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Research article

Stability of hyper homomorphisms and hyper derivations in complex Banach algebras

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Abstract: In this paper, we introduce the concept of hyper homomorphisms and hyper derivations in Banach algebras and we establish the stability of hyper homomorphisms and hyper derivations in Banach algebras for the following 3-additive functional equation:

$$g(x_1 + x_2, y_1 + y_2, z_1 + z_2) = \sum_{i,j,k=1}^{2} g(x_i, y_j, z_k).$$

Keywords: Hyers-Ulam stability; 3-additive functional equation; hyper derivation; hyper homomrphism; 3-linear mapping
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1. Introduction

The stability problem of functional equations originated from a question of Ulam [26] concerning the stability of group homomorphisms in 1940. Hyers [8] gave the first partial solution to Ulam's question for the case of approximate additive mappings in Banach spaces. Aoki [1] generalized Hyers' theorem for approximately additive mappings. In 1978, Rassias [22] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. Rassias' influential paper [22] played a key role in the development of what we call Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations.

Theorem 1.1. *[22] Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(a + b) - f(a) - f(b)\| \leq \epsilon(\|a\|^p + \|b\|^p), \tag{1.1}$$

for all $a, b \in E$, *where* $\epsilon > 0$ *and* $p < 1$ *are constants. Then, there exists a unique additive mapping* $T : E \to E'$ *such that*

$$\|f(a) - T(a)\| \leq \frac{2\epsilon}{2 - 2^p}\|a\|^p, \tag{1.2}$$

for all $a \in E$. *If* $p < 0$, *then* (1.1) *holds for all* $a, b \neq 0$, *and* (1.2) *holds for* $a \neq 0$. *Also, if the function* $t \mapsto f(ta)$ *from* $\mathcal{R}$ *into* E' *is continuous for each fixed* $a \in X$, *then* T *is* $\mathbb{R}$*-linear.*

Note that if $\epsilon(\|a\|^p + \|b\|^p)$ is replaced by ϵ in (1.1), the resulting conclusion is Hyers' theorem.

In 1991, Gajda [6], following the same approach as that by Rassias [22], gave an affirmative solution to the stability question for $p > 1$. It was shown by Gajda [6] as well as by Rassias and Šemrl [23], that one cannot prove a Rassias-type theorem when $p = 1$. Găvruta [7] obtained a generalized result of the Rassias theorem which allows the Cauchy difference to be controlled by a general unbounded function.

The method provided by Hyers [8] which produces the additive function will be called a direct method. This method is the most important and useful tool to study the stability of different functional equations. The stability problems for several functional equations or inequalities have been extensively investigated by a number of authors and there are many interesting results, containing ternary homomorphisms and ternary derivations, concerning this problem (see [3–5, 9–12, 17–21]). In this paper, we introduce hyper homomorphisms and hyperderivations in Banach algebras and we prove the Hyers-Ulam stability of hyper homomorphisms and hyper derivations in Banach algebras.

The theory of stability of homomorphisms and derivations in Banach algebras is an important part of functional equations theory. It has several applications in dynamical systems, physics and connections with other parts of mathematics. With an increasing amount of theory and applications concerning Banach algebras, it is becoming necessary to ascertain which tools are applicable for handling them.

Moreover, in modern industry, various analytical approaches for solving mathematical equations are widely applied in analysis of problems in packaging engineering, and so mathematical modeling and computation methods by using mathematical equations play an important role in application of packaging engineering. From now, we wish to note that mathematical equations for stability properties in this paper can have applications to engineering.

During the last two decades, a number of articles and research monographs have been published on various generalizations and applications of the Hyers-Ulam stability to a number of functional equations and mappings, for example, Cauchy-Jensen mappings, k-additive mappings, invariant means, multiplicative mappings, bounded nth differences, convex functions, generalized orthogonality mappings, Euler-Lagrange functional equations, differential equations, and Navier-Stokes equations (see [2, 13–15, 24, 25, 27]).

