_{\mathfrak{F}}$ in Eq (3.32), we obtain

$$|\mathcal{T}(\mathscr{U}(t))-\mathcal{T}(\mathscr{U}^*(t))|$$

$$
\begin{aligned}
&\leq \|\mathscr{U}_1 - \mathscr{U}_2\| \Bigg[ \left( \frac{|b_1|}{|a_1+b_1|} + 1 \right) m \mathscr{L}_I + \left( \frac{|b_1|}{|a_1+b_1|} + |\sigma_1| \right) \tau (2m-1) \mathscr{L}_{I^*} + \frac{\mathscr{L}_{g_1}}{|a_1+b_1|} \\
&+ \frac{|b_2|}{|a_2+b_2|} \sup_{t\in J} \left\{ \left| t - \frac{\tau b_1}{a_1+b_1} \right| \right\} \mathscr{L}_{g_2} + \Bigg\{ \frac{\tau^{\alpha_1}}{|\sigma_1|\Gamma(\alpha_1)} \left( \frac{|\sigma_1| m+1}{\alpha_1} + 2m - 1 \right) \\
&+ \frac{|b_2|}{|\sigma_1||a_2+b_2|} \sup_{t\in J} \left\{ \left| t - \frac{\tau b_1}{a_1+b_1} \right| \right\} \frac{(m+1)\tau^{\alpha_1-1}}{\Gamma(\alpha_1)} + \frac{|b_1|}{|\sigma_1||a_1+b_1|} \frac{\tau^{\alpha_1}}{\Gamma(\alpha_1)} \left( \frac{m+1}{\alpha_1} + (2m-1) \right) \Bigg\} \mathscr{L}_{\mathfrak{F}} \\
&+ \Bigg\{ \sum_{i=2}^{p} \frac{|\sigma_i| \tau^{\alpha_1-\alpha_i}}{|\sigma_1|\Gamma(\alpha_1-\alpha_i)} \left( \frac{m+1}{\alpha_1-\alpha_i} + 2m - 1 \right) \\
&+ \frac{|b_2|}{|\sigma_1||a_2+b_2|} \sup_{t\in J} \left\{ \left| t - \frac{\tau b_1}{a_1+b_1} \right| \right\} \sum_{i=2}^{p} \frac{(m+1)|\sigma_i|}{\Gamma(\alpha_1-\alpha_i)} \tau^{\alpha_1-\alpha_i-1} \\
&+ \frac{|b_1|}{|\sigma_1||a_1+b_1|} \sum_{i=2}^{p} |\sigma_i| \frac{1}{\Gamma(\alpha_1-\alpha_i)} \tau^{\alpha_1-\alpha_i} \left( \frac{m+1}{\alpha_1-\alpha_i} + 2m - 1) \right) \Bigg\} \Bigg]. \qquad (3.33)
\end{aligned}
$$

Now using (3.22)–(3.27) in (3.33), we have

$$
\begin{aligned}
|\mathcal{T}(\mathscr{U}(t)) - \mathcal{T}(\mathscr{U}^*(t))| \leq& \|\mathscr{U}_1 - \mathscr{U}_2\| \left[ \mathscr{B}_I \mathscr{L}_I + \mathscr{B}_{I^*} \mathscr{L}_{I^*} + \frac{\mathscr{L}_{g_1}}{|a_1+b_1|} + \mathscr{B}_{g_2} \mathscr{L}_{g_2} + \mathscr{B}_{\mathscr{L}_{\mathscr{F}}} \mathscr{L}_{\mathfrak{F}} + \mathscr{B}_{\mathscr{C}} \right] \\
\leq& \mathscr{L} \|\mathscr{U} - \mathscr{U}^*\|.
\end{aligned}
$$

Therefore, by Banach contraction principal the mapping $\mathcal{T}$ has fixed point. Thus, the consider problem (1.2) has solution, which is unique. □

**Theorem 3.3.** *The consider problem (1.2) has at least one solution, if* $(H_1)$ *and* $(H_5)$–$(H_8)$ *holds and* $\mathscr{L}_d < 1$*, where* $\mathscr{L}_d$ *is defined in* (3.28).

*Proof.* In order to prove existence of at least one solution, we define operator $\mathfrak{T}_1, \mathfrak{T}_2 : PC(J,R) \mapsto PC(J,R)$ given by

$$
\begin{aligned}
\mathfrak{T}_1(\mathscr{U}) =& \sum_{j=1}^{k} \mathcal{I}_j \mathscr{U}(t_j) + \sum_{j=1}^{k-1} (t_k - t_j) \mathcal{I}_j^* \mathscr{U}(t_j) + \sum_{j=1}^{k} (t - t_k) \mathcal{I}_j^* \mathscr{U}(t_j) \Bigg] + \frac{b_2}{a_2+b_2} \left( t - \frac{\tau b_1}{a_1+b_1} \right) g_2(\mathscr{U}) \\
&+ \frac{1}{a_1+b_1} g_1(\mathscr{U}) - \frac{b_1}{a_1+b_1} \left[ \sum_{j=1}^{m} \mathcal{I}_j \mathscr{U}(t_j) + \sum_{j=1}^{m-1} (t_m - t_j) \mathcal{I}_j^* \mathscr{U}(t_j) + \sum_{j=1}^{m} (t - t_m) \mathcal{I}_j^* \mathscr{U}(t_j) \right]
\end{aligned}
$$

and

$$
\begin{aligned}
\mathfrak{T}_2 \mathscr{U}(t) =& \frac{1}{\sigma_1} \Bigg[ \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \left( \frac{\sigma_1 (t_j - \mathscr{X})^{\alpha_1-1}}{\Gamma(\alpha_1)} + \frac{t-t_k}{\Gamma(\alpha_1-1)} (t_j - \mathscr{X})^{\alpha_1-2} \right) \mathscr{H}(\mathscr{X}, \mathscr{U}(\mathscr{X}), \mathscr{U}(\lambda \mathscr{X})) d\mathscr{X} \\
&+ \sum_{j=1}^{k-1} \frac{t_k - t_j}{\Gamma(\alpha_1-1)} \int_{t_{j-1}}^{t_j} (t_1 - \mathscr{X})^{\alpha_1-2} \mathscr{H}(\mathscr{X}, \mathscr{U}(\mathscr{X}), \mathscr{U}(\lambda \mathscr{X})) d\mathscr{X} \\
&- \sum_{j=1}^{k} \sum_{i=2}^{p} \int_{t_{j-1}}^{t_j} \frac{\sigma_i (t_j - \mathscr{X})^{\alpha_1-\alpha_i-1}}{\Gamma(\alpha_1-\alpha_i)} \mathscr{U}(\mathscr{X}) d\mathscr{X}
\end{aligned}
$$

