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Research article

Hybrid pair of multivalued mappings in modular-like metric spaces and applications

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Abstract: Our aim is to prove some new fixed point theorems for a hybrid pair of multivalued α_*-dominated mappings involving a generalized Q-contraction in a complete modular-like metric space. Further results involving graphic contractions for a pair of multi-graph dominated mappings have been considered. Applying our obtained results, we resolve a system of nonlinear integral equations.

Keywords: fixed point; generalized Q-contraction; α_*-dominated multivalued mapping; graphic contraction; integral equation
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1. Introduction and preliminaries

If the image of a point x under two single-valued mappings is x itself, then x is said to be a fixed point of these mappings. Banach [7] proved a meaningful result for contraction mappings. Due to its significance, several authors, like Acar *et al.* [3], Altun *et al.* [5], Aslantas *et al.* [6], Sahin *et al.* [27], Hussain *et al.* [17], Hammad *et al.* [14–16] and Ceng *et al.* [8–11] presented many related useful applications in fixed point theory. In [23,31], the authors showed a new iterative scheme for the solution of nonlinear mixed Volterra Fredholm type fractional delay integro-differential equations of different orders. Chistyakov [13] introduced the notion of a modular metric space. Mongkolkeha *et al.* [21] established some results in modular metric spaces for contraction mappings. Chaipunya *et al.* [12], Abdou *et al.* [2] and Alfuraidan *et al.* [4] showed fixed point results for multivalued mappings in modular metric spaces. Abdou *et al.* [1] proved fixed point theorems of pointwise contractions in modular metric spaces. Hussain *et al.* [19] discussed some fixed point theorems for generalized F-contractions in fuzzy metric and modular metric spaces. Later, Padcharoen *et al.* [22] introduced the concept of α-type F-contractions in modular metric spaces and showed fixed point and periodic point results for such a contraction. Recently, Rasham *et al.* [26] introduced a modular-like metric space and proved results for families of mappings in such spaces. In this research work, we prove existence of fixed point results for a hybrid pair of multivalued maps fulfilling generalized rational type F-contractions, by using a weaker class of strictly increasing mappings F rather than the class of mappings introduced by Wardowski [30].

Let us state the following preliminary concepts.

Definition 1.1. [26] Let B be a non-empty set. A function $\upsilon : (0, \infty) \times B \times B \to [0, \infty)$ is said to be a modular-like metric on B, if for each $e, i, o \in B$ and $\upsilon(a, i, o) = \upsilon_a(i, o)$, the following hold:
(i) $\upsilon_a(i, o) = \upsilon_a(o, i)$ for all $a > 0$;
(ii) $\upsilon_a(i, o) = 0$ for all $a > 0$ implies $i = o$;
(iii) $\upsilon_{l+n}(i, o) \leq \upsilon_l(i, e) + \upsilon_n(e, o)$ for all $l, n > 0$.
The pair (B, υ) is said to be a modular-like metric space. If we change (ii) by "$\upsilon_l(i, o) = 0$ for each $l > 0$ iff $i = o$", then (B, υ) becomes a modular metric space. While, by changing (ii) with "$\upsilon_l(i, o) = 0$ for some $l > 0$, such that $i = o$", we obtain a regular modular-like metric space. For $s \in B$ and $\varepsilon > 0$, $\overline{C_{\upsilon_l}(s, \varepsilon)} = \{t \in B : |\upsilon_l(s, t) - \upsilon_l(t, t)| \leq \varepsilon\}$ is a closed ball in (B, υ).

Example 1.2. Let $B = [0, \infty) \times [0, \infty)$. Define $\upsilon : (0, \infty) \times B \times B \to [0, \infty)$ as

$$(i)\ \upsilon(a, (e, p), (i, o)) = \frac{e + p + i + o}{a},$$
$$(ii)\ \upsilon(a, (e, p), (i, o)) = \frac{\max\{e, p, i, o\}}{a}.$$

The functions given in (i) and (ii) are examples of a modular-like metric on B.

Definition 1.3. [26] Let (B, υ) be a modular-like metric space.
(i) A sequence $(a_n)_{n\in\mathbb{N}}$ in B is said to be υ-convergent to a point $a \in B$ for some $l > 0$ if $\lim_{n\to+\infty} \upsilon_l(a_n, a) = \upsilon_l(a, a)$.
(ii) A sequence $(a_n)_{n\in\mathbb{N}}$ in B is said to be an υ-Cauchy sequence for some $l > 0$ if $\lim_{n,m\to\infty} \upsilon_l(a_m, a_n)$ exists and is finite.
(iii) B is called υ-complete if each υ-Cauchy sequence $(a_n)_{n\in\mathbb{N}}$ in B is υ-convergent to some $a \in B$, that is,

$$\lim_{n\to+\infty} \upsilon_l(a_n, a) = \upsilon_l(a, a).$$

(iv) If every sequence has a convergent subsequence, then B is called compact.

Definition 1.4. [26] Let (B, υ) be a modular-like metric space and $U \subseteq B$. An element p_0 in U verifying

$$\upsilon_l(s, U) = \inf_{p_0 \in U} \upsilon_l(s, p_0)$$

is called a best approximation in U for $s \in B$. If each $s \in B$ possesses a best approximation in U, then U is called a proximinal set.

From now on, let $P(B)$ represent the set of proximinal compact subsets in B.

Example 1.5. Let $B = [0, \infty)$ and $\upsilon_l(s, r) = \frac{1}{w}(s + r)$ with $w > 0$. Take $U = [7, 8]$. Then for any $m \in B$,

$$\upsilon_l(m, U) = \upsilon_l(m, [7, 8]) = \inf_{n \in [7,8]} \upsilon_l(m, n) = \upsilon_l(m, 7).$$

So 7 is a best approximation in U for any $m \in B$. Moreover, $[7, 8]$ is a proximinal set.

