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Research article

Generalized contraction theorems approach to fuzzy differential equations in fuzzy metric spaces

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Abstract: Fixed point theory is one of the most interesting areas of research in mathematics. In this direction, we study some unique common fixed point results for a pair of self-mappings without continuity on fuzzy metric spaces under the generalized contraction conditions by using "the triangular property of fuzzy metric". Moreover, we present weak-contraction and generalized Ćirić-contraction theorems. The results are supported by suitable examples. Further, we establish a supportable application of the fuzzy differential equations to ensure the existence of a unique common solution to validate our main work.

Keywords: fixed point; common fixed point; fuzzy differential equations; contraction conditions; fuzzy metric space
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

In 1965, Zadeh [1], introduced the concept of a fuzzy set which is defined as: "a set constructed from a function having a domain is a nonempty set W and range in $[0, 1]$ is called a fuzzy set, that is, if $G : W \to [0, 1]$, then the set constructing from the mapping G is called a fuzzy set". Later on, the theory of fuzzy sets has been extensively developed and investigated in many directions with different types of applications. Kramosil and Michalek [2], introduced the notion of fuzzy metric spaces (FM spaces) by using the concept of fuzzy set and some more derived concepts from the one in order.

They compared the FM concepts with the statistical metric space and proved that both the spaces are equivalent in some cases. After that, the modified form of the FM space was given by George and Veeramani [3] and proved that every metric induces an FM. They proved some basic properties and Baire's theorem for FM spaces.

In 1988, Grabiec [4] proved two fixed point theorems of "Banach and Edelstein contraction mapping theorems on complete and compact FM spaces, respectively" by using the concept of Kramosil and Michalek [2]. In [5], Kiany et al. proved some fixed point results on FM spaces for set-valued contractive type mappings. Aubin and Siegel [6], Fakhar [7], Gregori and Sapena [8], Harandi [9], Hussain et al. [10], Mizoguchi and Takahashi [11], Rehman et al. [12], and Wlodarczyk et al. [13] proved some set-valued and multi-valued contractive type mapping results in different spaces.

Bari and Vetro [14] proved fixed point theorems for a family of mappings on FM spaces. While Beg et al. [15] established some invariant approximation results for fuzzy non-expansive mappings defined on FM spaces. As an application, they obtained a fixed point result on the best approximation in a fuzzy normed space. Further, they defined the strictly convex fuzzy normed space and obtained a necessary condition for the set of all t-best approximations which contained a fixed point of the arbitrary mappings. While Beg et al. [16] established some fixed point theorems on complete FM spaces for self-mappings satisfying an implicit relation. Bari and Vetro [17], Imdad and Ali [18], Hierro et al. [19], Jleli et al. [20], Li et al. [21], Pant and Chauhan [22], Lopez and Romaguera [23], Rehman et al. [24], Roldan et al. [25, 26], Sadeghi et al. [27], Shamas et al. [28, 29] and Som [30] proved some fixed point and common fixed point results on FM spaces by using different contractive type mappings with applications.

In this paper, we present some unique common fixed point theorems for a pair of self-mappings on FM spaces without continuity by using "the triangular property of fuzzy metric". We use the concept of Li et al. [21] and Rehman et al. [31] and establish different contractive types of common fixed point theorems on FM spaces with illustrative examples. Further, we present weak contraction and a generalized Ćirić-contraction theorems on FM space. In addition, we present an application of fuzzy differential equations to support our work. This paper is organized as: Section 2 presents the preliminary concepts. Section 3 deals with different contractive types of unique common fixed point theorems on complete FM spaces with examples. While in Section 4, we define a generalized Ćirić-contraction and will prove a unique common fixed point theorem on complete FM spaces. Section 5, is the most important section of this paper which deals with the application of fuzzy differential equations (FDEs) to increase the validity of our work. Finally, in section 6 we discussed the conclusion.

2. Preliminaries

Definition 2.1 ([32]). An operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is known as a continuous t-norm if it holds the following;

(1) $*$ is associative, commutative and continuous.

(2) $1 * \rho_1 = \rho_1$ and $\rho_1 * \rho_2 \leq \rho_3 * \rho_4$, whenever $\rho_1 \leq \rho_3$ and $\rho_2 \leq \rho_4$, for each $\rho_1, \rho_2, \rho_3, \rho_4 \in [0, 1]$.

The basic continuous t-norms are (see [32]): The minimum, the product and the Lukasiewicz t-norms are defined respectively as following;

$$\rho_1 * \rho_2 = \min\{\rho_1, \rho_2\}, \quad \rho_1 * \rho_2 = \rho_1\rho_2 \quad \text{and} \quad \rho_1 * \rho_2 = \max\{\rho_1 + \rho_2 - 1, 0\}.$$

Definition 2.2 ([3]). A 3-tuple $(W, M_F, *)$ is said to be an FM space, if W is a nonempty arbitrary set, $*$ is a continuous t-norm and M_F is a fuzzy set on $W \times W \times (0, \infty)$ satisfying the following;

(i) $M_F(w, x, t) > 0$ and $M_F(w, x, t) = 1 \Leftrightarrow\ w = x$.

(ii) $M_F(w, x, t) = M_F(x, w, t)$.

(iii) $M_F(w, y, t) * M_F(y, x, s) \leq M_F(w, x, t + s)$.

(iv) $M_F(w, x, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

for all $w, x, y \in W$ and $t, s > 0$.

Definition 2.3 ([3,8]). Let $(W, M_F, *)$ be an FM space, $w \in W$ and $\{w_m\}$ be a sequence in W. Then,

(i) $\{w_m\}$ is said to be convergent to a point $w \in W$ if $\lim\limits_{m\to\infty} M_F(w_m, w, t) = 1$ for $t > 0$.

(ii) $\{w_m\}$ is said to be a Cauchy sequence, if for each $0 < \epsilon < 1$ and $t > 0$, there is $m_0 \in \mathbb{N}$ such that $M_F(w_k, w_m, t) > 1 - \epsilon$, $\forall\ k, m \geq m_0$.

(iii) $(W, M_F, *)$ is said to be an FM space if every Cauchy sequence is convergent in W.

(iv) $\{w_m\}$ is called a fuzzy contractive, if $\exists\, \beta \in (0, 1)$ so that

$$\frac{1}{M_F(w_m, w_{m+1}, t)} - 1 \leq \beta\left(\frac{1}{M_F(w_{m-1}, w_m, t)} - 1\right) \quad \text{for } t > 0, \text{ and } m \geq 1.$$

Definition 2.4. [17] Let $(W, M_F, *)$ be an FM space. Then fuzzy metric M_F is triangular if,

$$\frac{1}{M_F(w, x, t)} - 1 \leq \left(\frac{1}{M_F(w, y, t)} - 1\right) + \left(\frac{1}{M_F(y, x, t)} - 1\right) \quad \forall w, x \in W, \text{ and } t > 0.$$

Note: A fuzzy metric F_M is triangular, if $M_F : W \times W \times (0, \infty) \to [0, 1]$ is defined by

$$M_F(w, x, t) = \frac{t}{t + |w - x|} \quad \forall w, x \in W, \text{ and } t > 0.$$

Lemma 2.5 ([17]). *Let $(W, M_F, *)$ be an FM space. Let $w \in W$ and $\{w_m\}$ be a sequence in W. Then $w_m \to w$ iff $\lim\limits_{i\to\infty} M_F(w_m, w, t) = 1$, for $t > 0$.*

Definition 2.6 ([8]). Let $(W, M_F, *)$ be an FM space and $G : W \to W$. Then F is called a fuzzy contraction, if $\exists\, h \in (0, 1)$ so that

$$\frac{1}{M_F(F_1 w, F_1 x, t)} - 1 \leq h\left(\frac{1}{M_F(w, x, t)} - 1\right) \quad \forall w, x \in W, \text{ and } t > 0. \tag{2.1}$$

3. Generalized common fixed point results on FM spaces

Now we present our first main result.

Theorem 3.1. *Let $(W, M_F, *)$ be a complete FM space in which M_F is triangular and a pair of self-mappings $F_1, F_2 : W \to W$ satisfies,*

$$\frac{1}{M_F(F_1w, F_2x, t)} - 1 \leq a\left(\frac{1}{M_F(w, x, t)} - 1\right) + b\left(\begin{array}{c} \frac{1}{M_F(w, F_1w, t)} - 1 + \frac{1}{M_F(x, F_2x, t)} - 1 \\ +\frac{1}{M_F(x, F_1w, t)} - 1 + \frac{1}{M_F(w, F_2x, t)} - 1 \end{array}\right) \tag{3.1}$$
$$+ c\left(\frac{1}{\min\{M_F(w, F_1w, t), M_F(x, F_2x, t)\}} - 1\right) + d\left(\frac{1}{\max\{M_F(w, F_2x, t), M_F(x, F_1w, t)\}} - 1\right),$$

$\forall\ w, x \in W,\ t > 0,$ and $a, b, c, d \in [0, 1)$. Then F_1 and F_2 have a common fixed point in W if $(a + 4b + c) < 1$. Moreover, if $(a + 2b + d) < 1$, then F_1 and F_2 have a unique common fixed point in W.

Proof. Fix $w_0 \in W$ and define a sequence $\{w_m\}$ in W such that

$$w_{2m+1} = F_1w_{2m} \quad \text{and} \quad w_{2m+2} = F_2w_{2m+1} \quad \text{for all } m \geq 0.$$

Now by a view of (3.19), we have

$$\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1 = \frac{1}{M_F(F_1w_{2m}, F_2w_{2m+1}, t)} - 1$$
$$\leq a\left(\frac{1}{M_F(w_{2m}, w_{2m+1}, t)} - 1\right) + b\left(\begin{array}{c} \frac{1}{M_F(w_{2m}, F_1w_{2m}, t)} - 1 + \frac{1}{M_F(w_{2m+1}, F_2w_{2m+1}, t)} - 1 \\ \frac{1}{M_F(w_{2m+1}, F_1w_{2m}, t)} - 1 + \frac{1}{M_F(w_{2m}, F_2w_{2m+1}, t)} - 1 \end{array}\right)$$
$$+ c\left(\frac{1}{\min\{M_F(w_{2m}, F_1w_{2m}, t), M_F(w_{2m+1}, F_2w_{2m+1}, t)\}} - 1\right)$$
$$+ d\left(\frac{1}{\max\{M_F(w_{2m}, F_2w_{2m+1}, t), M_F(w_{2m+1}, F_1w_{2m}, t)\}} - 1\right)$$

$$\leq a\left(\frac{1}{M_F(w_{2m}, w_{2m+1}, t)} - 1\right) + b\left(\begin{array}{c} \frac{1}{M_F(w_{2m}, w_{2m+1}, t)} - 1 + \frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1 \\ +\frac{1}{M_F(w_{2m}, w_{2m+2}, t)} - 1 \end{array}\right)$$
$$+ c\left(\frac{1}{\min\{M_F(w_{2m}, w_{2m+1}, t), M_F(w_{2m+1}, w_{2m+2}, t)\}} - 1\right) + d\left(\frac{1}{\max\{M_F(w_{2m}, w_{2m+2}, t), 1\}} - 1\right)$$

Then, for $t > 0$, we have

$$\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1 \leq a\left(\frac{1}{M_F(w_{2m}, w_{2m+1}, t)} - 1\right)$$
$$+ b\left(\frac{1}{M_F(w_{2m}, w_{2m+1}, t)} - 1 + \frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1 + \frac{1}{M_F(w_{2m}, w_{2m+2}, t)} - 1\right) \tag{3.2}$$
$$+ c\left(\frac{1}{\min\{M_F(w_{2m}, w_{2m+1}, t), M_F(w_{2m+1}, w_{2m+2}, t)\}} - 1\right)$$

Now two possibilities arise,

(i) If $M_F(w_{2m}, w_{2m+1}, t)$ is a minimum term in $\{M_F(w_{2m}, w_{2m+1}, t), M_F(w_{2m+1}, w_{2m+2}, t)\}$, then after simplification, (3.2) can be written as;

$$\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1 \leq \frac{a+2b+c}{1-2b}\left(\frac{1}{M_F(w_{2m}, w_{2m+1}, t)} - 1\right) \quad \text{for } t > 0. \tag{3.3}$$

(ii) If $M_F(w_{2m+1}, w_{2m+2}, t)$ is a minimum term in $\{M_F(w_{2m}, w_{2m+1}, t), M_F(w_{2m+1}, w_{2m+2}, t)\}$, then after simplification, (3.2) can be written as;

$$\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t) - 1} \leq \frac{a+2b}{1-2b-c}\left(\frac{1}{M_F(w_{2m}, w_{2m+1}, t)} - 1\right) \quad \text{for } t > 0. \tag{3.4}$$

Let us define $\beta := \max\{\frac{a+2b+c}{1-2b}, \frac{a+2b}{1-2b-c}\} < 1$, then from the above two cases, we get that

$$\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1 \leq \beta\left(\frac{1}{M_F(w_{2m}, w_{2m+1}, t)} - 1\right) \quad \text{for } t > 0. \tag{3.5}$$