$$
\begin{aligned}
&-\Big(\sum_{j=1}^{k}\sum_{i=2}^{p}\frac{\sigma_i(t-t_k)}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\\
&-\sum_{j=1}^{k-1}\sum_{i=2}^{p}\frac{\sigma_i(t_k-t_j)}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\Big)\mathscr{U}(\mathscr{X})d\mathscr{X}\\
&+\frac{1}{\Gamma(\alpha_1)}\int_{t_k}^{t}(t-\mathscr{X})^{\alpha_1-1}\mathscr{H}(\mathscr{X},\mathscr{U}(\mathscr{X}),\mathscr{U}(\lambda\mathscr{X}))d\mathscr{X}\\
&-\sum_{i=2}^{p}\frac{\sigma_i}{\Gamma(\alpha_1-\alpha_i)}\int_{t_k}^{t}(t-\mathscr{X})^{\alpha_1-\alpha_i-1}\mathscr{U}(\mathscr{X})d\mathscr{X}\Big]\\
&+\frac{b_2}{\sigma_1(a_2+b_2)}\Big(t-\frac{\tau b_1}{a_1+b_1}\Big)\Big[-\sum_{j=1}^{m}\frac{1}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-2}\mathscr{H}(\mathscr{X},\mathscr{U}(\mathscr{X}),\mathscr{U}(\lambda\mathscr{X}))d\mathscr{X}\\
&+\sum_{j=1}^{m}\sum_{i=2}^{p}\int_{t_{j-1}}^{t_j}\frac{\sigma_i(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}}{\Gamma(\alpha_1-\alpha_i-1)}\mathscr{U}(\mathscr{X})d\mathscr{X}\\
&-\frac{1}{\Gamma(\alpha_1-1)}\int_{t_m}^{\tau}(\tau-\mathscr{X})^{\alpha_1-2}\mathscr{H}(\mathscr{X},\mathscr{U}(\mathscr{X}),\mathscr{U}(\lambda\mathscr{X}))d\mathscr{X}\\
&+\sum_{i=2}^{p}\int_{t_m}^{\tau}\frac{\sigma_i(\tau-\mathscr{X})^{\alpha_1-\alpha_i-2}}{\Gamma(\alpha_1-\alpha_i-1)}\mathscr{U}(\mathscr{X})d\mathscr{X}\Big]\\
&+\frac{b_1}{\sigma_1(a_1+b_1)}\Big[\sum_{j=1}^{m}\sum_{i=2}^{p}\frac{\sigma_i}{\Gamma(\alpha_1-\alpha_i)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-1}\mathscr{U}(\mathscr{X})d\mathscr{X}\\
&-\sum_{j=1}^{m}\Big(\frac{1}{\Gamma(\alpha_1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-1}+\frac{\tau-t_m}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-2}\Big)\mathscr{H}(\mathscr{X},\mathscr{U}(\mathscr{X}),\mathscr{U}(\lambda\mathscr{X}))d\mathscr{X}\\
&-\Big(\sum_{j=1}^{m-1}\frac{t_m-t_j}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_1-\mathscr{X})^{\alpha_1-2}-\frac{1}{\Gamma(\alpha_1)}\int_{t_m}^{\tau}(\tau-\mathscr{X})^{\alpha_1-1}\Big)\mathscr{H}(\mathscr{X},\mathscr{U}(\mathscr{X}),\mathscr{U}(\lambda\mathscr{X}))d\mathscr{X}\\
&+\sum_{j=1}^{m}\sum_{i=2}^{p}\frac{\sigma_i(\tau-t_m)}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\mathscr{U}(\mathscr{X})d\mathscr{X}\\
&-\sum_{i=2}^{p}\int_{t_m}^{\tau}\frac{\sigma_i(\tau-\mathscr{X})^{\alpha_1-\alpha_i-1}}{\Gamma(\alpha_1-\alpha_i)}\mathscr{U}(\mathscr{X})d\mathscr{X}\\
&+\sum_{j=1}^{m-1}\sum_{i=2}^{p}\sigma_i\frac{t_m-t_j}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\mathscr{U}(\mathscr{X})d\mathscr{X}\Big].
\end{aligned}
$$

Moreover, we construct a ball $\mathcal{H}=\{\mathscr{U}(t)\in PC(J,R):\|\mathscr{U}\|\leq\mathcal{R}\}$, with positive radius $\mathcal{R}$ chosen as

$$
\mathcal{R}\geq\frac{1}{(1-\mathscr{B}_{\mathscr{C}})}\Big(\mathscr{B}_I\|\mathscr{B}_I(t)\|+\mathscr{B}_{I^*}\|\mathscr{B}_{I^*}(t)\|+\frac{\|\mathscr{B}_{g_1}(t)\|}{|a_1+b_1|}+\mathscr{B}_{g_2}\|\mathscr{B}_{g_2}(t)\|+\mathscr{B}_{\mathscr{L}_{\mathscr{F}}}\|\mathscr{B}_{\mathfrak{F}}(t)\|\Big).
$$

**Step 1:** We claim that $\mathfrak{T}_1\mathscr{U}_1(t)+\mathfrak{T}_2\mathscr{U}_2(t)\in\mathcal{H}\subseteq PC(J,R)$ for every $\mathscr{U}_1(t),\mathscr{U}_2(t)\in\mathcal{H}\subseteq PC(J,R)$.

Now for the proof of desired results, consider

$$
\begin{aligned}
&\|\mathfrak{T}_1\mathscr{U}_1(t)+\mathfrak{T}_2\mathscr{U}_2(t)\|\\
&\leq \sum_{j=1}^{k}\|\mathscr{B}_I(t)\|+\sum_{j=1}^{k-1}(t_k-t_j)\|\mathscr{B}_{I^*}(t)\|+\sum_{j=1}^{k}\sup_{t\in J}\left\{|t-t_k|\right\}\|\mathscr{B}_{I^*}(t)\|+\frac{1}{|a_1+b_1|}\|\mathscr{B}_{g_1}(t)\|\\
&+\frac{|b_2|}{|a_2+b_2|}\left|t-\frac{\tau b_1}{a_1+b_1}\right|\|\mathscr{B}_{g_2}(t)\|\\
&+\frac{|b_1|}{|a_1+b_1|}\left[\sum_{j=1}^{m}\|\mathscr{B}_I(t)\|+\sum_{j=1}^{m-1}(t_m-t_j)\|\mathscr{B}_{I^*}(t)\|+\sum_{j=1}^{m}\sup_{t\in J}\left\{|t-t_m|\right\}\|\mathscr{B}_{I^*}(t)\|\right]\\
&+\frac{1}{|\sigma_1|}\left[|\sigma_1|\sum_{j=1}^{k}\frac{1}{\Gamma(\alpha_1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-1}\|\mathscr{B}_{\mathscr{F}}(t)\|d\mathscr{X}\right.\\
&+\sum_{j=1}^{k}\frac{|t-t_k|}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-2}\|\mathscr{B}_{\mathscr{F}}(t)\|d\mathscr{X}\\
&+\sum_{j=1}^{k-1}\frac{t_k-t_j}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_1-\mathscr{X})^{\alpha_1-2}\|\mathscr{B}_{\mathscr{F}}(t)\|d\mathscr{X}\\
&+\sum_{j=1}^{k}\sum_{i=2}^{p}|\sigma_i|\frac{1}{\Gamma(\alpha_1-\alpha_i)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-1}\mathcal{R}d\mathscr{X}\\
&+\sum_{j=1}^{k}\sum_{i=2}^{p}|\sigma_i|\frac{|t-t_k|}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\mathcal{R}d\mathscr{X}\\
&+\sum_{j=1}^{k-1}\sum_{i=2}^{p}|\sigma_i|\frac{t_k-t_j}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\mathcal{R}d\mathscr{X}\\
&+\sup_{t\in J}\left\{\frac{1}{\Gamma(\alpha_1)}\int_{t_k}^{t}(t-\mathscr{X})^{\alpha_1-1}\|\mathscr{B}_{\mathscr{F}}(t)\|d\mathscr{X}\right\}\\
&\left.+\sup_{t\in J}\left\{\sum_{i=2}^{p}|\sigma_i|\frac{1}{\Gamma(\alpha_1-\alpha_i)}\int_{t_k}^{t}(t-\mathscr{X})^{\alpha_1-\alpha_i-1}\mathcal{R}d\mathscr{X}\right\}\right]\\
&+\frac{|b_2|}{|\sigma_1||a_2+b_2|}\sup_{t\in J}\left\{\left|t-\frac{\tau b_1}{a_1+b_1}\right|\right\}\left[\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\frac{(t_j-\mathscr{X})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\|\mathscr{B}_{\mathscr{F}}(t)\|d\mathscr{X}\right.\\
&+\int_{t_m}^{\tau}\frac{(\tau-\mathscr{X})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\|\mathscr{B}_{\mathscr{F}}(t)\|d\mathscr{X}\\
&+\sum_{j=1}^{m}\sum_{i=2}^{p}|\sigma_i|\frac{1}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\mathcal{R}d\mathscr{X}\\
&\left.+\sum_{i=2}^{p}|\sigma_i|\frac{1}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_m}^{\tau}(\tau-\mathscr{X})^{\alpha_1-\alpha_i-2}\mathcal{R}d\mathscr{X}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{|b_1|}{|\sigma_1||a_1+b_1|}\bigg[\sum_{j=1}^{m}\frac{1}{\Gamma(\alpha_1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-1}\|\mathscr{B}_{\mathscr{F}}(t)\|d\mathscr{X}\\
&+ \sum_{j=1}^{m}\frac{\tau-t_m}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-2}\|\mathscr{B}_{\mathscr{F}}(t)\|d\mathscr{X}\\
&+ \sum_{j=1}^{m-1}\frac{t_m-t_j}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_1-\mathscr{X})^{\alpha_1-2}\|\mathscr{B}_{\mathscr{F}}(t)\|d\mathscr{X}\\
&+ \sum_{j=1}^{m}\sum_{i=2}^{p}|\sigma_i|\frac{1}{\Gamma(\alpha_1-\alpha_i)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-1}\mathcal{R}d\mathscr{X}\\
&+ \sum_{j=1}^{m}\sum_{i=2}^{p}|\sigma_i|\frac{\tau-t_m}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\mathcal{R}d\mathscr{X}\\
&+ \frac{1}{\Gamma(\alpha_1)}\int_{t_m}^{\tau}(\tau-\mathscr{X})^{\alpha_1-1}\|\mathscr{B}_{\mathscr{F}}(t)\|d\mathscr{X}\\
&+ \sum_{j=1}^{m-1}\sum_{i=2}^{p}|\sigma_i|\frac{t_m-t_j}{\Gamma(\alpha_1-\alpha_i-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-\alpha_i-2}\mathcal{R}d\mathscr{X}\\
&+ \sum_{i=2}^{p}|\sigma_i|\frac{1}{\Gamma(\alpha_1-\alpha_i)}\int_{t_m}^{\tau}(\tau-\mathscr{X})^{\alpha_1-\alpha_i-1}\mathcal{R}d\mathscr{X}\bigg].
\end{aligned}
\tag{3.34}
$$