Definition 1.6. [26] The mapping $H_{\upsilon_l} : P(B) \times P(B) \to [0, \infty)$, given by

$$H_{\upsilon_l}(X, Y) = \max\{\sup_{\sigma \in X} \upsilon_l(\sigma, Y), \ \sup_{\varsigma \in Y} \upsilon_l(\varsigma, X)\},$$

is known as an υ_l- Hausdorff metric. Note that $(P(B), H_{\upsilon_l})$ is named as an υ_l- Hausdorff metric space.

Example 1.7. Let $B = [0, \infty)$ and $\upsilon_l(\theta, \vartheta) = \frac{1}{l}(\theta + \vartheta)$ with $l > 0$. Taking $W = [5, 6]$ and $Q = [9, 10]$ we get $H_{\upsilon_l}(W, Q) = \frac{15}{l}$.

Definition 1.8. [26] Let (X, υ) be a modular-like metric space. υ is said to satisfy the $\triangle_M$-condition if $\lim_{n,m\to\infty} \upsilon_p(x_n, x_m) = 0$, where $p \in \mathbb{N}$ implies $\lim_{n,m\to\infty} \upsilon_l(x_n, x_m) = 0$, for some $l > 0$.

Definition 1.9. [28] Let $C \neq \Phi$, $Y : C \to P(C)$ be a multivalued mapping, $E \subseteq C$ and $\alpha : C \times C \to [0, +\infty)$ be a function. Then Y is said to be α_*-admissible on E if $\alpha_*(Ye, Yz) = \inf\{\alpha(l, m) : l \in Ye, m \in Yz\} \geq 1$, whenever $\alpha(e, z) \geq 1$ for all $e, z \in E$.

Definition 1.10. [29] Let $B \neq \Phi$, $Y : B \to P(B)$ be a multi-valued mapping, $R \subseteq B$ and $\alpha : B \times B \to [0, \infty)$ be a function. Then Y is said to be α_*-dominated on R if for all $v \in R$, $\alpha_*(v, Yv) = \inf\{\alpha(v, j) : j \in Yv\} \geq 1$.

Definition 1.11. [30] Let (C, d) be a metric space. A self mapping $H : C \to C$ is said to be a Q-contraction if for each $g, k \in C$, there is $\tau > 0$ such that $d(Ca, Cg) > 0$ implies

$$\tau + Q\left(d(Ca, Cg)\right) \leq Q\left(d(a, g)\right),$$

where $Q : (0, \infty) \to \mathbb{R}$ satisfies the following:
(F1) For any $k \in (0, 1)$, $\lim_{\sigma\to 0^+} \sigma^k Q(\sigma) = 0$;
(F2) For each $u, v > 0$ such that $u < v$, $Q(u) < Q(v)$;
(F3) $\lim_{n\to+\infty} \sigma_n = 0$ if and only if $\lim_{n\to+\infty} Q(\sigma_n) = -\infty$ for every positive sequence $\{\sigma_n\}_{n=1}^{\infty}$.

Let F denote the set of mappings such that (F1)–(F3) hold.

Lemma 1.12. *[26] Let $(£, \upsilon)$ be a modular-like metric space. Let $(P(£), H_{\upsilon_l})$ be a Hausdorff υ_l−metric-like space. Then, for all $b \in U$ and for each $U, Y \in P(£)$, there is $b_a \in Y$ such that $H_{\upsilon_l}(U, Y) \geq \upsilon_l(a, b_a)$.*

Example 1.13. [24] Let $W = \mathbb{R}$. Consider $\alpha : W \times W \to [0, \infty)$ as

$$\alpha(s, r) = \begin{cases} 1 \text{ if } s > r \\ \frac{1}{4} \text{ if } s \ngtr r \end{cases}.$$

Define $L, N : W \to P(W)$ by

$$Ls = [-4 + s, -3 + s] \text{ and } Nr = [-2 + r, -1 + r].$$

The α_*-dominated property for L and N holds. Note that L and N are not α_*-admissible.

2. Main results

Let $(£, \upsilon)$ be a modular-like metric space, $\delta_0 \in £$, and $R, C : £ \to P(£)$ be two multifunctions on £. For $\delta_1 \in R\delta_0$ with $\upsilon_1(\delta_0, R\delta_0) = \upsilon_1(\delta_0, \delta_1)$, take $\delta_2 \in C\delta_1$ such that $\upsilon_1(\delta_1, C\delta_1) = \upsilon_1(\delta_1, \delta_2)$. Choose $\delta_3 \in R\delta_2$ such that $\upsilon_1(\delta_2, R\delta_2) = \upsilon_1(\delta_2, \delta_3)$. In this way, we get a sequence $\{CR(\delta_n)\}$ in £, where

$$\delta_{2n+1} \in R\delta_{2n},\ \delta_{2n+2} \in C\delta_{2n+1},$$

for all $n \in \mathbb{N} \cup \{0\}$. Note that $\upsilon_1(\delta_{2n}, R\delta_{2n}) = \upsilon_1(\delta_{2n}, \delta_{2n+1})$ and $\upsilon_1(\delta_{2n+1}, C\delta_{2n+1}) = \upsilon_1(\delta_{2n+1}, \delta_{2n+2})$. $\{CR(\delta_n)\}$ is said to be a sequence in £ generated by δ_0. If $R = C$, then we denote $\{£R(\delta_n)\}$ instead of $\{CR(\delta_n)\}$.

Theorem 2.1. *Let $(£, \upsilon)$ be a complete modular-like metric space. Suppose that υ is regular and verifies the $\triangle_M$-condition. Let $\delta_0 \in £$, $\alpha : £ \times £ \to [0, \infty)$ and $R, C : £ \to P(£)$ be α_*-dominated multifunctions on £. Assume there are $\tau > 0$ and $Q \in F$ such that*

$$\tau + Q(H_{\upsilon_1}(Rt, C\delta)) \leq Q\left(\max\left\{\upsilon_1(t, \delta), \upsilon_1(t, Rt), \frac{\upsilon_2(t, C\delta)}{2}, \frac{\upsilon_1(t, Rt).\upsilon_1(\delta, C\delta)}{1 + \upsilon_1(t, \delta)}\right\}\right) \tag{2.1}$$

where $t, \delta \in \{CR(\delta_n)\}$, $\alpha(t, \delta) \geq 1$ or $\alpha(\delta, t) \geq 1$, and $H_{\upsilon_1}(Rt, C\delta) > 0$. Then the sequence $\{CR(\delta_n)\}$ generated by δ_0 converges to $e \in £$ and for each $n \in \mathbb{N}$, $\alpha(\delta_n, \delta_{n+1}) \geq 1$. Furthermore, if e satisfies (2.1), $\alpha(\delta_n, e) \geq 1$ and $\alpha(e, \delta_n) \geq 1$ for all integers $n \geq 0$, then R and C have a common fixed point e in £.

