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*Research article*

# A remark for Gauss sums of order 3 and some applications for cubic congruence equations

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**Abstract:** In this paper, we give some relations between Gauss sums of order 3. As application, we give the number of solutions of some cubic diagonal equations. These generalize the earlier results obtained by Hong-Zhu and solve the sign problem raised by Zhang-Zhang.

**Keywords:** Gauss sum; exponential sum; diagonal cubic form; congruence equations; prime
**Mathematics Subject Classification:** 11T23, 11T24

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## 1. Introduction

For a prime $p \equiv 1(\text{mod}3)$, let $\mathbb{F}_p$ be the finite field of residues $(\text{mod}p)$, let $G$ be the multiplicative group of non-zero residues $(\text{mod}p)$ and let $H$ be the subgroup of non-zero cubic residues $(\text{mod}p)$. For any $a \in G$, we defined the sums

$$S(a) = \sum_{k=0}^{p-1} e(ak^3/p)$$

and

$$G(\chi) = \sum_{k=1}^{p-1} \chi(k)e(k/p),$$

where $\chi$ is a multiplicative character of order 3 over $\mathbb{F}_p$ and $e(x) = e^{2\pi i x}$ in this paper. Both $S(a)$ and $G(\chi)$ are called Gauss sums of order 3. Gauss sums is very important in the analytic number theory and related research filed. Many scholars studied its properties and obtained a series of interesting results (see [5, 6, 8–11, 13]).

Let $z \in G \setminus H$. By a classical result of Gauss [4] (also see Theorem 4.1.2 of [1]), $S(1), S(z)$ and $S(z^2)$ are three roots of the cubic equation

$$x^3 - 3px - pc = 0,$$

where $c$ is uniquely determined by

$$4p = c^2 + 27d^2, \quad c \equiv 1(\bmod 3). \tag{1.1}$$

However, how to determine which of the three roots corresponds to $S(1)$ is still an open problem.

In this paper, for a fixed $z \in G \setminus H$, we find a relation between $S(1), S(z)$ and $S(z^2)$.

**Theorem 1.1.** *Let $p \equiv 1(\bmod 3)$ and $z \in G \setminus H$. Then*

$$S(1) = 2\sqrt{p}\cos\left(\theta_p\right), \ S(z) = 2\sqrt{p}\cos\left(\theta_p - sgn(d)\frac{2}{3}\pi\right), \ S(z^2) = 2\sqrt{p}\cos\left(\theta_p + sgn(d)\frac{2}{3}\pi\right),$$

*where $\theta_p = \frac{1}{3}\arccos\left(-\frac{c}{2\sqrt{p}}\right) + j_p\frac{2}{3}\pi$; $j_p$ is one of three values $-1, 0, 1$ and only dependent on $p$; $c$ and $d$ are uniquely determined by*

$$4p = c^2 + 27d^2, \quad c \equiv 1(\bmod 3), \quad 9d \equiv c(2z^{\frac{p-1}{3}} + 1)(\bmod p). \tag{1.2}$$

*Moreover, there is a unique multiplicative character $\chi$ of order 3 over $\mathbb{F}_p$ such that*

$$\chi(z) = \frac{-1 + \sqrt{3}\mathrm{i}}{2}, \ G(\chi) = \sqrt{p}e^{\mathrm{i}sgn(d)\theta_p}.$$

As application, we consider some congruence equations $\bmod p$. For $a_1, a_2, a_3 \in G$, let $M(a_1, a_2, a_3)$ be the number of solutions of

$$a_1x_1^3 + a_2x_2^3 + a_3x_3^3 \equiv 0(\bmod p),$$

and let $N(a_1, a_2, a_3)$ be the number of solutions of

$$a_1x_1^3 + a_2x_2^3 \equiv a_3(\bmod p).$$

In [2], Chowla, Cowles and Cowles showed that $M(1, 1, 1) = p^2 + c(p - 1)$. As pointed out in [3], the following is essentially included in the derivation of the cubic equation of periods by Gauss [4]: For a prime $p \equiv 1(\bmod 3)$ and for $z \in G \setminus H$, then one has

$$M(1, 1, z) = p^2 + \frac{1}{2}(p - 1)(9d - c),$$

where $c$ and $d$ are uniquely determined by (1.1) (except for the sign of $d$).