Let X be a complex Banach algebra. A mapping $g : X \times X \times X \to X$ is 3-additive if

$$g(x_1 + x_2, y_1 + y_2, z_1 + z_2) = \sum_{i,j,k=1}^{2} g(x_i, y_j, z_k), \tag{1.3}$$

for all $x_1, y_1, z_1, x_2, y_2, z_2 \in X$. A mapping $g : X \times X \times X \to X$ is called 3-linear if g is 3-additive and $\mathbb{C}$-linear for each variable. For an example, let $g(x, y, z) = ax + by + cz$ for constants a, b, c and for all $x, y, z \in X$. Then $g : X \times X \times X \to X$ is clearly 3-additive and 3-linear.

Assume that X is a complex Banach algebra and that s and t are fixed complex numbers with $0 < s < 1$ and $0 < t < 1$ in the whole paper.

2. Stability of hyper homomorphisms in complex Banach algebras

In this section, we introduce the concept of hyper homomorphisms in Banach algebras and we establish the stability of hyper homomorphisms in Banach algebras by using the direct method.

Definition 2.1. Let X be a complex Banach algebra. A mapping $h : X \times X \times X \to X$ is called a hyper quadratic mapping if h satisfies

$$8h(x_1, y_1, z_1) = \sum_{i,j,k=1}^{2} h(x_1 + (-1)^i x_2, y_1 + (-1)^j y_2, z_1 + (-1)^k z_2), \tag{2.1}$$

for all $x_1, y_1, z_1, x_2, y_2, z_2 \in X$.

Definition 2.2. Let X be a complex Banach algebra. A 3-linear mapping $h : X \times X \times X \to X$ is called a hyper homomorphism if h satisfies

$$h(x_1x_2x_3, y_1y_2y_3, z_1z_2z_3) = h(x_1, x_2, x_3)h(y_1, y_2, y_3)h(z_1, z_2, z_3),$$

for all $x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3 \in X$.

Example 2.3. Let X be a complex Banach algebra and $h_1, h_2, h_3 : X \to X$ be homomorphisms. Let $h(x_1x_2x_3, y_1y_2y_3, z_1z_2z_3) = h_1(x_1y_1z_1)h_2(x_2y_2z_2)h_3(x_3, y_3, z_3)$ for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in X$. Then $h : X^3 \to X$ is a hyper homomorphism in case that X is commutative, but it may not be a hyper homomorphism in case that X is not commutative.

Lemma 2.4. *Let $h : X \times X \times X \to X$ be a hyper quadratic mapping and $h(2x, 2y, 2z) = 8h(x, y, z)$ for all $x, y, z \in X^3$, then h is 3-additive.*

Proof. Let x_1, x_2, y_1, y_2, z_1 and z_2 be arbitrary members in X. Define

$$u_1 := \frac{x_1 + x_2}{2},\ u_2 := \frac{x_1 - x_2}{2},\ v_1 := \frac{y_1 + y_2}{2},$$
$$v_2 := \frac{y_1 - y_2}{2},\ w_1 := \frac{z_1 + z_2}{2} \text{ and } w_2 := \frac{z_1 - z_2}{2}.$$

By the use of (2.1) we have

$$\begin{aligned} h(x_1 + x_2, y_1 + y_2, z_1 + z_2) &= h(2u_1, 2v_1, 2w_1) = 8h(u_1, v_1, w_1) \\ &= \sum_{i,j,k=1}^{2} h(u_1 + (-1)^i u_2, v_1 + (-1)^j v_2, w_1 + (-1)^k w_2) = \sum_{i,j,k=1}^{2} h(x_i, y_j, z_k), \end{aligned}$$