Similarly,

$$\begin{aligned}
&\frac{1}{M_F(w_{2m+2}, w_{2m+3}, t)} - 1 = \frac{1}{M_j(F_2 w_{2m+1}, F_1 w_{2m+2}, t)} - 1 \\
&\leq a\left(\frac{1}{M_F(w_{2m+2}, w_{2m+1}, t)} - 1\right) + b\left(\begin{array}{c} \frac{1}{M_F(w_{2m+2}, F_1 w_{2m+2}, t)} - 1 + \frac{1}{M_F(w_{2m+1}, F_2 w_{2m+1}, t)} - 1 \\ + \frac{1}{M_F(w_{2m+1}, F_1 w_{2m+2}, t)} - 1 + \frac{1}{M_F(w_{2m+2}, F_2 w_{2m+1}, t)} - 1 \end{array}\right) \\
&+ c\left(\frac{1}{\min\{M_F(w_{2m+2}, F_1 w_{2m+2}, t), M_F(w_{2m+1}, F_2 w_{2m+1}, t)\}} - 1\right) \\
&+ d\left(\frac{1}{\max\{M_F(w_{2m+1}, F_1 w_{2m+2}, t), M_F(w_{2m+2}, F_2 w_{2m+1}, t)\}} - 1\right) \\
&= a\left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1\right) + b\left(\begin{array}{c} \frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1 + \frac{1}{M_F(w_{2m+2}, w_{2m+3}, t)} - 1 \\ + \frac{1}{M_F(w_{2m+1}, w_{2m+3}, t)} - 1 \end{array}\right) \\
&+ c\left(\frac{1}{\min\{M_F(w_{2m+1}, w_{2m+2}, t), M_F(w_{2m+2}, w_{2m+3}, t)\}} - 1\right) \\
&+ d\left(\frac{1}{\max\{M_F(w_{2m+1}, w_{2m+3}, 1\}} - 1\right)
\end{aligned}$$

Then, for $t > 0$, we have

$$\begin{aligned}
\frac{1}{M_F(w_{2m+2}, w_{2m+3}, t)} - 1 &\leq a\left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1\right) \\
&+ b\left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1 + \frac{1}{M_F(w_{2m+2}, w_{2m+3}, t)} - 1 + \frac{1}{M_F(w_{2m+1}, w_{2m+3}, t)} - 1\right) \\
&+ c\left(\frac{1}{\min\{M_F(w_{2m+1}, w_{2m+2}, t), M_F(w_{2m+2}, w_{2m+3}, t)\}} - 1\right)
\end{aligned} \tag{3.6}$$

Now again there are two possibilities that arises,

(i) If $M_F(w_{2m+1}, w_{2m+2}, t)$ is a minimum term in $\{M_F(w_{2m+1}, w_{2m+2}, t), M_F(w_{2m+2}, w_{2m+3}, t)\}$, then after simplification, (3.6) can be written as;

$$\frac{1}{M_F(w_{2m+2}, w_{2m+3}, t)} - 1 \leq \frac{a+2b+c}{1-2b}\left(\frac{1}{M_F(w_{2j+1}, w_{2m+2}, t)} - 1\right) \quad \text{for } t > 0. \tag{3.7}$$

(ii) If $M_F(w_{2m+2}, w_{2m+3}, t)$ is a minimum term in $\{M_F(_{2j+1}, w_{2m+2}, t), M_F(w_{2m+2}, w_{2m+3}, t)\}$, then after simplification, (3.6) can be written as;

$$\frac{1}{M_F(w_{2m+2}, w_{2m+3}, t) - 1} \leq \frac{a+2b}{1-2b-c}\left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1\right) \quad \text{for } t > 0. \tag{3.8}$$

Then

$$\frac{1}{M_F(w_{2m+2}, w_{2m+3}, t)} - 1 \leq \beta\left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1\right) \quad \text{for } t > 0, \tag{3.9}$$

where β is similar as in (3.5). Then, from (3.5) and (3.9), and by induction, for $t > 0$, we have

$$\begin{aligned}
\frac{1}{M_F(w_{2m+2}, w_{2m+3}, t)} - 1 &\leq \beta\left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1\right) \\
&\leq \beta^2\left(\frac{1}{M_F(w_{2m}, w_{2m+1}, t)} - 1\right) \\
&\leq \cdots \leq \beta^{2m+2}\left(\frac{1}{M_F(w_0, w_1, t)} - 1\right) \to 0, \quad \text{as } j \to \infty.
\end{aligned}$$

Hence, proved that $\{w_m\}_{m\geq 0}$ is a fuzzy contractive sequence in W, that is,

$$\lim_{m\to\infty} M_F(w_m, w_{m+1}, t) = 1 \quad \text{for } t > 0. \tag{3.10}$$

Since M_F is triangular, for $k > m$ and $t > 0$, then we have

$$\begin{aligned}
&\frac{1}{M_F(w_m, w_k, t)} - 1 \\
&\leq \left(\frac{1}{M_F(w_m, w_{m+1}, t)} - 1\right) + \left(\frac{1}{M_F(w_{m+1}, w_{m+2}, t)} - 1\right) + \cdots + \left(\frac{1}{M_F(w_{k-1}, w_k, t)} - 1\right) \\
&\leq \left(\beta^m + \beta^{m+1} + \cdots + \beta^{k-1}\right)\left(\frac{1}{M_F(w_0, w_1, t)} - 1\right) \\
&\leq \left(\frac{\beta^m}{1-\beta}\right)\left(\frac{1}{M_F(w_0, w_1, t)} - 1\right) \to 0, \quad \text{as } m \to \infty,
\end{aligned}$$

which shows that $\{w_m\}$ is a Cauchy sequence. Since, by the completeness of $(W, M_F, *)$, $\exists\, \kappa \in W$ such that

$$\lim_{m\to\infty} M_F(\kappa, w_m, t) = 1 \quad \text{for } t > 0. \tag{3.11}$$

Now we have to show that $F_1\kappa = \kappa$, since M_F is triangular, therefore

$$\frac{1}{M_F(\kappa, F_1\kappa, t)} - 1 \leq \left(\frac{1}{M_F(\kappa, w_{2m+2}, t)} - 1\right) + \left(\frac{1}{M_F(w_{2m+2}, F_1\kappa, t)} - 1\right) \quad \text{for } t > 0. \tag{3.12}$$

Now by using (3.19), (3.11) and (3.10), for $t > 0$, we have

$$\begin{aligned}
&\frac{1}{M_F(w_{2m+2}, F_1\kappa, t)} - 1 = \frac{1}{M_F(F_2 w_{2m+1}, F_1\kappa, t)} - 1 \\
&\leq a\left(\frac{1}{M_F(w_{2m+1}, \kappa, t)} - 1\right) + b\left(\begin{array}{c} \frac{1}{M_F(w_{2m+1}, F_2 w_{2m+1}, t)} - 1 + \frac{1}{M_F(\kappa, F_1\kappa, t)} - 1 \\ \frac{1}{M_F(\kappa, F_2 w_{2m+1}, t)} - 1 + \frac{1}{M_F(w_{2m+1}, F_1\kappa, t)} - 1 \end{array}\right) \\
&+ c\left(\frac{1}{\min\{M_F(w_{2m+1}, F_2 w_{2m+1}, t), M_F(\kappa, F_1\kappa, t)\}} - 1\right) \\
&+ d\left(\frac{1}{\max\{M_F(w_{2m+1}, F_1\kappa, t), M_F(\kappa, F_2 w_{2z+1}, t)\}} - 1\right) \\
&= a\left(\frac{1}{M_F(w_{2m+1}, \kappa, t)} - 1\right) + b\left(\begin{array}{c} \frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1 + \frac{1}{M_F(\kappa, F_1\kappa, t)} - 1 \\ \frac{1}{M_F(\kappa, w_{2m+2}, t)} - 1 + \frac{1}{M_F(w_{2m+1}, F_1\kappa, t)} - 1 \end{array}\right) \\
&+ c\left(\frac{1}{\min\{M_F(w_{2m+1}, w_{2m+2}, t), M_F(\kappa, F_1\kappa, t)\}} - 1\right) \\
&+ d\left(\frac{1}{\max\{M_F(w_{2m+1}, F_1\kappa, t), M_F(\kappa, w_{2m+2}, t)\}} - 1\right) \\
&\to (2b + c)\left(\frac{1}{M_F(\kappa, F_1\kappa, t)} - 1\right) \quad \text{as } j \to \infty.
\end{aligned}$$

Then,

$$\limsup_{m\to\infty}\left(\frac{1}{M_F(w_{2m+2}, F_1\kappa, t)} - 1\right) \leq (2b + c)\left(\frac{1}{M_F(\kappa, F_1\kappa, t)} - 1\right) \quad \text{for } t > 0. \tag{3.13}$$

The above (3.13) is together with (3.11) and (3.12), we get that

$$\frac{1}{M_F(\kappa, F_1\kappa, t)} - 1 \leq (2b + c)\left(\frac{1}{M_F(\kappa, F_1\kappa, t)} - 1\right) \quad \text{for } t > 0.$$

As $(2b + c) < 1$, where $(a + 4b + c) < 1$, therefore $M_F(\kappa, F_1\kappa, t) = 1$, this implies that $F_1\kappa = \kappa$. Similarly, again by triangular property of M_F,

$$\frac{1}{M_F(\kappa, F_2\kappa, t)} - 1 \leq \left(\frac{1}{M_F(\kappa, w_{2m+1}, t)} - 1\right) + \left(\frac{1}{M_F(w_{2m+1}, F_2\kappa, t)} - 1\right) \quad \text{for } t > 0. \tag{3.14}$$

Again by using (3.19), (3.10) and (3.11), similar as above, after simplification, we get

$$\limsup_{m\to\infty}\left(\frac{1}{M_F(w_{2m+1}, F_2\kappa, t)} - 1\right) \leq (2b + c)\left(\frac{1}{M_F(\kappa, F_2\kappa, t)} - 1\right) \quad \text{for } t > 0. \tag{3.15}$$

The above (3.15) is together with (3.11) and (3.14), we get that

$$\frac{1}{M_F(\kappa, F_2\kappa, t)} - 1 \leq (2b + c)\left(\frac{1}{M_F(\kappa, F_2\kappa, t)} - 1\right) \quad \text{for } t > 0.$$

As $(2b + c) < 1$, where $(a + 4b + c) < 1$, therefore $M_F(\kappa, F_2\kappa, t) = 1$, this implies that $F_2\kappa = \kappa$. Hence proved that κ is a common fixed point of F_1 and F_2.
Uniqueness: let $\kappa^* \in W$ be the other common fixed point of F_1 and F_2 such that $F_1\kappa^* = F_2\kappa^* = \kappa^*$, then again by the view of (3.19), for $t > 0$, we have

$$\begin{aligned}
&\frac{1}{M_F(\kappa, \kappa^*, t)} - 1 = \left(\frac{1}{M_F(F_1\kappa, F_2\kappa^*, t)} - 1\right) \\
&\leq a\left(\frac{1}{M_F(\kappa, \kappa^*, t)} - 1\right) + b\left(\begin{array}{c} \frac{1}{M_F(\kappa, F_1\kappa, t)} - 1 + \frac{1}{M_F(\kappa^*, F_2\kappa^*, t)} - 1 \\ + \frac{1}{M_F(\kappa, F_2\kappa^*, t)} - 1 + \frac{1}{M_m(\kappa^*, F_1\kappa, t)} - 1 \end{array}\right) \\
&+ c\left(\frac{1}{\min\{M_F(\kappa, F_1\kappa, t), M_F(\kappa^*, F_2\kappa^*, t)\}} - 1\right) \\
&+ d\left(\frac{1}{\min\{M_F(\kappa, F_2\kappa^*, t), M_F(\kappa^*, F_1\kappa, t)\}} - 1\right) \\
&= (a + 2b + d)\left(\frac{1}{M_F(\kappa, \kappa^*, t)} - 1\right) = (a + 2b + d)\left(\frac{1}{M_F(F_1\kappa, F_2\kappa^*, t)} - 1\right) \\
&\leq (a + 2b + d)^2\left(\frac{1}{M_F(\kappa, \kappa^*, t)} - 1\right) \leq \cdots \leq (a + 2b + d)^m\left(\frac{1}{M_F(\kappa, \kappa^*, t)} - 1\right) \to 0, \quad \text{as } m \to \infty,
\end{aligned}$$

where $(a + 4b + d) < 1$. Hence we get that $M_F(\kappa, \kappa^*, t) = 1$, this implies that $\kappa = \kappa^*$. Thus, F_1 and F_2 have a unique common fixed point in W.