By using $(H_5)$–$(H_8)$, and evaluating the integral in (3.34), we obtain

$$
\begin{aligned}
&\|\mathfrak{T}_1\mathscr{U}_1(t)+\mathfrak{T}_2\mathscr{U}_2(t)\|\\
&\le \|\mathscr{B}_I(t)\||m+\|\mathscr{B}_{I^*}(t)\|\tau(2m-1)+\frac{\|\mathscr{B}_{g_1}(t)\|}{|a_1+b_1|}+\frac{|b_2|\|\mathscr{B}_{g_2}(t)\|}{|a_2+b_2|}\sup_{t\in J}\left\{\left|t-\frac{\tau b_1}{a_1+b_1}\right|\right\}\\
&+ \frac{|b_1|}{|a_1+b_1|}\Big(m\|\mathscr{B}_I(t)\|+(2m-1)\tau\|\mathscr{B}_{I^*}(t)\|\Big)+\frac{1}{|\sigma_1|}\bigg[\|\mathscr{B}_{\mathfrak{F}}(t)\|\frac{\tau^{\alpha_1}}{\Gamma(\alpha_1)}\Big(\frac{|\sigma_1|m+1}{\alpha_1}+2m-1\Big)\\
&+ \sum_{i=2}^{p}\frac{\mathcal{R}|\sigma_i|\tau^{\alpha_1-\alpha_i}}{\Gamma(\alpha_1-\alpha_i)}\Big(\frac{m+1}{\alpha_1-\alpha_i}+2m-1\Big)\bigg]\\
&+ \frac{|b_2|}{|\sigma_1||a_2+b_2|}\sup_{t\in J}\left\{\left|t-\frac{\tau b_1}{a_1+b_1}\right|\right\}\bigg[\frac{(m+1)\tau^{\alpha_1-1}}{\Gamma(\alpha_1)}\|\mathscr{B}_{\mathfrak{F}}(t)\|+\sum_{i=2}^{p}\frac{(m+1)\mathcal{R}|\sigma_i|}{\Gamma(\alpha_1-\alpha_i)}\tau^{\alpha_1-\alpha_i-1}\bigg]\\
&+ \frac{|b_1|}{|\sigma_1||a_1+b_1|}\bigg[\frac{\tau^{\alpha_1}}{\Gamma(\alpha_1)}\Big(\frac{m+1}{\alpha_1}+(2m-1)\Big)\|\mathscr{B}_{\mathfrak{F}}(t)\|+\sum_{i=2}^{p}|\sigma_i|\frac{\mathcal{R}\tau^{\alpha_1-\alpha_i}}{\Gamma(\alpha_1-\alpha_i)}\Big(\frac{m+1}{\alpha_1-\alpha_i}+2m-1)\Big)\bigg].
\end{aligned}
$$

By rearranging term containing $\mathscr{L}_I$, $\mathscr{L}_{I^*}$, $\mathscr{L}_{g_1}$, $\mathscr{L}_{g_2}$, and $\mathscr{L}_{\mathfrak{F}}$ in (3.34), we obtain

$$
\begin{aligned}
&\|\mathfrak{T}_1\mathscr{U}_1(t)+\mathfrak{T}_2\mathscr{U}_2(t)\|\\
&\le \Big(\frac{|b_1|}{|a_1+b_1|}+1\Big)m\|\mathscr{B}_I(t)\|+\Big(\frac{|b_1|}{|a_1+b_1|}+1\Big)\tau(2m-1)\|\mathscr{B}_{I^*}(t)\|\\
&+ \frac{\|\mathscr{B}_{g_1}(t)\|}{|a_1+b_1|}+\frac{|b_2|}{|a_2+b_2|}\sup_{t\in J}\left\{\left|t-\frac{\tau b_1}{a_1+b_1}\right|\right\}\|\mathscr{B}_{g_2}(t)\|
\end{aligned}
$$

$$+\left\{\frac{\tau^{\alpha_1}}{|\sigma_1|\Gamma(\alpha_1)}\left(\frac{|\sigma_1|m+1}{\alpha_1}+2m-1\right)+\frac{|b_2|}{|\sigma_1||a_2+b_2|}\sup_{t\in J}\left\{\left|t-\frac{\tau b_1}{a_1+b_1}\right|\right\}\frac{(m+1)\tau^{\alpha_1-1}}{\Gamma(\alpha_1)}\right.$$
$$\left.+\frac{|b_1|}{|\sigma_1||a_1+b_1|}\frac{\tau^{\alpha_1}}{\Gamma(\alpha_1)}\left(\frac{m+1}{\alpha_1}+(2m-1)\right)\right\}\|\mathscr{B}_{\mathfrak{F}}(t)\|$$
$$+\mathcal{R}\left\{\sum_{i=2}^{p}\frac{\tau^{\alpha_1-\alpha_i}}{\Gamma(\alpha_1-\alpha_i)}\left(\frac{m+1}{\alpha_1-\alpha_i}+2m-1\right)\right.$$
$$+\frac{|b_2|}{|\sigma_1||a_2+b_2|}\sup_{t\in J}\left\{\left|t-\frac{\tau b_1}{a_1+b_1}\right|\right\}\sum_{i=2}^{p}\frac{(m+1)|\sigma_i|}{\Gamma(\alpha_1-\alpha_i)}\tau^{\alpha_1-\alpha_i-1}$$
$$\left.+\frac{|b_1|}{|\sigma_1||a_1+b_1|}\sum_{i=2}^{p}|\sigma_i|\frac{1}{\Gamma(\alpha_1-\alpha_i)}\tau^{\alpha_1-\alpha_i}\left(\frac{m+1}{\alpha_1-\alpha_i}+2m-1\right)\right)\right\}. \tag{3.35}$$