which completes the proof. □

Theorem 2.5. *If a mapping $h : X^3 \to X$ satisfies*

$$h(0, a, b) = h(a, 0, b) = h(a, b, 0) = 0,$$

for all $a, b \in X$ and

$$\left\| 8h(x_1, y_1, z_1) - \sum_{i,j,k=1}^{2} h(x_1 + (-1)^i x_2, y_1 + (-1)^j y_2, z_1 + (-1)^k z_2) \right\| \tag{2.2}$$

$$\leq \left\| 8t \left(\sum_{i,j,k=1}^{2} h\left(\frac{x_1 + (-1)^i x_2}{2}, \frac{y_1 + (-1)^j y_2}{2}, \frac{z_1 + (-1)^k z_2}{2} \right) - h(x_1, y_1, z_1) \right) \right\|,$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$, then the mapping $h : X^3 \longrightarrow X$ is hyper quadratic.

Proof. Letting $x_1 = x_2 := x$, $y_1 = y_2 := y$ and $z_1 = z_2 := z$ in (2.2), we get

$$\|h(2x, 2y, 2z) - 8h(x, y, z)\| \leq 0,$$

for all $x, y, z \in X$. So $h(2x, 2y, 2z) = 8h(x, y, z)$ for all $x, y, z \in X$. It follows from (2.2) that

$$\begin{aligned} &\left\| 8h(x_1, y_1, z_1) - \sum_{i,j,k=1}^{2} h(x_1 + (-1)^i x_2, y_1 + (-1)^j y_2, z_1 + (-1)^k z_2) \right\| \\ &\leq \left\| t \left(8h(x_1, y_1, z_1) - \sum_{i,j,k=1}^{2} h(x_1 + (-1)^i x_2, y_1 + (-1)^j y_2, z_1 + (-1)^k z_2) \right) \right\|, \end{aligned}$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$. Thus

$$8h(x_1, y_1, z_1) = \sum_{i,j,k=1}^{2} h(x_1 + (-1)^i x_2, y_1 + (-1)^j y_2, z_1 + (-1)^k z_2),$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$, since $0 < t < 1$. So the mapping h is hyper quadratic. □

Theorem 2.6. *Let $\phi : X^6 \longrightarrow [0, \infty)$ be a function such that*

$$\sum_{j=1}^{\infty} 8^{3j} \phi\left(\frac{x}{8^j}, \frac{y}{8^j}, \frac{z}{8^j}, \frac{x}{8^j}, \frac{y}{8^j}, \frac{z}{8^j} \right) < \infty, \tag{2.3}$$

for all $x, y, z \in X$. Let $h : X^3 \longrightarrow X$ be a mapping satisfying

$$h(0, a, b) = h(a, 0, b) = h(a, b, 0) = 0,$$

for all $a, b \in X$ and

$$\begin{aligned} &\left\| 8\lambda h(x_1, y_1, z_1) - \sum_{i,j,k=1}^{2} h(\lambda x_1 + (-1)^i \lambda x_2, \lambda y_1 + (-1)^j \lambda y_2, \lambda z_1 + (-1)^k \lambda z_2) \right\| \\ &\leq \left\| 8t \left(\lambda h(x_1, y_1, z_1) - \sum_{i,j,k=1}^{2} h\left(\lambda \frac{x_1 + (-1)^i x_2}{2}, \lambda \frac{y_1 + (-1)^j y_2}{2}, \lambda \frac{z_1 + (-1)^k z_2}{2} \right) \right) \right\| \\ &+ \phi(x_1, y_1, z_1, x_2, y_2, z_2), \end{aligned} \tag{2.4}$$

for all $\lambda \in \mathbb{T}^1 := \{\xi \in \mathbb{C} : |\xi| = 1\}$ and all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$. If the mapping $h : X^3 \longrightarrow X$ satisfies

$$\|h(x_1 x_2 x_3, y_1 y_2 y_3, z_1 z_2 z_3) - h(x_1, x_2, x_3) h(y_1, y_2, y_3) h(z_1, z_2, z_3)\| \tag{2.5}$$