Proof. Consider a sequence $\{CR(\delta_n)\}$. Obviously, $\delta_n \in £$ for each integer $n \geq 0$. If j is odd, then $j = 2\imath + 1$ for some $\imath \in \mathbb{N}$. By definition of α_*-dominated mappings, one has $\alpha_*(\delta_{2\imath}, R\delta_{2\imath}) \geq 1$ and $\alpha_*(\delta_{2\imath+1}, C\delta_{2\imath+1}) \geq 1$. Since $\alpha_*(\delta_{2\imath}, R\delta_{2\imath}) \geq 1$, one gets $\inf\{\alpha(\delta_{2\imath}, b) : b \in R\delta_{2\imath}\} \geq 1$. Also, $\delta_{2\imath+1} \in R\delta_{2\imath}$ and so $\alpha(\delta_{2\imath}, \delta_{2\imath+1}) \geq 1$. Moreover, $\delta_{2\imath+2} \in C\delta_{2\imath+1}$ and so $\alpha(\delta_{2\imath+1}, \delta_{2\imath+2}) \geq 1$. In view of Lemma 1.12, we have

$$\begin{aligned}
\tau + Q(\upsilon_1(\delta_{2\imath+1}, \delta_{2\imath+2})) &\leq \tau + Q(H_{\upsilon_1}(R\delta_{2\imath}, C\delta_{2\imath+1})) \\
&\leq Q\left(\max\left\{\begin{array}{c} \upsilon_1(\delta_{2\imath}, \delta_{2\imath+1}), \upsilon_1(\delta_{2\imath}, R\delta_{2\imath}), \frac{\upsilon_2(\delta_{2\imath}, C\delta_{2\imath+1})}{2}, \\ \frac{\upsilon_1(\delta_{2\imath}, R\delta_{2\imath}).\upsilon_1(\delta_{2\imath+1}, C\delta_{2\imath+1})}{1+\upsilon_1(\delta_{2\imath}, \delta_{2\imath+1})} \end{array}\right\}\right) \\
&\leq Q\left(\max\left\{\begin{array}{c} \upsilon_1(\delta_{2\imath}, \delta_{2\imath+1}), \upsilon_1(\delta_{2\imath}, \delta_{2\imath+1}), \frac{\upsilon_1(\delta_{2\imath}, \delta_{2\imath+1})+\upsilon_1(\delta_{2\imath+1}, \delta_{2\imath+2})}{2}, \\ \frac{\upsilon_1(\delta_{2\imath}, \delta_{2\imath+1}).\upsilon_1(\delta_{2\imath+1}, \delta_{2\imath+2})}{1+\upsilon_1(\delta_{2\imath}, \delta_{2\imath+1})} \end{array}\right\}\right) \\
&\leq Q(\max\{\upsilon_1(\delta_{2\imath}, \delta_{2\imath+1}), \upsilon_1(\delta_{2\imath+1}, \delta_{2\imath+2})\}).
\end{aligned}$$

This implies

$$\tau + Q(\upsilon_1(\delta_{2\grave{\imath}+1}, \delta_{2\grave{\imath}+2})) \leq Q(\max\{\upsilon_1(\delta_{2\grave{\imath}}, \delta_{2\grave{\imath}+1}), \upsilon_1(\delta_{2\grave{\imath}+1}, \delta_{2\grave{\imath}+2})\}). \quad (2.2)$$

Now, if

$$\max\{\upsilon_1(\delta_{2\grave{\imath}}, \delta_{2\grave{\imath}+1}), \upsilon_1(\delta_{2\grave{\imath}+1}, \delta_{2\grave{\imath}+2})\} = \upsilon_1(\delta_{2\grave{\imath}+1}, \delta_{2\grave{\imath}+2}),$$

then from (2.2), we have

$$Q(\upsilon_1(\delta_{2\grave{\imath}+1}, \delta_{2\grave{\imath}+2})) \leq Q(\upsilon_1(\delta_{2\grave{\imath}+1}, \delta_{2\grave{\imath}+2})) - \tau,$$

which is a contradiction. Therefore,

$$\max\{\upsilon_1(\delta_{2\grave{\imath}}, \delta_{2\grave{\imath}+1}), \upsilon_1(\delta_{2\grave{\imath}+1}, \delta_{2\grave{\imath}+2})\} = \upsilon_1(\delta_{2\grave{\imath}}, \delta_{2\grave{\imath}+1})$$

for all $\grave{\imath} \geq 0$. Hence, from (2.2), we have

$$Q(\upsilon_1(\delta_{2\grave{\imath}+1}, \delta_{2\grave{\imath}+2})) \leq Q(\upsilon_1(\delta_{2\grave{\imath}}, \delta_{2\grave{\imath}+1})) - \tau. \quad (2.3)$$

Similarly, we have

$$Q(\upsilon_1(\delta_{2\grave{\imath}}, \delta_{2\grave{\imath}+1})) \leq Q(\upsilon_1(\delta_{2\grave{\imath}-1}, \delta_{2\grave{\imath}})) - \tau \quad (2.4)$$

for all $\grave{\imath} \geq 0$. By (2.3) and (2.4), we have

$$Q(\upsilon_1(\delta_{2\grave{\imath}+1}, \delta_{2\grave{\imath}+2})) \leq Q(\upsilon_1(\delta_{2\grave{\imath}-1}, \delta_{2\grave{\imath}})) - 2\tau.$$