If we put $d = 0$ in Theorem 3.1, we get the following corollary;

Corollary 3.2. *Let $(W, M_F, *)$ be a complete FM space in which M_F is triangular and a pair of self-mappings $F_1, F_2 : W \to W$ satisfies,*

$$\begin{aligned}
\frac{1}{M_F(F_1w, F_2x, t)} - 1 \leq & a\left(\frac{1}{M_F(w, x, t)} - 1\right) + b\left(\begin{array}{c} \frac{1}{M_F(w, F_1w, t)} - 1 + \frac{1}{M_F(x, F_2x, t)} - 1 \\ + \frac{1}{M_F(x, F_1w, t)} - 1 + \frac{1}{M_F(w, F_2x, t)} - 1 \end{array}\right) \\
& + c\left(\frac{1}{\min\{M_F(w, F_1w, t), M_F(x, F_2x, t)\}} - 1\right)
\end{aligned} \tag{3.16}$$

$\forall\ w, x \in W,\ t > 0,$ *and $a, b, c \in [0, 1)$ with $(a + 4b + c) < 1$. Then F_1 and F_2 have a unique common fixed point in W.*

If we put $c = 0$ in Theorem 3.1, we get the following corollary;

Corollary 3.3. *Let $(W, M_F, *)$ be a complete FM space in which M_F is triangular and a pair of self-mappings $F_1, F_2 : W \to W$ satisfies,*

$$\frac{1}{M_F(F_1w, F_2x, t)} - 1 \leq a\left(\frac{1}{M_F(w, x, t)} - 1\right) + b\left(\begin{array}{c} \frac{1}{M_F(w, F_1w, t)} - 1 + \frac{1}{M_F(x, F_2x, t)} - 1 \\ + \frac{1}{M_F(x, F_1w, t)} - 1 + \frac{1}{M_F(w, F_2x, t)} - 1 \end{array}\right) + d\left(\frac{1}{\max\{M_F(w, F_2x, t), M_F(x, F_1w, t)\}} - 1\right), \tag{3.17}$$

$\forall\ w, x \in W,\ t > 0$, *and $a, b, d \in [0, 1)$ with $(a + 4b + d) < 1$. Then F_1 and F_2 have a unique common fixed point in W.*

If we put $c = d = 0$ in Theorem 3.1, we get the following corollary;

Corollary 3.4. *Let $(W, M_F, *)$ be a complete FM space in which M_F is triangular and a pair of self-mappings $F_1, F_2 : W \to W$ satisfies,*

$$\frac{1}{M_F(F_1w, F_2x, t)} - 1 \leq a\left(\frac{1}{M_F(w, x, t)} - 1\right) + c\left(\frac{1}{\min\{M_F(w, F_1w, t), M_F(x, F_2x, t)\}} - 1\right) + d\left(\frac{1}{\max\{M_F(w, F_2x, t), M_F(x, F_1w, t)\}} - 1\right) \tag{3.18}$$

$\forall\ w, x \in W,\ t > 0$, *and $a, c, d \in [0, 1)$. Then F_1 and F_2 have a common fixed point if $(a + c) < 1$. Moreover, if $(a + d) < 1$, then F_1 and F_2 have a unique common fixed point in W.*

If the mapping $F_1 = F_2$ in Theorem 3.1, we get the following corollary;

Corollary 3.5. *Let $(W, M_F, *)$ be a complete FM space in which M_F is triangular and a self-mapping $F_1 : W \to W$ satisfies,*

$$\frac{1}{M_F(F_1w, F_1x, t)} - 1 \leq a\left(\frac{1}{M_F(w, x, t)} - 1\right) + b\left(\begin{array}{c} \frac{1}{M_F(w, F_1w, t)} - 1 + \frac{1}{M_F(x, F_1x, t)} - 1 \\ + \frac{1}{M_F(x, F_1w, t)} - 1 + \frac{1}{M_F(w, F_1x, t)} - 1 \end{array}\right) + c\left(\frac{1}{\min\{M_F(w, F_1w, t), M_F(x, F_1x, t)\}} - 1\right) + d\left(\frac{1}{\max\{M_F(w, F_1x, t), M_F(x, F_1w, t)\}} - 1\right) \tag{3.19}$$

$\forall\ w, x \in W,\ t > 0$, *and $a, b, c, d \in [0, 1)$. Then F_1 has a fixed point if $(a + 4b + c) < 1$. Moreover, if $(a + 4b + d) < 1$, then F_1 has a unique fixed point in W.*

Example 3.6. Let $W = [0, \infty)$ and t-norm is a product continuous t-norm. Let $M_F : W \times W \times (0, \infty) \to [0, 1]$ be defined as

$$M_F(w, x, t) = \frac{t}{t + d(w, x)}, \quad \text{where } d(w, x) = \frac{2\,|w - x|}{3},$$

$\forall\ w, x \in W$ and $t > 0$. Then $(W, M_F, *)$ is complete. The mappings $F_1, F_2 : W \to W$ be defined as

$$F_1 w = \begin{cases} \dfrac{2w}{5} + \dfrac{1}{10}, & \text{if } w \in [0,1], \\ \dfrac{3w}{4} + 3, & \text{if } w \in (1,\infty). \end{cases}$$

And,

$$F_2 x = \begin{cases} \dfrac{2x}{5} + \dfrac{1}{10}, & \text{if } x \in [0,1], \\ \dfrac{2x}{7} + \dfrac{60}{7}, & \text{if } x \in (1,\infty). \end{cases}$$

Then we have

$$\begin{aligned} \frac{1}{M_F(F_1w, F_2x, t)} - 1 &= \frac{2|F_1w - F_2x|}{3t} = \frac{4|w-x|}{15t} = \frac{2}{5}\left(\frac{1}{M_F(w,x,t)} - 1\right) \\ &\leq \frac{2}{5}\left(\frac{1}{M_F(w,x,t)} - 1\right) + \frac{1}{20}\left(\begin{array}{c} \dfrac{1}{M_F(w,F_1w,t)} - 1 + \dfrac{1}{M_F(x,F_2x,t)} - 1 \\ + \dfrac{1}{M_F(x,F_1w,t)} - 1 + \dfrac{1}{M_F(w,F_2x,t)} - 1 \end{array}\right) \\ &+ \frac{2}{7}\left(\frac{1}{\min\{M_F(w,F_1w,t), M_F(x,F_2x,t)\}} - 1\right) + \frac{2}{7}\left(\frac{1}{\max\{M_F(w,F_2x,t), M_F(x,F_1w,t)\}} - 1\right). \end{aligned}$$

Hence all the conditions of Theorem 3.1 are satisfied with $a = \frac{2}{5}, b = \frac{1}{20}$, and $c = d = \frac{2}{7}$, where $(a + 4b + c) = \frac{31}{35} < 1$ and $(a + 2b + d) = \frac{55}{70} < 1$, the self mappings F_1 and F_2 have a unique common fixed point, that is, $F_1(12) = F_2(12) = 12 \in [1,\infty)$.

In the following theorem, we use a function $\psi : [0,\infty) \to [0,\infty)$ such that $\psi(0) = 0$, and $\psi(\xi) < \xi$, for $\xi > 0$, and prove a unique common fixed point result in FM spaces.

Theorem 3.7. *Let $F_1, F_2 : W \to W$ be a pair of self-mappings on a complete FM space $(W, M_F, *)$ in which M_F is triangular. Suppose that there exists a non-decreasing function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$, $\psi(\xi) < \xi$, for $\xi > 0$ and $\sum_{m=0}^{\infty} \psi^m(\xi) < \infty$, $\xi \geq 0$ such that the following inequality holds;*

$$\begin{aligned} \frac{1}{M_F(F_1w, F_2x, t)} - 1 \leq{}& \psi\left(\frac{1}{N(F_1,F_2,w,x,t)} - 1\right) \\ &+ \ell \min\left\{ \begin{array}{c} \left(\dfrac{1}{M_F(w,F_1w,t)} - 1\right), \left(\dfrac{1}{M_F(x,F_2x,t)} - 1\right), \\ \left(\dfrac{1}{M_F(x,F_1w,t)} - 1\right), \left(\dfrac{1}{M_F(w,F_2x,t)} - 1\right) \end{array} \right\}, \end{aligned} \tag{3.20}$$

where

$$\frac{1}{N(F_1,F_2,w,x,t)} - 1 = \max\left\{ \begin{array}{c} \left(\dfrac{1}{M_F(w,x,t)} - 1\right), \\ \left(\dfrac{1}{M_F(w,F_1w,t)} - 1\right), \left(\dfrac{1}{M_F(x,F_2x,t)} - 1\right), \\ \dfrac{1}{4}\left(\dfrac{1}{M_F(x,F_1w,t)} - 1 + \dfrac{1}{M_F(w,F_2x,t)} - 1\right) \end{array} \right\},$$

for all $w, x \in W$, $\ell \in (0,1)$, then F_1 and F_2 have a unique common fixed point in W.

Proof. Fix $w_0 \in W$ and define a sequence $\{w_m\}$ in W such that

$$w_{2m+1} = F_1 w_{2m} \quad \text{and} \quad w_{2m+2} = F_2 w_{2m+1} \quad \text{for all } m \geq 0.$$

Now by the view of (3.20), for $t > 0$,

$$\begin{aligned}
\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1 &= \frac{1}{M_F(F_1 w_{2m}, F_2 w_{2m+1}, t)} - 1 \\
&\leq \psi\left(\frac{1}{N(F_1, F_2, w_{2m}, w_{2m+1}, t)} - 1\right) \\
&\quad + \ell \min\left\{\begin{array}{c} \left(\frac{1}{M_F(w_{2m}, F_1 w_{2m}, t)} - 1\right), \left(\frac{1}{M_F(w_{2m+1}, F_2 w_{2m+1}, t)} - 1\right), \\ \left(\frac{1}{M_F(w_{2m+1}, F_1 w_{2m}, t)} - 1\right), \left(\frac{1}{M_F(w_{2m}, F_2 w_{2m+1}, t)} - 1\right) \end{array}\right\} \\
&= \psi\left(\frac{1}{N(F_1, F_2, w_{2m}, w_{2m+1}, t)} - 1\right) \\
&\quad + \ell \min\left\{\begin{array}{c} \left(\frac{1}{M_F(w_{2m}, w_{2m+1}, t)} - 1\right), \left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1\right), \\ \left(\frac{1}{M_F(w_{2m+1}, w_{2m+1}, t)} - 1\right), \left(\frac{1}{M_F(w_{2m}, w_{2m+2}, t)} - 1\right) \end{array}\right\},
\end{aligned}$$

where

$$\begin{aligned}
&\frac{1}{N(F_1, F_2, w_{2m}, w_{2m+1}, t)} - 1 \\
&= \max\left\{\begin{array}{c} \left(\frac{1}{M_F(w_{2m}, w_{2m+1}, t)} - 1\right), \\ \left(\frac{1}{M_F(w_{2m}, F_1 w_{2m}, t)} - 1\right), \left(\frac{1}{M_F(w_{2m+1}, F_2 w_{2m+1}, t)} - 1\right), \\ \frac{1}{4}\left(\frac{1}{M_F(w_{2m+1}, F_1 w_{2m}, t)} - 1 + \frac{1}{M_F(w_{2m}, F_2 w_{2m+1}, t)} - 1\right) \end{array}\right\} \\
&= \max\left\{\begin{array}{c} \left(\frac{1}{M_F(w_{2m}, w_{2m+1}, t)} - 1\right), \\ \left(\frac{1}{M_F(w_{2m}, w_{2m+1}, t)} - 1\right), \left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1\right), \\ \frac{1}{4}\left(\frac{1}{M_F(w_{2m+1}, w_{2m+1}, t)} - 1 + \frac{1}{M_F(w_{2m}, w_{2m+2}, t)} - 1\right) \end{array}\right\} \\
&= \max\left\{\left(\frac{1}{M_F(w_{2m}, w_{2m+1}, t)} - 1\right), \left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1\right)\right\},
\end{aligned}$$

which is further implies that

$$\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1 \leq \psi\left(\max\left\{\left(\frac{1}{M_F(w_{2m}, w_{2m+1}, t)} - 1\right), \left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1\right)\right\}\right).$$

Now if,

$$\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1 > \left(\frac{1}{M_F(w_{2m}, w_{2m+1}, t)} - 1\right) \quad \text{for } t > 0,$$

then for $t > 0$, we have

$$\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1 \leq \psi\left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1\right) < \left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1\right),$$

which is a contradiction. Hence,

$$\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1 \leq \psi\left(\frac{1}{M_F(w_{2m}, w_{2m+1}, t)} - 1\right) \quad \text{for } t > 0.$$

Similarly, it can be shown that

$$\frac{1}{M_F(w_{2m+2}, w_{2m+3}, t)} - 1 \leq \psi\left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1\right) \quad \text{for } t > 0.$$

Thus, by induction for all $m \geq 0$ and $t > 0$, we have that

$$\begin{aligned}\frac{1}{M_F(w_m, w_{m+1}, t)} - 1 &\leq \psi\left(\frac{1}{M_F(w_{m-1}, w_m, t)} - 1\right)\\ &\leq \psi^2\left(\frac{1}{M_F(w_{m-2}, w_{m-1}, t)} - 1\right)\\ &\leq \cdots \leq \psi^m\left(\frac{1}{M_F(w_0, w_1, t)} - 1\right).\end{aligned}$$

Hence, for $k > m$ and $t > 0$,

$$\begin{aligned}&\frac{1}{M_F(w_m, w_k, t)} - 1\\ &\leq \left(\frac{1}{M_F(w_m, w_{m+1}, t)} - 1\right) + \left(\frac{1}{M_F(w_{m+1}, w_{m+2}, t)} - 1\right) + \cdots + \left(\frac{1}{M_F(w_{k-1}, w_k, t)} - 1\right)\\ &\leq \psi^m\left(\frac{1}{M_F(w_0, w_1, t)} - 1\right) + \psi^{m+1}\left(\frac{1}{M_F(w_0, w_1, t)} - 1\right) + \cdots + \psi^{k-1}\left(\frac{1}{M_F(w_0, w_1, t)} - 1\right)\\ &\leq \sum_{n=m}^{k-1} \psi^n\left(\frac{1}{M_F(w_0, w_1, t)} - 1\right).\end{aligned}$$