Chowla, Cowles and Cowles [3] determined the sign of $d$ for the case $2 \in G \setminus H$ as the following result shows.

**Proposition 1.2.** *[3] Let a prime $p \equiv 1(\text{mod}3)$. If $2 \in G \setminus H$, then for any $z \in G \setminus H$, one has*

$$M(1,1,z) = p^2 + \frac{1}{2}(p-1)(9d-c),$$

*where c and d are uniquely determined by (1.1) with*

$$d \equiv c(\text{mod}4) \text{ for } z \equiv 2(\text{mod}H)$$

*and*

$$d \equiv -c(\text{mod}4) \text{ for } z \equiv 4(\text{mod}H).$$

Recently, Hong and Zhu [7] solve the Gauss sign problem. In fact, they gave the following result.

**Proposition 1.3.** *[7] Let a prime $p \equiv 1(\text{mod}3)$ and $z \in G \setminus H$. Let g be a generator of the multiplicative group G. one has*

$$M(1,1,z) = p^2 + \frac{1}{2}(p-1)(-c - \delta_z(p)d),$$

*where c and d are uniquely determined by (1.1) with $d > 0$ and*

$$\delta_z(p) = (-1)^{\langle ind_g(d)\rangle_3} \cdot sgn\left(Im(r_1 + 3\sqrt{3}r_2\mathrm{i})\right).$$

*Here $r_1$ and $r_2$ are uniquely determined by*

$$4p = r_1^2 + 27r_2^2, \ \ r_1 \equiv 1(\text{mod}3), \ \ 9r_2 \equiv (2g^{\frac{p-1}{3}} + 1)r_1(\text{mod}p).$$

Indeed, their result need to use the generator of group $G$ (that is the primitive root of module $p$). However, for a large prime $p$, it is not easy to find the primitive root of module $p$. In this paper, we consider $M(a_1, a_2, a_3)$, $N(a_1, a_2, a_3)$ and determine the sign of $d$ immediately by the coefficients $a_1, a_2$ and $a_3$. We have the following three more general results.

**Theorem 1.4.** *Let a prime $p \equiv 1(\text{mod}3)$ and $a_1, a_2, a_3 \in G$.*
*(1) For the case $a_1a_2a_3 \in H$, $M(a_1, a_2, a_3) = p^2 + c(p-1)$;*
*(2) For the case $a_1a_2a_3 \notin H$, $M(a_1, a_2, a_3) = p^2 + \frac{1}{2}(p-1)(9d-c)$,*
*where c and d are uniquely determined by*

$$4p = c^2 + 27d^2, \ \ c \equiv 1(\text{mod}3), \ \ 9d \equiv c(2(a_1a_2a_3)^{\frac{p-1}{3}} + 1)(\text{mod}p). \tag{1.3}$$

**Theorem 1.5.** *Let $p \equiv 1(\text{mod}3)$ and $a_1, a_2, a_3 \in G$.*
*(1) For the case $a_1a_2a_3 \in H$,*

$$N(a_1, a_2, a_3) = \begin{cases} p - 2 + c, & \text{if } a_1 \equiv a_2(\text{mod}H); \\ p + 1 + c, & \text{otherwise.} \end{cases}$$

*(2) For the case $a_1a_2a_3 \notin H$,*

$$N(a_1, a_2, a_3) = \begin{cases} p - 2 + \frac{1}{2}(9d-c), & \text{if } a_1 \equiv a_2(\text{mod}H); \\ p + 1 + \frac{1}{2}(9d-c), & \text{otherwise,} \end{cases}$$

*where c and d are uniquely determined by (1.3).*

**Corollary 1.6.** *Let* $p \equiv 1(\bmod 3)$ *and* $a_1, a_2, a_3 \in G$. *Then*

$$M(a_1, a_2, a_3) \equiv -c(a_1 a_2 a_3)^{\frac{p-1}{3}} (\bmod p).$$

In [14], H. Zhang and W. P. Zhang proposed the following open problem:

Can the number of solutions to the cubic congruence equation

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 \equiv z(\bmod p) \tag{1.4}$$

be calculated when $z \in G$?