By using (3.22)–(3.26) in (3.35), we obtain

$$\|\mathfrak{T}_1\mathscr{U}_1(t)+\mathfrak{T}_2\mathscr{U}_2(t)\|\leq\mathscr{B}_I\|\mathscr{B}_I(t)\|+\mathscr{B}_{I^*}\|\mathscr{B}_{I^*}(t)\|+\frac{\|\mathscr{B}_{g_1}(t)\|}{|a_1+b_1|}+\mathscr{B}_{g_2}\|\mathscr{B}_{g_2}(t)\|+\mathscr{B}_{\mathscr{L}_{\mathscr{F}}}\|\mathscr{B}_{\mathfrak{F}}(t)\|+\mathcal{R}\mathscr{B}_{\mathscr{C}}.$$

Hence it show that $\mathfrak{T}_1\mathscr{U}_1(t)+\mathfrak{T}_2\mathscr{U}_2(t)\in\mathcal{H}\subseteq PC(J,R)$.

**Step 2:** Here we claim that $\mathfrak{F}_g$ is uniformly bounded for confirmation, we proceed as

$$\|\mathfrak{T}_1\mathscr{U}_1(t)+\mathfrak{T}_2\mathscr{U}_2(t)\|\leq\sum_{j=1}^{k}\|\mathscr{B}_I(t)\|+\sum_{j=1}^{k-1}(t_k-t_j)\|\mathscr{B}_{I^*}(t)\|+\sum_{j=1}^{k}\sup_{t\in J}\left\{|t-t_k|\right\}\|\mathscr{B}_{I^*}(t)\|$$
$$+\frac{1}{|a_1+b_1|}\|\mathscr{B}_{g_1}(t)\|+\frac{|b_2|}{|a_2+b_2|}\left|t-\frac{\tau b_1}{a_1+b_1}\right|\|\mathscr{B}_{g_2}(t)\|$$
$$+\frac{|b_1|}{|a_1+b_1|}\left[\sum_{j=1}^{m}\|\mathscr{B}_I(t)\|+\sum_{j=1}^{m-1}(t_m-t_j)\|\mathscr{B}_{I^*}(t)\|+\sum_{j=1}^{m}\sup_{t\in J}\left\{|t-t_m|\right\}\|\mathscr{B}_{I^*}(t)\|\right]. \tag{3.36}$$

As we clearly see that $t-t_m\leq\tau$. So substitute $\tau$ instead of $t-t_m$ in (3.36), then we obtain

$$\|\mathfrak{T}_1\mathscr{U}_1(t)+\mathfrak{T}_2\mathscr{U}_2(t)\|\leq\left(\frac{|b_1|}{|a_1+b_1|}+1\right)m\|\mathscr{B}_I(t)\|+\left(\frac{|b_1|}{|a_1+b_1|}+|\sigma_1|\right)\tau(2m-1)\|\mathscr{B}_{I^*}(t)\|+\frac{\|\mathscr{B}_{g_1}(t)\|}{|a_1+b_1|}$$
$$+\frac{|b_2|}{|a_2+b_2|}\sup_{t\in J}\left\{\left|t-\frac{\tau b_1}{a_1+b_1}\right|\right\}\|\mathscr{B}_{g_2}(t)\|. \tag{3.37}$$

Using (3.22)–(3.24) in (3.37), we obtain

$$\|\mathfrak{T}_1\mathscr{U}_1(t)+\mathfrak{T}_2\mathscr{U}_2(t)\|\leq\mathscr{B}_I\|\mathscr{B}_I(t)\|+\mathscr{B}_{I^*}\|\mathscr{B}_{I^*}(t)\|+\frac{\|\mathscr{B}_{g_1}(t)\|}{|a_1+b_1|}+\mathscr{B}_{g_2}\|\mathscr{B}_{g_2}(t)\|.$$

Hence $\mathfrak{T}_1$ is uniformly bounded.

**Step 3:** Suppose $\mathscr{U}_n(t)$ is a sequence in $\mathcal{H}$ which converge to $\mathscr{U}\in\mathcal{H}$ for the continuity of $\mathfrak{T}_g$ we have to prove $\mathfrak{F}_g(\mathscr{U}_n(t))\mapsto\mathfrak{T}_g(\mathscr{U}(t))$. For the proof we precede as

$$\|\mathfrak{T}_1\mathscr{U}_n(t)-\mathfrak{T}_1\mathscr{U}(t)\|$$

$$\leq \sum_{j=1}^{k}\Big|\mathcal{I}_j\mathscr{U}_n(t_j)-\mathcal{I}_j\mathscr{U}(t_j)\Big|+\sum_{j=1}^{k-1}(t_k-t_j)\Big|\mathcal{I}^*{}_j\mathscr{U}_n(t_j)-\mathcal{I}^*{}_j\mathscr{U}(t_j)\Big|$$
$$+\sum_{j=1}^{k}\sup_{t\in J}\Big\{|t-t_k|\Big\}\Big|\mathcal{I}^*{}_j\mathscr{U}_n(t_j)-\mathcal{I}^*{}_j\mathscr{U}(t_j)\Big|\Big]+\frac{1}{|a_1+b_1|}\Big|g_1(\mathscr{U}_n)-g_1(\mathscr{U})\Big|$$
$$+\frac{|b_2|}{|a_2+b_2|}\sup_{t\in J}\Big\{\Big|t-\frac{\tau b_1}{a_1+b_1}\Big|\Big\}\Big|g_2(\mathscr{U}_n)-g_2(\mathscr{U})\Big|+\frac{|b_1|}{|a_1+b_1|}\Big[\sum_{j=1}^{m}\Big|\mathcal{I}_j\mathscr{U}_n(t_j)-\mathcal{I}_j\mathscr{U}(t_j)\Big|$$
$$+\sum_{j=1}^{m-1}(t_m-t_j)\Big|\mathcal{I}^*{}_j\mathscr{U}_n(t_j)-\mathcal{I}^*{}_j\mathscr{U}(t_j)\Big|+\sum_{j=1}^{m}\sup_{t\in J}\Big\{|t-t_m|\Big\}\Big|\mathcal{I}^*{}_j\mathscr{U}_n(t_j)-\mathcal{I}^*{}_j\mathscr{U}(t_j)\Big|\Big].$$

Hence clearly from the continuity of $g_1,\ g_2,\ \mathcal{I}_j$ and $\mathcal{I}^*_j$, we get that $\mathfrak{T}_1$ is continuous.