Since, $\sum_{m=0}^{\infty} \psi^m\left(\frac{1}{M_F(w_0,w_1,t)} - 1\right) < \infty$, hence $\{w_m\}$ is a Cauchy sequence and from the completeness of $(W, M_F, *)$, it follows that $w_m \to \kappa \in W$, as $m \to \infty$. This can be written as

$$\lim_{m\to\infty} M_F(w_m, \kappa, t) = 1 \quad \text{for } t > 0. \tag{3.21}$$

Now we have to show that $F_1\kappa = \kappa$, since M_F is triangular, therefore

$$\frac{1}{M_F(\kappa, F_1\kappa, t)} - 1 \leq \left(\frac{1}{M_F(\kappa, w_{2m+2}, t)} - 1\right) + \left(\frac{1}{M_F(w_{2m+2}, F_1\kappa, t)} - 1\right) \quad \text{for } t > 0. \tag{3.22}$$

Now from (3.20), for $t > 0$, we have

$$\begin{aligned}\frac{1}{M_F(w_{2m+2}, F_1\kappa, t)} - 1 &= \frac{1}{M_F(F_1\kappa, F_2w_{2m+1}, t)} - 1\\ &\leq \psi\left(\frac{1}{N(F_1, F_2, \kappa, w_{2m+1}, t)} - 1\right)\\ &+ \ell \min\left\{\begin{array}{c}\left(\frac{1}{M_F(\kappa, F_1\kappa, t)} - 1\right), \left(\frac{1}{M_F(w_{2m+1}, F_2w_{2m+1}, t)} - 1\right),\\ \left(\frac{1}{M_F(w_{2m+1}, F_1\kappa, t)} - 1\right), \left(\frac{1}{M_F(\kappa, F_2w_{2m+1}, t)} - 1\right)\end{array}\right\}.\end{aligned}$$

Now we substitute the value of $\left(\frac{1}{N(F_1,F_2,\kappa,w_{2m+1},t)} - 1\right)$ in the above inequality and then from (3.21), for $t > 0$, we have that

$$\begin{aligned}&\frac{1}{M_F(w_{2m+2}, F_1\kappa, t)} - 1\\ &\leq \psi\left(\max\left\{\begin{array}{c}\left(\frac{1}{M_F(w_{2m+1}, \kappa, t)} - 1\right),\\ \left(\frac{1}{M_F(\kappa, F_1\kappa, t)} - 1\right), \left(\frac{1}{M_F(w_{2m+1}, F_2w_{2m+1}, t)} - 1\right),\\ \frac{1}{4}\left(\frac{1}{M_F(w_{2m+1}, F_1\kappa, t)} - 1 + \frac{1}{M_F(\kappa, F_2w_{2m+1}, t)} - 1\right)\end{array}\right\}\right)\\ &+ \ell \min\left\{\begin{array}{c}\left(\frac{1}{M_F(\kappa, F_1\kappa, t)} - 1\right), \left(\frac{1}{M_F(w_{2m+1}, F_2w_{2m+1}, t)} - 1\right),\\ \left(\frac{1}{M_F(w_{2m+1}, F_1\kappa, t)} - 1\right), \left(\frac{1}{M_F(\kappa, F_2w_{2m+1}, t)} - 1\right)\end{array}\right\}\\ &= \psi\left(\max\left\{\begin{array}{c}\left(\frac{1}{M_F(w_{2m+1}, \kappa, t)} - 1\right),\\ \left(\frac{1}{M_F(\kappa, F_1\kappa, t)} - 1\right), \left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1\right),\\ \frac{1}{4}\left(\frac{1}{M_F(w_{2m+1}, F_1\kappa, t)} - 1 + \frac{1}{M_F(\kappa, w_{2m+2}, t)} - 1\right)\end{array}\right\}\right)\\ &+ \ell \min\left\{\begin{array}{c}\left(\frac{1}{M_F(\kappa, F_1\kappa, t)} - 1\right), \left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1\right),\\ \left(\frac{1}{M_F(w_{2m+1}, F_1\kappa, t)} - 1\right), \left(\frac{1}{M_F(\kappa, w_{2m+2}, t)} - 1\right)\end{array}\right\}\\ &\to \psi\left(\frac{1}{M_F(\kappa, F_1\kappa, t)} - 1\right).\end{aligned}$$

Then,

$$\limsup_{j\to\infty}\left(\frac{1}{M_F(w_{2m+2}, F_1\kappa, t)} - 1\right) \leq \psi\left(\frac{1}{M_F(\kappa, F_1\kappa, t)} - 1\right) \quad \text{for } t > 0. \tag{3.23}$$

The above (3.23) is together with (3.21) and (3.22), for $t > 0$, we have that

$$\frac{1}{M_F(\kappa, F_1\kappa, t)} - 1 \leq \psi\left(\frac{1}{M_F(\kappa, F_1\kappa, t)} - 1\right) < \left(\frac{1}{M_F(\kappa, F_1\kappa, t)} - 1\right),$$

which is a contradiction. Hence, $M_F(\kappa, F_1\kappa, t) = 1 \Rightarrow F_1\kappa = \kappa$ for $t > 0$. Similarly, by the triangular property of M_F,

$$\frac{1}{M_F(\kappa, F_2\kappa, t)} - 1 \leq \left(\frac{1}{M_F(\kappa, w_{2m+1}, t)} - 1\right) + \left(\frac{1}{M_F(w_{2m+1}, F_2\kappa, t)} - 1\right) \quad \text{for } t > 0. \tag{3.24}$$

Again by using (3.20) and (3.21), similar as above, after simplification, we get that

$$\lim_{m\to\infty} \sup\left(\frac{1}{M_F(w_{2m+1}, F_2\kappa, t)} - 1\right) \leq \psi\left(\frac{1}{M_F(\kappa, F_2\kappa, t)} - 1\right) \quad \text{for } t > 0. \tag{3.25}$$

The above (3.25) is together with (3.21) and (3.24), we have that

$$\frac{1}{M_F(\kappa, F_2\kappa, t)} - 1 \leq \psi\left(\frac{1}{M_F(\kappa, F_2\kappa, t)} - 1\right) < \left(\frac{1}{M_F(\kappa, F_2\kappa, t)} - 1\right),$$

which is a contradiction. Hence, $M_F(\kappa, F_2\kappa, t) = 1 \Rightarrow F_2\kappa = \kappa$ for $t > 0$. Hence proved that κ is a common fixed point of F_1 and F_2.
Uniqueness: let $\kappa^* \in W$ be the other common fixed point of F_1 and F_2 such that $F_1\kappa^* = F_2\kappa^* = \kappa^*$, then again by the view of (3.20), for $t > 0$, we have

$$\begin{aligned}
\frac{1}{M_F(\kappa, \kappa^*, t)} - 1 &= \frac{1}{M_F(F_1\kappa, F_2\kappa^*, t)} - 1 \\
&\leq \psi\left(\frac{1}{N(F_1, F_2, \kappa, \kappa^*, t)} - 1\right) \\
&\quad + \ell \min\left\{\begin{array}{c} \left(\frac{1}{M_F(\kappa, F_1\kappa, t)} - 1\right), \left(\frac{1}{M_F(\kappa^*, F_2\kappa^*, t)} - 1\right), \\ \left(\frac{1}{M_F(\kappa^*, F_1\kappa, t)} - 1\right), \left(\frac{1}{M_F(\kappa, F_2\kappa^*, t)} - 1\right) \end{array}\right\} \\
&= \psi\left(\max\left\{\begin{array}{c} \left(\frac{1}{M_F(\kappa, \kappa^*, t)} - 1\right), \\ \left(\frac{1}{M_F(\kappa, F_1\kappa, t)} - 1\right), \left(\frac{1}{M_F(\kappa^*, F_2\kappa^*, t)} - 1\right), \\ \frac{1}{4}\left(\frac{1}{M_F(\kappa^*, F_1\kappa, t)} - 1 + \frac{1}{M_F(\kappa, F_2\kappa^*, t)} - 1\right) \end{array}\right\}\right) \\
&\quad + \ell \min\left\{\begin{array}{c} \left(\frac{1}{M_F(\kappa, F_1\kappa, t)} - 1\right), \left(\frac{1}{M_F(\kappa^*, F_2\kappa^*, t)} - 1\right), \\ \left(\frac{1}{M_F(\kappa^*, F_1\kappa, t)} - 1\right), \left(\frac{1}{M_F(\kappa, F_2\kappa^*, t)} - 1\right) \end{array}\right\} \\
&= \psi\left(\max\left\{\begin{array}{c} \left(\frac{1}{M_F(\kappa, \kappa^*, t)} - 1\right), \\ \left(\frac{1}{M_F(\kappa, \kappa, t)} - 1\right), \left(\frac{1}{M_F(\kappa^*, \kappa^*, t)} - 1\right), \\ \frac{1}{4}\left(\frac{1}{M_F(\kappa^*, \kappa, t)} - 1 + \frac{1}{M_F(\kappa, \kappa^*, t)} - 1\right) \end{array}\right\}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\ell\min\left\{\begin{array}{c}\left(\dfrac{1}{M_F(\kappa,\kappa,t)}-1\right),\left(\dfrac{1}{M_F(\kappa^*,\kappa^*,t)}-1\right),\\ \left(\dfrac{1}{M_F(\kappa^*,\kappa,t)}-1\right),\left(\dfrac{1}{M_F(\kappa,\kappa^*,t)}-1\right)\end{array}\right\}\\
&=\psi\left(\dfrac{1}{M_F(\kappa,\kappa^*,t)}-1\right),
\end{aligned}
$$

Hence, we get that

$$
\frac{1}{M_F(\kappa,\kappa^*,t)}-1\leq\psi\left(\frac{1}{M_F(\kappa,\kappa^*,t)}-1\right)<\frac{1}{M_F(\kappa,\kappa^*,t)}-1,\quad\text{for } t>0,
$$

which is a contradiction. Hence, $M_F(\kappa,\kappa^*,t)=1 \Rightarrow \kappa=\kappa^*$ for $t>0$.

If we define a mapping ψ by $\psi(\xi)=\lambda\xi$ in Theorem 3.7, where $\lambda\in(0,1)$, we get the following corollary;

Corollary 3.8. *Let $(W,M_F,*)$ be a complete FM space in which M_F is triangular and a pair of self mappings $F_1,F_2:W\to W$ satisfies,*

$$
\begin{aligned}
\frac{1}{M_F(F_1w,F_2x,t)}-1\leq{}&\lambda\left(\frac{1}{N(F_1,F_2,w,x,t)}-1\right)\\
&+\ell\min\left\{\begin{array}{c}\left(\dfrac{1}{M_F(w,F_1w,t)}-1\right),\left(\dfrac{1}{M_F(x,F_2x,t)}-1\right),\\ \left(\dfrac{1}{M_F(x,F_1w,t)}-1\right),\left(\dfrac{1}{M_F(w,F_2x,t)}-1\right)\end{array}\right\},
\end{aligned}
\tag{3.26}
$$

where

$$
\frac{1}{N(F_1,F_2,w,x,t)}-1=\max\left\{\begin{array}{c}\left(\dfrac{1}{M_F(w,x,t)}-1\right),\\ \left(\dfrac{1}{M_F(w,F_1w,t)}-1\right),\left(\dfrac{1}{M_F(x,F_2x,t)}-1\right),\\ \dfrac{1}{4}\left(\dfrac{1}{M_F(x,F_1w,t)}-1+\dfrac{1}{M_F(w,F_2x,t)}-1\right)\end{array}\right\},
$$

$\forall\ w,x\in W$, $\lambda\in(0,1)$ *and* $\ell\geq0$. *Then F_1 and F_2 have a unique common fixed point in W.*

If we put $\ell=0$ in Corollary 3.8, we get the following corollary;

Corollary 3.9. *Let $(W,M_F,*)$ be a complete FM space in which M_F is triangular and a pair of self mappings $F_1,F_2:W\to W$ satisfies,*

$$
\frac{1}{M_F(F_1w,F_2x,t)}-1\leq\lambda\max\left\{\begin{array}{c}\left(\dfrac{1}{M_F(w,x,t)}-1\right),\\ \left(\dfrac{1}{M_F(w,F_1w,t)}-1\right),\left(\dfrac{1}{M_F(x,F_2x,t)}-1\right),\\ \dfrac{1}{4}\left(\dfrac{1}{M_F(x,F_1w,t)}-1+\dfrac{1}{M_F(w,F_2x,t)}-1\right)\end{array}\right\}
\tag{3.27}
$$

$\forall\ w,x\in W$, $\lambda\in(0,1)$. *Then F_1 and F_2 have a unique common fixed point in W.*

Definition 3.10. A self-mapping F_1 will be called weakly contractive on a complete FM space $(W, M_F, *)$, i.e., $F_1 : W \to W$, if there exists a continuous and non-decreasing function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(\tau) = 0$ if and only if $\tau = 0$, $\lim_{\tau\to\infty} \varphi(\tau) = \infty$ and satisfying

$$\frac{1}{M_F(w,x,t)} - 1 \le \left(\frac{1}{M_F(w,x,t)} - 1\right) - \varphi\left(\frac{1}{M_F(w,x,t)} - 1\right), \quad \forall\, w, x \in W \text{ and } t > 0. \tag{3.28}$$

Theorem 3.11. *Let a pair of weakly self-contractive on a complete FM space $(W, M_F, *)$, that is, $F_1, F_2 : W \to W$ in which a fuzzy metric M_F is triangular and satisfies,*

$$\frac{1}{M_F(F_1w, F_2x, t)} - 1 \le \left(\frac{1}{M_F(w,x,t)} - 1\right) - \varphi\left(\frac{1}{M_F(w,x,t)} - 1\right), \quad \forall\, w, x \in W \text{ and } t > 0, \tag{3.29}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous and monotone non-decreasing function with $\varphi(\tau) = 0$ if and only if $\tau = 0$ and $\lim_{\tau\to\infty} \varphi(\tau) = \infty$. Then F_1 and F_2 have a unique common fixed point in W.