Repeating these steps, we get

$$Q(\upsilon_1(\delta_{2\grave{\imath}+1}, \delta_{2\grave{\imath}+2})) \leq Q(\upsilon_1(\delta_0, \delta_1)) - (2\grave{\imath} + 1)\tau. \quad (2.5)$$

Similarly, we have

$$Q(\upsilon_1(\delta_{2\grave{\imath}}, \delta_{2\grave{\imath}+1})) \leq Q(\upsilon_1(\delta_0, \delta_1)) - 2\grave{\imath}\tau. \quad (2.6)$$

By (2.5) and (2.6), we obtain

$$Q(\upsilon_1(\delta_n, \delta_{n+1})) \leq Q(\upsilon_1(\delta_0, \delta_1)) - n\tau. \quad (2.7)$$

Letting $n \to \infty$ in (2.7), one obtains

$$\lim_{n\to\infty} Q(\upsilon_1(\delta_n, \delta_{n+1})) = -\infty.$$

Since $Q \in F$,

$$\lim_{n\to\infty} \upsilon_1(\delta_n, \delta_{n+1}) = 0. \quad (2.8)$$

Due to (F1) of F, there is $k \in (0, 1)$ such that

$$\lim_{n\to\infty} (\upsilon_1(\delta_n, \delta_{n+1}))^k (Q(\upsilon_1(\delta_n, \delta_{n+1})) = 0. \quad (2.9)$$

By (2.7), for all $n \in \mathbb{N}$, we obtain

$$(\upsilon_1(\delta_n, \delta_{n+1}))^k (Q(\upsilon_1(\delta_n, \delta_{n+1})) - Q(\upsilon_1(\delta_0, \delta_1)) \leq -(\upsilon_1(\delta_n, \delta_{n+1}))^k n\tau \leq 0. \quad (2.10)$$

Using (2.8), (2.9) and taking $n \to \infty$ in (2.10), we have

$$\lim_{n\to\infty} n(\upsilon_1(\delta_n, \delta_{n+1}))^k = 0. \tag{2.11}$$

By (2.11), there is $n_1 \in \mathbb{N}$ such that $n(\upsilon_1(\delta_n, \delta_{n+1}))^k \leq 1$ for all $n \geq n_1$, or

$$\upsilon_1(\delta_n, \delta_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}} \text{ for all } n \geq n_1.$$

Letting $p > 0$ and $m = n + p > n > n_1$, we get

$$\upsilon_p(\delta_n, \delta_m) \leq \upsilon_1(\delta_n, \delta_{n+1}) + \upsilon_1(\delta_{n+1}, \delta_{n+2}) + \cdots + \upsilon_1(\delta_m, \delta_{m+1}) \leq \sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{k}}}.$$

Since $k \in (0, 1)$, $\frac{1}{k} > 1$ and so the series $\sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{k}}}$ converges. Thus,

$$\lim_{n,m\to\infty} \upsilon_p(\delta_n, \delta_m) = 0.$$

Since υ satisfies the $\triangle_M$-condition, we have

$$\lim_{n,m\to\infty} \upsilon_1(\delta_n, \delta_m) = 0. \tag{2.12}$$

Hence $\{CR(\delta_n)\}$ is Cauchy in the regular complete modular-like metric space $(£, \upsilon)$ and so there is $e \in £$ such that $\{CR(\delta_n)\} \to e$ as $n \to \infty$ and thus

$$\lim_{n\to\infty} \upsilon_1(\delta_n, e) = 0. \tag{2.13}$$

Now, by Lemma 1.12, one obtains

$$\tau + Q(\upsilon_1(\delta_{2n+1}, Ce) \leq \tau + Q(H_{\upsilon_1}(R\delta_{2n}, Ce)). \tag{2.14}$$

Now, there exists $\delta_{2n+1} \in R\delta_{2n}$ such that $\upsilon_1(\delta_{2n}, R\delta_{2n}) = \upsilon_1(\delta_{2n}, \delta_{2n+1})$. From assumption, $\alpha(\delta_n, e) \geq 1$. Assume that $\upsilon_1(e, Ce) > 0$. Then there is an integer $p > 0$ such that $\upsilon_1(\delta_{2n+1}, Ce) > 0$ for $n \geq p$. Now, if $H_{\upsilon_1}(R\delta_{2n}, Ce) > 0$, then by (2.1), we have

$$\tau + Q(\upsilon_1(\delta_{2n+1}, Ce)) \leq Q\left(\max\left\{\begin{array}{c} \upsilon_1(\delta_{2n}, e), \upsilon_1(\delta_{2n}, e), \\ \frac{\upsilon_1(\delta_{2n},\delta_{2n+1})+\upsilon_1(\delta_{2n+1},Ce)}{2}, \\ \frac{\upsilon_1(\delta_{2n},R\delta_{2n}).\upsilon_1(Q,Ce)}{1+\upsilon_1(\delta_{2n},e)} \end{array}\right\}\right).$$

Letting $n \to \infty$ and using (2.13), we get

$$\tau + Q(\upsilon_1(e, Ce)) \leq Q(\upsilon_1(e, Ce)).$$

Since Q is strictly increasing, (2.14) implies

$$\upsilon_1(e, Ce) < \upsilon_1(e, Ce).$$

This is a contradiction. Hence $\upsilon_1(e, Ce) = 0$ and so $e \in Ce$.