$$\leq \phi(x_1x_2x_3, y_1y_2y_3, z_1z_2z_3, x_1x_2x_3, y_1y_2y_3, z_1z_2z_3),$$

for all $x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3 \in X$*, then there exists a unique hyper homomorphism* $H : X \times X \times X \longrightarrow X$ *such that*

$$\|h(x, y, z) - H(x, y, z)\| \leq \sum_{j=0}^{\infty} 8^j \phi\left(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, \frac{z}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, \frac{z}{2^{j+1}}\right). \tag{2.6}$$

for all $x, y, z \in X$.

Proof. Letting $\lambda = 1$, $x_1 = x_2 := x$, $y_1 = y_2 := y$ and $z_1 = z_2 := z$ in (2.4), we get

$$\|h(2x, 2y, 2z) - 8h(x, y, z)\| \leq \phi(x, y, z, x, y, z),$$

and so

$$\left\|h(x, y, z) - 8h\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)\right\| \leq \phi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right),$$

for all $x, y, z \in X$. Hence

$$\begin{aligned}
&\left\|2^l h\left(\frac{x}{2^l}, \frac{y}{2^l}, \frac{z}{2^l}\right) - 2^{l+3m} h\left(\frac{x}{2^{l+m}}, \frac{y}{2^{l+m}}, \frac{z}{2^{l+m}}\right)\right\| \qquad (2.7)\\
&\leq \sum_{j=0}^{m-1} \left\|2^{l+3j} h\left(\frac{x}{2^{l+j}}, \frac{y}{2^{l+j}}, \frac{z}{2^{l+j}}\right) - 2^{l+3(j+1)} h\left(\frac{x}{2^{l+j+1}}, \frac{y}{2^{l+j+1}}, \frac{z}{2^{l+j+1}}\right)\right\| \\
&= \sum_{j=0}^{m-1} 2^{l+3j} \left\|h\left(\frac{x}{2^{l+j}}, \frac{y}{2^{l+j}}, \frac{z}{2^{l+j}}\right) - 8h\left(\frac{x}{2^{l+j+1}}, \frac{y}{2^{l+j+1}}, \frac{z}{2^{l+j+1}}\right)\right\| \\
&\leq \sum_{j=0}^{m-1} 2^{l+3j} \phi\left(\frac{x}{2^{l+j+1}}, \frac{y}{2^{l+j+1}}, \frac{z}{2^{l+j+1}}, \frac{x}{2^{l+j+1}}, \frac{y}{2^{l+j+1}}, \frac{z}{2^{l+j+1}}\right),
\end{aligned}$$

for all positive integers l, m and $x, y, z \in X$. It follows from (2.7) that the sequence $\{2^{l+3k} h(\frac{x}{2^{l+k}}, \frac{y}{2^{l+k}}, \frac{z}{2^{l+k}})\}$ is Cauchy for all natural number l and for all $(x, y, z) \in X^3$. Since X is a Banach space, the sequence $\{2^{l+3k} h(\frac{x}{2^{l+k}}, \frac{y}{2^{l+k}}, \frac{z}{2^{l+k}})\}$ converges. So one can define the mapping $H : X^3 \to X$ by

$$H(x, y, z) := \lim_{n\to\infty} 2^{3k} h\left(\frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k}\right),$$

for all $(x, y, z) \in X^3$. Moreover, letting $l = 0$ and passing to the limit $m \to \infty$ in (2.7), we get (2.6). It folllows from (2.4) that