Proof. Fix $w_0 \in W$ and define a sequence $\{w_m\}$ in W such that

$$w_{2m+1} = F_1 w_{2m} \quad \text{and} \quad w_{2m+2} = F_2 w_{2m+1} \quad \text{for all } m \ge 0.$$

Now by view of (3.29), for $t > 0$,

$$\begin{aligned} \frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1 &= \frac{1}{M_F(F_1 w_{2m}, F_2 w_{2m+1}, t)} - 1 \\ &\le \left(\frac{1}{M_F(w_{2m}, w_{2m+1}, t)} - 1\right) - \varphi\left(\frac{1}{M_F(w_{2m}, w_{2m}, t)} - 1\right) \\ &< \left(\frac{1}{M_F(w_{2m}, w_{2m+1}, t)} - 1\right). \end{aligned} \tag{3.30}$$

Similarly, for $t > 0$,

$$\begin{aligned} \frac{1}{M_F(w_{2m+2}, w_{2m+3}, t)} - 1 &= \frac{1}{M_F(F_1 w_{2m+2}, F_2 w_{2m+1}, t)} - 1 \\ &\le \left(\frac{1}{M_F(w_{2m+2}, w_{2m+1}, t)} - 1\right) - \varphi\left(\frac{1}{M_F(w_{2m+2}, w_{2m}, t)} - 1\right) \\ &< \left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1\right) \\ &< \left(\frac{1}{M_F(w_{2m}, w_{2m+1}, t)} - 1\right), \quad \text{by using (3.30).} \end{aligned} \tag{3.31}$$

Thus $\left(\frac{1}{M_F(w_{2m+3}, w_{2m+2}, t)} - 1\right)$ is a monotone decreasing sequence of non-negative real numbers and convergent to some point ϱ as $j \to \infty$. Let we denote $\left(\frac{1}{M_F(w_{2m+3}, w_{2m+2}, t)} - 1\right)$ by ϱ_{2m+2}. Then, we have that $\varrho_{2m+2} \to \varrho$ as $m \to \infty$. Now we have to prove that $\varrho = 0$. If not, then on taking $m \to \infty$, we have

$$\left(\frac{1}{M_F(w_m, w_{m+1}, t)} - 1\right) \le \left(\frac{1}{M_F(w_{m-1}, w_m, t)} - 1\right) + \varphi\left(\frac{1}{M_F(w_{m-1}, w_m, t)} - 1\right) \quad \text{for } t > 0,$$

which gives that

$$\varrho \leq \varrho - \varphi(\varrho) < \varrho,$$

a contradiction. Hence, we conclude that $\left(\frac{1}{M_F(w_m, w_{m+1}, t)} - 1\right) = \varrho_m \to 0$ as $m \to \infty$, this can be written as

$$\lim_{m\to\infty} M_F(w_m, w_{m+1}, t) = 1 \quad \text{for } t > 0. \tag{3.32}$$

Next we have to prove that $\{w_m\}$ is a Cauchy sequence. Let $\{m(i)\}$ and $\{k(i)\}$ be the increasing sequences of integers and there exists ε such that for all integers i and $p(i), n(i) \geq 0$,

$$m(i) = 2p(i) + 1 > k(i) = 2n(i) > i, \quad \text{or} \quad m(i) = 2p(i) > k(i) = 2n(i) + 1 > i.$$

This implies that

$$\begin{aligned} &\left(\frac{1}{M_F(w_{k(i)}, w_{m(i)}, t)} - 1\right) \geq \varepsilon \quad \text{for } t > 0, \\ \Rightarrow &\left(\frac{1}{M_F(w_{2n(i)}, w_{2p(i)+1}, t)} - 1\right) \geq \varepsilon \text{ or } \left(\frac{1}{M_F(w_{2n(i)+1}, w_{2p(i)}, t)} - 1\right) \geq \varepsilon \text{ for } t > 0, \end{aligned} \tag{3.33}$$

and

$$\begin{aligned} &\left(\frac{1}{M_F(w_{k(i)}, w_{m(i)-1}, t)} - 1\right) < \varepsilon \quad \text{for } t > 0, \\ \Rightarrow &\left(\frac{1}{M_F(w_{2n(i)-1}, w_{2p(i)}, t)} - 1\right) < \varepsilon \text{ or } \left(\frac{1}{M_F(w_{2n(i)}, w_{2p(i)-1}, t)} - 1\right) < \varepsilon \text{ for } t > 0. \end{aligned} \tag{3.34}$$

By taking limit $i \to \infty$ on the above (3.34) and by using (3.32), we have that

$$\begin{aligned} &\lim_{i\to\infty}\left(\frac{1}{M_F(w_{k(i)}, w_{m(i)-1}, t)} - 1\right) = \varepsilon \quad \text{for } t > 0, \\ \Rightarrow &\lim_{i\to\infty}\left(\frac{1}{M_F(w_{2n(i)-1}, w_{2p(i)}, t)} - 1\right) = \varepsilon \text{ or } \lim_{i\to\infty}\left(\frac{1}{M_F(w_{2n(i)}, w_{2p(i)-1}, t)} - 1\right) = \varepsilon \text{ for } t > 0. \end{aligned} \tag{3.35}$$

Then, from (3.33), (3.29) and (3.34), for $t > 0$, if

$$\begin{aligned} \varepsilon &\leq \frac{1}{M_F(w_{k(i)}, w_{m(i)}, t)} - 1 \\ &= \frac{1}{M_F(w_{2n(i)}, w_{2n(i)+1}, t)} - 1 \\ &= \frac{1}{M_F(F_1 w_{2p(i)}, F_2 w_{2n(i)-1}, t)} - 1 \\ &\leq \left(\frac{1}{M_F(w_{2n(i)-1}, w_{2p(i)}, t)} - 1\right) - \varphi\left(\frac{1}{M_F(w_{2n(i)-1}, w_{2p(i)}, t)} - 1\right). \end{aligned}$$

Now by applying limit $i \to \infty$ and from (3.35), we get

$$\varepsilon \leq \varepsilon - \varphi(\varepsilon) < \varepsilon, \tag{3.36}$$

which is a contradiction. Similarly, again from (3.33), (3.29) and (3.34), for $t > 0$, if

$$\begin{aligned}
\varepsilon &\leq \frac{1}{M_F(w_{k(i)}, w_{m(i)}, t)} - 1 \\
&= \frac{1}{M_F(w_{2n(i)+1}, w_{2p(i)}, t)} - 1 \\
&= \frac{1}{M_F(F_1 w_{2n(i)}, F_2 w_{2p(i)-1}, t)} - 1 \\
&\leq \left(\frac{1}{M_F(w_{2n(i)}, w_{2p(i)-1}, t)} - 1\right) - \varphi\left(\frac{1}{M_F(w_{2n(i)}, w_{2p(i)-1}, t)} - 1\right).
\end{aligned}$$

By taking limit $i \to \infty$ and from (3.35), we get

$$\varepsilon \leq \varepsilon - \varphi(\varepsilon) < \varepsilon, \tag{3.37}$$

which is also a contradiction. Therefore, in both cases, that is (3.36) and (3.37), we got a contradiction. Hence proved that $\{w_m\}$ is a Cauchy sequence in W. Now from the completeness of $(W, M_F, *)$, it follows that $w_m \to \kappa \in W$, as $m \to \infty$. This can be written as

$$\lim_{m\to\infty} M_F(w_m, \kappa, t) = 1 \quad \text{for } \; t > 0. \tag{3.38}$$

Now we have to show that $F_1\kappa = \kappa$. Since M_F is triangular and from (3.29) and (3.38) for $t > 0$, we have that

$$\begin{aligned}
\frac{1}{M_F(\kappa, F_1\kappa, t)} - 1 &\leq \left(\frac{1}{M_F(\kappa, w_{2m+2}, t)} - 1\right) + \left(\frac{1}{M_F(w_{2m+2}, F_1\kappa, t)} - 1\right) \\
&= \left(\frac{1}{M_F(\kappa, w_{2m+2}, t)} - 1\right) + \left(\frac{1}{M_F(F_2 w_{2m+1}, F_1\kappa, t)} - 1\right) \\
&\leq \left(\frac{1}{M_F(\kappa, w_{2m+2}, t)} - 1\right) + \left(\frac{1}{M_F(w_{2m+1}, \kappa, t)} - 1\right) - \varphi\left(\frac{1}{M_F(w_{2m+1}, \kappa, t)} - 1\right) \\
&\to 0 \quad \text{as } j \to \infty.
\end{aligned}$$

Hence, $M_F(\kappa, F_1\kappa, t) = 1 \;\Rightarrow\; F_1\kappa = \kappa$ for $t > 0$. Similarly, by the triangular property of M_F, and again from (3.29) and (3.38) for $t > 0$, we have that

$$\begin{aligned}
\frac{1}{M_F(\kappa, F_2\kappa, t)} - 1 &\leq \left(\frac{1}{M_F(\kappa, w_{2m+1}, t)} - 1\right) + \left(\frac{1}{M_F(w_{2m+1}, F_2\kappa, t)} - 1\right) \\
&= \left(\frac{1}{M_F(\kappa, w_{2m+1}, t)} - 1\right) + \left(\frac{1}{M_F(F_1 w_{2m}, F_2\kappa, t)} - 1\right) \\
&\leq \left(\frac{1}{M_F(\kappa, w_{2m+1}, t)} - 1\right) + \left(\frac{1}{M_F(w_{2m}, \kappa, t)} - 1\right) - \varphi\left(\frac{1}{M_F(w_{2m}, \kappa, t)} - 1\right) \\
&\to 0 \quad \text{as } m \to \infty.
\end{aligned}$$

Hence, $M_F(\kappa, F_2\kappa, t) = 1 \;\Rightarrow\; F_2\kappa = \kappa$ for $t > 0$, which shows that κ is a common fixed point of the mappings F_1 and F_2.

Uniqueness: let $\kappa^* \in W$ be the other common fixed point of F_1 and F_2 such that $F_1\kappa^* = F_2\kappa^* = \kappa^*$, then again by the view of (3.29), for $t > 0$, we have

$$\begin{aligned}\frac{1}{M_F(\kappa, \kappa^*, t)} - 1 &= \frac{1}{M_F(F_1\kappa, F_2\kappa^*, t)} - 1\\ &\leq \left(\frac{1}{M_F(\kappa, \kappa^*, t)} - 1\right) - \varphi\left(\frac{1}{M_F(\kappa, \kappa^*, t)} - 1\right),\end{aligned}$$

which by the property of φ is contradiction unless $M_F(\kappa, \kappa^*, t) = 1, \Rightarrow \kappa = \kappa^*$. Hence proved that F_1 and F_2 have a unique common fixed point in W.

Example 3.12. Let $W = [0, \infty)$ and t-norm is a product continuous t-norm. Let $M_F : W\times W\times(0, \infty) \to [0, 1]$ be defined as

$$M_F(w, x, t) = \frac{t}{t + |w - x|}, \quad \forall w, x \in W, \text{ and } t > 0.$$

Then $(W, M_F, *)$ is complete and M_F is triangular. Now we define $F_1, F_2 : W \to W$ by $F_1(w) = F_2(w) = \frac{w^2+4}{4}$, $\forall\ w \in [0, 1]$. Further, a mapping $\varphi : [0, \infty) \to [0, \infty)$ be defined as $\varphi(\tau) = \frac{\tau}{2}$, for $\tau > 0$. Then, from (3.29), for $t > 0$, we have

$$\begin{aligned}\frac{1}{M_F(F_1w, F_2x, t)} - 1 &= \frac{|F_1w - F_2x|}{t}\\ &= \frac{|w^2 - x^2|}{4t}\\ &\leq \frac{|w - x|}{2t}\\ &= \frac{|w - x|}{t} - \frac{|w - x|}{2t}\\ &= \left(\frac{1}{M_F(w, x, t)} - 1\right) - \varphi\left(\frac{1}{M_F(w, x, t)} - 1\right),\end{aligned}$$

for all $w, x \in W$. Hence the conditions of Theorem 3.11 are satisfied and the mappings F_1 and F_2 have a unique common fixed point in W, that is, $F_1(2) = F_2(2) = 2 \in W$.