Let $L(z)$ be the number of solutions of the above Eq (1.4). In [12], W. P. Zhang and J. Y. Hu proved that

$$L(z) = \begin{cases} p^3 - 6p - \frac{1}{2}p(5c \pm 27d), & \text{if } z \in G \setminus H; \\ p^3 - 6p + 5cp, & \text{if } z \in H. \end{cases} \tag{1.5}$$

However, in [12], they also proposed an interesting open problem: How to determine the choice of sign in (1.5). In this paper, we solve the sign problem in (1.5), and get the following result.

**Theorem 1.7.** *Let p be a prime number and* $p \equiv 1(\bmod 3)$, *let* $z \in G \setminus H$. *Then*

$$L(z) = p^3 - 6p - \frac{1}{2}p(5c - 27d),$$

*where c and d are uniquely determined by*

$$4p = c^2 + 27d^2, \ \ c \equiv 1(\bmod 3), \ \ 9d \equiv c(2z^{\frac{p-1}{3}} + 1)(\bmod p).$$

## 2. Some useful lemmas

**Lemma 2.1** (Theorem 3.1.3 of [1])**.** *Let* $p \equiv 1(\bmod 3)$ *and* $\chi$ *be a multiplicative character of order 3 over* $\mathbb{F}_p$. *Then*

$$J(\chi, \chi) = \frac{c + 3\sqrt{3}d\mathrm{i}}{2},$$

*where the Jacobi sum* $J(\chi, \chi) = \sum_{a=1}^{p-1} \chi(a)\chi(1-a)$, *c and d are uniquely determined by*

$$4p = c^2 + 27d^2, \ \ c \equiv 1(\bmod 3), \ \ 9d \equiv c(2g^{\frac{p-1}{3}} + 1)(\bmod p)$$

*with g being the generator of the multiplicative group G of non-zero residues* $(\bmod p)$ *such that* $\chi(g) = \frac{-1+\sqrt{3}\mathrm{i}}{2}$.

**Lemma 2.2** (Lemma 4.1.1 of [1])**.** *Let* $p \equiv 1(\bmod 3)$. *Let g be a generator of the multiplicative group G of non-zero residues* $(\bmod p)$ *with* $\chi(g) = \frac{-1+\sqrt{3}\mathrm{i}}{2}$. *Then*

$$G^3(\chi) = pJ(\chi, \chi).$$

**Lemma 2.3.** *Let* $p \equiv 1(\mathrm{mod}3)$ *and* $z \in G \setminus H$*. Then there is a unique multiplicative character* $\chi$ *of order 3 over* $\mathbb{F}_p$ *such that*

$$\chi(z) = \frac{-1 + \sqrt{3}\mathrm{i}}{2}, \ \ G^3(\chi) = p \cdot \frac{c + 3\sqrt{3}d\mathrm{i}}{2},$$

*where c and d are uniquely determined by (1.2).*

*Proof.* Let $g'$ be a generator of the group $G$. Note that $z \in G \setminus H$. So we have $\mathrm{ind}_{g'}z \equiv \pm 1(\mathrm{mod}3)$. If $\mathrm{ind}_{g'}z \equiv 1(\mathrm{mod}3)$, we take $g = g'$; If $\mathrm{ind}_{g'}z \equiv -1(\mathrm{mod}3)$, we take $g = (g')^{-1}$. Hence $g$ also is a generator of the group $G$ and $\mathrm{ind}_g z \equiv 1(\mathrm{mod}3)$. Thus we have

$$z^{\frac{p-1}{3}} \equiv \left(g^{\mathrm{ind}_g z}\right)^{\frac{p-1}{3}} \equiv g^{\frac{p-1}{3}\mathrm{ind}_g z} \equiv g^{\frac{p-1}{3}} (\mathrm{mod} p).$$