Similarly, we can show that $\upsilon_1(e, Re) = 0$, that is, $e \in Re$. Hence e is a common fixed point of both mappings R and C in £. □

Corollary 2.2. *Let* $(£, \upsilon)$ *be a complete modular-like metric space. Suppose that* υ *is regular and verifies the* $\triangle_M$*-condition. Let* $\alpha : £ \times £ \to [0, \infty)$ *and* $R, C : £ \to P(£)$ *be* α_**-dominated multifunctions on* £. *Assume there are* $\tau > 0$ *and* $Q \in F$ *such that*

$$\tau + Q(H_{\upsilon_1}(Rt, C\delta)) \leq Q\left(\max\left\{\upsilon_1(t,\delta), \upsilon_1(t, Rt), \frac{\upsilon_2(t, C\delta)}{2}, \frac{\upsilon_1(t, Rt).\upsilon_1(\delta, C\delta)}{1 + \upsilon_1(t,\delta)}\right\}\right),$$

where $t, \delta \in £$, $\alpha(t,\delta) \geq 1$ *or* $\alpha(\delta, t) \geq 1$, *and* $H_{\upsilon_1}(Rt, C\delta) > 0$. *Then there exists a sequence* $\{\delta_n\}$ *in* £ *converging to* $e \in £$ *and for each* $n \in \mathbb{N}$, $\alpha(\delta_n, \delta_{n+1}) \geq 1$. *Also, if* $\alpha(\delta_n, e) \geq 1$ *and* $\alpha(e, \delta_n) \geq 1$ *for all integers* $n \geq 0$, *then* R *and* C *have a common fixed point* e *in* £.

Example 2.3. Let $£ = \mathbb{R}_+ \cup \{0\}$. Take $\upsilon_2(r, m) = r + m$ and $\upsilon_1(e, t) = \frac{1}{2}(e + t)$ for all $e, t \in £$. Define $R, C : £ \to P(£)$ by

$$Rv = \begin{cases} \left[\dfrac{v}{3}, \dfrac{2v}{3}\right] \text{ if } v \in \{1, \frac{1}{3}, \frac{1}{12}, \frac{1}{36}, \frac{1}{144}, \cdots\} \\ [7v, 10v] \text{ if } v \notin \{1, \frac{1}{3}, \frac{1}{12}, \frac{1}{36}, \frac{1}{144}, \cdots\} \end{cases}$$

and

$$Cv = \begin{cases} \left[\dfrac{v}{4}, \dfrac{3v}{4}\right] \text{ if } v \in \{1, \frac{1}{3}, \frac{1}{12}, \frac{1}{36}, \frac{1}{144}, \cdots\} \\ [5v, 13v] \text{ if } v \notin \{1, \frac{1}{3}, \frac{1}{12}, \frac{1}{36}, \frac{1}{144}, \cdots\} \end{cases}.$$

Suppose that $v_0 = 1$. Then $\upsilon_1(v_0, Rv_0) = \upsilon_1(1, R1) = \upsilon_1(1, \frac{1}{3})$ and so $v_1 = \frac{1}{3}$. Now, $\upsilon_1(v_1, Cv_1) = \upsilon_1(\frac{1}{3}, C\frac{1}{3}) = \upsilon_1(\frac{1}{3}, \frac{1}{12})$ and thus $v_2 = \frac{1}{12}$. Now, $\upsilon_1(v_2, Rv_2) = \upsilon_1(\frac{1}{12}, R\frac{1}{12}) = \upsilon_1(\frac{1}{12}, \frac{1}{36})$ and so $v_3 = \frac{1}{36}$. Continuing in this way, we have $\{CR(v_n)\} = \{1, \frac{1}{3}, \frac{1}{12}, \frac{1}{36}, \frac{1}{144}, \cdots\}$. Define $\alpha : £ \times £ \to [0, \infty)$ as

$$\alpha(r, t) = \begin{cases} 1 & \text{if } r > t \\ \frac{1}{2} & \text{otherwise} \end{cases}.$$

Let $v, y \in \{CR(v_n)\}$ with $\alpha(v, y) \geq 1$. Then

$$\begin{aligned} H_{\upsilon_1}(Rv, Cy) &= \max\{\sup_{a \in Rv} \upsilon_1(a, Cy), \sup_{b \in Cy} \upsilon_1(Rv, b)\} \\ &= \max\left\{\begin{array}{c} \upsilon_1\left(\frac{2v}{3}, \left[\frac{y}{4}, \frac{3y}{4}\right]\right), \\ \upsilon_1\left(\left[\frac{v}{3}, \frac{2v}{3}\right], \frac{3y}{4}\right) \end{array}\right\} \\ &= \max\left\{\upsilon_1\left(\frac{2v}{3}, \frac{y}{4}\right), \upsilon_1\left(\frac{v}{3}, \frac{3y}{4}\right)\right\} \\ &= \max\left\{\frac{2v}{3} + \frac{y}{4}, \frac{v}{3} + \frac{3y}{4}\right\}. \end{aligned}$$

Also,

$$\max\left\{\upsilon_1(v, y), \upsilon_1(v, Rv), \frac{\upsilon_2(v, Cy)}{2}, \frac{\upsilon_1(v, Rv).\upsilon_1(y, Cy)}{1 + \upsilon_1(v, y)}\right\} = \max\left\{\begin{array}{c} v + y, v + \frac{v}{3}, \\ \frac{1}{4}\left(v + \frac{y}{4}\right), \frac{\left(v + \frac{v}{3}\right).\left(y + \frac{y}{4}\right)}{1 + v + y} \end{array}\right\}.$$

If $Q(t) = \ln t$ and $\tau = \ln(1.2)$, then we have

$$\tau + Q(H_{\upsilon_1}(Rv, Cy)) \leq Q\left(\max\left\{\upsilon_1(v, y), \upsilon_1(v, Rv), \frac{\upsilon_2(v, Cy)}{2}, \frac{\upsilon_1(v, Rv).\upsilon_1(y, Cy)}{1 + \upsilon_1(v, y)}\right\}\right).$$

Hence all the conditions in Theorem 2.1 hold and so R and C possess a common fixed point.