4. Generalized Ćirić-contraction results on FM spaces

In this section, we define a generalized Ćirić type of fuzzy contraction on FM spaces and present a unique common fixed point theorem for a pair of self-mappings on a complete FM space.

Definition 4.1. Let $(W, M_F, *)$ be an FM space. A self-mapping $F_1 : W \to W$ is said to be a generalized Ćirić type fuzzy-contraction if $\exists\ \alpha \in (0, 1)$ such that

$$\frac{1}{M_F(F_1w, F_1x, t)} - 1 \leq \alpha \max \left\{ \begin{array}{c} \left(\frac{1}{M_F(w, x, t)} - 1\right), \\ \left(\frac{1}{M_F(w, F_1w, t)} - 1\right), \left(\frac{1}{M_F(x, F_1x, t)} - 1\right), \\ \left(\frac{1}{M_F(x, F_1w, t)} - 1\right), \left(\frac{1}{M_F(w, F_1x, t)} - 1\right), \\ \frac{1}{2}\left(\frac{1}{M_F(w, F_1w, t)} - 1 + \frac{1}{M_F(x, F_1x, t)} - 1\right), \\ \frac{1}{2}\left(\frac{1}{M_F(x, F_1w, t)} - 1 + \frac{1}{M_F(w, F_1x, t)} - 1\right) \end{array} \right\} \tag{4.1}$$

$\forall\ w, x \in W$ and $t > 0$.

In the following, we present a more generalized Ćirić type fuzzy contraction result for a pair of self-mappings to prove that a pair of self-mappings on a complete FM space have a unique common fixed point.

Theorem 4.2. *Let $(W, M_F, *)$ be a complete FM space in which M_F is triangular and a pair of self-mappings $F_1, F_2 : W \to W$ satisfies,*

$$\begin{aligned} \frac{1}{M_F(F_1w, F_2x, t)} - 1 \leq{} & \alpha\left(\frac{1}{M_F(w, x, t)} - 1\right) \\ & + \beta \max \left\{ \begin{array}{c} \left(\frac{1}{M_F(w, x, t)} - 1\right), \\ \left(\frac{1}{M_F(w, F_1w, t)} - 1\right), \left(\frac{1}{M_F(x, F_2x, t)} - 1\right), \\ \left(\frac{1}{M_F(x, F_1w, t)} - 1\right), \left(\frac{1}{M_F(w, F_2x, t)} - 1\right), \\ \frac{1}{2}\left(\frac{1}{M_F(w, F_1w, t)} - 1 + \frac{1}{M_F(x, F_2x, t)} - 1\right), \\ \frac{1}{2}\left(\frac{1}{M_F(x, F_1w, t)} - 1 + \frac{1}{M_F(w, F_2x, t)} - 1\right) \end{array} \right\} \end{aligned} \tag{4.2}$$

for all $w, x \in W$, $t > 0$, $\alpha \in (0, 1)$ and $\beta \geq 0$ with $(\alpha + 2\beta) < 1$. Then F_1 and F_2 have a unique common fixed point in W.

Proof. Fix $w_0 \in W$ and define a sequence $\{w_m\}$ in W such that

$$w_{2m+1} = F_1 w_{2m} \quad \text{and} \quad w_{2m+2} = F_2 w_{2m+1} \quad \text{for } m \geq 0. \tag{4.3}$$

Now, from (4.2), for $t > 0$, we have

$$\frac{1}{M(w_{2m+1}, w_{2m+2}, t)} - 1 = \frac{1}{M(F_1 w_{2m}, F_2 w_{2m+1}, t)} - 1 \leq \alpha\left(\frac{1}{M_F(w_{2m}, w_{2m+1}, t)} - 1\right)$$

$$
+\beta\max\left\{\begin{array}{c}
\left(\dfrac{1}{M_F(w_{2m},w_{2m+1},t)}-1\right),\\
\left(\dfrac{1}{M_F(w_{2m},F_1w_{2m},t)}-1\right),\left(\dfrac{1}{M_F(w_{2m+1},F_2w_{2m+1},t)}-1\right),\\
\left(\dfrac{1}{M_F(w_{2m+1},F_1w_{2m},t)}-1\right),\left(\dfrac{1}{M_F(w_{2m},F_2w_{2m+1},t)}-1\right),\\
\dfrac{1}{2}\left(\dfrac{1}{M_F(w_{2m},F_1w_{2m},t)}-1+\dfrac{1}{M_F(w_{2m+1},F_2w_{2m+1},t)}-1\right),\\
\dfrac{1}{2}\left(\dfrac{1}{M_F(w_{2m+1},F_1w_{2m},t)}-1+\dfrac{1}{M_F(w_{2m},F_2w_{2m+1},t)}-1\right)
\end{array}\right\},
$$

after simplification, we get that

$$
\begin{aligned}
\frac{1}{M(w_{2m+1},w_{2m+2},t)}-1=&\ \alpha\left(\frac{1}{M_F(w_{2m},w_{2m+1},t)}-1\right)\\
&+\beta\max\left\{\begin{array}{c}
\left(\dfrac{1}{M_F(w_{2m},w_{2m+1},t)}-1\right),\\
\left(\dfrac{1}{M_F(w_{2m+1},w_{2m+2},t)}-1\right),\left(\dfrac{1}{M_F(w_{2m},w_{2m+2},t)}-1\right),\\
\dfrac{1}{2}\left(\dfrac{1}{M_F(w_{2m},w_{2m+1},t)}-1+\dfrac{1}{M_F(w_{2m+1},w_{2m+2},t)}-1\right),
\end{array}\right\}.
\end{aligned}
\tag{4.4}
$$

Then, we may have the following four cases;

(i) If $\frac{1}{M_F(w_{2m},w_{2m+1},t)}-1$ is the maximum in (4.4), then after simplification for $t>0$, we obtain

$$
\frac{1}{M(w_{2m+1},w_{2m+2},t)}-1\leq\lambda_1\left(\frac{1}{M_F(w_{2m},w_{2m+1},t)}-1\right),\quad \text{where } \lambda_1=\alpha+\beta<1. \tag{4.5}
$$

(ii) If $\frac{1}{M_F(w_{2m+1},w_{2m+2},t)}-1$ is the maximum in (4.4), then after simplification for $t>0$, we obtain

$$
\frac{1}{M(w_{2m+1},w_{2m+2},t)}-1\leq\lambda_2\left(\frac{1}{M_F(w_{2m},w_{2m+1},t)}-1\right),\quad \text{where } \lambda_2=\frac{\alpha}{1-\beta}<1. \tag{4.6}
$$

(iii) If $\frac{1}{M_F(w_{2m},w_{2m+2},t)}-1$ is the maximum in (4.4), then after simplification for $t>0$, we obtain

$$
\frac{1}{M(w_{2m+1},w_{2m+2},t)}-1\leq\lambda_3\left(\frac{1}{M_F(w_{2m},w_{2m+1},t)}-1\right),\quad \text{where } \lambda_3=\frac{\alpha+\beta}{1-\beta}<1. \tag{4.7}
$$

(iv) If $\frac{1}{2}\left(\frac{1}{M_F(w_{2m},w_{2m+1},t)}-1+\frac{1}{M_F(w_{2m+1},w_{2m+2},t)}-1\right)$ is the maximum in (4.4), then after simplification for $t>0$, we obtain

$$
\frac{1}{M(w_{2m+1},w_{2m+2},t)}-1\leq\lambda_4\left(\frac{1}{M_F(w_{2m},w_{2m+1},t)}-1\right),\quad \text{where } \lambda_4=\frac{2\alpha+\beta}{2-\beta}<1. \tag{4.8}
$$

Let us define $\mu_1 := \max\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} < 1$, then from (4.5)–(4.8), we get that

$$\frac{1}{M(w_{2m+1}, w_{2m+2}, t)} - 1 \leq \mu_1 \left(\frac{1}{M_F(w_{2m}, w_{2m+1}, t)} - 1 \right) \quad \text{for } t > 0. \tag{4.9}$$

Similarly, again by the view of (4.2), for $t > 0$, we have

$$\begin{aligned}
\frac{1}{M(w_{2m+2}, w_{2m+3}, t)} - 1 = \frac{1}{M(F_1 w_{2m+2}, F_2 w_{2m+1}, t)} - 1 \leq \alpha \left(\frac{1}{M_F(w_{2m+2}, w_{2m+1}, t)} - 1 \right) \\
+ \beta \max \left\{ \begin{array}{c}
\left(\frac{1}{M_F(w_{2m+2}, w_{2m+1}, t)} - 1 \right), \\
\left(\frac{1}{M_F(w_{2m+2}, F_1 w_{2m+2}, t)} - 1 \right), \left(\frac{1}{M_F(w_{2m+1}, F_2 w_{2m+1}, t)} - 1 \right), \\
\left(\frac{1}{M_F(w_{2m+1}, F_1 w_{2m+2}, t)} - 1 \right), \left(\frac{1}{M_F(w_{2m+2}, F_2 w_{2m+1}, t)} - 1 \right), \\
\frac{1}{2} \left(\frac{1}{M_F(w_{2m+2}, F_1 w_{2m+2}, t)} - 1 + \frac{1}{M_F(w_{2m+1}, F_2 w_{2m+1}, t)} - 1 \right), \\
\frac{1}{2} \left(\frac{1}{M_F(w_{2m+1}, F_1 w_{2m+2}, t)} - 1 + \frac{1}{M_F(w_{2m+2}, F_2 w_{2m+1}, t)} - 1 \right)
\end{array} \right\},
\end{aligned}$$

after simplification, we get that

$$\begin{aligned}
\frac{1}{M_F(w_{2m+2}, w_{2m+3}, t)} - 1 = \alpha \left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1 \right) \\
+ \beta \max \left\{ \begin{array}{c}
\left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1 \right), \\
\left(\frac{1}{M_F(w_{2m+2}, w_{2m+3}, t)} - 1 \right), \left(\frac{1}{M_F(w_{2m+1}, w_{2m+3}, t)} - 1 \right), \\
\frac{1}{2} \left(\frac{1}{M_F(w_{2m+2}, w_{2m+3}, t)} - 1 + \frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1 \right),
\end{array} \right\}.
\end{aligned} \tag{4.10}$$

Again we may have the following four cases;

(i) If $\left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1 \right)$ is the maximum term in (4.10), then after simplification for $t > 0$, we obtain

$$\frac{1}{M(w_{2m+2}, w_{2m+3}, t)} - 1 \leq \lambda_1 \left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1 \right), \quad \text{where } \lambda_1 = \alpha + \beta < 1. \tag{4.11}$$

(ii) If $\left(\frac{1}{M_F(w_{2m+2}, w_{2m+3}, t)} - 1 \right)$ is the maximum term in (4.10), then after simplification for $t > 0$, we obtain

$$\frac{1}{M(w_{2m+2}, w_{2m+3}, t)} - 1 \leq \lambda_2 \left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1 \right), \quad \text{where } \lambda_2 = \frac{\alpha}{1 - \beta} < 1. \tag{4.12}$$

(iii) If $\left(\frac{1}{M_F(w_{2m+1}, w_{2m+3}, t)} - 1 \right)$ is the maximum term in (4.10), then after simplification for $t > 0$, we obtain

$$\frac{1}{M(w_{2m+2}, w_{2m+3}, t)} - 1 \leq \lambda_3 \left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1 \right), \quad \text{where } \lambda_3 = \frac{\alpha + \beta}{1 - \beta} < 1. \tag{4.13}$$

(iv) If $\frac{1}{2}\left(\frac{1}{M_F(w_{2m+2},w_{2m+3},t)} - 1 + \frac{1}{M_F(w_{2m+1},w_{2m+2},t)} - 1\right)$ is the maximum term in (4.10), then after simplification for $t > 0$, we obtain

$$\frac{1}{M(w_{2m+2}, w_{2m+3}, t)} - 1 \leq \lambda_4 \left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1\right), \quad \text{where } \lambda_4 = \frac{2\alpha + \beta}{2 - \beta} < 1. \tag{4.14}$$

Hence, from (4.11)–(4.14), we get that

$$\frac{1}{M(w_{2m+2}, w_{2m+3}, t)} - 1 \leq \mu_1 \left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1\right) \quad \text{for } t > 0, \tag{4.15}$$

where $\mu_1 = \max\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} < 1$. Now from (4.9) and (4.15), we have that

$$\begin{aligned}
\frac{1}{M(w_{2m+2}, w_{2m+3}, t)} - 1 &\leq \mu_1 \left(\frac{1}{M_F(w_{2m+1}, w_{2m+2}, t)} - 1\right) \\
&\leq (\mu_1)^2 \left(\frac{1}{M_F(w_{2m}, w_{2m+1}, t)} - 1\right) \\
&\leq \cdots \leq (\mu_1)^{2m+2} \left(\frac{1}{M_F(w_0, w_1, t)} - 1\right) \to 0, \quad \text{as } m \to \infty.
\end{aligned}$$

