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Research article

Conditionally oscillatory linear differential equations with coefficients containing powers of natural logarithm

Petr Hasil and Michal Veselý*

Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Kotlářská 2, CZ 611 37 Brno, Czech Republic

* **Correspondence:** Email: michal.vesely@mail.muni.cz; Tel: +420549493105;
Fax: +420549491421.

Abstract: In this paper, we study linear differential equations whose coefficients consist of products of powers of natural logarithm and very general continuous functions. Recently, using the Riccati transformation, we have identified a new type of conditionally oscillatory linear differential equations together with the critical oscillation constant. The studied equations are a generalization of these equations. Applying the modified Prüfer angle, we prove that they remain conditionally oscillatory with the same critical oscillation constant.

Keywords: linear equation; differential equation; conditional oscillation; non-oscillation; logarithm
Mathematics Subject Classification: 34C10

1. Introduction

We analyze the oscillation of linear differential equations in the form

$$[r(t)x'(t)]' + s(t)x(t) = 0, \tag{1.1}$$

where the coefficients $r > 0, s$ are continuous on an interval $[\tau, \infty)$. We point out that Eq (1.1) is called *oscillatory* if all solutions are oscillatory (i.e., any solution has infinitely many zero points in any neighbourhood of ∞). In the other case (any non-trivial solution has the biggest zero point), we say that Eq (1.1) is *non-oscillatory*. Concerning the basics of the oscillation theory of Eq (1.1), we refer, e.g., to [3, 51] with references cited therein. We focus on the study of conditionally oscillatory equations which are treated as “ideal testing equations” (it is described, e.g., in [21]).

To recall the notion of the conditional oscillation, we consider Eq (1.1) in the following modified form

$$[r(t)x'(t)]' + \gamma s(t)x(t) = 0, \tag{1.2}$$

where $\gamma \in \mathbb{R}$ is a parameter. We say that Eq (1.2) is *conditionally oscillatory* if there exists the so-called *critical oscillation constant* $\Gamma > 0$ such that Eq (1.2) is oscillatory for $\gamma > \Gamma$ and non-oscillatory for $\gamma < \Gamma$. Note that the case $\gamma = \Gamma$ is not covered by the definition above. Some simple equations are non-oscillatory in the borderline case (see, e.g., [27, 35, 37, 48]). Nevertheless, many equations can be oscillatory in the borderline case (see, e.g., [11, 13, 14, 33] and also [28, 50] in the discrete case).

In this paragraph, we mention examples of conditionally oscillatory equations together with the corresponding critical oscillation constants Γ. We add that log denotes the natural logarithm and $p > 0$ stands for an arbitrarily given number in the whole paper. In [47] (see also [46, 49]), there is shown that the equation

$$x''(t) + \gamma \frac{\sin t}{t} x(t) = 0$$

is conditionally oscillatory for $\Gamma = 1/\sqrt{2}$ and the equation

$$x''(t) + \gamma \sin\left(t^2\right) x(t) = 0$$

is conditionally oscillatory for $\Gamma = \sqrt{2}$. In [16, 38], it is proved that the equation

$$[r(t)x'(t)]' + \gamma \frac{1}{t^2} s(t)x(t) = 0,$$

where the coefficients r, s are α-periodic (for some $\alpha > 0$) and positive, is conditionally oscillatory for

$$\Gamma = \frac{1}{4}\left(\frac{1}{\alpha}\int_0^\alpha \frac{\mathrm{d}\tau}{r(\tau)}\right)^{-1}\left(\frac{1}{\alpha}\int_0^\alpha s(\tau)\,\mathrm{d}\tau\right)^{-1}. \tag{1.3}$$

Analogously (see [24]), for α-periodic and positive functions r, s, the equation

$$[tr(t)x'(t)]' + \gamma \frac{1}{t\log^2 t} s(t)x(t) = 0$$

is conditionally oscillatory for the critical oscillation constant in (1.3) and, from [8] (see also [27, Corollary 4.2]), it follows that the equation

$$[t^q r(t)x'(t)]' + \gamma \frac{t^q}{t^2} s(t)x(t) = 0, \qquad q \in \mathbb{R} \setminus \{1\},$$

is conditionally oscillatory for

$$\Gamma = \frac{(q-1)^2}{4}\left(\frac{1}{\alpha}\int_0^\alpha \frac{\mathrm{d}\tau}{r(\tau)}\right)^{-1}\left(\frac{1}{\alpha}\int_0^\alpha s(\tau)\,\mathrm{d}\tau\right)^{-1}.$$

For other relevant results about the considered conditional oscillation, we refer at least to [15, 31, 34, 39] (and also [19, 40]). However, the main motivation for our current research comes from [21], where the following theorem is proved.

Theorem 1.1. *Let us consider the equation*

$$\left[\frac{\log^p t}{r(t)}\, x'(t)\right]' + \frac{\log^p t}{t^2}\, s(t)x(t) = 0, \tag{1.4}$$

where $r, s : \mathbb{R} \to (0, \infty)$ *are continuous and periodic functions with period* $\alpha > 0$.

(A) If

$$4\left(\frac{1}{\alpha}\int_0^\alpha r(\tau)\,\mathrm{d}\tau\right)\left(\frac{1}{\alpha}\int_0^\alpha s(\tau)\,\mathrm{d}\tau\right) > 1,$$

then Eq (1.4) *is oscillatory.*

(B) If

$$4\left(\frac{1}{\alpha}\int_0^\alpha r(\tau)\,\mathrm{d}\tau\right)\left(\frac{1}{\alpha}\int_0^\alpha s(\tau)\,\mathrm{d}\tau\right) < 1,$$

then Eq (1.4) *is non-oscillatory.*

The aim of this paper is to extend Theorem 1.1, i.e., to identify a more general class of conditionally oscillatory linear equations. In fact, we consider equations in the form of Eq (1.4) for non-periodic functions r, s, for periodic r, s having different periods and for periodic s changing its sign. All these cases are covered. To prove such a result, we use the modified Prüfer angle in this paper. Note that Theorem 1.1 is proved using the Riccati transformation, i.e., the methods are dissimilar.

To complete the literature overview, we add that conditionally oscillatory equations are studied in the field of difference equations and dynamic equations on time scales (see, e.g., [4, 12, 20, 45] and also [1, 2, 18, 25, 52, 53] for possible research directions). Concerning more general half-linear conditionally oscillatory equations, we refer to [9,17,23,29] in the continuous case and to [22,26,36,44] in the discrete case; concerning non-linear equations, we point out at least [5,32,41,42] (see also [6,43] for possible research directions).