$$\begin{aligned}
&\left\|8\lambda H(x_1, y_1, z_1) - \sum_{i,j,k=1}^{2} H(\lambda x_1 + (-1)^i \lambda x_2, \lambda y_1 + (-1)^j \lambda y_2, \lambda z_1 + (-1)^k \lambda z_2)\right\| \\
&= \lim_{n\to\infty} 8^n \left\|8\lambda h\left(\frac{x_1}{2^n}, \frac{y_1}{2^n}, \frac{z_1}{2^n}\right) - \sum_{i,j,k=1}^{2} h\left(\frac{\lambda x_1 + (-1)^i \lambda x_2}{2^n}, \frac{\lambda y_1 + (-1)^j \lambda y_2}{2^n}, \frac{\lambda z_1 + (-1)^k \lambda z_2}{2^n}\right)\right\|
\end{aligned}$$

$$\leq \lim_{n\to\infty} 8^n \left\| 8t \left(\lambda h\left(\frac{x_1}{2^n}, \frac{y_1}{2^n}, \frac{z_1}{2^n}\right) - \sum_{i,j,k=1}^{2} h\left(\frac{\lambda x_1 + (-1)^i \lambda x_2}{2^{n+1}}, \frac{\lambda y_1 + (-1)^j \lambda y_2}{2^{n+1}}, \frac{\lambda z_1 + (-1)^k \lambda z_2}{2^{n+1}}\right)\right)\right\|$$
$$+ \lim_{n\to\infty} 8^n \phi\left(\frac{x_1}{2^n}, \frac{y_1}{2^n}, \frac{z_1}{2^n}, \frac{x_2}{2^n}, \frac{y_2}{2^n}, \frac{z_2}{2^n}\right)$$
$$= \left\| t\left(8\lambda H(x_1, y_1, z_1) - \sum_{i,j,k=1}^{2} H(\lambda x_1 + (-1)^i \lambda x_2, \lambda y_1 + (-1)^j \lambda y_2, \lambda z_1 + (-1)^k \lambda z_2)\right)\right\|,$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$ and $\lambda \in \mathbb{T}^1$. Thus

$$\left\| 8\lambda H(x_1, y_1, z_1) - \sum_{i,j,k=1}^{2} H(\lambda x_1 + (-1)^i \lambda x_2, \lambda y_1 + (-1)^j \lambda y_2, \lambda z_1 + (-1)^k \lambda z_2)\right\| \tag{2.8}$$
$$\leq \left\| t\left(8\lambda H(x_1, y_1, z_1) - \sum_{i,j,k=1}^{2} H(\lambda x_1 + (-1)^i \lambda x_2, \lambda y_1 + (-1)^j \lambda y_2, \lambda z_1 + (-1)^k \lambda z_2)\right)\right\|,$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$ and $\lambda \in \mathbb{T}^1$. Let $\lambda = 1$ in (2.8). By Theorem 2.5, the mapping $H : X^3 \to X$ is 3-additive. Since $0 < t < 1$,

$$8\lambda H(x_1, y_1, z_1) = \sum_{i,j,k=1}^{2} H(\lambda x_1 + (-1)^i \lambda x_2, \lambda y_1 + (-1)^j \lambda y_2, \lambda z_1 + (-1)^k \lambda z_2),$$

and $H(\lambda(x_1, y_1, z_1)) = \lambda H(x_1, y_1, z_1)$ for all $(x_1, y_1, z_1) \in X^3$ and $\lambda \in \mathbb{T}^1$. Since H is 3-additive, $H : X^3 \to X$ is 3-linear (see [16]). It follows from (2.5) and the 3-additivity of H that

$$\|H(x_1x_2x_3, y_1y_2y_3, z_1z_2z_3) - H(x_1, x_2, x_3)H(y_1, y_2, y_3)H(z_1, z_2, z_3)\|$$
$$= 8^{3k} \left\| h\left(\frac{x_1x_2x_3}{8^k}, \frac{y_1y_2y_3}{8^k}, \frac{z_1z_2z_3}{8^k}\right) - h\left(\frac{x_1}{2^k}, \frac{x_2}{2^k}, \frac{x_3}{2^k}\right) h\left(\frac{y_1}{2^k}, \frac{y_2}{2^k}, \frac{y_3}{2^k}\right) h\left(\frac{z_1}{2^k}, \frac{z_2}{2^k}, \frac{z_3}{2^k}\right)\right\|$$
$$\leq 8^{3k} \phi\left(\frac{x_1x_2x_3}{8^k}, \frac{y_1y_2y_3}{8^k}, \frac{z_1z_2z_3}{8^k}, \frac{x_1x_2x_3}{8^k}, \frac{y_1y_2y_3}{8^k}, \frac{z_1z_2z_3}{8^k}\right),$$