Note that

$$Rv = \begin{cases} \left[\dfrac{v}{3}, \dfrac{2v}{3}\right] \text{ if } v \in \{1, \frac{1}{3}, \frac{1}{12}, \frac{1}{36}, \frac{1}{144}, \cdots\} \\ [7v, 10v] \text{ if } v \notin \{1, \frac{1}{3}, \frac{1}{12}, \frac{1}{36}, \frac{1}{144}, \cdots\} \end{cases}$$

and

$$Cv = \begin{cases} \left[\dfrac{v}{4}, \dfrac{3v}{4}\right] \text{ if } v \in \{1, \frac{1}{3}, \frac{1}{12}, \frac{1}{36}, \frac{1}{144}, \cdots\} \\ [5v, 13v] \text{ if } v \notin \{1, \frac{1}{3}, \frac{1}{12}, \frac{1}{36}, \frac{1}{144}, \cdots\} \end{cases}.$$

If $v = 2$ and $y = 3$, then we have

$$\begin{aligned} H_{\upsilon_1}(R2, C3) &= \max\left\{\sup_{a\in R2} \upsilon_1(a, C3), \sup_{b\in C3} \upsilon_1(R2, b)\right\} \\ &= \max\left[\left\{\sup_{a\in[14,20]} \upsilon_1(a, [15, 39]), \sup_{b\in[15,39]} \upsilon_1([14, 20], b)\right\}\right] \\ &= \max\left[\left\{\sup_{a\in[14,20]} \upsilon_1(a, 15), \sup_{b\in[15,39]} \upsilon_1(14, b)\right\}\right] \\ &= \max\{\upsilon_1(20, 15), \upsilon_1(14, 39)\} \\ &= \max\{20 + 15, 14 + 39\} = 53. \end{aligned}$$

Also

$$\begin{aligned} &\max\left\{\upsilon_1(v, y), \upsilon_1(v, Rv), \frac{\upsilon_2(v, Cy)}{2}, \frac{\upsilon_1(v, Rv).\upsilon_1(y, Cy)}{1 + \upsilon_1(v, y)}\right\} \\ &= \max\left\{\upsilon_1(2, 3), \upsilon_1(2, [14, 20]), \frac{\upsilon_2(2, [15, 39])}{2}, \frac{\upsilon_1(2, [14, 20]).\upsilon_1(3, [15, 39])}{1 + \upsilon_1(2, 3)}\right\} \\ &= \max\left\{5, 16, \frac{17}{4}, \frac{(16)(18)}{6}\right\} = 48. \end{aligned}$$

Now,

$$\ln(1.2) + \ln(53) > \ln(48).$$

This implies that

$$\tau + F(H_{\upsilon_1}(R2, C3) > F(\upsilon_1(2, 3)).$$

So the condition (2.1) does not hold on the whole space. Hence Corollary 2.2 and the other existing results in modular metric spaces cannot be applied to ensure the existence of a common fixed point. However, Theorem 2.1 is valid here.

Taking $R = C$ in Theorem 2.1, we may state the following corollary.

Corollary 2.4. *Let* $(£, \upsilon)$ *be a complete modular-like metric space. Suppose* υ *is regular and the* $\triangle_M$*-condition holds. Let* $\delta_0 \in £$, $\alpha : £ \times £ \to [0, \infty)$ *and* $R : £ \to P(£)$ *be a* α_**-dominated set-valued function on* $£$*. Assume there are* $\tau > 0$ *and* $Q \in F$ *such that*

$$\tau + Q(H_{\upsilon_l}(Rt, R\delta)) \leq Q\left(\max\left\{\begin{array}{c} \upsilon_1(t, \delta), \upsilon_1(t, Rt), \upsilon_2(t, R\delta), \\ \frac{\upsilon_1(t,Rt).\upsilon_1(\delta,R\delta)}{1+\upsilon_1(t,\delta)} \end{array}\right\}\right), \tag{2.15}$$

where $t, \delta \in \{£R(\delta_n)\}$, $\alpha(t, \delta) \geq 1$, *and* $H_{\upsilon_1}(Rt, R\delta) > 0$. *Then, the sequence* $\{£R(\delta_n)\}$ *generated by* δ_0 *converges to* $e \in £$ *and for each integer* $n \geq 0$, $\alpha(\delta_n, \delta_{n+1}) \geq 1$. *Also, if* e *satisfies (2.15) and either* $\alpha(\delta_n, e) \geq 1$ *or* $\alpha(e, \delta_n) \geq 1$ *for all integers* $n \geq 0$, *then* R *has a fixed point* e *in* £.

3. Applications to graph theory

Jachymski [20] initiated the graph concept in fixed point theory. Hussain *et al.* [18] gave new results for graphic contractions. Recently, Younis *et al.* [32] discussed a significant result on the graphical structure of extended *b*-metric spaces and Shoaib *et al.* [29] established some results on graph dominated set-valued mappings in the setting of *b*-metric like spaces. Further results on graph theory can be seen in [24, 25, 28].

Definition 3.1. [29] Let A be a non-empty set and $\Upsilon = (\mathcal{V}(\Upsilon), \mathcal{L}(\Upsilon))$ be a graph with $\mathcal{V}(\Upsilon) = A$. A mapping P from A into $P(A)$ is said to be multi-graph dominated on A if for each $\imath \in A$, we have $(\imath, \jmath) \in \mathcal{L}(\Upsilon)$, where $\jmath \in Pa$.

Theorem 3.2. *Let* (U, υ) *be a complete modular-like metric space endowed with a graph* Υ *and* $\delta_0 \in R$ *satisfying the following:*
(i) R *and* C *are multi-graph dominated functions on* $\{CR(\delta_n)\}$*;*
(ii) There are $\tau > 0$ *and* $Q \in F$ *such that*

$$\tau + Q(H_{\upsilon_1}(Rw, Ch)) \leq Q\left(\max\left\{\begin{array}{c} \upsilon_1(w, h), \upsilon_1(w, Rw), \frac{\upsilon_2(w,Ch)}{2}, \\ \frac{\upsilon_1(w,Rw).\upsilon_1(h,Ch)}{1+\upsilon_1(w,h)} \end{array}\right\}\right), \tag{3.1}$$

where $w, h \in \{CR(\delta_n)\}$, $(w,h) \in \mathcal{L}(\Upsilon)$ *or* $(h, w) \in \mathcal{L}(\Upsilon)$, *and* $H_{\upsilon_1}(Rw, Ch) > 0$.*Suppose that the regularity of* R *and the* $\triangle_M$*-condition are verified. Then* $(\delta_n, \delta_{n+1}) \in \mathcal{L}(\Upsilon)$ *and* $\{CR(\delta_n)\} \to \delta^*$. *Also, if* δ^* *satisfies (3.1),* $(\delta_n, \delta^*) \in \mathcal{L}(\Upsilon)$ *and* $(\delta^*, \delta_n) \in \mathcal{L}(\Upsilon)$ *for all integers* $n \geq 0$, *then* R *and* C *have a common fixed point in* U.