We take the multiplicative character $\chi(\cdot) = e\left(\frac{\mathrm{ind}_g(\cdot)}{3}\right)$. Obviously, we have

$$\chi(z) = e\left(\frac{\mathrm{ind}_g z}{3}\right) = e\left(\frac{1}{3}\right) = \frac{-1 + \sqrt{3}\mathrm{i}}{2} = \chi(g).$$

Obviously, all of the multiplicative non-principal characters of order 3 over $\mathbb{F}_p$ are $\chi$ and $\overline{\chi}$, $\overline{\chi}(z) = \overline{\chi(z)} = \frac{-1-\sqrt{3}\mathrm{i}}{2}$. Thus $\chi$ is the unique multiplicative character of order 3 over $\mathbb{F}_p$ with $\chi(z) = \frac{-1+\sqrt{3}\mathrm{i}}{2}$.

Note that $G^3(\chi) = pJ(\chi, \chi)$ by Lemma 2.2. Finally, using the Lemma 2.1, one immediately arrive the Lemma 2.3 as required. □

**Lemma 2.4.** *Let* $\chi$ *be a multiplicative character of order 3. Then for any* $a \in G$*, we have*

$$S(a) = \overline{\chi}(a)G(\chi) + \chi(a)G(\overline{\chi}). \tag{2.1}$$

*Proof.* Let $\chi$ be any multiplicative character of order 3. Then we have

$$1 + \chi(k) + \overline{\chi}(k) = \begin{cases} 3, & \text{if } k \in H; \\ 0, & \text{if } k \in G \setminus H. \end{cases}$$

Thus for any $a \in G$, we have

$$\begin{aligned} S(a) = \sum_{k=0}^{p-1} e(ak^3/p) &= 1 + \sum_{k=1}^{p-1}(1 + \chi(k) + \overline{\chi}(k))e(ak/p) \\ &= 1 + \sum_{k=1}^{p-1} e(ak/p) + \sum_{k=1}^{p-1}\chi(k)e(ak/p) + \sum_{k=1}^{p-1}\overline{\chi}(k)e(ak/p) \\ &= \overline{\chi}(a)\sum_{k=1}^{p-1}\chi(ak)e(ak/p) + \chi(a)\sum_{k=1}^{p-1}\overline{\chi}(ak)e(ak/p) \\ &= \overline{\chi}(a)G(\chi) + \chi(a)G(\overline{\chi}). \end{aligned}$$

□

## 3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. First, by Lemma 2.3, there is a unique multiplicative character $\chi$ of order 3 such that

$$\chi(z) = \frac{-1 + \sqrt{3}\mathrm{i}}{2}, \ \ G^3(\chi) = p \cdot \frac{c + 3\sqrt{3}d\mathrm{i}}{2},$$

where $c$ and $d$ are uniquely determined by (1.2). We can rewrite $G^3(\chi)$ by argument, and get

$$G^3(\chi) = p^{\frac{3}{2}} e^{3\mathrm{i}\theta sgn(d)},$$

where $\theta = \frac{1}{3}\arccos(-\frac{c}{2\sqrt{p}})$. Thus we have

$$G(\chi) = \sqrt{p} e^{\mathrm{i}(sgn(d)\theta + j\frac{2}{3}\pi)} = \sqrt{p} e^{\mathrm{i}sgn(d)(\theta + sgn(d)j\frac{2}{3}\pi)},$$