which tends to zero as $k \to \infty$, by (2.3). So

$$H(x_1x_2x_3, y_1y_2y_3, z_1z_2z_3) = H(x_1, x_2, x_3)H(y_1, y_2, y_3)H(z_1, z_2, z_3),$$

for all $x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3 \in X$. This completes the proof. □

3. Stability of hyper derivations in complex Banach algebras

In this section, we introduce the concept of hyper derivation on Banach algebras and we establish the stability of hyper derivation on Banach algebras by using the direct method.

Definition 3.1. Let X be a complex Banach algebra. A 3-linear mapping $g : X \times X \times X \to X$ is called a hyper derivation if g satisfies

$$g(x_1x_2, y_1y_2, z_1z_2) = x_1y_1z_1g(x_2, y_2, z_2) + g(x_1, y_1, z_1)x_2y_2z_2,$$

for all $x_1, y_1, z_1, x_2, y_2, z_2 \in X$.

Lemma 3.2. *If a mapping* $g : X^3 \longrightarrow X$ *satisfies*

$$\left\| g(x_1 + x_2, y_1 + y_2, z_1 + z_2) - \sum_{i,j,k=1}^{2} g(x_i, y_j, z_k) \right\| \le \left\| s\left(8g\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right) - \sum_{i,j,k=1}^{2} g(x_i, y_j, z_k)\right) \right\|, \tag{3.1}$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$*, then the mapping* $g : X^3 \longrightarrow X$ *is 3-additive.*

Proof. Letting $x_1 = x_2 := x$, $y_1 = y_2 := y$ and $z_1 = z_2 := z$ in (3.1), we get

$$\|g(2x, 2y, 2z) - 8g(x, y, z)\| \le 0,$$

for all $x, y, z \in X$. So $g(2x, 2y, 2z) = 8g(x, y, z)$ for all $x, y, z \in X$. It follows from (3.1) that

$$\left\| g(x_1 + x_2, y_1 + y_2, z_1 + z_2) - \sum_{i,j,k=1}^{2} g(x_i, y_j, z_k) \right\| \le \left\| s\left(g(x_1 + x_2, y_1 + y_2, z_1 + z_2) - \sum_{i,j,k=1}^{2} g(x_i, y_j, z_k)\right) \right\|,$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$. Thus

$$g(x_1 + x_2, y_1 + y_2, z_1 + z_2) = \sum_{i,j,k=1}^{2} g(x_i, y_j, z_k),$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$, since $0 < s < 1$. So the mapping $g : X^3 \to X$ is 3-additive. □

Theorem 3.3. *Let* $\phi : X^6 \to [0, \infty)$ *be a function such that*

$$\sum_{j=1}^{\infty} 64^j \phi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}, \frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty, \tag{3.2}$$

for all $x, y, z \in X$*. Let* $g : X^3 \to X$ *be a mapping satisfying*

$$\left\| g(\lambda(x_1 + x_2, y_1 + y_2, z_1 + z_2)) - \lambda \sum_{i,j,k=1}^{2} g(x_i, y_j, z_k) \right\| \le \left\| s\left(8g\left(\lambda\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)\right) - \lambda \sum_{i,j,k=1}^{2} g(x_i, y_j, z_k)\right) \right\| + \phi(x_1, y_1, z_1, x_2, y_2, z_2), \tag{3.3}$$