This paper is organized as follows. In the next section, we present the main tool of our paper, i.e., the used version of the Prüfer angle. In Section 3, we collect all auxiliary results. The main result is proved in Section 4, where we also mention an example of conditionally oscillatory equations. This example demonstrates how our result substantially generalizes Theorem 1.1. In the last section, we present new corollaries together with simple illustrative examples of linear equations whose oscillation properties can be deduced from the corollaries and do not follow from any previously known results.

Since we study the oscillation and non-oscillation of differential equations, it suffices to consider all equations only for t large enough. For simplicity, we consider $t \geq \mathrm{e}$, where e is the base of the natural logarithm. We also put $\mathbb{R}_\mathrm{e} := [\mathrm{e}, \infty)$.

2. Modified Prüfer angle

We consider linear second order differential equations of the form

$$[R(t)x'(t)]' + S(t)x(t) = 0, \tag{2.1}$$

where $R, S : \mathbb{R}_e \to \mathbb{R}$ are continuous functions and R is positive. Let x be a non-trivial solution of Eq (2.1). For $x(t) \neq 0$, the well-known Riccati transformation

$$w(t) = R(t)\frac{x'(t)}{x(t)} \tag{2.2}$$

gives

$$w'(t) + S(t) + R^{-1}(t)w^2(t) = 0. \tag{2.3}$$

For details, we can refer, e.g., to [30].

Applying the substitution

$$v(t) = \frac{t}{\log^p t}w(t), \tag{2.4}$$

from Eq (2.3), we have

$$\begin{aligned} v'(t) &= \left(\frac{t}{\log^p t}\right)' w(t) + \frac{t}{\log^p t}w'(t) \\ &= \frac{\log t - p}{\log^{p+1} t}w(t) - \frac{t}{\log^p t}\left[S(t) + R^{-1}(t)w^2(t)\right] \\ &= \frac{\log t - p}{t\log t}v(t) - \frac{t}{\log^p t}S(t) - R^{-1}(t)\frac{\log^p t}{t}v^2(t). \end{aligned} \tag{2.5}$$

Then, for a non-trivial solution x of Eq (2.1), we consider the modified Prüfer transformation in the form

$$x(t) = \rho(t)\sin\varphi(t), \qquad x'(t) = \rho(t)R^{-1}(t)\frac{\log^p t}{t}\cos\varphi(t). \tag{2.6}$$

We have (see (2.2), (2.4) and (2.6))

$$\begin{aligned} v(t) &= \frac{t}{\log^p t}w(t) = \frac{t}{\log^p t}R(t)\frac{x'(t)}{x(t)} \\ &= \frac{t}{\log^p t}R(t)\frac{\rho(t)R^{-1}(t)\frac{\log^p t}{t}\cos\varphi(t)}{\rho(t)\sin\varphi(t)} = \frac{\cos\varphi(t)}{\sin\varphi(t)} = \cot\varphi(t), \end{aligned} \tag{2.7}$$

i.e.,

$$v'(t) = \frac{-1}{\sin^2\varphi(t)}\varphi'(t). \tag{2.8}$$

Thus (see (2.5) and (2.8)), we obtain

$$\frac{-1}{\sin^2\varphi(t)}\varphi'(t) = \frac{\log t - p}{t\log t}v(t) - \frac{t}{\log^p t}S(t) - R^{-1}(t)\frac{\log^p t}{t}v^2(t) \tag{2.9}$$

and (see (2.7) and (2.9))

$$\varphi'(t) = -\frac{\log t - p}{t\log t}\cos\varphi(t)\sin\varphi(t) + \frac{t}{\log^p t}S(t)\sin^2\varphi(t) + R^{-1}(t)\frac{\log^p t}{t}\cos^2\varphi(t),$$

i.e.,

$$\varphi'(t) = \frac{\log^p t}{tR(t)}\cos^2\varphi(t) - \frac{\log t - p}{t\log t}\cos\varphi(t)\sin\varphi(t) + \frac{tS(t)}{\log^p t}\sin^2\varphi(t). \tag{2.10}$$

Let the functions R and S have the forms

$$R(t) = \frac{\log^p t}{r(t)}, \qquad S(t) = \frac{\log^p t}{t^2} s(t), \qquad t \in \mathbb{R}_e, \tag{2.11}$$

i.e., let us consider the equation

$$\left[\frac{\log^p t}{r(t)} x'(t)\right]' + \frac{\log^p t}{t^2} s(t)x(t) = 0, \tag{2.12}$$

where continuous functions $r : \mathbb{R}_e \to (0, \infty)$ and $s : \mathbb{R}_e \to \mathbb{R}$ satisfy

$$\lim_{t\to\infty} \frac{\int_t^{t+1} r(\tau)\,\mathrm{d}\tau}{\sqrt{t}} = \lim_{t\to\infty} \frac{\int_t^{t+1} |s(\tau)|\,\mathrm{d}\tau}{\sqrt{t}} = 0. \tag{2.13}$$

Note that (2.13) implies

$$\lim_{t\to\infty} \frac{\int_t^{t+\alpha} r(\tau)\,\mathrm{d}\tau}{\sqrt{t}} = \lim_{t\to\infty} \frac{\int_t^{t+\alpha} |s(\tau)|\,\mathrm{d}\tau}{\sqrt{t}} = 0 \tag{2.14}$$

for any $\alpha > 0$.

Finally, for the coefficients in (2.11), Eq (2.10) has the form

$$\varphi'(t) = \frac{1}{t}\left[r(t)\cos^2\varphi(t) - \frac{\log t - p}{\log t}\cos\varphi(t)\sin\varphi(t) + s(t)\sin^2\varphi(t)\right], \tag{2.15}$$

which is the considered form of the equation for the modified Prüfer angle, i.e., we use Eq (2.15) to study Eq (2.12). In addition, for given $\alpha > 0$ and a solution φ of Eq (2.15) on $\mathbb{R}_e$, we use the auxiliary average function $\psi_\alpha : \mathbb{R}_e \to \mathbb{R}$ defined by the formula

$$\psi_\alpha(t) := \frac{1}{\alpha}\int_t^{t+\alpha} \varphi(\tau)\,\mathrm{d}\tau, \qquad t \in \mathbb{R}_e. \tag{2.16}$$

3. Auxiliary results

In this section, we collect all used lemmas. At first, we mention a known result (in the form which plays a crucial role in the proof of our main result).