**Step 4:** To prove $\mathfrak{T}_1$ is equi-continuous. Consider

$$\|\mathfrak{T}_1\mathscr{U}(t_2)-\mathfrak{T}_1\mathscr{U}(t_1)\|\leq\sum_{j=1}^{k}\sup_{t\in J}\Big\{|t_2-t_1|\Big\}\Big|\mathcal{I}^*{}_j\mathscr{U}(t_j)\Big|\Big]+\frac{1}{|a_1+b_1|}\Big|g_1(\mathscr{U}(t_2))-g_1(\mathscr{U}(t_1))\Big|$$
$$+\frac{|b_2|}{|a_2+b_2|}\sup_{t\in J}\{\Big|t_2-t_1\Big|\Big\}\Big|g_2(\mathscr{U})\Big|+\frac{|b_1|}{|a_1+b_1|}+\sum_{j=1}^{m}\sup_{t\in J}\Big\{|t_2-t_1|\Big\}\Big|\mathcal{I}^*{}_j\mathscr{U}(t_j)\Big|$$

clearly as $t_1\mapsto t_2$ we have $\|\mathfrak{T}_1\mathscr{U}(t_2)-\mathfrak{T}_1\mathscr{U}(t_1)\|=0$. Hence $\mathfrak{T}_1$ is equi-continuous.

**Step 5:** To prove $\mathfrak{T}_2$ is contraction, one can get help from Theorem 3.2 and obtain the following expression,

$$\|\mathfrak{T}_2\mathscr{U}_1(t)-\mathfrak{T}_2\mathscr{U}_2(t)\|$$
$$\leq\|\mathscr{U}_1-\mathscr{U}_2\|\Big[\Big\{\tau^{\alpha_1}\Big(\frac{|\sigma_1|m+1}{\alpha_1}+2m-1\Big)+\frac{|b_2|}{|a_2+b_2|}\sup_{t\in J}\Big\{\Big|t-\frac{\tau b_1}{a_1+b_1}\Big|\Big\}\frac{(m+1)\tau^{\alpha_1-1}}{|\sigma_1|\Gamma(\alpha_1)}$$
$$+\frac{|b_1|}{|\sigma_1||a_1+b_1|}\frac{\tau^{\alpha_1}}{\Gamma(\alpha_1)}\Big(\frac{m+1}{\alpha_1}+(2m-1)\Big)\Big\}\mathscr{L}_{\mathfrak{F}}+\Big\{\sum_{i=2}^{p}\frac{\tau^{\alpha_1-\alpha_i}}{\Gamma(\alpha_1-\alpha_i)}\Big(\frac{m+1}{\alpha_1-\alpha_i}+2m-1\Big)$$
$$+\frac{|b_2|}{|\sigma_1||a_2+b_2|}\sup_{t\in J}\Big\{\Big|t-\frac{\tau b_1}{a_1+b_1}\Big|\Big\}\sum_{i=2}^{p}\frac{(m+1)|\sigma_i|}{\Gamma(\alpha_1-\alpha_i)}\tau^{\alpha_1-\alpha_i-1}$$
$$+\frac{|b_1|\|\mathscr{U}-\mathscr{U}^*\|}{|\sigma_1||a_1+b_1|}\sum_{i=2}^{p}|\sigma_i|\frac{1}{\Gamma(\alpha_1-\alpha_i)}\tau^{\alpha_1-\alpha_i}\Big(\frac{m+1}{\alpha_1-\alpha_i}+2m-1)\Big)\Big\}\Big].\tag{3.38}$$

By using (3.25), (3.26) and (3.28) in (3.38), we obtain

$$\|\mathfrak{T}_2\mathscr{U}_1(t)-\mathfrak{T}_2\mathscr{U}_2(t)\|\leq\|\mathscr{U}_1-\mathscr{U}_2\|\Big[\mathscr{B}_{\mathscr{L}_{\mathscr{F}}}\mathscr{L}_{\mathfrak{F}}+\mathscr{B}_{\mathscr{C}}\Big],$$
$$\leq\mathscr{L}_d\|\mathscr{U}-\mathscr{U}^*\|.$$

Thus all assumption of Krasnoselskii's fixed point theorem are satisfied. So the problem (1.2) has at least one solution. □

## 4. Stability analysis

The authors motivated by the literature [10, 29] and present some specific findings for the stability analysis of the proposed problem (1.2).

**Definition 4.1.** *The solution $\mathscr{U}(t)$ of proposed problem will be Ulam-Hyres (UH) stable, if for unique solution $\mathscr{U}^*(t)$ one can find $\mathcal{B}_1 > 0$, such that for each solution $\mathscr{U} \in PC(J, R)$ of the following differential inequality and $\epsilon > 0$*

$$\begin{cases} \left| \sum_{i=1}^{p} \sigma_i {}^c D^{\alpha_i} \mathscr{U}(t) - f(t, \mathscr{U}(t), \mathscr{U}(\lambda t)) \right| \le \epsilon, \quad t \in [0, \tau] \\ \left| \Delta(\mathscr{U}(t_k)) - \mathcal{I}_k \mathscr{U}(t_k) \right| \le \epsilon, \quad k = 1, 2, \ldots, m, \\ \left| \Delta(\mathscr{U}'(t_k)) - \mathcal{I}_k^* \mathscr{U}(t_k) \right| \le \epsilon, \quad k = 1, 2, \ldots, m, \end{cases} \tag{4.1}$$

*and a unique solution $\mathscr{U}^* \in PC(J, R)$ of the given problem (1.2), such that $|\mathscr{U} - \mathscr{U}^*| \le \mathcal{B}_1 \epsilon$ and solution will be generalized Ulam-Hyers (GUH) stable, if there exist a positive function $\mathcal{K} : (0, \infty) \mapsto (0, \infty)$ with $\mathcal{K}(0) = 0$, such that $|\mathscr{U} - \mathscr{U}^*| \le \mathcal{B}_1 \mathcal{K}(t)$.*

**Definition 4.2.** *The solution of consider problem is UH Rassias stable, with respect to continuous function $\chi \in X$ and a positive constant $\psi > 0$ if we have $\mathcal{B}_2$ (positive constant) $> 0$, and $\epsilon > 0$, for each solution $\mathscr{U} \in PC(J, R)$ of the following differential inequality*

$$\begin{cases} \left| \sum_{i=1}^{p} \sigma_i {}^c D^{\alpha_i} \mathscr{U}(t) - f(t, \mathscr{U}(t), \mathscr{U}(\lambda t)) \right| \le \chi(t)\epsilon, \quad t \in [0, \tau] \\ \left| \Delta(\mathscr{U}(t_k)) - \mathcal{I}_k \mathscr{U}(t_k) \right| \le \psi\epsilon, \quad k = 1, 2, \ldots, m, \\ \left| \Delta(\mathscr{U}'(t_k)) - \mathcal{I}_k^* \mathscr{U}(t_k) \right| \le \psi\epsilon, \quad k = 1, 2, \ldots, m, \end{cases} \tag{4.2}$$

*and a unique solution $\mathscr{U}^* \in PC(J, R)$ of the given problem (1.2), such that $|\mathscr{U} - \mathscr{U}^*| \le \mathcal{B}_2(\chi(t) + \psi)\epsilon$.*