for all $\lambda \in \mathbb{T}^1$ *and all* $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$*. Here* $g(\lambda(x, y, z)) := g(\lambda x, \lambda y, \lambda z)$*. If the mapping* $g : X^3 \to X$ *satisfies*

$$\|g(x_1x_2, y_1y_2, z_1z_2) - x_1y_1z_1 g(x_2, y_2, z_2) - g(x_1, y_1, z_1)x_2y_2z_2\| \le \phi(x_1, x_2, y_1, y_2, z_1, z_2), \tag{3.4}$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$*, then there exists a unique hyper derivation* $D : X \times X \times X \longrightarrow X$ *such that*

$$\|g(x,y,z) - D(x,y,z)\| \leq \sum_{j=0}^{\infty} 8^j \phi\left(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, \frac{z}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, \frac{z}{2^{j+1}}\right), \tag{3.5}$$

for all $x, y, z \in X$.

Proof. Letting $\lambda = 1$, $x_1 = x_2 := x$, $y_1 = y_2 := y$ and $z_1 = z_2 := z$ in (3.3), we get

$$\|g(2x, 2y, 2z) - 8g(x,y,z)\| \leq \phi(x,y,z,x,y,z),$$

and so

$$\left\|g(x,y,z) - 8g\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)\right\| \leq \phi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right),$$

for all $x, y, z \in X$. Hence

$$\begin{aligned}\left\|8^l g\left(\frac{x}{2^l}, \frac{y}{2^l}, \frac{z}{2^l}\right) - 8^m g\left(\frac{x}{2^m}, \frac{y}{2^m}, \frac{z}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1} \left\|8^j g\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) - 8^{j+1} g\left(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, \frac{z}{2^{j+1}}\right)\right\| \\ &\leq \sum_{j=l}^{m-1} 8^j \phi\left(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, \frac{z}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, \frac{z}{2^{j+1}}\right), \end{aligned} \tag{3.6}$$

for all natural numbers $l, m (m > l)$ and $x, y, z \in X$. It follows from (3.6) that the sequence $\{8^k g(\frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k})\}$ is Cauchy for all $(x, y, z) \in X^3$. Since X is a Banach space, the sequence $\{8^k g(\frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k})\}$ converges. So one can define the mapping $D : X^3 \to X$ by

$$D(x,y,z) := \lim_{n\to\infty} 8^k g\left(\frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k}\right),$$

for all $(x, y, z) \in X^3$. Moreover, letting $l = 0$ and passing to the limit $m \to \infty$ in (2.7), we get (3.5).

It folllows from (3.3) that

$$\begin{aligned}&\left\|D(\lambda(x_1 + x_2, y_1 + y_2, z_1 + z_2)) - \lambda \sum_{i,j,k=1}^{2} D(x_i, y_j, z_k)\right\| \\ &= \lim_{n\to\infty} 8^n \left\|g\left(\lambda\left(\frac{x_1 + x_2}{2^n}, \frac{y_1 + y_2}{2^n}, \frac{z_1 + z_2}{2^n}\right)\right) - \lambda \sum_{i,j,k=1}^{2} g\left(\frac{x_i}{2^n}, \frac{y_j}{2^n}, \frac{z_k}{2^n}\right)\right\| \\ &\leq \lim_{n\to\infty} 8^n \left\|s\left(8g\left(\lambda\left(\frac{x_1 + x_2}{2^{n+1}}, \frac{y_1 + y_2}{2^{n+1}}, \frac{z_1 + z_2}{2^{n+1}}\right)\right) - \lambda \sum_{i,j,k=1}^{2} g\left(\frac{x_i}{2^n}, \frac{y_j}{2^n}, \frac{z_k}{2^n}\right)\right)\right\| \\ &+ \lim_{n\to\infty} 8^n \phi\left(\frac{x_1}{2^n}, \frac{y_1}{2^n}, \frac{z_1}{2^n}, \frac{x_2}{2^n}, \frac{y_2}{2^n}, \frac{z_2}{2^n}\right) \\ &= \left\|s\left(8D\left(\lambda\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)\right) - \lambda \sum_{i,j,k=1}^{2} D(x_i, y_j, z_k)\right)\right\|,\end{aligned}$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$ and $\lambda \in \mathbb{T}^1$. Thus