Lemma 3.1. *Let us consider Eq* (2.12) *together with Eq* (2.15). *Let* $\varphi : \mathbb{R}_e \to \mathbb{R}$ *be a solution of Eq* (2.15).

(A) If $\lim_{t\to\infty} \varphi(t) = \infty$*, then Eq* (2.12) *is oscillatory.*
(B) If $\limsup_{t\to\infty} \varphi(t) < \infty$*, then Eq* (2.12) *is non-oscillatory.*

Proof. The non-oscillation of Eq (2.12) is equivalent to the boundedness from above of the Prüfer angle φ given by Eq (2.15). See, e.g., [37] or directly consider the transformation in (2.6) and Eq (2.15) for $\sin\varphi(t) = 0$. In addition (see the form of Eq (2.15)), the set of all values of φ is unbounded if and only if $\lim_{t\to\infty}\varphi(t) = \infty$. □

One can easily reformulate Lemma 3.1 as follows.

Lemma 3.2. *Let us consider Eq* (2.12) *together with Eq* (2.15). *Let* $\varphi : \mathbb{R}_e \to \mathbb{R}$ *be a solution of Eq* (2.15).

(A) If Eq (2.12) *is oscillatory, then* $\lim_{t\to\infty} \varphi(t) = \infty$.
(B) If Eq (2.12) *is non-oscillatory, then* $\limsup_{t\to\infty} \varphi(t) < \infty$.

Now we mention a simple corollary of Theorem 1.1.

Lemma 3.3. *Let* $C, D > 0$. *Let us consider the equation*

$$\left[\frac{\log^p t}{C}\, x'(t)\right]' + \frac{\log^p t}{t^2}\, Dx(t) = 0. \tag{3.1}$$

(A) If $4CD > 1$, *then Eq* (3.1) *is oscillatory.*
(B) If $4CD < 1$, *then Eq* (3.1) *is non-oscillatory.*

Proof. It suffices to consider Theorem 1.1 for constant functions r, s. □

The two previous lemmas give the following result.

Lemma 3.4. *Let* $C, D > 0$. *Let* $\eta : \mathbb{R}_e \to \mathbb{R}$ *be a solution of the equation*

$$\eta'(t) = \frac{1}{t}\left[C\cos^2\eta(t) - \frac{\log t - p}{\log t}\,\cos\eta(t)\sin\eta(t) + D\sin^2\eta(t)\right]. \tag{3.2}$$

(A) If $4CD > 1$, *then* $\lim_{t\to\infty} \eta(t) = \infty$.
(B) If $4CD < 1$, *then* $\limsup_{t\to\infty} \eta(t) < \infty$.

Proof. The statement of the lemma follows immediately from Lemmas 3.2 and 3.3, because Eq (3.2) has the form of the equation for the modified Prüfer angle η which corresponds to Eq (3.1). □

Next, we deal with the auxiliary average function ψ_α.

Lemma 3.5. *Let* $\varphi : \mathbb{R}_e \to \mathbb{R}$ *be a solution of Eq* (2.15) *and* ψ_α *be defined in* (2.16). *The limit*

$$\lim_{t\to\infty} \sqrt{t}\,|\varphi(\tau) - \psi_\alpha(t)| = 0 \tag{3.3}$$

exists uniformly with respect to $\tau \in [t, t+\alpha]$.

Proof. The continuity of φ gives

$$|\varphi(\tau) - \psi_\alpha(t)| \le \max_{\tau_1,\tau_2\in[0,\alpha]} |\varphi(t+\tau_1) - \varphi(t+\tau_2)|, \qquad t \in \mathbb{R}_e,\ \tau \in [t, t+\alpha].$$

Hence, we obtain (see (2.14) and Eq (2.15))

$$\begin{aligned}\limsup_{t\to\infty} \sqrt{t}\,|\varphi(\tau) - \psi_\alpha(t)| &\le \limsup_{t\to\infty} \sqrt{t} \max_{\tau_1,\tau_2\in[0,\alpha]} |\varphi(t+\tau_1) - \varphi(t+\tau_2)| \\ &\le \limsup_{t\to\infty} \sqrt{t} \int_t^{t+\alpha} |\varphi'(\tau)|\,\mathrm{d}\tau\end{aligned}$$

$$
\begin{aligned}
&= \limsup_{t\to\infty} \sqrt{t} \int_t^{t+\alpha} \left| \frac{1}{\tau}\Big[r(\tau)\cos^2\varphi(\tau) \right.\\
&\qquad \left. - \frac{\log\tau - p}{\log\tau} \cos\varphi(\tau)\sin\varphi(\tau) + s(\tau)\sin^2\varphi(\tau)\Big] \right| \mathrm{d}\tau \\
&\le \limsup_{t\to\infty} \frac{\sqrt{t}}{t} \int_t^{t+\alpha} \left[r(\tau) + 1 + \frac{p}{\log t} + |s(\tau)| \right] \mathrm{d}\tau \\
&= \limsup_{t\to\infty} \left(\frac{1}{\sqrt{t}} \int_t^{t+\alpha} r(\tau)\,\mathrm{d}\tau + \frac{1}{\sqrt{t}} \int_t^{t+\alpha} |s(\tau)|\,\mathrm{d}\tau \right) = 0
\end{aligned}
$$

uniformly with respect to $\tau \in [t, t+\alpha]$. □

Lemma 3.6. *Let $\varphi : \mathbb{R}_e \to \mathbb{R}$ be a solution of Eq* (2.15) *and ψ_α be defined in* (2.16). *Then, there exists a continuous function $F : (\mathrm{e}, \infty) \to \mathbb{R}$ satisfying*

$$
\lim_{t\to\infty} F(t) = 0 \tag{3.4}
$$

and

$$
\begin{aligned}
\psi_\alpha'(t) = \frac{1}{t} \Bigg[\left(\frac{1}{\alpha} \int_t^{t+\alpha} r(\tau)\,\mathrm{d}\tau \right) \cos^2\psi_\alpha(t) - \frac{\log t - p}{\log t} \cos\psi_\alpha(t)\sin\psi_\alpha(t) \\
+ \left(\frac{1}{\alpha} \int_t^{t+\alpha} s(\tau)\,\mathrm{d}\tau \right) \sin^2\psi_\alpha(t) + F(t) \Bigg]
\end{aligned} \tag{3.5}
$$

for all $t > \mathrm{e}$.