Proof. Define $\alpha : U \times U \to [0, \infty)$ as $\alpha(w, h) = 1$ if $w \in U$ and $(w, h) \in \mathcal{L}(\Upsilon)$, and $\alpha(w, h) = 0$, otherwise. The graph domination on U yields that $(w, h) \in \mathcal{L}(\Upsilon)$ for all $h \in Rw$ and $(w, h) \in \mathcal{L}(\Upsilon)$ for each $h \in Cw$. So $\alpha(w, h) = 1$ for all $h \in Rw$ and $\alpha(w, h) = 1$ for each $h \in Cw$. Thus $\inf\{\alpha(w, h) : h \in Rw\} = 1$ and $\inf\{\alpha(w, h) : h \in Cw\} = 1$. Hence $\alpha_*(w, Rw) = 1$ and $\alpha_*(w, Cw) = 1$ for any $w \in R$. So R and C are α_*-dominated on U. Furthermore,

$$\tau + Q(H_{\upsilon_1}(Rw, Ch)) \leq Q\left(\max\left\{\begin{array}{c} \upsilon_1(w, h), \upsilon_1(w, Rw), \frac{\upsilon_2(w,Ch)}{2}, \\ \frac{\upsilon_1(w,Rw).\upsilon_1(h,Ch)}{1+\upsilon_1(w,h)} \end{array}\right\}\right),$$

where $w, h \in U \cap \{CR(\delta_n)\}$, $\alpha(w, h) \geq 1$ and $H_{\upsilon_1}(Rw, Ch) > 0$. Also, (ii) is fulfilled. Due to Theorem 2.1, $\{CR(\delta_n)\}$ is a sequence in U and $\{CR(\delta_n)\} \to \delta^* \in U$. Here, $\delta_n, \delta^* \in U$ and either $(\delta_n, \delta^*) \in \mathcal{L}(\Upsilon)$ or $(\delta^*, \delta_n) \in \mathcal{L}(\Upsilon)$ yields that either $\alpha(\delta_n, \delta^*) \geq 1$ or $\alpha(\delta^*, \delta_n) \geq 1$. So all the hypotheses of Theorem 2.1 hold. Thus δ^* is a common fixed point of R and C in U and $\upsilon_1(\delta^*, \delta^*) = 0$. □

4. On single-valued mappings

In this section, some corollaries related to single-valued mappings in modular-like metric space are derived. Let $(£, \upsilon)$ be a modular-like metric space, $\delta_0 \in £$ and $R, C : £ \to £$ be a pair of mappings. Let

$\delta_1 = R\delta_0$, $\delta_2 = C\delta_1$, $\delta_3 = R\delta_2$. Consider a sequence $\{\delta_n\}$ in £ such that $\delta_{2n+1} = R\delta_{2n}$ and $\delta_{2n+2} = C\delta_{2n+1}$, for integers $n \geq 0$. We represent this type of iteration by $\{CR(\delta_n)\}$. $\{CR(\delta_n)\}$ is a sequence in £ generated by δ_0. If $R = C$, then we use $\{£R(\delta_n)\}$ instead of $\{CR(\delta_n)\}$.

Theorem 4.1. *Let $(£, \upsilon)$ be a complete modular-like metric space. Suppose that the regularity of υ and the $\triangle_M$-condition hold. Take $r > 0$, $\delta_0 \in £$, $\alpha : £ \times £ \to [0, \infty)$ and let $R, C : £ \to £$ be α_*-dominated multifunctions on £. Then there are $\tau > 0$ and $Q \in F$ such that*

$$\tau + Q(\upsilon_1(Rt, C\delta)) \leq Q\left(\max\left\{\begin{array}{c} \upsilon_1(t,\delta), \upsilon_1(t,Rt), \frac{\upsilon_2(t,C\delta)}{2}, \\ \frac{\upsilon_1(t,Rt).\upsilon_1(\delta,C\delta)}{1+\upsilon_1(t,\delta)} \end{array}\right\}\right), \tag{4.1}$$

where $t, \delta \in \{CR(\delta_n)\}$, $\alpha(t,\delta) \geq 1$, or $\alpha(\delta, t) \geq 1$, and $\upsilon_1(Rt, C\delta) > 0$. Then $\alpha(\delta_n, \delta_{n+1}) \geq 1$ for all integers $n \geq 0$ and $\{CR(\delta_n)\} \to h \in £$. Also, if h verifies (4.1), $\alpha(\delta_n, h) \geq 1$ and $\alpha(h, \delta_n) \geq 1$ for all integers $n \geq 0$, then R and C admit a common fixed point h in £.

Proof. The proof is similar to the proof of Theorem 2.1. □

Letting $R = C$ in Theorem 4.1, we have the following corollary.

Corollary 4.2. *Let $(£, \upsilon)$ be a complete modular like metric space. Suppose that the regularity of υ and the $\triangle_M$-condition hold. Choose $\delta_0 \in £$, $\alpha : £ \times £ \to [0, \infty)$ and let $R : £ \to £$ be a single-valued function on £. Then there are $\tau > 0$ and $Q \in F$ such that*

$$\tau + Q(\upsilon_1(Rt, R\delta)) \leq Q\left(\max\left\{\begin{array}{c} \upsilon_1(t,\delta), \upsilon_1(t,Rt), \frac{\upsilon_2(t,R\delta)}{2}, \\ \frac{\upsilon_1(t,Rt).\upsilon_1(\delta,R\delta)}{1+\upsilon_1(t,\delta)} \end{array}\right\}\right), \tag{4.2}$$

where $t, \delta \in \{£R(\delta_n)\}$, $\alpha(t,\delta) \geq 1$, or $\alpha(\delta, t) \geq 1$, and $\upsilon_1(Rt, R\delta) > 0$.Then $\alpha(\delta_n, \delta_{n+1}) \geq 1$ for all integers $n \geq 0$ and $\{\delta_n\} \to h \in £$. Also, if (4.2) holds for h, $\alpha(\delta_n, h) \geq 1$ and $\alpha(h, \delta_n) \geq 1$ for all integers $n \geq 0$, then R has a fixed point h.