**Definition 4.3.** *The solution of consider problem is GUH Rassias stable, with respect to continuous function $\chi \in X$ and a positive constant $\psi > 0$ if we have $\mathcal{B}_2$(positive constant) $> 0$, for each solution $\mathscr{U} \in PC(J, R)$ of the following differential inequality*

$$\begin{cases} \left| \sum_{i=1}^{p} \sigma_i {}^c D^{\alpha_i} \mathscr{U}(t) - f(t, \mathscr{U}(t), \mathscr{U}(\lambda t)) \right| \le \chi(t), \quad t \in [0, \tau] \\ \left| \Delta(\mathscr{U}(t_k)) - \mathcal{I}_k \mathscr{U}(t_k) \right| \le \psi, \quad k = 1, 2, \ldots, m, \\ \left| \Delta(\mathscr{U}'(t_k)) - \mathcal{I}_k^* \mathscr{U}(t_k) \right| \le \psi, \quad k = 1, 2, \ldots, m. \end{cases} \tag{4.3}$$

*and a unique solution $\mathscr{U}^* \in PC(J, R)$ of the given problem (1.2), such that $|\mathscr{U} - \mathscr{U}^*| \le \mathcal{B}_2 \chi(t)\epsilon$.*

**Remark 1.** *The solution of the inequality (4.1) is* $\mathscr{U}^* \in PC(J, R)$*, iff one can find a function* $\zeta \in PC(J, R)$*, and a sequence* $\zeta_k,\ k = 1, 2, \ldots, m$*. Depend on* $\mathscr{U}$*, such that*

$$\begin{cases} (i)\ \zeta(t) \le \epsilon,\ \zeta_k \le \epsilon \quad \text{where } k = 1, 2, 3, \ldots, m,\ t \in J \\ (ii)\ \sum_{i=1}^{p} \sigma_i {}^c D^{\alpha_i} \mathscr{U}(t) = f(t, \mathscr{U}(t), \mathscr{U}(\lambda t)) + \zeta(t), \quad t \in [0, \tau] \\ (iii)\ \Delta(\mathscr{U}(t_k)) = \mathcal{I}_k \mathscr{U}(t_k) + \zeta_k, \quad k = 1, \ldots, m, \\ (iv)\ \Delta(\mathscr{U}'(t_k)) = \mathcal{I}_k^* \mathscr{U}(t_k) + \zeta_k, \quad k = 1, \ldots, m. \end{cases} \tag{4.4}$$

**Remark 2.** *Let* $\mathscr{U}^* \in PC(J, R)$ *be the solution of (4.2), iff one can find a function* $\zeta \in PC(J, R)$*, and a sequence* $\zeta_k,\ k = 1, 2, \ldots, m$*. Depend on* $\mathscr{U}$*, such that*

$$\begin{cases} (i)\ \zeta(t) \le \chi(t)\epsilon,\ \zeta_k \le \psi\epsilon \quad \text{where } k = 1, 2, 3, \ldots, m,\ t \in J \\ (ii)\ \sum_{i=1}^{p} \sigma_i {}^c D^{\alpha_i} \mathscr{U}(t) = f(t, \mathscr{U}(t), \mathscr{U}(\lambda t)) + \zeta(t), \quad t \in [0, \tau] \\ (iii)\ \Delta(\mathscr{U}(t_k)) = \mathcal{I}_k \mathscr{U}(t_k) + \zeta_k, \quad k = 1, \ldots, m, \\ (iv)\ \Delta(\mathscr{U}'(t_k)) = \mathcal{I}_k^* \mathscr{U}(t_k) + \zeta_k, \quad k = 1, \ldots, m. \end{cases} \tag{4.5}$$

**Remark 3.** *Let* $\mathscr{U}^* \in PC(J, R)$ *be the solution of (4.3), iff one can find a function* $\xi \in PC(J, R)$*, and a sequence* $\zeta_k,\ k = 1, 2, \ldots, m$*. Depend on* $\mathscr{U}$*, such that*

$$\begin{cases} (i)\ \zeta(t) \le \chi(t),\ \zeta_k \le \psi \quad \text{where } k = 1, 2, 3, \ldots, m,\ t \in J \\ (ii)\ \sum_{i=1}^{p} \sigma_i {}^c D^{\alpha_i} \mathscr{U}(t) = f(t, \mathscr{U}(t), \mathscr{U}(\lambda t)) + \zeta(t), \quad t \in [0, \tau] \\ (iii)\ \Delta(\mathscr{U}(t_k)) = \mathcal{I}_k \mathscr{U}(t_k) + \zeta_k, \quad k = 1, \ldots, m, \\ (iv)\ \Delta(\mathscr{U}'(t_k)) = \mathcal{I}_k^* \mathscr{U}(t_k) + \zeta_k, \quad k = 1, \ldots, m. \end{cases} \tag{4.6}$$

**Lemma 4.1.** *Consider* $\mathscr{U} \in PC(J, R)$ *is solution of FDDE,*

$$\begin{cases} \sum_{i=1}^{p} \sigma_i {}^c D^{\alpha_i} \mathscr{U}(t) = f(t, \mathscr{U}(t), \mathscr{U}(\lambda t)) + \zeta(t), \quad \alpha_1 \in (1, 2] \quad \alpha_i \in (0, 1], \text{ for } i = 2, 3, \ldots, p, \quad t \in [0, \tau] \\ \Delta(\mathscr{U}(t_k)) = \mathcal{I}_k \mathscr{U}(t_k), \quad \Delta(\mathscr{U}'(t_k)) = \mathcal{I}_k^* \mathscr{U}(t_k), \quad k = 0, 1, \ldots, m, \\ a_1 \mathscr{U}(0) + b_1 \mathscr{U}(\tau) = g_1(\mathscr{U}), \quad a_2 \mathscr{U}'(0) + b_2 \mathscr{U}'(\tau) = g_2(\mathscr{U}), \quad a_l, b_l \in R \quad \text{for} \quad l = 1, 2 \end{cases} \tag{4.7}$$

*satisfy the following relation,*

$$\left| \mathscr{U}(t) - \mathcal{T}\mathscr{U}(t) \right| \le \left( \mathscr{B}_I + \mathscr{B}_{I^*} + \mathscr{B}_{\mathscr{L}_{\mathscr{F}}} \right) \epsilon.$$

*Proof.* In light of Theorem 3.1, the solution of (4.7), is given as

$$\mathscr{U}(t) - \mathcal{T}\mathscr{U}(t)$$
$$= \frac{1}{\sigma_1} \Bigg[ \sigma_1 \sum_{j=1}^{k} \zeta_i + \sigma_1 \sum_{j=1}^{k-1} (t_k - t_j) \zeta_i + \sigma_1 \sum_{j=1}^{k} (t - t_k) \zeta_i + \sigma_1 \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha_1)} \int_{t_{j-1}}^{t_j} (t_j - \mathscr{X})^{\alpha_1 - 1} \zeta(\mathscr{X}) d\mathscr{X}$$