$$\begin{aligned} &\left\| D(\lambda(x_1 + x_2, y_1 + y_2, z_1 + z_2)) - \lambda \sum_{i,j,k=1}^{2} D(x_i, y_j, z_k) \right\| \\ &\leq \left\| s\left(8D\left(\lambda\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)\right) - \lambda \sum_{i,j,k=1}^{2} D(x_i, y_j, z_k) \right) \right\|, \end{aligned} \tag{3.7}$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$ and $\lambda \in \mathbb{T}^1$. Let $\lambda = 1$ in (3.7). By Theorem 3.2, the mapping $D : X^3 \to X$ is 3-additive. It follows from (3.7) and the 3-additivity of D that

$$\begin{aligned} &\left\| D(\lambda(x_1 + x_2, y_1 + y_2, z_1 + z_2)) - \lambda \sum_{i,j,k=1}^{2} D(x_i, y_j, z_k) \right\| \\ &\leq \left\| s\left(D(\lambda(x_1 + x_2, y_1 + y_2, z_1 + z_2)) - \lambda \sum_{i,j,k=1}^{2} D(x_i, y_j, z_k) \right) \right\|, \end{aligned}$$

for all $\lambda \in \mathbb{T}^1$, (x_1, y_1, z_1) and $(x_2, y_2, z_2) \in X^3$. Since $0 < s < 1$,

$$D(\lambda(x_1 + x_2, y_1 + y_2, z_1 + z_2)) - \lambda \sum_{i,j,k=1}^{2} D(x_i, y_j, z_k) = 0,$$

and $D(\lambda(x_1, y_1, z_1)) = \lambda D(x_1, y_1, z_1)$ for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$ and $\lambda \in \mathbb{T}^1$. Since D is 3-additive, $D : X^3 \to X$ is 3-linear (see [16]). It follows from (3.4) and the 3-additivity of D that

$$\begin{aligned} &\|D(x_1x_2, y_1y_2, z_1z_2) - x_1y_1z_1D(x_2, y_2, z_2) - D(x_1, y_1, z_1)x_2y_2z_2\| \\ &= 64^k \left\| g\left(\frac{x_1x_2}{4^k}, \frac{y_1y_2}{4^k}, \frac{z_1z_2}{4^k}\right) - \frac{x_1}{2^k}\frac{y_1}{2^k}\frac{z_1}{2^k} g\left(\frac{x_2}{2^k}, \frac{y_2}{2^k}, \frac{z_2}{2^k}\right) - g\left(\frac{x_1}{2^k}, \frac{y_1}{2^k}, \frac{z_1}{2^k}\right)\frac{x_2}{2^k}\frac{y_2}{2^k}\frac{z_2}{2^k} \right\| \\ &\leq 64^k \phi\left(\frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k}, \frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k}\right), \end{aligned}$$

which tends to zero as $n \to \infty$, by (3.2). So

$$D(x_1x_2, y_1y_2, z_1z_2) = x_1y_1z_1D(x_2, y_2, z_2) + D(x_1, y_1, z_1)x_2y_2z_2,$$

for all $x_1, y_1, z_1, x_2, y_2, z_2 \in X$. This completes the proof. □

4. Conclusions

We have introduced the concept of hyper homomorphisms and hyper derivations in Banach algebras and we have established the Hyers-Ulam stability of hyper homomorphisms and hyper derivations in Banach algebras for the 3-additive functional Eq (1.3).

Conflict of interest

The authors declare that they have no competing interests.

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