5. Integral equations

In this section, we apply our work to solve integral equations.

Theorem 5.1. *Let $(£, \upsilon)$ be a complete modular-like metric space. Suppose that the regularity of υ and the $\triangle_M$-condition hold. Take $r > 0$, $\delta_0 \in £$ and let $R, C : £ \to £$ be α_*-dominated multifunctions on £. Then there are $\tau > 0$ and $Q \in F$ such that*

$$\tau + Q(\upsilon_1(Rt, C\delta)) \leq Q\left(\max\left\{\begin{array}{c} \upsilon_1(t,\delta), \upsilon_1(t,Rt), \frac{\upsilon_2(t,C\delta)}{2}, \\ \frac{\upsilon_1(t,Rt).\upsilon_1(\delta,C\delta)}{1+\upsilon_1(t,\delta)} \end{array}\right\}\right), \tag{5.1}$$

where $t, \delta \in \{CR(\delta_n)\}$, and $\upsilon_1(Rt, C\delta) > 0$. Then $\{CR(\delta_n)\} \to f \in £$. Also, if f verifies (5.1), then R and C admit a unique common fixed point f in £.

Let $W = C([0, 1], \mathbb{R}_+)$ be the family of continuous functions defined on $[0, 1]$. The following are two integral equations:

$$u(e) = \int_0^e H(e, f, u(f))df, \tag{5.2}$$

$$c(e) = \int_0^e G(e, f, c(f))df \tag{5.3}$$

for all $e \in [0, 1]$, where $H, G : [0, 1] \times [0, 1] \times W \to \mathbb{R}$. For $\delta \in C([0, 1], \mathbb{R}_+)$, define supremum norm as $\|\delta\|_\eta = \sup_{s\in[0,1]} \{|\delta(s)| e^{-\tau s}\}$, and take $\tau > 0$ arbitrarily. For all $c, w \in C([0, 1], \mathbb{R}_+)$, define

$$\upsilon_1(\delta, w) = \frac{1}{2} \sup_{s\in[0,1]} \{|\delta(s) + w(s)| e^{-\tau s}\} = \frac{1}{2}\|\delta + w\|_\tau.$$

It is clear that $(C([0, 1], \mathbb{R}_+), d_\tau)$ is a complete modular-like metric space. So we have the following result.

Theorem 5.2. *Suppose that*
(i) $H, G : [0, 1] \times [0, 1] \times C([0, 1], \mathbb{R}_+) \to \mathbb{R}$*;*
(ii) Define

$$\begin{aligned}(Ru)(e) &= \int_0^e H(e, f, u(f))df,\\ (C\delta)(e) &= \int_0^e G(e, f, \delta(f))df.\end{aligned}$$

Assume that there is $\tau > 0$ *such that*

$$|H(e, f, u) + G(e, f, \delta)| \leq \frac{\tau M(u, \delta)}{\tau M(u, \delta) + 1}$$

for all $e, f \in [0, 1]$ *and* $u, \delta \in C([0, 1], \mathbb{R}^+)$*, where*

$$M(u, \delta) = \max\left(\frac{1}{2}\left\{\begin{array}{c}\|u + \delta\|_\tau , \|u + Ru\|_\tau ,\\ \frac{\|u+Ru\|_\tau+\|\delta+C\delta\|_\tau}{2},\\ \frac{\|u+Ru\|_\tau.\|\delta+C\delta\|_\tau}{1+\|u+\delta\|_\tau}\end{array}\right\}\right).$$

Then (5.2) and (5.3) possess a unique solution.

Proof. By (ii),

$$\begin{aligned}|Ru + C\delta| &= \int_0^e |H(e, f, u) + G(e, f, \delta)|\, df \leq \int_0^e \frac{\tau M(u, \delta)}{\tau M(u, \delta) + 1} e^{\tau f} df\\ &\leq \frac{\tau M(u, \delta)}{\tau M(u, \delta) + 1} \int_0^e e^{\tau f} df \leq \frac{M(u, \delta)}{\tau M(u, \delta) + 1} e^{\tau e}.\end{aligned}$$

This implies

$$|Ru + C\delta|\, e^{-\tau e} \leq \frac{M(u, \delta)}{\tau M(u, \delta) + 1},$$

$$\|Ru + C\delta\|_\tau \leq \frac{M(u,\delta)}{\tau M(u,\delta)+1},$$
$$\frac{\tau M(u,\delta)+1}{M(u,\delta)} \leq \frac{1}{\|Ru + C\delta\|_\tau},$$
$$\tau + \frac{1}{M(u,\delta)} \leq \frac{1}{\|Ru + C\delta\|_\tau}.$$

Thus

$$\tau - \frac{1}{\|Ru(e) + C\delta(e)\|_\tau} \leq \frac{-1}{M(u,\delta)}.$$

All the conditions of Theorem 5.1 hold for $Q(f) = \frac{-1}{f}$ for $f > 0$ and $\upsilon_1(f,\delta) = \frac{1}{2}\|f + \delta\|_\tau$. Hence both the integral Eqs (5.2) and (5.3) admit a unique common solution. □

6. Conclusions

In this article, we have achieved some new results for a pair of set-valued mappings verifying a generalized rational Wardowski type contraction. Dominated mappings are applied to obtain some fixed point theorems. Applications on integral equations and graph theory are given. Moreover, we investigate our results in a more better new framework. New results in ordered spaces, modular metric space, dislocated metric space, partial metric space, b-metric space and metric space can be obtained as corollaries of our results. One can further extend our results to fuzzy mappings, bipolar fuzzy mappings and fuzzy neutrosophic soft mappings.

Conflict of interest

The authors declare that we have no conflict of interest.

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