Proof. In fact, the considered continuous function $F : (\mathrm{e}, \infty) \to \mathbb{R}$ can be introduced as the function for which (3.5) holds, i.e., one can put

$$
\begin{aligned}
F(t) := t\psi_\alpha'(t) - \left(\frac{1}{\alpha} \int_t^{t+\alpha} r(\tau)\,\mathrm{d}\tau \right) \cos^2\psi_\alpha(t) + \frac{\log t - p}{\log t} \cos\psi_\alpha(t)\sin\psi_\alpha(t) \\
- \left(\frac{1}{\alpha} \int_t^{t+\alpha} s(\tau)\,\mathrm{d}\tau \right) \sin^2\psi_\alpha(t)
\end{aligned} \tag{3.6}
$$

for all $t > \mathrm{e}$. The aim is to prove (3.4) for this function F. For all $t > \mathrm{e}$, we have (see Eq (2.15) and (2.16))

$$
\begin{aligned}
\psi_\alpha'(t) &= \frac{1}{\alpha} \left(\varphi(t+\alpha) - \varphi(t) \right) = \frac{1}{\alpha} \int_t^{t+\alpha} \varphi'(\tau)\,\mathrm{d}\tau \\
&= \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{\tau} \left[r(\tau)\cos^2\varphi(\tau) - \frac{\log\tau - p}{\log\tau} \cos\varphi(\tau)\sin\varphi(\tau) + s(\tau)\sin^2\varphi(\tau) \right] \mathrm{d}\tau
\end{aligned}
$$

and

$$\left| t\psi_\alpha'(t) - \frac{1}{\alpha}\int_t^{t+\alpha}\left[r(\tau)\cos^2\varphi(\tau) - \frac{\log\tau - p}{\log\tau}\cos\varphi(\tau)\sin\varphi(\tau) + s(\tau)\sin^2\varphi(\tau)\right]\mathrm{d}\tau\right|$$

$$= \frac{t}{\alpha}\left|\int_t^{t+\alpha}\frac{1}{\tau}\left[r(\tau)\cos^2\varphi(\tau) - \frac{\log\tau - p}{\log\tau}\cos\varphi(\tau)\sin\varphi(\tau) + s(\tau)\sin^2\varphi(\tau)\right]\mathrm{d}\tau\right.$$

$$\left. - \int_t^{t+\alpha}\frac{1}{t}\left[r(\tau)\cos^2\varphi(\tau) - \frac{\log\tau - p}{\log\tau}\cos\varphi(\tau)\sin\varphi(\tau) + s(\tau)\sin^2\varphi(\tau)\right]\mathrm{d}\tau\right|$$

$$\leq \frac{t}{\alpha}\int_t^{t+\alpha}\left(\frac{1}{t} - \frac{1}{\tau}\right)\left[r(\tau) + 1 + \frac{p}{\log t} + |s(\tau)|\right]\mathrm{d}\tau \leq \frac{1}{t}\int_t^{t+\alpha}\left[r(\tau) + 1 + p + |s(\tau)|\right]\mathrm{d}\tau$$

$$= \frac{1}{\sqrt{t}}\left[\frac{1}{\sqrt{t}}\int_t^{t+\alpha} r(\tau)\,\mathrm{d}\tau + \frac{\alpha(1+p)}{\sqrt{t}} + \frac{1}{\sqrt{t}}\int_t^{t+\alpha}|s(\tau)|\,\mathrm{d}\tau\right],$$

which implies (see (2.14))

$$\lim_{t\to\infty}\left| t\psi_\alpha'(t) - \frac{1}{\alpha}\int_t^{t+\alpha}\left[r(\tau)\cos^2\varphi(\tau) - \frac{\log\tau - p}{\log\tau}\cos\varphi(\tau)\sin\varphi(\tau) + s(\tau)\sin^2\varphi(\tau)\right]\mathrm{d}\tau\right| = 0.$$

Therefore (see directly (3.6)), (3.4) can be proved by

$$\lim_{t\to\infty}\left(\frac{1}{\alpha}\int_t^{t+\alpha} r(\tau)\cos^2\psi_\alpha(t)\,\mathrm{d}\tau - \frac{1}{\alpha}\int_t^{t+\alpha} r(\tau)\cos^2\varphi(\tau)\,\mathrm{d}\tau\right) = 0, \tag{3.7}$$

$$\lim_{t\to\infty}\left(\frac{\log t - p}{\log t}\cos\psi_\alpha(t)\sin\psi_\alpha(t) - \frac{1}{\alpha}\int_t^{t+\alpha}\frac{\log\tau - p}{\log\tau}\cos\varphi(\tau)\sin\varphi(\tau)\,\mathrm{d}\tau\right) = 0, \tag{3.8}$$

$$\lim_{t\to\infty}\left(\frac{1}{\alpha}\int_t^{t+\alpha} s(\tau)\sin^2\psi_\alpha(t)\,\mathrm{d}\tau - \frac{1}{\alpha}\int_t^{t+\alpha} s(\tau)\sin^2\varphi(\tau)\,\mathrm{d}\tau\right) = 0. \tag{3.9}$$

To obtain (3.7) and (3.9), we use Lemma 3.5 and the Lipschitz continuity of $y = \cos^2 x$ and $y = \sin^2 x$ which gives $L > 0$ such that

$$\left|\cos^2 x_1 - \cos^2 x_2\right| \leq L\,|x_1 - x_2|, \qquad x_1, x_2 \in \mathbb{R}, \tag{3.10}$$

and that

$$\left|\sin^2 x_1 - \sin^2 x_2\right| \leq L\,|x_1 - x_2|, \qquad x_1, x_2 \in \mathbb{R}. \tag{3.11}$$