where $j$ is one of three values $-1, 0, 1$. Let $j_p = sgn(d)j$. Thus we have

$$G(\chi) = \sqrt{p} e^{\mathrm{i}sgn(d)(\theta + j_p\frac{2}{3}\pi)}.$$

Next, we will prove that $j_p$ does not depend on the sign of $d$. Note that $G(\overline{\chi}) = \chi(-1)\overline{G(\chi)} = \sqrt{p} e^{-\mathrm{i}sgn(d)(\theta + j_p\frac{2}{3}\pi)}$. By Lemma 2.4, we have

$$\begin{aligned} S(1) &= \overline{\chi}(1)G(\chi) + \chi(1)G(\overline{\chi}) = G(\chi) + G(\overline{\chi}) \\ &= 2\sqrt{p}\cos[sgn(d)(\theta + j_p\frac{2}{3}\pi)] \\ &= 2\sqrt{p}\cos(\theta + j_p\frac{2}{3}\pi). \end{aligned}$$

Obviously, by the definition of $S(1)$, the value of $S(1)$ doesn't depend on the sign of $d$. Thus we have that $j_p$ does not depend on the sign of $d$.

Take $\theta_p = \theta + j_p\frac{2}{3}\pi$. We have $G(\chi) = \sqrt{p}e^{\mathrm{i}sgn(d)\theta_p}$ and $S(1) = 2\sqrt{p}\cos(\theta_p)$. By Lemma 2.4, we have

$$\begin{aligned} S(z) &= \overline{\chi}(z)G(\chi) + \chi(z)G(\overline{\chi}) \\ &= \frac{-1 - \sqrt{3}\mathrm{i}}{2} \cdot \sqrt{p}e^{\mathrm{i}sgn(d)\theta_p} + \frac{-1 + \sqrt{3}\mathrm{i}}{2} \cdot \sqrt{p}e^{-\mathrm{i}sgn(d)\theta_p} \\ &= \sqrt{p}e^{\mathrm{i}(sgn(d)\theta_p - \frac{2\pi}{3})} + \sqrt{p}e^{-\mathrm{i}(sgn(d)\theta_p - \frac{2\pi}{3})} \\ &= 2\sqrt{p}\cos(sgn(d)\theta_p - \frac{2\pi}{3}) = 2\sqrt{p}\cos(\theta_p - sgn(d)\frac{2\pi}{3}). \end{aligned}$$

Similarly, we have

$$S(z^2) = 2\sqrt{p}\cos(\theta_p + sgn(d)\frac{2}{3}\pi).$$

This completes the proof of the Theorem 1.1.

## 4. Some applications for cubic congruence equations and an example

In this section, we prove Theorem 1.4, 1.5 and 1.7. First, we begin with the proof of Theorem 1.4.
*Proof of Theorem 1.4.* By the orthogonality of additive character, we have

$$\begin{aligned} M(a_1, a_2, a_3) &= \frac{1}{p}\sum_{m=0}^{p-1}\sum_{x_1=0}^{p-1}\sum_{x_2=0}^{p-1}\sum_{x_3=0}^{p-1} e\left(\frac{m(a_1x_1^3 + a_2x_2^3 + a_3x_3^3)}{p}\right) \\ &= p^2 + \frac{1}{p}\sum_{m=1}^{p-1} S(ma_1)S(ma_2)S(ma_3). \end{aligned}$$