Hence proved that $\{w_m\}_{m \geq 0}$ is a fuzzy contractive sequence, therefore

$$\lim_{m \to \infty} M_F(w_m, w_{m+1}, t) = 1 \quad \text{for } t > 0. \tag{4.16}$$

Since M_F is triangular, for $k > m$ and $t > 0$, then we have

$$\begin{aligned}
&\frac{1}{M_F(w_m, w_k, t)} - 1 \\
&\leq \left(\frac{1}{M_F(w_m, w_{m+1}, t)} - 1\right) + \left(\frac{1}{M_F(w_{m+1}, w_{m+2}, t)} - 1\right) + \cdots + \left(\frac{1}{M_F(w_{k-1}, w_k, t)} - 1\right) \\
&\leq \left((\mu_1)^m + (\mu_1)^{m+1} + \cdots + (\mu_1)^{k-1}\right)\left(\frac{1}{M_F(w_0, w_1, t)} - 1\right) \\
&\leq \left(\frac{(\mu_1)^m}{1 - \mu_1}\right)\left(\frac{1}{M_F(w_0, w_1, t)} - 1\right) \to 0, \quad \text{as } m \to \infty,
\end{aligned}$$

which shows that $\{w_m\}$ is a cauchy sequence. By the completeness of $(W, M_F, *)$, $\exists\, \kappa \in W$ such that

$$\lim_{m \to \infty} M_F(\kappa, w_m, t) = 1 \quad \text{for } t > 0. \tag{4.17}$$

Now we have to show that $F_1\kappa = \kappa$. Since, M_F is triangular, therefore

$$\frac{1}{M_F(\kappa, F_1\kappa, t)} - 1 \leq \left(\frac{1}{M_F(\kappa, w_{2m+2}, t)} - 1\right) + \left(\frac{1}{M_F(w_{2m+2}, F_1\kappa, t)} - 1\right) \quad \text{for } t > 0. \tag{4.18}$$

Now, by the view of (4.2), (4.16) and (4.17) for $t > 0$, we have that

$$\frac{1}{M_F(w_{2m+2}, F_1\kappa, t)} - 1 = \frac{1}{M_F(F_2 w_{2m+1}, F_1\kappa, t)} - 1 \leq \alpha\left(\frac{1}{M_F(w_{2m+1}, \kappa, t)} - 1\right)$$

$$
\begin{aligned}
&+\beta\max\left\{\begin{array}{c}\left(\dfrac{1}{M_F(w_{2m+1},\kappa,t)}-1\right),\\ \left(\dfrac{1}{M_F(\kappa,F_1\kappa,t)}-1\right),\left(\dfrac{1}{M_F(w_{2m+1},F_2w_{2m+1},t)}-1\right),\\ \left(\dfrac{1}{M_F(w_{2m+1},F_1\kappa,t)}-1\right),\left(\dfrac{1}{M_F(\kappa,F_2w_{2m+1},t)}-1\right),\\ \dfrac{1}{2}\left(\dfrac{1}{M_F(\kappa,F_1\kappa,t)}-1+\dfrac{1}{M_F(w_{2m+1},F_2w_{2m+1},t)}-1\right),\\ \dfrac{1}{2}\left(\dfrac{1}{M_F(w_{2m+1},F_1\kappa,t)}-1+\dfrac{1}{M_F(\kappa,F_2w_{2m+1},t)}-1\right)\end{array}\right\}\\
&=\alpha\left(\dfrac{1}{M_F(w_{2m+1},\kappa,t)}-1\right)\\
&+\beta\max\left\{\begin{array}{c}\left(\dfrac{1}{M_F(w_{2m+1},\kappa,t)}-1\right),\\ \left(\dfrac{1}{M_F(\kappa,F_1\kappa,t)}-1\right),\left(\dfrac{1}{M_F(w_{2m+1},w_{2m+2},t)}-1\right),\\ \left(\dfrac{1}{M_F(w_{2m+1},F_1\kappa,t)}-1\right),\left(\dfrac{1}{M_F(\kappa,w_{2m+2},t)}-1\right),\\ \dfrac{1}{2}\left(\dfrac{1}{M_F(\kappa,F_1\kappa,t)}-1+\dfrac{1}{M_F(w_{2m+1},w_{2m+2},t)}-1\right),\\ \dfrac{1}{2}\left(\dfrac{1}{M_F(w_{2m+1},F_1\kappa,t)}-1+\dfrac{1}{M_F(\kappa,w_{2m+2},t)}-1\right)\end{array}\right\}\\
&\to\beta\max\left\{\dfrac{1}{M_F(\kappa,F_1\kappa,t)}-1,\dfrac{1}{2}\left(\dfrac{1}{M_F(\kappa,F_1\kappa,t)}-1\right)\right\},\quad \text{as } j\to\infty.
\end{aligned}
$$

Then,

$$
\limsup_{m\to\infty}\left(\frac{1}{M_F(w_{2m+2},F_1\kappa,t)}-1\right)\le\beta\left(\frac{1}{M_F(\kappa,F_1\kappa,t)}-1\right)\quad \text{for } t>0. \tag{4.19}
$$

The above (4.19) is together with (4.18) and (4.17), we get that

$$
\frac{1}{M_F(\kappa,F_1\kappa,t)}-1\le\beta\left(\frac{1}{M_F(\kappa,F_1\kappa,t)}-1\right)\quad \text{for } t>0.
$$

Since $(1-\beta)\neq 0$, therefore we get that $M_F(\kappa,F_1\kappa,t)=1$, this implies that $F_1\kappa=\kappa$. Similarly, we can show $F_2\kappa=\kappa$. Hence proved that κ is a common fixed point of F_1 and F_2, that is, $F_1\kappa=F_2\kappa=\kappa$. Uniqueness: let $\kappa^*\in W$ be the other common fixed point of F_1 and F_2 such that $F_1\kappa^*=F_2\kappa^*=\kappa^*$, then by the view of (4.2), for $t>0$, we have

$$
\frac{1}{M_F(\kappa,\kappa^*,t)}-1=\left(\frac{1}{M_F(F_1\kappa,F_2\kappa^*,t)}-1\right)\le\alpha\left(\frac{1}{M_F(\kappa,\kappa^*,t)}-1\right)
$$

$$+\beta\max\left\{\begin{array}{c}\left(\frac{1}{M_F(\kappa,\kappa^*,t)}-1\right),\\ \left(\frac{1}{M_F(\kappa,F_1\kappa,t)}-1\right),\left(\frac{1}{M_F(\kappa^*,F_2\kappa^*,t)}-1\right),\\ \left(\frac{1}{M_F(\kappa^*,F_1\kappa,t)}-1\right),\left(\frac{1}{M_F(\kappa,F_2\kappa^*,t)}-1\right),\\ \frac{1}{2}\left(\frac{1}{M_F(\kappa,F_1\kappa,t)}-1+\frac{1}{M_F(\kappa^*,F_2\kappa^*,t)}-1\right),\\ \frac{1}{2}\left(\frac{1}{M_F(\kappa^*,F_1\kappa,t)}-1+\frac{1}{M_F(\kappa,F_2\kappa^*,t)}-1\right)\end{array}\right\}$$
$$=(\alpha+\beta)\left(\frac{1}{M_F(\kappa,\kappa^*,t)}-1\right)=(\alpha+\beta)\left(\frac{1}{M_F(F_1\kappa,F_2\kappa^*,t)}-1\right)$$
$$\leq(\alpha+\beta)^2\left(\frac{1}{M_F(\kappa,\kappa^*,t)}-1\right)\leq\cdots\leq(\alpha+\beta)^m\left(\frac{1}{M_F(\kappa,\kappa^*,t)}-1\right)\to 0,\quad \text{as } m\to\infty.$$

Hence we get that $M_F(\kappa,\kappa^*,t)=1$, this implies that $\kappa=\kappa^*$. Thus, F_1 and F_2 have a unique common fixed point in W.

If the mapping $F_1=F_2$ or one of them considers an identity map in Theorem 4.2, then we get the following corollary:

Corollary 4.3. *Let $(W,M_F,*)$ be a complete FM space in which M_F is triangular and a pair of self-mappings $F_1:W\to W$ satisfies,*

$$\frac{1}{M_F(F_1w,F_1x,t)}-1\leq\alpha\left(\frac{1}{M_F(w,x,t)}-1\right)$$
$$+\beta\max\left\{\begin{array}{c}\left(\frac{1}{M_F(w,x,t)}-1\right),\\ \left(\frac{1}{M_F(w,F_1w,t)}-1\right),\left(\frac{1}{M_F(x,F_1x,t)}-1\right),\\ \left(\frac{1}{M_F(x,F_1w,t)}-1\right),\left(\frac{1}{M_F(w,F_1x,t)}-1\right),\\ \frac{1}{2}\left(\frac{1}{M_F(w,F_1w,t)}-1+\frac{1}{M_F(x,F_1x,t)}-1\right),\\ \frac{1}{2}\left(\frac{1}{M_F(x,F_1w,t)}-1+\frac{1}{M_F(w,F_1x,t)}-1\right)\end{array}\right\} \tag{4.20}$$

$\forall\, w,x\in W,\ t>0,\ \alpha\in(0,1)$ *and* $\beta\geq 0$ *with* $(\alpha+2\beta)<1$. *Then F_1 has a unique fixed point.*

Remark 4.4. If we put $\beta=0$ in Theorem 4.2, we get "a fuzzy Banach contraction theorem for FP" on a complete FM space.

Example 4.5. Let $W=[0,1]$ and from Example 3.6, the mappings $F_1,F_2:W\to W$ be defined as $F_1u=F_2u=\frac{7u}{10}+\frac{4}{15}$ for all $u\in W$. Then, we have

$$\frac{1}{M_F(F_1w,F_2x,t)}-1=\frac{2\,|F_1w-F_2x|}{3t}=\frac{7\,|w-x|}{15t}=\frac{7}{10}\left(\frac{1}{M_F(w,x,t)}-1\right)\quad\forall\, w,x\in W \text{ and } t>0.$$

Hence, the mappings F_1 and F_2 are contractive and satisfied the conditions of Theorem 4.2 with $\alpha = \frac{7}{10}$, $\beta = \frac{1}{7}$. The mappings F_1 and F_2 have a common fixed point, that is, $F_1(\frac{8}{9}) = F_2(\frac{8}{9}) = \frac{8}{9} \in [0,1]$.

5. Application

In this section, we present an application of fuzzy differential equations (FDEs) to support our results. Some differential equation results in different directions can be found in (see [33–37] the references are therein). From the book of Lakshmikantham et al. [38], we have the following FDEs.

Let $\mathbb{E}$ be the space of all fuzzy subsets w of $\mathbb{R}$ where $w : \mathbb{R} \to \mathbb{I} = [0,1]$.

$$\begin{aligned} & w^{''}(s) = h(s, w(s), w^{'}(s)), \quad s \in \mathbb{J} = [a,b], \\ & w(s_1) = w_1,\ w(s_2) = w_2, \quad s_1, s_2 \in \mathbb{J} = [a,b], \end{aligned} \tag{5.1}$$

where $h : \mathbb{J} \times \mathbb{E} \times \mathbb{E} \to \mathbb{E}$ is a continuous function. This problem is equivalent to the integral equation

$$w(s) = \int_{s_1}^{s_2} K(s,\tau) \left(h(\tau, w(\tau), w^{'}(\tau)) \right) d\tau + \mathcal{B}(s),$$

where Green's function K is given by

$$K(s,\tau) = \begin{cases} \dfrac{(s_2 - s)(\tau - s_1)}{s_2 - s_1}, & s_1 \le \tau \le s \le s_2, \\ \dfrac{(s_2 - \tau)(s - s_1)}{s_2 - s_1}, & s_1 \le s \le \tau \le s_2. \end{cases}$$

And $\mathcal{B}(s)$ satisfies $\mathcal{B}^{''} = 0$, $\mathcal{B}(s_1) = w_1$, $\mathcal{B}(s_2) = w_2$. Let here we recall some properties of $K(s,\tau)$, that are;

$$\int_{s_1}^{s_2} K(s,\tau) d\tau \le \frac{(s_2 - s_1)^2}{8},$$

and

$$\int_{s_1}^{s_2} K_s(s,\tau) d\tau \le \frac{s_2 - s_1}{2}.$$

Let $C = C^1(\mathbb{J}, \mathbb{E})$, $*$ is a continuous t-norm, and $M_F : C \times C \times (0,\infty) \to [0,1]$ be defined as

$$M_F(w,x,t) = \frac{t}{t + D(w,x)} \quad \text{where } D(w,x) = |w - x|, \tag{5.2}$$

for all $w, x \in C$ and $t > 0$. Then one can verify that M_F is triangular and $(C, M_F, *)$ is complete.

Now, we are in the position to prove the existing result for the above boundary value problem by applying Theorem 3.1.