Thus, we have (see (2.14), (3.3) and (3.10))

$$\begin{aligned} &\limsup_{t\to\infty}\left|\frac{1}{\alpha}\int\limits_t^{t+\alpha} r(\tau)\cos^2\psi_\alpha(t)\,\mathrm{d}\tau-\frac{1}{\alpha}\int\limits_t^{t+\alpha} r(\tau)\cos^2\varphi(\tau)\,\mathrm{d}\tau\right| \\ &\le \limsup_{t\to\infty}\frac{1}{\alpha}\int\limits_t^{t+\alpha} r(\tau)\cdot\left|\cos^2\psi_\alpha(t)-\cos^2\varphi(\tau)\right|\mathrm{d}\tau \\ &\le \limsup_{t\to\infty}\frac{1}{\sqrt{t}}\int\limits_t^{t+\alpha} r(\tau)\cdot\frac{L}{\alpha}\sqrt{t}\,|\psi_\alpha(t)-\varphi(\tau)|\,\mathrm{d}\tau\le\limsup_{t\to\infty}\frac{1}{\sqrt{t}}\int\limits_t^{t+\alpha} r(\tau)\,\mathrm{d}\tau=0 \end{aligned} \tag{3.12}$$

and (see (2.14), (3.3) and (3.11))

$$\begin{aligned} &\limsup_{t\to\infty}\left|\frac{1}{\alpha}\int\limits_t^{t+\alpha} s(\tau)\sin^2\psi_\alpha(t)\,\mathrm{d}\tau-\frac{1}{\alpha}\int\limits_t^{t+\alpha} s(\tau)\sin^2\varphi(\tau)\,\mathrm{d}\tau\right| \\ &\le \limsup_{t\to\infty}\frac{1}{\alpha}\int\limits_t^{t+\alpha} |s(\tau)|\cdot\left|\sin^2\psi_\alpha(t)-\sin^2\varphi(\tau)\right|\mathrm{d}\tau \\ &\le \limsup_{t\to\infty}\frac{1}{\sqrt{t}}\int\limits_t^{t+\alpha} |s(\tau)|\cdot\frac{L}{\alpha}\sqrt{t}\,|\psi_\alpha(t)-\varphi(\tau)|\,\mathrm{d}\tau\le\limsup_{t\to\infty}\frac{1}{\sqrt{t}}\int\limits_t^{t+\alpha} |s(\tau)|\,\mathrm{d}\tau=0. \end{aligned} \tag{3.13}$$

It is seen that (3.12) and (3.13) affirms (3.7) and (3.9), respectively.

Similarly, we use the Lipschitz continuity of $y=\cos x\sin x$ which implies that there exists $M>0$ for which

$$|\cos x_1\sin x_1-\cos x_2\sin x_2|\le M\,|x_1-x_2|\,,\qquad x_1,x_2\in\mathbb{R}. \tag{3.14}$$

Now (3.8) follows from

$$\begin{aligned} &\limsup_{t\to\infty}\left|\frac{\log t-p}{\log t}\cos\psi_\alpha(t)\sin\psi_\alpha(t)-\frac{1}{\alpha}\int\limits_t^{t+\alpha}\frac{\log\tau-p}{\log\tau}\cos\varphi(\tau)\sin\varphi(\tau)\,\mathrm{d}\tau\right| \\ &\le \limsup_{t\to\infty}\left|\frac{\log t-p}{\log t}\cos\psi_\alpha(t)\sin\psi_\alpha(t)-\frac{1}{\alpha}\int\limits_t^{t+\alpha}\frac{\log t-p}{\log t}\cos\varphi(\tau)\sin\varphi(\tau)\,\mathrm{d}\tau\right| \\ &\quad+\limsup_{t\to\infty}\left|\frac{1}{\alpha}\int\limits_t^{t+\alpha}\frac{\log t-p}{\log t}\cos\varphi(\tau)\sin\varphi(\tau)\,\mathrm{d}\tau-\frac{1}{\alpha}\int\limits_t^{t+\alpha}\frac{\log\tau-p}{\log\tau}\cos\varphi(\tau)\sin\varphi(\tau)\,\mathrm{d}\tau\right| \\ &\le (p+1)\limsup_{t\to\infty}\frac{1}{\alpha}\int\limits_t^{t+\alpha}|\cos\psi_\alpha(t)\sin\psi_\alpha(t)-\cos\varphi(\tau)\sin\varphi(\tau)|\,\mathrm{d}\tau \\ &\quad+\limsup_{t\to\infty}\frac{1}{\alpha}\int\limits_t^{t+\alpha}\left|\frac{\log t-p}{\log t}-\frac{\log\tau-p}{\log\tau}\right|\mathrm{d}\tau \end{aligned}$$

$$\leq M(p+1)\limsup_{t\to\infty}\frac{1}{\sqrt{t}}\left(\frac{1}{\alpha}\int_t^{t+\alpha}\sqrt{t}\,|\psi_\alpha(t)-\varphi(\tau)|\,\mathrm{d}\tau\right)$$
$$+\,p\limsup_{t\to\infty}\frac{1}{\alpha}\int_t^{t+\alpha}\left(\frac{1}{\log t}-\frac{1}{\log(t+\alpha)}\right)\mathrm{d}\tau$$
$$\leq M(p+1)\limsup_{t\to\infty}\frac{1}{\sqrt{t}}+p\limsup_{t\to\infty}\frac{\alpha}{\log t\cdot\log(t+\alpha)}=0,$$

where (3.3) (see Lemma 3.5) and (3.14) are applied. □

4. Conditional oscillation

We prove the conditional oscillation of Eq (2.12), i.e., the announced generalization of Theorem 1.1. This main result follows.

Theorem 4.1. *Let us consider Eq* (2.12) *with continuous coefficients* $r:\mathbb{R}_e\to(0,\infty)$ *and* $s:\mathbb{R}_e\to\mathbb{R}$ *satisfying* (2.13)*. Let* $A,B,\alpha>0$*.*

(A) If $4AB>1$ *and if the inequalities*

$$\frac{1}{\alpha}\int_t^{t+\alpha}r(\tau)\,\mathrm{d}\tau\geq A,\qquad\frac{1}{\alpha}\int_t^{t+\alpha}s(\tau)\,\mathrm{d}\tau\geq B\tag{4.1}$$

are valid for all large t, then Eq (2.12) *is oscillatory.*

(B) If $4AB<1$ *and if the inequalities*

$$\frac{1}{\alpha}\int_t^{t+\alpha}r(\tau)\,\mathrm{d}\tau\leq A,\qquad\frac{1}{\alpha}\int_t^{t+\alpha}s(\tau)\,\mathrm{d}\tau\leq B\tag{4.2}$$

are valid for all large t, then Eq (2.12) *is non-oscillatory.*

Proof. In both parts of the proof, we use the equation for the modified Prüfer angle which corresponds to Eq (2.12), i.e., we consider Eq (2.15). Let $\varphi:\mathbb{R}_e\to\mathbb{R}$ be a solution of Eq (2.15) and let $\psi_\alpha:\mathbb{R}_e\to\mathbb{R}$ be the corresponding average function introduced in (2.16). Based on Lemma 3.1, we analyze the limit behaviour of φ. In addition, due to Lemma 3.5, it suffices to show that ψ_α is unbounded in the first case and bounded from above in the second case.