Then by Lemma 2.4, for any multiplicative character $\chi$ of order 3, we have

$$\begin{aligned} M(a_1, a_2, a_3) &= p^2 + \frac{1}{p}\sum_{m=1}^{p-1}\left[\prod_{j=1}^{3}\left(\overline{\chi}(ma_j)G(\chi) + \chi(ma_j)G(\overline{\chi})\right)\right] \\ &= p^2 + \frac{1}{p}\sum_{m=1}^{p-1}\left[\overline{\chi}(a_1a_2a_3)G^3(\chi) + \chi(a_1a_2a_3)G^3(\overline{\chi})\right] \\ &+ G(\chi)(\chi(\overline{a}_1\overline{a}_2a_3) + \chi(\overline{a}_1a_2\overline{a}_3) + \chi(a_1\overline{a}_2\overline{a}_3))\sum_{m=1}^{p-1}\overline{\chi}(m) \\ &+ G(\overline{\chi})(\chi(\overline{a}_1a_2a_3) + \chi(a_1\overline{a}_2a_3) + \chi(a_1a_2\overline{a}_3)))\sum_{m=1}^{p-1}\chi(m) \\ &= p^2 + \frac{p-1}{p}\left[\overline{\chi}(a_1a_2a_3)G^3(\chi) + \chi(a_1a_2a_3)G^3(\overline{\chi})\right]. \end{aligned}$$

If $a_1a_2a_3 \in H$, thus we have $\chi(a_1a_2a_3) = \overline{\chi}(a_1a_2a_3) = 1$. Then by Lemma 2.3, we have

$$\begin{aligned} M(a_1, a_2, a_3) &= p^2 + \frac{p-1}{p}(G^3(\chi) + G^3(\overline{\chi})) \\ &= p^2 + (p-1)\left[\frac{c + 3\sqrt{3}d\mathrm{i}}{2} + \frac{c - 3\sqrt{3}d\mathrm{i}}{2}\right] \\ &= p^2 + c(p-1). \end{aligned}$$

If $a_1a_2a_3 \in G \setminus H$, then by Lemma 2.3, we can take multiplicative character $\chi$ of order 3 satisfying

$$\chi(a_1a_2a_3) = \frac{-1 + \sqrt{3}\mathrm{i}}{2}, \quad G^3(\chi) = p \cdot \frac{c + 3\sqrt{3}d\mathrm{i}}{2},$$

where $c$ and $d$ are uniquely determined by (1.3). Thus we have

$$\begin{aligned} M(a_1, a_2, a_3) &= p^2 + (p-1)\left(\frac{-1 - \sqrt{3}\mathrm{i}}{2} \cdot \frac{c + 3\sqrt{3}d\mathrm{i}}{2} + \frac{-1 + \sqrt{3}\mathrm{i}}{2} \cdot \frac{c - 3\sqrt{3}d\mathrm{i}}{2}\right) \\ &= p^2 + \frac{1}{2}(p-1)(9d - c). \end{aligned}$$

This completes the proof of the Theorem 1.4.

*Proof of Theorem 1.5.* We have

$$\begin{aligned}
M(a_1,a_2,a_3) &= \sum_{\substack{x_1,x_2,x_3=0\\ a_1x_1^3+a_2x_2^3+a_3x_3^3\equiv 0(\bmod p)}}^{p-1} 1\\
&= \sum_{x_3=1}^{p-1}\sum_{\substack{x_1,x_2=0\\ a_1x_1^3+a_2x_2^3+a_3x_3^3\equiv 0(\bmod p)}}^{p-1} 1 + \sum_{\substack{x_1,x_2=0\\ a_1x_1^3+a_2x_2^3\equiv 0(\bmod p)}}^{p-1} 1\\
&= \sum_{x_3=1}^{p-1}\sum_{\substack{x_1,x_2=0\\ a_1(-x_1\overline{x}_3)^3+a_2(x_2\overline{x}_3)^3\equiv a_3(\bmod p)}}^{p-1} 1 + 1 + \sum_{x_1=1}^{p-1}\sum_{\substack{x_2=1\\ (-\overline{x}_1x_2)^3\equiv a_1\overline{a}_2(\bmod p)}}^{p-1} 1\\
&= (p-1)\sum_{\substack{x_1,x_2=0\\ a_1x_1^3+a_2x_2^3\equiv a_3(\bmod p)}}^{p-1} 1 + 1 + \sum_{x_1=1}^{p-1}\sum_{\substack{x=1\\ x^3\equiv a_1\overline{a}_2(\bmod p)}}^{p-1} 1\\
&= (p-1)N(a_1,a_2,a_3) + 1 + \sum_{x_1=1}^{p-1}\sum_{\substack{x=1\\ x^3\equiv a_1\overline{a}_2(\bmod p)}}^{p-1} 1.
\end{aligned}$$