$$+\sum_{j=1}^{k}\frac{t-t_k}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-2}\zeta(\mathscr{X})d\mathscr{X}+\sum_{j=1}^{k-1}\frac{t_k-t_j}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_1-\mathscr{X})^{\alpha_1-2}\zeta(\mathscr{X})d\mathscr{X}$$
$$+\frac{1}{\Gamma(\alpha_1)}\int_{t_k}^{t}(t-\mathscr{X})^{\alpha_1-1}\zeta(\mathscr{X})d\mathscr{X}\Bigg]-\frac{b_1}{a_1+b_1}\Bigg[\sum_{j=1}^{m}\zeta_i+\sum_{j=1}^{m-1}(t_m-t_j)\zeta_i+\sum_{j=1}^{m}(t-t_m)\zeta_i\Bigg]$$
$$+\frac{b_2}{\sigma_1(a_2+b_2)}\Big(t-\frac{\tau b_1}{a_1+b_1}\Big)\Bigg[\sum_{j=1}^{m}\frac{-1}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-2}\zeta(\mathscr{X})d\mathscr{X}$$
$$-\frac{1}{\Gamma(\alpha_1-1)}\int_{t_m}^{\tau}(\tau-\mathscr{X})^{\alpha_1-2}\zeta(\mathscr{X})d\mathscr{X}\Bigg]+\frac{b_1}{\sigma_1(a_1+b_1)}\Bigg[-\sum_{j=1}^{m}\frac{1}{\Gamma(\alpha_1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-1}\zeta(\mathscr{X})d\mathscr{X}$$
$$-\sum_{j=1}^{m}\frac{\tau-t_m}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_j-\mathscr{X})^{\alpha_1-2}\zeta(\mathscr{X})d\mathscr{X}-\sum_{j=1}^{m-1}\frac{t_m-t_j}{\Gamma(\alpha_1-1)}\int_{t_{j-1}}^{t_j}(t_1-\mathscr{X})^{\alpha_1-2}\zeta(\mathscr{X})d\mathscr{X}$$
$$-\frac{1}{\Gamma(\alpha_1)}\int_{t_m}^{\tau}(\tau-\mathscr{X})^{\alpha_1-1}\zeta(\mathscr{X})d\mathscr{X}\Bigg]. \tag{4.8}$$

By taking absolute on (4.8) and using Remark 1, we get

$$\begin{aligned}\Big|\mathscr{U}(t)-\mathcal{T}\mathscr{U}(t)\Big|\le\Bigg[&\Big(\Big(\frac{|b_1|}{|a_1+b_1|}+1\Big)m+\Big(\frac{|b_1|}{|a_1+b_1|}+|\sigma_1|\Big)\tau(2m-1)\\&+\frac{\tau^{\alpha_1}}{|\sigma_1|\Gamma(\alpha_1)}\Big(\frac{|\sigma_1|m+1}{\alpha_1}+2m-1\Big)+\frac{|b_2|}{|\sigma_1||a_2+b_2|}\sup_{t\in J}\Big\{\Big|t-\frac{\tau b_1}{a_1+b_1}\Big|\Big\}\frac{(m+1)\tau^{\alpha_1-1}}{\Gamma(\alpha_1)}\\&+\frac{|b_1|}{|\sigma_1||a_1+b_1|}\frac{\tau^{\alpha_1}}{\Gamma(\alpha_1)}\Big(\frac{m+1}{\alpha_1}+(2m-1)\Big)\Bigg]\epsilon.\end{aligned}$$

By using (3.22), (3.23) and (3.25), we obtain

$$\Big|\mathscr{U}(t)-\mathcal{T}\mathscr{U}(t)\Big|\le\Big(\mathscr{B}_I+\mathscr{B}_{I^*}+\mathscr{B}_{\mathscr{L}\mathscr{F}}\Big)\epsilon.$$

Which prove the required result. □

**Theorem 4.4.** *Under the assumptions ($H_1$–$H_4$), the problem (1.2) is UH as well as GUH stable, if* $\mathscr{L}<1$, *where* $\mathscr{L}$ *is defined in (3.27).*

*Proof.* For any solution $\mathscr{U}\in PC(J,R)$, and unique solution $\mathscr{U}^*$ of the the given problem (1.2), then

$$\begin{aligned}\|\mathscr{U}(t)-\mathscr{U}^*(t)\|&=\|\mathscr{U}(t)-\mathcal{T}\mathscr{U}^*(t)\|=\|\mathscr{U}(t)-\mathcal{T}\mathscr{U}^*+\mathcal{T}\mathscr{U}-\mathcal{T}\mathscr{U}\|,\\&\le\|\mathscr{U}(t)-\mathcal{T}\mathscr{U}\|+\|\mathfrak{T}\mathscr{U}-\mathfrak{T}\mathscr{U}^*\|.\end{aligned}$$

Using Theorem 3.2 and Lemma 4.1 we have

$$\begin{aligned}&\|\mathscr{U}(t)-\mathscr{U}^*(t)\|\le\mathscr{L}_U\epsilon+\mathscr{L}\|\mathscr{U}(t)-\mathscr{U}^*(t)\|,\\&\|\mathscr{U}(t)-\mathscr{U}^*(t)\|-\mathscr{L}\|\mathscr{U}(t)-\mathscr{U}^*(t)\|\le\mathscr{L}_U\epsilon,\\&\|\mathscr{U}(t)-\mathscr{U}^*(t)\|\le\frac{\mathscr{L}_U}{1-\mathscr{L}}\epsilon.\end{aligned}$$

Let $\mathcal{B}_1=\frac{\mathscr{L}_U}{1-\mathscr{L}}$, then the solution of the consider problem (1.2) is UH stable further set $\mathcal{K}(\epsilon)=\epsilon$ then the consider problem (1.2) is GUH stable. □

To prove the next stability result we need the following assumption given as ($H_9$). For any non decreasing function $\chi \in PC(J, R^+)$ there exist a positive constant $\mathscr{G}$, such that

$$\frac{1}{\Gamma(\alpha_1)}\int_{t_k}^{t}(t-\mathscr{X})^{\alpha_1-1}\chi(\mathscr{X})d\mathscr{X} \le \mathscr{G}\chi(t).$$

**Lemma 4.2.** *If assumption ($H_9$) holds then for any solution $x \in PC(J, R)$ the muti-term fractional delay differential equation*

$$\begin{cases}\displaystyle\sum_{i=1}^{p}\sigma_i{}^cD^{\alpha_i}\mathscr{U}(t) = f(t,\mathscr{U}(t),\mathscr{U}(\lambda t)) + \zeta(t),\\ \qquad \alpha_1 \in (1,2] \quad \alpha_i \in (0,1],\ for\ \ i = 2,3,\ldots,p, \quad t\in[0,\tau]\\ \Delta(\mathscr{U}(t_k)) = \mathcal{I}_k\mathscr{U}(t_k), \quad \Delta(\mathscr{U}'(t_k)) = \mathcal{I}^*_k\mathscr{U}(t_k), \quad k = 0,1,\ldots,m,\\ a_1\mathscr{U}(0) + b_1\mathscr{U}(\tau) = g_1(\mathscr{U}), \quad a_2\mathscr{U}'(0) + b_2\mathscr{U}'(\tau) = g_2(\mathscr{U}), \quad a_l, b_l \in R \quad for \qquad l = 1,2\end{cases} \tag{4.9}$$

*satisfy the following relation,*

$$\left|\mathscr{U}(t) - \mathcal{T}\mathscr{U}(t)\right| \le \Big(\mathscr{B}_I + \mathscr{B}_{I^*} + \mathscr{B}_{\mathscr{L}_\mathscr{R}}\Big)\mathscr{A}\epsilon + \frac{\mathscr{G}}{\sigma_1}\chi(t)\epsilon.$$

*Proof.* We omit the proof as it is straightforward and may be derived like Lemma 4.1. □

**Theorem 4.5.** *Under the assumption ($H_1$–$H_4$) and $H_9$, the problem (1.2) is UHR stable and GUHR stable, if* $\mathscr{L} < 1$.