We begin with the first part (i.e., the oscillation part) of the theorem. Considering (3.5) in Lemma 3.6 and (4.1), we obtain

$$\psi'_\alpha(t)=\frac{1}{t}\left[\left(\frac{1}{\alpha}\int_t^{t+\alpha}r(\tau)\,\mathrm{d}\tau\right)\cos^2\psi_\alpha(t)-\frac{\log t-p}{\log t}\cos\psi_\alpha(t)\sin\psi_\alpha(t)\right.$$
$$\left.+\left(\frac{1}{\alpha}\int_t^{t+\alpha}s(\tau)\,\mathrm{d}\tau\right)\sin^2\psi_\alpha(t)+F(t)\right]$$

$$\geq \frac{1}{t}\left[A\cos^2\psi_\alpha(t) - \frac{\log t - p}{\log t}\cos\psi_\alpha(t)\sin\psi_\alpha(t) + B\sin^2\psi_\alpha(t) + F(t)\right]$$
$$= \frac{1}{t}\left[(A + F(t))\cos^2\psi_\alpha(t) - \frac{\log t - p}{\log t}\cos\psi_\alpha(t)\sin\psi_\alpha(t) + (B + F(t))\sin^2\psi_\alpha(t)\right]$$

for all large t, where $\lim_{t\to\infty} F(t) = 0$ (see (3.4)). Let $\delta > 0$ be so small that

$$A - \delta, B - \delta > 0, \qquad 4(A - \delta)(B - \delta) > 1 \tag{4.3}$$

and t so large that $|F(t)| < \delta$. Then,

$$\psi'_\alpha(t) > \frac{1}{t}\left[(A - \delta)\cos^2\psi_\alpha(t) - \frac{\log t - p}{\log t}\cos\psi_\alpha(t)\sin\psi_\alpha(t) + (B - \delta)\sin^2\psi_\alpha(t)\right] \tag{4.4}$$

for all large t. Putting $C = A - \delta$ and $D = B - \delta$, from (4.3), we have $C, D > 0$ and $4CD > 1$. We apply Lemma 3.4, (A). Thus, any solution $\zeta : \mathbb{R}_e \to \mathbb{R}$ of the equation

$$\zeta'(t) = \frac{1}{t}\left[(A - \delta)\cos^2\zeta(t) - \frac{\log t - p}{\log t}\cos\zeta(t)\sin\zeta(t) + (B - \delta)\sin^2\zeta(t)\right] \tag{4.5}$$

satisfies $\lim_{t\to\infty}\zeta(t) = \infty$. Let $T > e$ be such that (4.4) is valid for all $t \geq T$. Since $y = \sin^2 x$, $y = \cos^2 x$, $y = \cos x \sin x$ are π-period functions, comparing Eq (4.5) with the right-hand side of (4.4), we have

$$\psi_\alpha(t) \geq \zeta(t) - |\psi_\alpha(T) - \zeta(T)| - \pi$$

for all $t \geq T$. Therefore,

$$\liminf_{t\to\infty}\psi_\alpha(t) \geq \lim_{t\to\infty}\zeta(t) = \infty.$$

Now we prove the second part. We proceed analogously as in the first part. Applying Lemma 3.6 and (4.2), we have

$$\psi'_\alpha(t) = \frac{1}{t}\left[\left(\frac{1}{\alpha}\int_t^{t+\alpha} r(\tau)\,\mathrm{d}\tau\right)\cos^2\psi_\alpha(t) - \frac{\log t - p}{\log t}\cos\psi_\alpha(t)\sin\psi_\alpha(t)\right.$$
$$\left. + \left(\frac{1}{\alpha}\int_t^{t+\alpha} s(\tau)\,\mathrm{d}\tau\right)\sin^2\psi_\alpha(t) + F(t)\right]$$
$$\leq \frac{1}{t}\left[A\cos^2\psi_\alpha(t) - \frac{\log t - p}{\log t}\cos\psi_\alpha(t)\sin\psi_\alpha(t) + B\sin^2\psi_\alpha(t) + F(t)\right]$$
$$= \frac{1}{t}\left[(A + F(t))\cos^2\psi_\alpha(t) - \frac{\log t - p}{\log t}\cos\psi_\alpha(t)\sin\psi_\alpha(t) + (B + F(t))\sin^2\psi_\alpha(t)\right]$$

for all large t, where $\lim_{t\to\infty} F(t) = 0$ (see (3.4)). Let $\delta > 0$ satisfy $4(A + \delta)(B + \delta) < 1$ and t be such that $|F(t)| < \delta$. Then,

$$\psi'_\alpha(t) < \frac{1}{t}\left[(A + \delta)\cos^2\psi_\alpha(t) - \frac{\log t - p}{\log t}\cos\psi_\alpha(t)\sin\psi_\alpha(t) + (B + \delta)\sin^2\psi_\alpha(t)\right] \tag{4.6}$$

for all large t. We put $C = A + \delta$, $D = B + \delta$. Considering $4CD < 1$, Lemma 3.4, (B) says that any solution $\theta : \mathbb{R}_e \to \mathbb{R}$ of the equation

$$\theta'(t) = \frac{1}{t}\left[(A+\delta)\cos^2\theta(t) - \frac{\log t - p}{\log t}\cos\theta(t)\sin\theta(t) + (B+\delta)\sin^2\theta(t)\right] \tag{4.7}$$

satisfies $\limsup_{t\to\infty}\theta(t) < \infty$. From (4.6) and (4.7), we have

$$\limsup_{t\to\infty}\psi_\alpha(t) \le \limsup_{t\to\infty}\theta(t) + |\psi_\alpha(T) - \theta(T)| + \pi < \infty,$$

where $T > \mathrm{e}$ is a sufficiently large number. □

Remark 1. Concerning the statement of Theorem 4.1, we remark that the requirement (see (2.13))

$$\lim_{t\to\infty}\frac{\int_t^{t+1} r(\tau)\,\mathrm{d}\tau}{\sqrt{t}} = 0 \tag{4.8}$$

can be omitted (in contrast to the second requirement about s in (2.13)), i.e., it is evident that Theorem 4.1 is valid also without this limitation provided we have proved its statement with (4.8). Note that we use (4.8) in the proofs of Lemmas 3.5 and 3.6.