Theorem 5.1. *Assume that* $h_1, h_2 : \mathbb{J} \times \mathbb{E} \times \mathbb{E} \to \mathbb{E}$ *and let there exists* $p, q \in (0,1)$ *with* $p \le q$ *such that for all* $w, x \in C^1(\mathbb{J}, \mathbb{E})$, *satisfies*

$$\left| h_1(s, w, w^{'}) - h_2(s, x, x^{'}) \right| \le p|w - x| + q \left| w^{'} - x^{'} \right|. \tag{5.3}$$

Let there exists $\eta \in (0,1)$ *such that*

$$D(w,x) \leq \eta \mathbf{M}(F_1,F_2,w,x), \tag{5.4}$$

where

$$\mathbf{M}(F_1,F_2,w,x) = \max\left\{ \begin{array}{c} |w-x|, |F_1w-w| + |F_2x-x| + |F_1w-x| + |F_2x-w| \\ \min\{|F_1w-w|, |F_2x-x|\}, \max\{|F_1w-x|, |F_2x-w|\} \end{array} \right\}. \tag{5.5}$$

Then the integral equations

$$w(s) = \int_{s_1}^{s_2} K(s,\tau)\left(h_1(\tau, w(\tau), w'(\tau)\right) d\tau + \mathcal{B}(s), \quad s \in \mathbb{J},$$

and

$$x(s) = \int_{s_1}^{s_2} K(s,\tau)\left(h_2(\tau, x(\tau), x'(\tau)\right) d\tau + \mathcal{B}(s), \quad s \in \mathbb{J},$$

have a unique common solution in $C^1[[s_1,s_2],\mathbb{E}]$.

Proof. Suppose that $C = [[s_1,s_2],\mathbb{E}]$ with metric

$$D(w,x) = \max_{s_1 \leq s \leq s_2} \left(p|w(s)-x(s)| + q\left|w'(s) - x'(s)\right|\right). \tag{5.6}$$

The space (C,D) is a complete metric space. Now, the operators $F_1, F_2 : C \to C$ are defined as

$$F_1(w)(s) = \int_{s_1}^{s_2} K(s,\tau)\left(h_1(\tau, w(\tau), w'(\tau)\right) d\tau + \mathcal{B}(s), \quad s \in \mathbb{J},$$

and

$$F_2(x)(s) = \int_{s_1}^{s_2} K(s,\tau)\left(h_2(\tau, x(\tau), x'(\tau)\right) d\tau + \mathcal{B}(s), \quad s \in \mathbb{J},$$

where $h_1, h_2 \in C\left(\mathbb{J} \times \mathbb{E} \times \mathbb{E}, \mathbb{E}\right)$, $w \in C^1(\mathbb{J},\mathbb{E})$, and $\mathcal{B} \in C(\mathbb{J},\mathbb{E})$. Now by the properties of $K(s,\tau)$ and by using our hypothesis,

$$\begin{aligned} |F_1w(s) - F_2x(s)| &\leq \int_{s_1}^{s_2} |K(s,\tau)| \left|h_1\left(\tau, w(\tau), w'(\tau)\right) - h_2\left(\tau, x(\tau), x'(\tau)\right)\right| d\tau \\ &\leq D(w,x) \int_{s_1}^{s_2} |K(s,\tau)| d\tau \\ &\leq \frac{(s_2-s_1)^2}{8} D(w,x) \\ &\leq \frac{D(w,x)}{8}. \end{aligned}$$

And

$$\left|(F_1w)'(s) - (F_2x)'(s)\right| \leq \int_{s_1}^{s_2} |K_s(s,\tau)| \left|h_1\left(\tau, w(\tau), w'(\tau)\right) - h_2\left(\tau, x(\tau), x'(\tau)\right)\right| d\tau$$

$$
\begin{aligned}
&\leq D(w,x)\int_{s_1}^{s_2}|K_s(s,\tau)|d\tau\\
&\leq \frac{s_2-s_1}{2}D(w,x)\\
&\leq \frac{D(w,x)}{2}.
\end{aligned}
$$

Now, from the above and by the view of (5.3) and (5.6), we have that

$$
\begin{aligned}
D(F_1w,F_2x) &= \max_{s_1\leq s\leq s_2}\left(p|F_1w(s)-F_2x(s)|+q\left|(F_1w)^{'}(s)-(F_2x)^{'}(s)\right|\right)\\
&\leq p\frac{D(w,x)}{8}+q\frac{D(w,x)}{2}\\
&\leq \left(\frac{5}{8}q\right)D(w,x)
\end{aligned}
$$

Now, from (5.4), we have that

$$
D(F_1w,F_2x)\leq\left(\frac{5}{8}q\right)D(w,x)\leq\xi\mathbf{M}(F_1,F_2,w,x), \tag{5.7}
$$

where $\xi=\frac{5}{8}q\eta<1$. Now we apply Theorem 3.1 to get that F_1 and F_2 have a unique common fixed point $w^*\in C$, i.e., w^* is a solution of the BVP. We may have the following main four cases:

1) If $|w-x|$ is the maximum term in (5.5), then $\mathbf{M}(F_1,F_2,w,x)=|w-x|$. Now from (5.2), (5.4) and (5.7), we have

$$
\begin{aligned}
\frac{1}{M_F(F_1w,F_2x,t)}-1 &= \frac{D(F_1w,F_2x)}{t}\\
&\leq\xi\frac{\mathbf{M}(F_1,F_2,w,x)}{t}\\
&=\xi\frac{|w-x|}{t}=\xi\left(\frac{1}{M_F(w,x,t)}-1\right).
\end{aligned}
$$

This implies that

$$
\frac{1}{M_F(F_1w,F_2x,t)}-1\leq\xi\left(\frac{1}{M_F(w,x,t)}-1\right)\quad\text{for } t>0, \tag{5.8}
$$

for all $w,x\in C$. Thus, the operators F_1 and F_2 satisfy the conditions of Theorem 3.1 with $\xi=a$ and $b=c=d=0$ in (3.19). Then the operators F_1 and F_2 have a unique common fixed point $w^*\in C$, i.e., w^* is a solution of the BVP (5.1).

2) If $|F_1w-w|+|F_2x-x|+|F_1w-x|+|F_2x-w|$ is the maximum term in (5.5), then $\mathbf{M}(F_1,F_2,w,x)=|F_1w-w|+|F_2x-x|+|F_1w-x|+|F_2x-w|$. Now from (5.2), (5.4) and (5.7), we have

$$
\frac{1}{M_F(F_1w,F_2x,t)}-1=\frac{D(F_1w,F_2x)}{t}
$$

$$
\begin{aligned}
&\leq \xi \frac{\mathbf{M}(F_1, F_2, w, x)}{t} \\
&= \xi \frac{|F_1 w - w| + |F_2 x - x| + |F_1 w - x| + |F_2 x - w|}{t} \\
&= \xi \left(\begin{array}{c} \frac{1}{M_F(w, F_1 w, t)} - 1 + \frac{1}{M_F(x, F_2 x, t)} - 1 \\ + \frac{1}{M_F(x, F_1 w, t)} - 1 + \frac{1}{M_F(w, F_2 x, t)} - 1 \end{array} \right).
\end{aligned}
$$

This implies that

$$
\frac{1}{M_F(F_1 w, F_2 x, t)} - 1 \leq \xi \left(\begin{array}{c} \frac{1}{M_F(w, F_1 w, t)} - 1 + \frac{1}{M_F(x, F_2 x, t)} - 1 \\ + \frac{1}{M_F(x, F_1 w, t)} - 1 + \frac{1}{M_F(w, F_2 x, t)} - 1 \end{array} \right) \quad \text{for } t > 0, \tag{5.9}
$$

for all $w, x \in C$. Thus, the operators F_1 and F_2 satisfy the conditions of Theorem 3.1 with $\xi = b$ and $a = c = d = 0$ in (3.19). Then the operators F_1 and F_2 have a unique common fixed point $w^* \in C$, i.e., w^* is a solution of the BVP (5.1).

3) If $\min\{|F_1 w - w|, |F_2 x - x|\}$ is the maximum term in (5.5), then $\mathbf{M}(F_1, F_2, w, x) = \min\{|F_1 w - w|, |F_2 x - x|\}$. Now, if $|F_1 w - w|$ is the minimum term in $\{|F_1 w - w|, |F_2 x - x|\}$, then $\mathbf{M}(F_1, F_2, w, x) = |F_1 w - w|$. Therefore, from (5.2), (5.4) and (5.7), we have

$$
\begin{aligned}
\frac{1}{M_F(F_1 w, F_2 x, t)} - 1 &= \frac{D(F_1 w, F_2 x)}{t} \\
&\leq \xi \frac{\mathbf{M}(F_1, F_2, w, x)}{t} \\
&= \xi \frac{|F_1 w - w|}{t} = \xi \left(\frac{1}{M_F(w, F_1 w, t)} - 1 \right).
\end{aligned}
$$

This implies that

$$
\frac{1}{M_F(F_1 w, F_2 x, t)} - 1 \leq \xi \left(\frac{1}{M_F(w, F_1 w, t)} - 1 \right) \quad \text{for } t > 0. \tag{5.10}
$$

Similarly, if $|F_2 x - x|$ is the minimum term in $\{|F_1 w - w|, |F_2 x - x|\}$, then $\mathbf{M}(F_1, F_2, w, x) = |F_2 x - x|$. Therefore, again from (5.2), (5.4) and (5.7), we have

$$
\begin{aligned}
\frac{1}{M_F(F_1 w, F_2 x, t)} - 1 &= \frac{D(F_1 w, F_2 x)}{t} \\
&\leq \xi \frac{\mathbf{M}(F_1, F_2, w, x)}{t} \\
&= \xi \frac{|F_2 x - x|}{t} = \xi \left(\frac{1}{M_F(x, F_2 x, t)} - 1 \right).
\end{aligned}
$$

This implies that

$$\frac{1}{M_F(F_1w, F_2x, t)} - 1 \leq \xi\left(\frac{1}{M_F(x, F_2x, t)} - 1\right) \quad \text{for } t > 0, \tag{5.11}$$

for all $w, x \in C$. Thus, from (5.10) and (5.11) the operators F_1 and F_2 satisfy the conditions of Theorem 3.1 with $\xi = c$ and $a = b = d = 0$ in (3.19). Then the operators F_1 and F_2 have a unique common fixed point $w^* \in C$, i.e., w^* is a solution of the BVP (5.1).

4) If $\max\{|F_1w - x|, |F_2x - w|\}$ is the maximum term in (5.5), then $\mathbf{M}(F_1, F_2, w, x) = \max\{|F_1w - x|, |F_2x-w|\}$. Now, if $|F_1w-x|$ is the maximum term in $\{|F_1w-x|, |F_2x-w|\}$, then $\mathbf{M}(F_1, F_2, w, x) = |F_1w - x|$. Therefore, from (5.2), (5.4) and (5.7), we have

$$\begin{aligned}\frac{1}{M_F(F_1w, F_2x, t)} - 1 &= \frac{D(F_1w, F_2x)}{t}\\ &\leq \xi\frac{\mathbf{M}(F_1, F_2, w, x)}{t}\\ &= \xi\frac{|F_1w - x|}{t} = \xi\left(\frac{1}{M_F(x, F_1w, t)} - 1\right).\end{aligned}$$

This implies that

$$\frac{1}{M_F(F_1w, F_2x, t)} - 1 \leq \xi\left(\frac{1}{M_F(x, F_1w, t)} - 1\right) \quad \text{for } t > 0. \tag{5.12}$$

Similarly, if $|F_2x-w|$ is the maximum term in $\{|F_1w-x|, |F_2x-w|\}$, then $\mathbf{M}(F_1, F_2, w, x) = |F_2x-w|$. Therefore, again from (5.2), (5.4) and (5.7), we have

$$\begin{aligned}\frac{1}{M_F(F_1w, F_2x, t)} - 1 &= \frac{D(F_1w, F_2x)}{t}\\ &\leq \xi\frac{\mathbf{M}(F_1, F_2, w, x)}{t}\\ &= \xi\frac{|F_2x - w|}{t} = \xi\left(\frac{1}{M_F(w, F_2x, t)} - 1\right).\end{aligned}$$

This implies that

$$\frac{1}{M_F(F_1w, F_2x, t)} - 1 \leq \xi\left(\frac{1}{M_F(w, F_2x, t)} - 1\right) \quad \text{for } t > 0. \tag{5.13}$$

for all $w, x \in C$. Thus, from (5.12) and (5.13) the operators F_1 and F_2 satisfy the conditions of Theorem 3.1 with $\xi = d$ and $a = b = c = 0$ in (3.19). Then the operators F_1 and F_2 have a unique common fixed point $w^* \in C$, i.e., w^* is a solution of the BVP (5.1).

6. Conclusions

In this paper, we presented some generalized unique common fixed point theorems for a pair of self-mappings on complete FM spaces. The triangular property of fuzzy metric is used as a basic tool throughout the complete paper and proved all the results without continuity of self-mappings. We defined weak-contraction and a generalized Ćirić-contraction on FM space and proved unique common fixed point theorems. The results are supported by suitable examples and showed the uniqueness of common fixed points. In addition, we presented an application of fuzzy differential equations and proved the existing result for a unique common solution to support our main work. By using this concept, one can prove more generalized different contractive type single-valued mapping results for fixed point, common fixed point, and coincidence point on FM spaces without the continuity of self-mappings by using different types of applications such as differential equations and integral equations applications.

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Conflict of interest

The authors declare that they have no conflict of interest.

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