*Proof.* For any solution $\mathscr{U} \in PC(J, R)$, and unique solution $\mathscr{U}^*$ of the the given problem (1.2), then

$$\begin{aligned}\|\mathscr{U}(t) - \mathscr{U}^*(t)\| &= \|\mathscr{U}(t) - \mathcal{T}\mathscr{U}^*(t)\| = \|\mathscr{U}(t) - \mathcal{T}\mathscr{U}^* + \mathcal{T}\mathscr{U} - \mathcal{T}\mathscr{U}\|\\ &\le \|\mathscr{U}(t) - \mathcal{T}\mathscr{U}\| + \|\mathcal{T}\mathscr{U} - \mathcal{T}\mathscr{U}^*\|.\end{aligned}$$

Using Theorem 3.2 and Lemma 4.2, we have

$$\begin{aligned}\|\mathscr{U}(t) - \mathscr{U}^*(t)\| &\le \Big(\mathscr{B}_I + \mathscr{B}_{I^*} + \mathscr{B}_{\mathscr{L}_\mathscr{R}}\Big)\mathscr{A}\epsilon + \frac{\mathscr{G}}{|\sigma_1|}\chi(t)\epsilon + \mathscr{L}\|\mathscr{U}(t) - \mathscr{U}^*(t)\|,\\ \|\mathscr{U}(t) - \mathscr{U}^*(t)\| - \mathscr{L}\|\mathscr{U}(t) - \mathscr{U}^*(t)\| &\le \Big(\mathscr{B}_I + \mathscr{B}_{I^*} + \mathscr{B}_{\mathscr{L}_\mathscr{R}}\Big)\mathscr{A}\epsilon + \frac{\mathscr{G}}{|\sigma_1|}\chi(t)\epsilon,\\ \|\mathscr{U}(t) - \mathscr{U}^*(t)\| &\le \frac{\Big(\mathscr{B}_I + \mathscr{B}_{I^*} + \mathscr{B}_{\mathscr{L}_\mathscr{R}}\Big)\mathscr{A} + \frac{\mathscr{G}}{|\sigma_1|}\chi(t)}{1-\mathscr{L}}\epsilon.\end{aligned}$$

Let $\mathscr{B}_1 = \frac{max\Big(\mathscr{B}_I + \mathscr{B}_{I^*} + \mathscr{B}_{\mathscr{L}_\mathscr{R}}, \frac{\mathscr{G}}{|\sigma_1|}\Big)}{1-\mathscr{L}}$, then the solution of the consider problem (1.2) is UHR stable further set $\epsilon = 1$ then the consider problem (1.2) is GUHR stable. □

## 5. Example

Here we present an example to demonstrate our results.

**Example 1.** *Consider the following multi-term impulsive fractional delay differential equation*

$$\begin{cases} \sum_{i=1}^{p} \frac{10}{10i-9} {}^{c}D^{\frac{4}{1+2i}} \mathscr{U}(t) = \frac{L}{(3t^3+44e^t+55t)^2} \left( \frac{2|\mathscr{U}(t)|}{cos(t)+|\mathscr{U}(t)|} \right. \\ \qquad \left. - \frac{t^3|\mathscr{U}(\frac{t}{6})|}{sec(t)+|\mathscr{U}(\frac{t}{8})|} - cosh(t) \right), \ t \in [0,1], \ t \neq \frac{1}{4}, \ L \in R^{+} \\ \Delta(\mathscr{U}(\frac{1}{4})) = \mathcal{I}_k \mathscr{U}(\frac{1}{4}) = \frac{|\mathscr{U}(\frac{1}{4})|}{32+|\mathscr{U}(\frac{1}{4})|}, \quad \Delta(\mathscr{U}'(\frac{1}{4})) = \mathcal{I}_k^{*} \mathscr{U}(\frac{1}{4}) = \frac{|\mathscr{U}(\frac{1}{4})|}{54+|\mathscr{U}(\frac{1}{4})|}, \\ 8\mathscr{U}(0) + \mathscr{U}(1) = g_1(\mathscr{U}) = \frac{|\mathscr{U}|}{100+|\mathscr{U}|}, \quad 6\mathscr{U}'(0) + \mathscr{U}'(1) = g_2(\mathscr{U}) = \frac{1}{42+|\mathscr{U}|}, \end{cases} \tag{5.1}$$

*here*

$$p = 5, \ m = 1, \ a_1 = 8, \ b_1 = b_2 = 1, \ a_2 = 6, \ \tau = 1, \ \tau = 1,$$

$$\alpha_1 = 4/3, \ \alpha_i = \frac{4}{1+2i}, \ \sigma_1 = 10, \ \sigma_i = \frac{10}{10i-9}.$$

$$\begin{aligned} &\left| \mathscr{H}(t, \mathscr{U}(t), \mathscr{U}(\frac{t}{6})) - f(t, \mathscr{U}_1(t), \mathscr{U}(\frac{t}{6}) \right| \\ &\leq \frac{L}{(3t^3+44e^t+55t)^2} \left( \left| \frac{2|\mathscr{U}_1(t)|}{cos(t)+|\mathscr{U}_1(t)|} - \frac{2|\mathscr{U}_2(t)|}{cos(t)+|\mathscr{U}_2(t)|} \right| \right. \\ &+ \left. \left| \frac{t^3|\mathscr{U}_1(\frac{t}{6})|}{sec(t)+|\mathscr{U}_1(\frac{t}{8})|} - \frac{t^3|\mathscr{U}_2(\frac{t}{6})|}{sec(t)+|\mathscr{U}_2(\frac{t}{8})|} \right| \right) \\ &\leq \frac{L}{44^2} \left( \frac{2}{cos(t)} |\mathscr{U}_1(t) - \mathscr{U}_2(t)| + \frac{t^3}{sec(t)} |\mathscr{U}_1(t) - \mathscr{U}_2(t)| \right). \end{aligned}$$

*Hence, we obtain* $\mathscr{L}_{\mathscr{F}} = \frac{L}{44^2}\left(\frac{2}{cos(1)} + 1\right)$, *and similarly by simple computation, one can calculate,*

$$\mathscr{L}_{g_1} = \frac{1}{100}, \quad \mathscr{L}_{g_2} = \frac{1}{1764}, \quad \mathscr{L}_{I} = \frac{1}{32}, \quad \textit{and} \quad \mathscr{L}_{I^*} = \frac{1}{54}.$$

*Hence, we can calculate that* $\mathfrak{L} < 1$ *if* $G < 24.608599$. *Now as a consequence of Theorem 3.2, we conclude that example 1, has unique solution.*

*Set* $\chi(t) = (4t-1)^2$, *and* $\psi = 3 \ \forall t \in J$. *since*

$$\frac{1}{\Gamma(\frac{4}{3})} \int_{\frac{1}{4}}^{t} (t - \mathscr{X})^{\frac{4}{3}-1} (4\mathscr{X} - 1)^2 d\mathscr{X} = \frac{32}{\Gamma(\frac{13}{3})} (t - \frac{1}{4})^{\frac{10}{3}} \leq \frac{2}{6.3103} (4t-1)^2$$

*condition* $H_9$ *is satisfied with* $\mathscr{G} = \frac{2}{6.3103}$. *Hence all assumption of Theorem 4.5 are satisfied so as a conclusion the solution of problem 1.2 is UHR and GUHR stable. Moreover problem* (1.2) *is UH and GUH stable due to Theorem 4.4.*

## 6. Conclusions

In this research work we have established a detailed analysis about existence theory of at least one solution to a class of multi term impulsive FODEs. The required theory has been established by using Krasnoselskii's fixed-point theorem and Banach contraction principle. Also keeping in mind the importance of stability, we have developed some results regarding different kinds of Ulam stability including HU, GHU, HUR and GHUR. The obtained analysis has been demonstrated by using pertinent examples.

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## Conflict of interest

No conflict of interest exists.

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