If $a_1 \equiv a_2(\bmod H)$, the number of solutions of the congruence equation $x^3 \equiv a_1\overline{a}_2(\bmod p)$ is exactly 3. Thus we have

$$M(a_1,a_2,a_3) = (p-1)N(a_1,a_2,a_3) + 1 + 3(p-1) = (p-1)N(a_1,a_2,a_3) + 3p - 2.$$

If $a_1 \not\equiv a_2(\bmod H)$, the congruence equation $x^3 \equiv a_1\overline{a}_2(\bmod p)$ has no solution. Thus we have

$$M(a_1,a_2,a_3) = (p-1)N(a_1,a_2,a_3) + 1.$$

Hence Theorem 1.5 immediately follows from Theorem 1.4.

*Proof of Theorem 1.7.* First, by Lemma 2.3, there is a unique multiplicative character $\chi$ of order 3 such that

$$\chi(z) = \frac{-1+\sqrt{3}\mathrm{i}}{2},\quad G^3(\chi) = p\cdot\frac{c+3\sqrt{3}d\mathrm{i}}{2},$$

where $c$ and $d$ are uniquely determined by (1.2).

Note that $\chi(-1) = 1$. By the orthogonality of additive character and Lemma 2.3, we have

$$\begin{aligned}
L(z) &= \frac{1}{p}\sum_{m=0}^{p-1}\sum_{x_1=0}^{p-1}\sum_{x_2=0}^{p-1}\sum_{x_3=0}^{p-1}\sum_{x_4=0}^{p-1} e\left(\frac{m(x_1^3+x_2^3+x_3^3+x_4^3-z)}{p}\right)\\
&= p^3 + \frac{1}{p}\sum_{m=1}^{p-1} S^4(m)e\left(\frac{-mz}{p}\right)\\
&= p^3 + \frac{1}{p}\sum_{m=1}^{p-1}[\overline{\chi}(m)G(\chi)+\chi(m)G(\overline{\chi})]^4 e\left(\frac{-mz}{p}\right)
\end{aligned}$$

$$
\begin{aligned}
&= p^3 - 6p + \frac{1}{p}\sum_{m=1}^{p-1}\left[\overline{\chi}(m)G^4(\chi) + 4p\chi(m)G^2(\chi) + 4p\overline{\chi}(m)G^2(\overline{\chi}) + \chi(m)G^4(\overline{\chi})\right] e\left(\frac{-mz}{p}\right)\\
&= p^3 - 6p + \frac{1}{p}G^4(\chi)\sum_{m=1}^{p-1}\overline{\chi}(m)e\left(\frac{-mz}{p}\right) + \frac{1}{p}G^4(\overline{\chi})\sum_{m=1}^{p-1}\chi(m)e\left(\frac{-mz}{p}\right)\\
&\quad + 4G^2(\chi)\sum_{m=1}^{p-1}\chi(m)e\left(\frac{-mz}{p}\right) + 4G^2(\overline{\chi})\sum_{m=1}^{p-1}\overline{\chi}(m)e\left(\frac{-mz}{p}\right)\\
&= p^3 - 6p + \frac{1}{p}G^4(\chi)\chi(-z)G(\overline{\chi}) + \frac{1}{p}G^4(\overline{\chi})\overline{\chi}(-z)G(\chi) + 4\overline{\chi}(-z)G^3(\chi) + 4\chi(-z)G^3(\overline{\chi})\\
&= p^3 - 6p + \chi(z)G^3(\chi) + \overline{\chi}(z)G^3(\overline{\chi}) + 4\overline{\chi}(z)G^3(\chi) + 4\chi(z)G^3(\overline{\chi})\\
&= p^3 - 6p + p\cdot\frac{-1+\sqrt{3}\mathrm{i}}{2}\cdot\frac{c+3\sqrt{3}d\mathrm{i}}{2} + p\cdot\frac{-1-\sqrt{3}\mathrm{i}}{2}\cdot\frac{c-3\sqrt{3}d\mathrm{i}}{2}\\
&\quad + 4p\cdot\frac{-1-\sqrt{3}\mathrm{i}}{2}\cdot\frac{c+3\sqrt{3}d\mathrm{i}}{2} + 4p\cdot\frac{-1+\sqrt{3}\mathrm{i}}{2}\cdot\frac{c-3\sqrt{3}d\mathrm{i}}{2}\\
&= p^3 - 6p - \frac{1}{2}p(5c - 27d).
\end{aligned}
$$