Remark 2. Concerning the conditional oscillation of perturbed Euler type equations (see, e.g., [7, 10]), we conjecture that it is not possible to decide the (non-)oscillation of Eq (2.12) for general functions r, s satisfying

$$\lim_{t\to\infty}\left(\frac{1}{\alpha}\int_t^{t+\alpha} r(\tau)\,\mathrm{d}\tau\right)\left(\frac{1}{\alpha}\int_t^{t+\alpha} s(\tau)\,\mathrm{d}\tau\right) = \frac{1}{4},$$

where $\alpha > 0$. We can also refer to [27].

Theorem 4.1 covers equations with unbounded coefficients which oscillate non-trivially. To illustrate this fact, we provide the following example.

Example 1. For $\mu > 0$ and $\nu \in \mathbb{R}$, we define

$$r(t) := \mu\left[1 + 3^n(t-n)\right], \quad t \in \left[n, n + \frac{1}{2^n}\right), \ n \in \mathbb{N};$$
$$r(t) := \mu\left[1 + 3^n\left(n + \frac{2}{2^n} - t\right)\right], \quad t \in \left[n + \frac{1}{2^n}, n + \frac{2}{2^n}\right], \ n \in \mathbb{N};$$
$$r(t) := \mu, \quad t \in \left(n + \frac{2}{2^n}, n + 1\right), \ n \in \mathbb{N},$$

and

$$s(t) := \nu + 4^n\frac{\sqrt{n}}{\log(n+1)}\left(t - n - \frac{4i}{4^n}\right),$$
$$t \in \left[n + \frac{4i}{4^n}, n + \frac{1+4i}{4^n}\right), \ i \in \{0, 1, \ldots, 4^{n-1} - 1\}, \ n \in \mathbb{N};$$

$$s(t) := \nu + 4^n \frac{\sqrt{n}}{\log(n+1)} \left(n + \frac{2+4i}{4^n} - t \right),$$
$$t \in \left[n + \frac{1+4i}{4^n}, n + \frac{3+4i}{4^n} \right), \ i \in \{0, 1, \ldots, 4^{n-1} - 1\}, \ n \in \mathbb{N};$$
$$s(t) := \nu + 4^n \frac{\sqrt{n}}{\log(n+1)} \left(t - n - \frac{4+4i}{4^n} \right),$$
$$t \in \left[n + \frac{3+4i}{4^n}, n + \frac{4+4i}{4^n} \right), \ i \in \{0, 1, \ldots, 4^{n-1} - 1\}, \ n \in \mathbb{N}.$$

For these functions, let us consider Eq (2.12). Let $\alpha = 1$, i.e., let the average ψ_α be taken over intervals of the length $\alpha = 1$. One can easily verify that

$$\lim_{t\to\infty} \int_t^{t+1} r(\tau)\,\mathrm{d}\tau = \mu, \qquad \lim_{t\to\infty} \int_t^{t+1} s(\tau)\,\mathrm{d}\tau = \nu, \qquad \lim_{t\to\infty} \frac{\int_t^{t+1} |s(\tau)|\,\mathrm{d}\tau}{\sqrt{t}} = 0.$$

Applying Theorem 4.1 (consider also Remark 1), we obtain the oscillation of the considered equation for $4\mu\nu > 1$ and its non-oscillation for $4\mu\nu < 1$.

5. Corollaries and examples

In this section, to explain the novelty of our main result, we mention its direct corollaries together with simple examples. We point out that none of the examples below is covered by previously known results and that all results below are new for any $p > 0$. At first, we recall the concept of the mean value for continuous functions which is used in the first corollary.

Definition 1. Let a continuous function $f : \mathbb{R}_e \to \mathbb{R}$ be such that the limit

$$M(f) := \lim_{a\to\infty} \frac{1}{a} \int_t^{t+a} f(\tau)\,\mathrm{d}\tau$$

is finite and exists uniformly with respect to $t \in \mathbb{R}_e$. The number $M(f)$ is called the *mean value* of f.

Corollary 5.1. *Let us consider Eq* (2.12)*, where continuous functions* $r : \mathbb{R}_e \to (0, \infty)$ *and* $s : \mathbb{R}_e \to \mathbb{R}$ *have mean values* $M(r), M(s)$ *and*

$$\lim_{t\to\infty} \frac{\int_t^{t+1} |s(\tau)|\,\mathrm{d}\tau}{\sqrt{t}} = 0.$$

(A) If $4M(r)M(s) > 1$*, then Eq* (2.12) *is oscillatory.*
(B) If $4M(r)M(s) < 1$*, then Eq* (2.12) *is non-oscillatory.*

Proof. According to the well-known Sturm theory (see, e.g., [51]), we can assume that $M(r) > 0$ and $M(s) > 0$. The corollary follows from Theorem 4.1 (and from Remark 1). Indeed, for $\delta > 0$ such that $M(r) - \delta, M(s) - \delta > 0$ and $4(M(r) - \delta)(M(s) - \delta) > 1$ or $4(M(r) + \delta)(M(s) + \delta) < 1$, from Definition 1, we obtain the existence of $\alpha > 0$ with the property that

$$\frac{1}{\alpha} \int_t^{t+\alpha} r(\tau)\,\mathrm{d}\tau \geq M(r) - \delta, \qquad \frac{1}{\alpha} \int_t^{t+\alpha} s(\tau)\,\mathrm{d}\tau \geq M(s) - \delta$$

or

$$\frac{1}{\alpha}\int\limits_{t}^{t+\alpha} r(\tau)\,\mathrm{d}\tau \le M(r)+\delta, \qquad \frac{1}{\alpha}\int\limits_{t}^{t+\alpha} s(\tau)\,\mathrm{d}\tau \le M(s)+\delta$$

for all t, respectively. □

There are known many types of continuous functions which have mean values. For example, we recall asymptotically almost periodic functions which are treated in the following example.

Example 2. For $\mu \ge 2$ and $\nu \in \mathbb{R}$, let us consider the equation

$$\begin{aligned}&\left[\frac{\log^p(t+2)}{\mu+\sin t+\cos\left(\sqrt{2}t\right)}\,x'(t)\right]' \\ &\qquad + \log^p t\,\frac{\nu+\sin\left(\sqrt{3}t\right)+\cos(2t)+\arctan\frac{t+1}{t^2}}{\left(t+\sqrt{t}\right)^2}\,x(t)=0,\end{aligned} \tag{5.1}$$

which has the form of Eq (2.12) for

$$r(t)=\left(\mu+\sin t+\cos\left(\sqrt{2}t\right)\right)\left(\frac{\log t}{\log(t+2)}\right)^p, \qquad t\in\mathbb{R}_{\mathrm{e}},$$

$$s(t)=\left(\nu+\sin\left(\sqrt{3}t\right)+\cos(2t)+\arctan\frac{t+1}{t^2}\right)\frac{t^2}{\left(t+\sqrt{t}\right)^2}, \qquad t\in\mathbb{R}_{\mathrm{e}}.$$

Since

$$M(r)=\mu+M(\sin t)+M\left(\cos\left(\sqrt{2}t\right)\right)=\mu$$

and

$$M(s)=\nu+M\left(\sin\left(\sqrt{3}t\right)\right)+M(\cos(2t))+M\left(\arctan\frac{t+1}{t^2}\right)=\nu,$$

Corollary 5.1 guarantees the oscillation of Eq (5.1) for $4\mu\nu > 1$ and the non-oscillation of Eq (5.1) for $4\mu\nu < 1$.

Now we concentrate on more concrete cases of the studied equations with periodic coefficients (nonpositive or with different periods). We highlight that these results are new as well.

Corollary 5.2. *Let us consider Eq* (2.12)*, where continuous functions* $r:\mathbb{R}_{\mathrm{e}}\to(0,\infty)$ *and* $s:\mathbb{R}_{\mathrm{e}}\to\mathbb{R}$ *are* α*-periodic for* $\alpha>0$.

(A) If

$$4\left(\frac{1}{\alpha}\int\limits_{\mathrm{e}}^{\mathrm{e}+\alpha} r(\tau)\,\mathrm{d}\tau\right)\left(\frac{1}{\alpha}\int\limits_{\mathrm{e}}^{\mathrm{e}+\alpha} s(\tau)\,\mathrm{d}\tau\right)>1,$$

then Eq (2.12) *is oscillatory.*

(B) If

$$4\left(\frac{1}{\alpha}\int_{\mathrm{e}}^{\mathrm{e}+\alpha} r(\tau)\,\mathrm{d}\tau\right)\left(\frac{1}{\alpha}\int_{\mathrm{e}}^{\mathrm{e}+\alpha} s(\tau)\,\mathrm{d}\tau\right) < 1,$$

then Eq (2.12) *is non-oscillatory.*

Proof. It suffices to consider Corollary 5.1 for

$$M(r) = \frac{1}{\alpha}\int_{\mathrm{e}}^{\mathrm{e}+\alpha} r(\tau)\,\mathrm{d}\tau, \qquad M(s) = \frac{1}{\alpha}\int_{\mathrm{e}}^{\mathrm{e}+\alpha} s(\tau)\,\mathrm{d}\tau$$

and the boundedness of s. □

Corollary 5.3. *Let us consider Eq* (2.12)*, where* $r : \mathbb{R}_{\mathrm{e}} \to (0, \infty)$ *is a continuous and* α*-periodic function and* $s : \mathbb{R}_{\mathrm{e}} \to (0, \infty)$ *is a continuous and* β*-periodic function for* $\alpha, \beta > 0$.

(A) If

$$4\left(\frac{1}{\alpha}\int_{\mathrm{e}}^{\mathrm{e}+\alpha} r(\tau)\,\mathrm{d}\tau\right)\left(\frac{1}{\beta}\int_{\mathrm{e}}^{\mathrm{e}+\beta} s(\tau)\,\mathrm{d}\tau\right) > 1,$$

then Eq (2.12) *is oscillatory.*

(B) If

$$4\left(\frac{1}{\alpha}\int_{\mathrm{e}}^{\mathrm{e}+\alpha} r(\tau)\,\mathrm{d}\tau\right)\left(\frac{1}{\beta}\int_{\mathrm{e}}^{\mathrm{e}+\beta} s(\tau)\,\mathrm{d}\tau\right) < 1,$$

then Eq (2.12) *is non-oscillatory.*

Proof. It suffices to consider Corollary 5.1 for

$$M(r) = \frac{1}{\alpha}\int_{\mathrm{e}}^{\mathrm{e}+\alpha} r(\tau)\,\mathrm{d}\tau, \qquad M(s) = \frac{1}{\beta}\int_{\mathrm{e}}^{\mathrm{e}+\beta} s(\tau)\,\mathrm{d}\tau$$

and the boundedness of s. □

Using the above corollaries, one can identify the critical oscillation constants for several equations. We mention at least the examples below.

Example 3. For $\mu > 1$, let us consider the equations

$$\left[\frac{\log^p t}{\mu + \sin t}\, x'(t)\right]' + \log^p t\,\frac{\frac{1}{8} + \sin t}{t^2}\, x(t) = 0,$$
$$\left[\frac{\log^p t}{\mu + \sin t}\, x'(t)\right]' + \log^p t\,\frac{\frac{1}{8} + \cos t}{t^2}\, x(t) = 0,$$
$$\left[\frac{\log^p t}{\mu + \cos t}\, x'(t)\right]' + \log^p t\,\frac{\frac{1}{8} + \sin t}{t^2}\, x(t) = 0,$$

$$\left[\frac{\log^p t}{\mu + \cos t}\, x'(t)\right]' + \log^p t\, \frac{\frac{1}{8} + \cos t}{t^2}\, x(t) = 0.$$

For these equations, one can apply Corollary 5.2. Therefore, the equations are oscillatory if $\mu > 2$, and they are non-oscillatory if $\mu < 2$.

Example 4. For $\nu > 1$, let us consider the equations

$$\left[\frac{\log^p t}{\nu + \sin t}\, x'(t)\right]' + \log^p t\, \frac{2 + \sin(\pi t)}{16t^2}\, x(t) = 0,$$
$$\left[\frac{\log^p t}{\nu + \sin t}\, x'(t)\right]' + \log^p t\, \frac{2 + \cos(\pi t)}{16t^2}\, x(t) = 0,$$
$$\left[\frac{\log^p t}{\nu + \cos t}\, x'(t)\right]' + \log^p t\, \frac{2 + \sin(\pi t)}{16t^2}\, x(t) = 0,$$
$$\left[\frac{\log^p t}{\nu + \cos t}\, x'(t)\right]' + \log^p t\, \frac{2 + \cos(\pi t)}{16t^2}\, x(t) = 0.$$

Now one can apply Corollary 5.3. Thus, the equations are oscillatory for $\nu > 2$ and non-oscillatory for $\nu < 2$.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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