This completes the proof of the Theorem 1.7.

**Example 4.1.** We take $\mathbb{F}_{31} := \{\overline{0}, \overline{1}, \cdots, \overline{30}\}$. Consider the cubic equations $x_1^3 + 2x_2^3 + 3x_3^3 \equiv 0(\bmod 31)$ and $x_1^3 + 2x_2^3 \equiv 3(\bmod 31)$.

If the integers $c$ and $d$ satisfying that $4\cdot 31 = c^2 + 27d^2, c \equiv 1(\bmod 3), 9d \equiv c(2\times 6^{\frac{31-1}{3}} + 1)(\bmod 31)$, then $c = 4, d = 2$. One can check that $2^{\frac{31-1}{3}} \equiv 1(\bmod 31)$ and $6^{\frac{31-1}{3}} \equiv 25(\bmod 31)$, so 6 is not a cubic element in $\mathbb{F}_{31}$ and 2 is a cubic element in $\mathbb{F}_{31}$. Thus $6 \notin H$ and $1 \equiv 2(\bmod H)$.

It then follows from Theorems 1.4 and 1.5 that the numbers $M(1, 2, 3)$ and $N(1, 2, 3)$ of the cubic equations $x_1^3 + 2x_2^3 + 3x_3^3 \equiv 0(\bmod 31)$ and $x_1^3 + 2x_2^3 \equiv 3(\bmod 31)$ are given by

$$M(1, 2, 3) = 31^2 + \frac{1}{2}(31 - 1)(9\times 2 - 4) = 1171$$

and

$$N(1, 2, 3) = 31 - 2 + \frac{1}{2}(9\times 2 - 4) = 36.$$

We list the solutions of equation $x_1^3 + 2x_2^3 \equiv 3(\bmod 31)$ as belove:

$$
\begin{aligned}
&(\overline{1},\overline{1}); (\overline{1},\overline{5}); (\overline{1},\overline{25}); (\overline{5},\overline{1}); (\overline{5},\overline{5}); (\overline{5},\overline{25}); (\overline{25},\overline{1}); (\overline{25},\overline{5}); (\overline{25},\overline{25});\\
&(\overline{6},\overline{4}); (\overline{6},\overline{7}); (\overline{6},\overline{20}); (\overline{26},\overline{4}); (\overline{26},\overline{7}); (\overline{26},\overline{20}); (\overline{30},\overline{4}); (\overline{30},\overline{7}); (\overline{30},\overline{20});\\
&(\overline{4},\overline{8}); (\overline{4},\overline{9}); (\overline{4},\overline{14}); (\overline{7},\overline{8}); (\overline{7},\overline{9}); (\overline{7},\overline{14}); (\overline{20},\overline{8}); (\overline{20},\overline{9}); (\overline{20},\overline{14});\\
&(\overline{16},\overline{17}); (\overline{16},\overline{22}); (\overline{16},\overline{23}); (\overline{18},\overline{17}); (\overline{18},\overline{22}); (\overline{18},\overline{23}); (\overline{28},\overline{17}); (\overline{28},\overline{22}); (\overline{28},\overline{23}).
\end{aligned}
$$

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